A GEOMETRIC APPROACH FOR COMPUTING THE DISTANCE BETWEEN TWO DISKS IN 3-D

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Abstract: We consider the problem of computing the distance between two circular disks in three-dimensional space. A geometric approach will be proposed for solving this problem. The approach is based on rotation transformations, projections, and geometric properties of disks. It will be shown that this problem has closed-form solutions for computing the distance and the optimal points. A fast computational algorithm will be proposed.

Keywords: Computational geometry, circular disks, distance computation, closed-form solutions.

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I. Introduction

The distance problem between two static objects has extensively been studied in the literature. Efficient algorithms have been developed for various types of objects including: line segments [1], boxes [2], polygons [3, 4, 5, and 6], and polytopes [7, 8, 9, 10, and 11]. Other references on the distance and intersection problems can be found in [12 and 13].

In this report, we are interested in computing the distance between two circular disks defined in three-dimensional space with arbitrary position and orientation. The approach for computing the distance depends on the representation used to describe a disk. For example, we can define a disk as a finite number of concentric circles, where each circle is defined by a finite number of points, i.e. discrete disks. In this case, it might be possible to find a polynomial time algorithm for computing the distance between both disks. This method, however, would provide approximate solutions which depend on the number of circles and points used to define a disk. It would require a significant number of iterations to achieve accurate solutions.

In this work, we consider a disk as a continuous area of points, i.e. a set of an infinite number of points, where a point can be represented by polar coordinates, Cartesian coordinates, or by some other equivalent parametric representation. Unlike the objects in the previously mentioned problems, which can be described by linear models, the geometry of a disk is nonlinear by nature. In general, distance problems are nonlinear problems, which can be solved using nonlinear mathematical programming methods [14 and 15]. However, the computation time of these methods may not be suitable for real-time applications. In addition, for objects having well defined shapes like disks [16], circles, spheres etc., the special geometric structure of these objects may be used to reduce the complexity of the distance problem and provide fast computational solutions.

Our approach for the distance problem between two disks is based on using rotation transformations and geometric properties of disks. The rotation transformations simplify the representation of the problem by eliminating some of the constants from the basic formulation of the distance problem. Using the special structure of the reduced representation and the geometric properties of disks, it will be shown that this problem can be reduced to the problem of computing the distance between a disk and a point. We will show that the latter has closed-form solutions. An algorithm for computing the distance between two disks will be proposed. The proposed algorithm (apparently the first in the literature) requires only a few tests to compute the distance and the optimal points.

II. Formulation of the Disks Distance Problem

Let F=(X,Y,Z) be the attached coordinates frame of a circular disk K of radius D, centered at the origin O and lying in the X-Y plane of F. A disk is a collection of points V which can be represented as follows:

$$K = \{V=(V_x,V_y,V_z)^T \mid V_x = r\cos(q),\ V_y = r\sin(q),\ V_z = 0,\ 0 \le r \le D\} \qquad (1)$$

An arbitrary 3D disk is obtained by rotating K and translating its origin with respect to a fixed coordinates frame $F_0 = (X_0,Y_0,Z_0)$. Let R be a 3 x 3 rotation matrix that defines the disk plane, and let $P=(P_x,P_y,P_z)^T$ be its origin with respect to F_0. Let W be the absolute coordinates of a point V on the disk, then K can be defined as follows:

$$K = \{W = R\,V + P \mid V = (V_x,V_y,V_z)^T,\ V_x = r\cos(q),\ V_y = r\sin(q),\ V_z = 0,\ 0 \le r \le D\} \qquad (2)$$

Let K_1 and K_2 be two disks of radius D_1 and D_2, centered at $P_1 = (P_{1x}, P_{1y}, P_{1z})^T$ and $P_2 = (P_{2x}, P_{2y}, P_{2z})^T$ respectively. Let $R_1 = \{R_{1ij}\}$, and $R_2 = \{R_{2ij}\}$, i, j = 1, 2, 3, be the rotation matrices that define the planes of K_1 and K_2 respectively. Determining the distance between K_1 and K_2 is equivalent to solving the following problem:

$$\begin{aligned} &\text{minimize} && \| (R_1 V_1 + P_1) - (R_2 V_2 + P_2) \|^2 \\ &\text{subject to} && \|V_1\| \le D_1,\ \|V_2\| \le D_2 \end{aligned} \qquad (3)$$

where $V_1 = (V_{1x}, V_{1y}, 0)^T$, $V_2 = (V_{2x}, V_{2y}, 0)^T$, and $\| \cdot \|^2$ denotes the L_2-norm. The best way for solving (3) is to find a method that provides closed-form solutions for the distance and the optimal points. In the following, we discuss some of the approaches that we have investigated, and then we develop our geometric approach based on the special properties of the problem under consideration.

Problem (3) is a nonlinear optimization problem where both the objective function and the inequality constraints are nonlinear. The above problem has an interesting property about the location of the optimal points that solve (3). We can show that one of the optimal points that minimize the distance between the two disks must be on the border of K_1 or K_2. This is also true when the disks have some intersection. Therefore, (3) can be reduced to the problem of finding the distance between a disk and a circle. Since the optimal point could be on the border of K_1 or K_2, one has to compute two distances, and the minimum of these two distances will give the optimum. This approach may lead to closed-form solutions, but as we will see later on, the method we propose is more efficient.

Problem (3) can also be viewed as a nonlinear programming problem, where techniques like the KKT (Karush-Khun-Tucker) with Lagragian multipliers [14] may be used. When applied to (3), the KKT produces a system of 10 nonlinear equations in 6 unknowns. Because of the L_2-norm and the convexity of disks, the Hessian of the objective function is positive semi-definite. Therefore, the KKT necessary and sufficient conditions are satisfied for the optimality of solutions. This guaranties that any solution of the nonlinear system is a global optimal solution of (3). However, because of the semidefiniteness of the objective function, the solution may not be unique. But this is not of great importance, since we are looking for any pair of points that solve (3).

Using the preceding property on the location of optimal points, and the system of nonlinear equations produced by KKT conditions, we have tried to combine these equations and to use induction and/or deduction procedures to find out closed-form solutions for the distance problem. With this formulation, one has to solve three problems, where each problem is described by a subset of nonlinear equations. The first two problems are based on the assumption that one of the optimal points is on the border of the first disk, and the other is interior to the second disk, and vice-versa. These two problems are equivalent to the distance problem between a circle and a disk. We can show that the KKT approach provides closed-form solutions for the distance between a disk and a circle. However, we failed to find closed-form solutions for the third problem, where both optimal points are assumed to be on the borders of the disks, i.e. distance problem between two circles. In this case, we found a system of 8 nonlinear equations in 6 unknowns, which does not apparently have a direct closed-form solution.

The approach we propose here is, instead, purely geometric. First, we show that using some rotation transformations the formulation of the distance problem (3) can be reduced to a simpler one. Second, using projections and the particular geometry of disks, we show that this problem can be transformed into a problem of computing the distance between a point and a disk. The latter has closed-form solutions. This geometric approach is developed in Sections III and IV.

III. Geometric Transformations

Our approach for the distance problem consists first of using rotation transformations to reduce (3) to a simpler form. For this, recall that a rotation matrix R is an orthogonal matrix satisfying $R R^T = R^T R = I$, where R^T denotes the transpose of R, and I is a 3 x 3 identity matrix. Let ρ be a three-dimensional vector defined as follows:

$$\rho = R_1 V_1 + P_1 - (R_2 V_2 + P_2) \tag{4}$$

Multiplying both sides of (4) by R_1^T, one obtains after rearrangement:

$$R_1^T \rho = V_1 - R_1^T R_2 V_2 - R_1^T (P_2 - P_1) \tag{5}$$

Let ${}^0P = R_1^T (P_2 - P_1)$, and ${}^0R = R_1^T R_2$, then (5) becomes:

$$R_1^T \rho = V_1 - {}^0R\, V_2 - {}^0P \tag{6}$$

Since $\| R_1^T \rho \|^2 = \| \rho \|^2$, the distance problem (3) becomes:

$$\begin{aligned} &\text{minimize} && \|V_1 - ({}^0R\, V_2 + {}^0P)\|^2 \\ &\text{subject to} && \|V_1\| \leq D_1,\ \|V_2\| \leq D_2 \end{aligned} \tag{7}$$

With the above transformation, (7) is the problem of finding the distance between a disk K_1 lying in the (X_0,Y_0) plane and centered at the origin O_0 of the fixed coordinates frame F_0, and a disk K_2 of center ${}^0P$ and rotation matrix ${}^0R$.

Further reductions can be applied on ${}^0R$ to make the X-axis of the frame attached to disk K_2 parallel to the X_0-axis of the fixed frame F_0. This means that ${}^0R$ can be reduced to a rotation matrix that rotates the plane of K_2 about the X_0-axis only. For this, we shall show that for any given matrix ${}^0R$, there exist three rotation matrices A, B, and R such that:

$$B\, {}^0R\, A^T = R \tag{8}$$

where A, B, and R are of the forms:

$$A = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} \cos(\beta) & -\sin(\beta) & 0 \\ \sin(\beta) & \cos(\beta) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\gamma) & -\sin(\gamma) \\ 0 & \sin(\gamma) & \cos(\gamma) \end{bmatrix}$$

The proof for relation (8) is conducted as follows: let C be a 3 x 3 rotation matrix such that $C = {}^0R\, A^T$. Post-multiply ${}^0R$ by A^T and set the entry C_{31} to zero, we obtain:

$$\begin{aligned} C_{31} &= {}^0R_{31} \cos(\alpha) - {}^0R_{32} \sin(\alpha) \\ &= 0 \end{aligned}$$

where α can be computed as follows:

$$\tan(\alpha) = \frac{{}^0R_{31}}{{}^0R_{32}} \quad \Rightarrow \quad \alpha = \operatorname{atan}\left(\frac{{}^0R_{31}}{{}^0R_{32}}\right) \tag{9}$$

The value of α can be computed in $[0, 2\pi]$ by checking the signs of ${}^0R_{31}$ and ${}^0R_{32}$. Since the disk is assumed to have two identical sides, the solutions $\alpha + \pi$ and $\alpha - \pi$ correspond to the same plane of the disk. Thus, we can restrict α to be in $[0, \pi]$. Note that, if ${}^0R_{32} = 0$, then we can permute column 1 and 2 of ${}^0R$ so that the entry $C_{31} = 0$. In this case, the permutation matrix is obtained by setting $\alpha = \pi/2$ in A. Thus, the resulting matrix C is of the form:

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ 0 & C_{32} & C_{33} \end{bmatrix}$$

Let D be a 3 x 3 rotation matrix such that D = B C. Post-multiply C by B and set the entry D_{21} to zero, we obtain:

$$\begin{aligned} D_{21} &= C_{11}\sin(\beta) + C_{21}\cos(\beta) \\ &= 0 \end{aligned}$$

which implies that:

$$\tan(\beta) = -\frac{C_{21}}{C_{11}} \quad \Rightarrow \quad \beta = -\operatorname{atan}\left(\frac{C_{21}}{C_{11}}\right) \qquad (10)$$

As in the case of α, the solution β can be computed in $[0, \pi]$. If $C_{11} = 0$, then we can permute rows 1 and 2 of C so that the entry $D_{21} = 0$. The resulting matrix D is of the form:

$$D = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ 0 & D_{22} & D_{23} \\ 0 & D_{32} & D_{33} \end{bmatrix}$$

Since D is an orthonormal matrix, then D_{11} must be equal to 1 or -1, and $D_{12} = D_{13} = 0$. If $D_{11} = -1$, then we can pre-multiply D by a matrix I_1 such that $R = I_1 D$, where I_1 is of the form:

$$I_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In this case, we have $R_{11} = 1$. Since D is a rotation matrix, we have $D_{22} = D_{33}$, and $D_{23} = -D_{32}$. Therefore, by setting $D_{22} = \cos(\gamma)$ and $D_{23} = -\sin(\gamma)$, the matrix R will be of the form:

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\gamma) & -\sin(\gamma) \\ 0 & \sin(\gamma) & \cos(\gamma) \end{bmatrix} \qquad (11)$$

where $R = D$, if $D_{11} = 1$, or $R = I_1 D$, if $D_{11} = -1$, and γ is determined in $[0, \pi]$ by checking the signs of D_{22} and D_{23}. Finally, the above transformations reduce ${}^0R$, to a rotation matrix R that rotates the plane of disk K_2 with an angle γ about the X_0-axis of the fixed frame F_0. Relation (8) is proved.

Consider now the objective function $\|V_1 - ({}^0R\ V_2 + {}^0P)\|^2$ of (7). Pre-multiply the expression inside the norm by B and post-multiply ${}^0R$ by $A^T A$, we obtain:

$$\| V_1 - ({}^0R\ V_2 + {}^0P) \|^2 = \| B\ V_1 - B\ {}^0R\ A^T A\ V_2 - B\ {}^0P \|^2$$

since the L_2-norm is invariant under rotation.

Let $M = B\ V_1$, $V = A\ V_2$, and $P = B\ {}^0P$, and substitute R for $B\ {}^0R\ A^T$ in the above equation, we have:

$$\| V_1 - ({}^0R\ V_2 + {}^0P) \|^2 = \| M - (R\ V + P) \|^2$$

where: $M = (r_1 \cos(q_1+ \beta), r_1 \sin(q_1+ \beta), 0)^T$, $V = (r_2 \cos(q_2+ \alpha), r_2 \sin(q_2+ \alpha), 0)^T$, and $P = (P_x, P_y, P_z)^T$. Replacing $q_1+ \beta$ by θ_1 in M, and $q_2+ \alpha$ by θ_2 in V, we have:

$$\begin{aligned} M &= (r_1 \cos(\theta_1), r_1 \sin(\theta_1), 0)^T \qquad , \theta_1 = q_1+ \beta \\ &= (M_x, M_y, 0)^T \end{aligned} \tag{12}$$

and,

$$\begin{aligned} V &= (r_2 \cos(\theta_2), r_2 \sin(\theta_2), 0)^T \qquad , \theta_2 = q_2+ \alpha \\ &= (V_x, V_y, 0)^T \end{aligned}$$

Finally, (7) is reduced to the following equivalent problem:

$$\begin{aligned} &\text{minimize} \quad \| M - (R\ V + P) \|^2 \\ &\text{subject to} \quad \|M\| \leq D_1,\ \|V\| \leq D_2 \end{aligned} \tag{13}$$

where disk K_1 is lying in the (X_0,Y_0) plane and centered at O_0, and disk K_2 is lying in the plane defined by the rotation matrix R and centered at point $P = (P_x,P_y,P_z)^T$. Note that, in the above formulation, the X-axis of the frame attached to K_2 is parallel to the X_0-axis of the fixed coordinates frame F_0, and R is a matrix that rotates K_2 by an angle γ about X_0-axis.

As we can see, the above rotations have no effect on the distance problem. They only change the phases of the optimal points, which can be restituted later by applying the inverse rotation transformations on the obtained solutions. Because of the reduced structure of R, the above formulation provides an optimal representation of the distance problem between two disks. Additional geometric transformations can be applied on the objective function of (13), to produce various possible configurations of the two disks in space. But the solution complexity of the resulting problems will be equivalent to that of (13).

IV. Geometric Properties

In this section, we show that the distance problem between two disks can be reduced to the problem of computing the distance between a disk and a fixed point. For this, we first proceed by presenting some preliminary results for computing the distance between a disk and a point.

Using the reduced formulation (13), we give in Proposition 1 a procedure for computing the distance between disk K_1 and a fixed point in 3D. Proposition 2 gives a similar procedure for computing the distance between disk K_2 and the origin O_0. Both procedures provide closed-form solutions for the problems under consideration.

Proposition 1: Let $W=(W_x,W_y,W_z)^T$ be a point in three-dimensional space (Fig. 1), and let $\|W\|_{xy}$ denote the distance between the origin O_0 and the projection of W on the (X_0,Y_0) plane. Let $d(M,W) = \| M - W \|^2$ denote the distance between a point $M \in K_1$ and W. Then, there exists an optimal point $M^* = (M_x^*, M_y^*, 0)^T$ that minimizes the distance between K_1 and W such that:

$$\min_{M \in K_1} d(M,W) = d(M^*,W)$$

The minimum distance $d(M^*,W)$ and the coordinates of the optimal point M^* depend on the following two cases:

<u>**Case 1:**</u> If $\|W\|_{xy} > D_1$, then $d(M^*,W) = (\|W\|_{xy} - D_1)^2 + W_z^2$, $M_x^* = \frac{D_1 W_x}{\|W\|_{xy}}$ and $M_y^* = \frac{D_1 W_y}{\|W\|_{xy}}$.

<u>**Case 2:**</u> If $0 \le \|W\|_{xy} \le D_1$, then $d(M^*,W) = W_z^2$, $M_x^* = W_x$, and $M_y^* = W_y$.

Proof: From simple geometry, we can see that if $\|W\|_{xy} > D_1$, which implies that W is outside the disk, then the optimal point M^* is the intersection point between the border of K_1 and the line joining the projection of W on the (X_0,Y_0) plane with the origin O_0. The proofs for the other two cases are straightforward. Proposition 1 is proved.

Several remarks are to be considered for the geometric interpretation of Proposition 1 (Fig. 1). First, we can easily see that if $\|W\|_{xy} > D_1$, then W is outside the disk. In this case, the nearest point M^* to W is the point that intersects the boundary of K_1 with the line joining the projection of W and the origin of the disk. Thus, the distance decreases when W gets closer to M^*. This means also that the closer W is to the origin O_0 of the disk, the less the distance between W and the disk is.

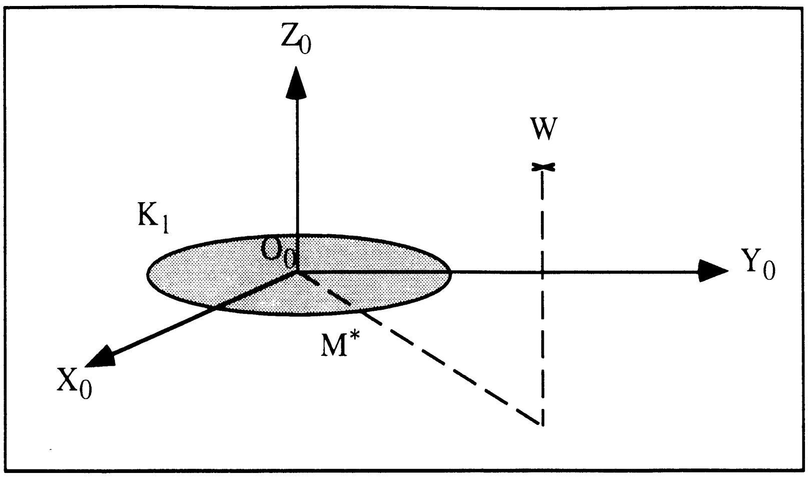

Fig. 1- Minimum distance between a fixed point W and disk K_1.

This remark shows also that, if W belongs to disk K_2, which is assumed here to be completely outside K_1, then the minimum distance between K_2 and K_1 can be obtained by computing the minimum distance between K_2 and the origin of K_1. As we can see from both cases of Proposition 1, the distance $d(M^*, W)$ depends only on the location of W with respect to the origin O_0, and it is independent of the location of the optimum point M^*.

Second, the minimum distance between K_1 and W is either the distance between W and a point on the border of K_1 ($\|W\|_{xy} > D_1$), or the distance between W and the plane of K_1 ($0 \leq \|W\|_{xy} \leq D_1$). Third, a necessary and sufficient condition for the intersection between K_1 and K_2 at W is that W be in the (X_0,Y_0) plane ($W_z = 0$) and its projection is inside the disk K_1 ($0 \leq \|W\|_{xy} \leq D_1$). Fourth, because of the symmetry of a disk, the optimal point M^* may not be unique. Therefore, there may also exist a point W' of K_2 that gives the same distance. However, this is not of great importance, since we are looking for any pair of points of K_1 and K_2 that minimize the distance.

These remarks are valid for any point W in the space, in particular, for the optimal point of K_2 that minimizes the distance between K_1 and K_2. The main result from the above remarks is that the optimal point of K_2 that minimizes the distance between K_1 and K_2 is the point that minimizes the distance between K_2 and the origin O_0 of K_1. This result is true for both cases of Proposition 1. Since the objective function of (13) is symmetric, in the sense that we can move K_2 to the origin of K_1 and vice-versa (by pre-multiplying the objective function by R^T), the above remarks are also valid when K_2 lies in the (X_0,Y_0) plane and centered at the origin O_0, and W is a point of K_1. Therefore, it is necessary to compute also the optimal point of K_1 that minimizes the distance between K_1 and the origin of K_2.

Thus, the distance between the two disks can be determined by computing the nearest point of disk K_2 to the center of K_1, and the nearest point of K_1 to the center of K_2. The minimum of these two distances will give the first optimal point for the distance problem. This optimal point will then be used to determine the second optimal point of the of the other disk. Thus, the distance between K_1 and K_2 can be computed in two steps, where in each step a point-disk distance is to be evaluated. Proposition 2 gives a procedure for computing the distance between K_2 and the origin O_0 of K_1.

Proposition 2: Let $W = R\,V + P$ be a point of K_2, and let $d(O_0,W) = \|R\,V + P\|^2$ denote the distance between W and O_0. Then, there exists an optimal point $W^* = (W_x^*, W_y^*, W_z^*)^T = R\,V^* + P$ that minimizes the distance between K_2 and O_0 such that:

$$\min_{W \in K_2} d(O_0,W) = d(O_0,W^*)$$

Depending on the following two cases, the minimum distance $d(O_0,W^*)$ and the optimal point V^* are:

__Case 1__: If $\|R^T P\|_{xy} > D_2$, then $d(O_0,W^*) = (\|R^T P\|_{xy} - D_2)^2 + (R^T P)_z^2$,

$$V_x^* = -\frac{D_2\,(R^T P)_x}{\|R^T P\|_{xy}}, \text{ and } V_y^* = -\frac{D_2\,(R^T P)_y}{\|R^T P\|_{xy}}.$$

__Case 2__: If $0 \le \|R^T P\|_{xy} \le D_2$, then $d(O_0,W^*) = (R^T P)_z^2$, $V_x^* = -(R^T P)_x$, and $V_y^* = -(R^T P)_y$.

where $\|R^T P\|_{xy}$ is the distance between the projection of point $(R^T P)$ on the (X_0,Y_0) plane and the origin O_0, and $(R^T P)_x$, $(R^T P)_y$, and $(R^T P)_z$ are the absolute coordinates of $(R^T P)$.

Proof: The distance between a point $W = R\,V + P$ and the origin O_0 is as follows:

$$d(O_0,W) = \|W\|^2$$

$$= \|R\,V + P\|^2$$

Since the L_2-norm is invariant under rotation, we have:

$$d(O_0,W) = \|V + R^T P\|^2$$

where K_2 is lying in the (X_0,Y_0) plane and centered at the origin O_0, and $R^T P$ is a fixed point in three-dimensional space (Fig. 2).

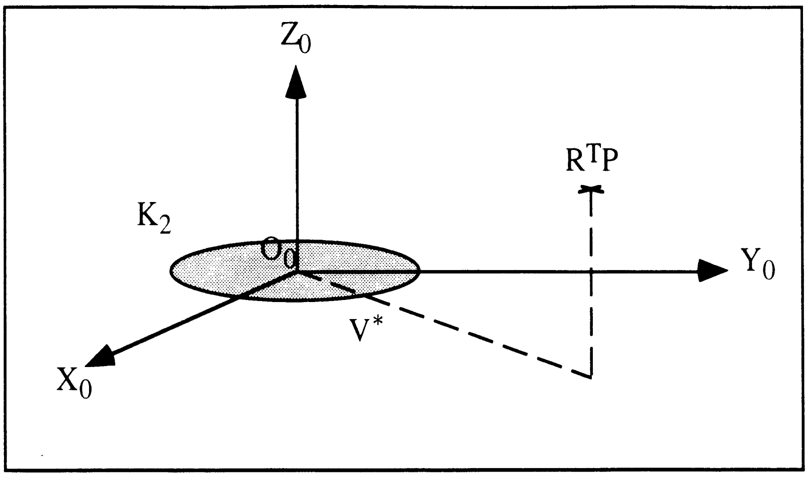

Fig. 2- Computing the minimum distance between K_2 at center P and the origin O_0. The problem is equivalent to computing the minimum distance between K_2 at center O_0 and the point $R^T P$.

Thus, we can use Proposition 1 to compute the minimum of d and the optimal point V^*. For this, let $\|R^T P\|_{xy}$ denote the distance between O_0 and the projection of $R^T P$ on the (X_0,Y_0) plane. From Proposition 1, there exists an optimum point $V^* = (V_x^*, V_y^*, 0)^T \in K_2$ that minimizes the distance between K_2 and $R^T P$, that is: If $\|R^T P\|_{xy} > D_2$, then $d(O_0,W^*) = (\|R^T P\|_{xy} - D_2)^2 + (R^T P)_z^2$, and

$$V_x^* = - \frac{D_2 (R^T P)_x}{\|R^T P\|_{xy}}, \quad V_y^* = - \frac{D_2 (R^T P)_y}{\|R^T P\|_{xy}}$$

If $0 \le \|R^T P\|_{xy} \le D_2$, then $d(O_0,W^*) = (R^T P)_z^2$, $V_x^* = - (R^T P)_x$, and $V_y^* = - (R^T P)_y$. For both cases, W^* can be computed from the relation $W^* = R V^* + P$. Proposition 2 is proved.

Consider now problem (13), that is:

$$\begin{array}{ll} \text{minimize} & \| M - (R V + P) \|^2 \\ \text{subject to} & \|M\| \le D_1, \ \|V\| \le D_2 \end{array}$$

Let $d(O_0, W_0^*)$ be the distance between O_0 and W_0^*, where W_0^* is the optimal point of K_2 that minimizes the distance between O_0 and K_2. Let $d(M_p^*, P)$ be the distance between M_p^* and P, where M_p^* is the optimal point of K_1 that minimizes the distance between K_1 and P. Then we have the following Lemma:

Lemma 1: The distance between K_1 and K_2 is as follows:

$$\min_{W \in K_2} \min_{M \in K_1} d(M,W) = \begin{cases} d(M_0^*,W_0^*) & \text{if } d(O_0,W_0^*) < d(M_p^*, P) \\ d(M_p^*,W_p^*) & \text{if } d(O_0,W_0^*) \geq d(M_p^*, P) \end{cases}$$

where W_0^* is the nearest point of K_2 to the origin O_0, M_0^* is the nearest point of K_1 to W_0^*, M_p^* is the nearest point of K_1 to the center P of K_2, and W_p^* is the nearest point of K_2 to M_p^*. The distances $d(M_0^*, W_0^*)$ and $d(M_p^*, W_p^*)$, and the corresponding optimal points can be computed from Propositions 1 and 2.

Proof: Let W_0^* be the nearest point of K_2 to the origin O_0 of K_1 (see Fig. 3). W_0^* can be computed from Proposition 2, we have:

$$\min_{W \in K_2} d(O_0,W) = d(O_0,W_0^*)$$

Since W_0^* is known, we can compute the nearest point M_0^* of K_1 to W_0^*. Proposition 1 gives:

$$\min_{M \in K_1} d(M,W_0^*) = d(M_0^*,W_0^*)$$

Similarly, let M_p^* be the nearest point of K_1 to the origin P of K_2, then from Proposition 1, we have:

$$\min_{M \in K_1} d(M,P) = d(M_p^*, P)$$

The nearest point W_p^* of K_2 to M_p^* can be computed by using Proposition 2 (see fig. 4), we obtain:

$$\min_{W \in K_2} d(M_p^*,W) = d(M_p^*, W_p^*)$$

Assume first that $d(O_0,W_0^*) < d(M_p^*,P)$. This means that W_0^* is closer to O_0 than M_p^* to P is. Since W_0^* is the closest point of K_2 to the origin O_0 of K_1, then W_0^* is the closest point to disk K_1. This is true because the minimum distance between a disk and a fixed point depends only on the location of the fixed point in space, or equivalently, the minimum distance depends on the distance between the point and the center of the disk. Consequently, from Proposition 1, there exists an optimal point M_0^* of K_1, that is the closest point to W_0^*. It follows that the points M_0^* and W_0^* are the optimal points that minimize the distance between both disks. This implies that

$$d(M_0^*,W_0^*) < d(M_p^*,W_p^*)$$

and consequently we have:

$$\min_{W \in K_2} \; \min_{M \in K_1} d(M,W) = d(M_0^*,W_0^*)$$

For the case where $d(O_0,W_0^*) \geq d(M_p^*,P)$, symmetric arguments can be used to show that M_p^* and W_p^* are the optimal points that satisfy the minimum distance between K_1 and K_2. Lemma 1 is proved.

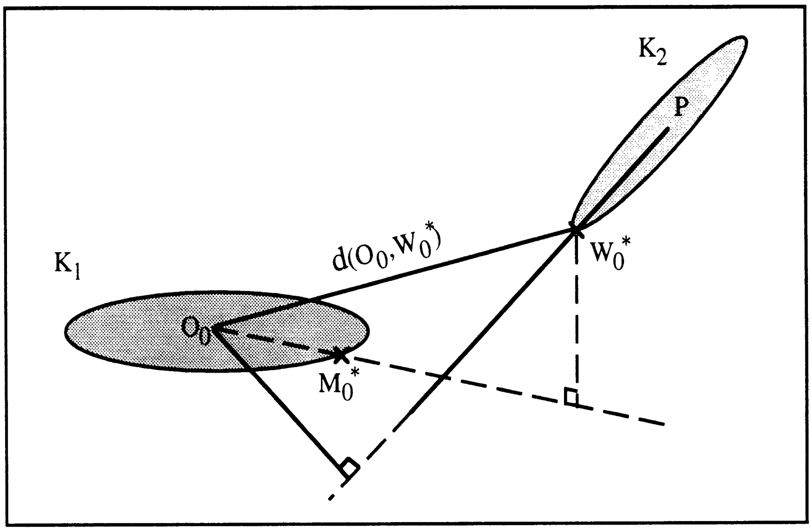

Fig. 3- Determining the optimal points M_0^* and W_0^*.

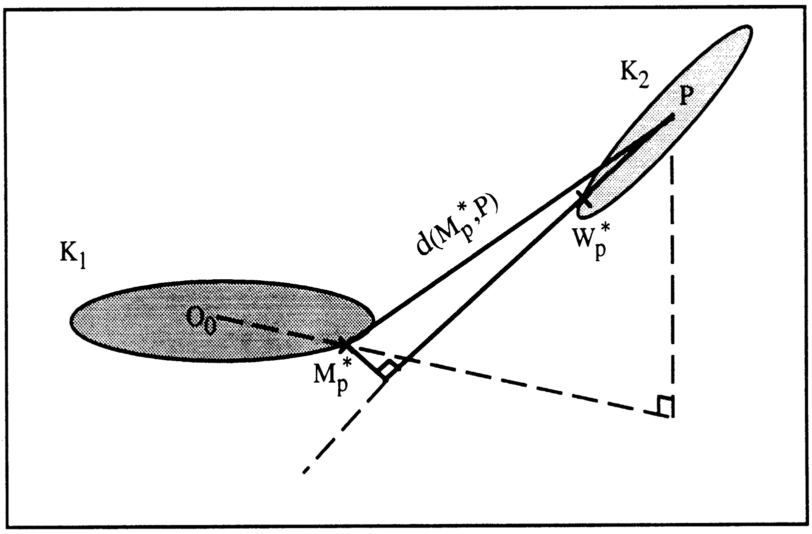

Fig. 4- Determining the optimal points M_p^* and W_p^*.

V. Algorithm:

The first two steps of the algorithm consist of computing the distances $d(O_0, W_0^*)$ and $d(M_p^*, P)$. The distance between the two disks and the coordinates of the optimal points are computed in Steps 3 or 4.

Step 1: Using Proposition 2, compute the distance between the origin O_0 and disk K_2. This gives $d(O_0, W_0^*)$ and the coordinates of the optimal point $V_0^* \in K_2$. Compute W_0^* from the relation $W_0^* = R\, V_0^* + P$.

Step 2: Using Proposition 1, compute the distance between disk K_1 and the center P of K_2. This gives $d(M_p^*, P)$ and the coordinates of the optimal point $M_p^* \in K_1$.

Step 3: If $d(O_0, W_0^*) < d(M_p^*, P)$, then use Proposition 1 for determining the distance between W_0^* and disk K_1. This gives the distance between the two disks $d(M_0^*, W_0^*)$, and the coordinates of the optimal point M_0^* of K_1. End.

Step 4: If $d(O_0, W_0^*) \geq d(M_p^*, P)$, then use Proposition 2 for computing the distance between M_p^* and disk K_2, that is, compute the minimum of $\|V - R^T(M_p^* - P)\|^2$. This gives the distance between the two disks $d(M_p^*, W_p^*)$, and the coordinates of the optimal point W_p^* of K_2. End.

Note that, Steps 1 and 2 require two comparison tests to find the distances $d(O_0, W_0^*)$ and $d(M_p^*, P)$. Steps 3 or 4 require 1 test for comparing both distances and two other tests to find the minimum distance between the two disks. Therefore, the algorithm terminates by finding the closed-form solutions of the distance and the optimal points in at most 5 comparison tests. Appendix A gives the detailed computational procedures for the algorithm. For all cases, we obtain simple expressions for the closed-form solutions, which can easily be computed with only a few operations.

V. Conclusion:

In this technical report, we presented an efficient algorithm for computing the distance between two circular disks in 3D. Some rotation transformations were used in Section 3 to simplify the formulation of the initial problem. These transformations may not be required for developing the algorithm. One may apply directly the proposed algorithm on the formulation given in (7). It was shown that the distance problem is reducible to a two-step problem, where each step involves the computation of the distance between a disk and a fixed point in space. Closed-form solutions were obtained for the point-disk distance problem. An algorithm that requires about 5 comparison tests was proposed. The algorithm provides closed-form solutions for the distance between two disks and the optimal points.

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Appendix A

Detailed Computations of the distance algorithm between two disks:

Step 1: Computing $d(O_0,W_0^*)$ from Proposition 2:

$$d(O_0,W_0^*) = \begin{cases} (\|R^T P\|_{xy} - D_1)^2 + (R^T P)_z^2 & \text{if } \|R^T P\|_{xy} > D_1 \\ (R^T P)_z^2 & \text{if } 0 \leq \|R^T P\|_{xy} \leq D_1 \end{cases}$$

Step 2: Computing $d(M_p^*, P)$ from Proposition 1:

$$d(M_p^*, P) = \begin{cases} (\|P\|_{xy} - D_2)^2 + P_z^2 & \text{if } \|P\|_{xy} > D_2 \\ P_z^2 & \text{if } 0 \leq \|P\|_{xy} \leq D_2 \end{cases}$$

At this stage of the algorithm, we assume that both distances $d(O_0,W_0^*)$ and $d(M_p^*, P)$ are known.

Step 3: If $d(O_0,W_0^*) < d(M_p^*, P)$, then from Proposition 2, the coordinates of the optimal point V_0^* $\in K_2$ are as follows:

Case 1: If $\|R^T P\|_{xy} > D_2$, then $V_{0x}^* = -\dfrac{D_2 (R^T P)_x}{\|R^T P\|_{xy}}$, $V_{0y}^* = -\dfrac{D_2 (R^T P)_y}{\|R^T P\|_{xy}}$, and $W_0^* = R$ $V_0^* + P$. Using proposition 1, the minimum distance $d(M_0^*, W_0^*)$ and the coordinates of the optimal point $M_0^* \in K_1$ depend on the following cases:

Case 1.1: If $\|W_0^*\|_{xy} > D_1$, then $d(M_0^*, W_0^*) = (\|W_0^*\|_{xy} - D_1)^2 + (W_0^*)_z^2$

$$M_{0x}^* = \frac{D_1 W_{0x}^*}{\|W_0^*\|_{xy}}, \text{ and } M_{0y}^* = \frac{D_1 W_{0y}^*}{\|W_0^*\|_{xy}}.$$

Case 1.2: If $0 \leq \|W_0^*\|_{xy} \leq D_1$, then $d(M_0^*, W_0^*) = (W_0^*)_z^2$, $M_{0x}^* = W_{0x}^*$, and $M_{0x}^* = W_{0y}^*$.

Case 2: If $0 \leq \|R^T P\|_{xy} \leq D_2$, then $V_{0x}^* = -(R^T P)_x$, $V_{0y}^* = -(R^T P)_y$, the distance $d(M_0^*, W_0^*)$ and the coordinates of M_0^* are:

Case 2.1: If $\|W_0^*\|_{xy} > D_1$, then $d(M_0^*, W_0^*) = (\|W_0^*\|_{xy} - D_1)^2 + (W_0^*)_z^2$

$$M^*_{0x} = \frac{D_1 W^*_{0x}}{\|W^*_0\|_{xy}}, \text{ and } M^*_{0y} = \frac{D_1 W^*_{0y}}{\|W^*_0\|_{xy}}.$$

<u>**Case 2.2**</u>: If $0 \le \|W^*_0\|_{xy} \le D_1$, then $d(M^*_0, W^*_0) = (W^*_0)^2_z$, $M^*_{0x} = W^*_{0x}$, and $M^*_{0y} = W^*_{0y}$.

<u>**Step 4**</u>: If $d(O_0, W^*_0) \ge d(M^*_p, P)$, then using Proposition 2, the coordinates of the optimal point $M^*_p \in K_1$ depend on the following cases:

<u>**Case 1**</u>: If $\|P\|_{xy} > D_1$, then $M^*_{px} = \frac{D_1 P_x}{\|P\|_{xy}}$, $M^*_{py} = \frac{D_1 P_y}{\|P\|_{xy}}$, and $d(M^*_p, W^*_p)$ is as follows:

<u>**Case 1.1**</u>: If $\|R^T(M^*_p - P)\|_{xy} > D_2$, then $d(M^*_p, W^*_p) = (\|R^T(M^*_p - P)\|_{xy} - D_2)^2 + (R^T(M^*_p - P))^2_z$ and the coordinates of the optimal point V^*_p are:

$$V^*_{px} = \frac{D_2 (R^T(M^*_p - P))_x}{\|R^T(M^*_p - P)\|_{xy}}, \text{ and } V^*_{py} = \frac{D_2 (R^T(M^*_p - P))_y}{\|R^T(M^*_p - P)\|_{xy}}.$$

<u>**Case 1.2**</u>: If $0 \le \|R^T(M^*_p - P)\|_{xy} \le D_2$, then $d(M^*_p, W^*_p) = (R^T(M^*_p - P))^2_z$ $M^*_{px} = (R^T(M^*_p - P))_x$, and $M^*_{py} = (R^T(M^*_p - P))_y$.

<u>**Case 2**</u>: If $0 \le \|P\|_{xy} \le D_1$, then $M^*_{px} = P_x$, $M^*_{py} = P_y$, and $d(M^*_p, W^*_p)$ is as follows:

<u>**Case 2.1**</u>: If $\|R^T(M^*_p - P)\|_{xy} > D_2$, then $d(M^*_p, W^*_p) = (\|R^T(M^*_p - P)\|_{xy} - D_1)^2 + (R^T(M^*_p - P))_z^2$ and the coordinates of the optimal point V^*_p are:

$$V^*_{px} = \frac{D_2 (R^T(M^*_p - P))_x}{\|R^T(M^*_p - P)\|_{xy}}, \text{ and } V^*_{py} = \frac{D_2 (R^T(M^*_p - P))_y}{\|R^T(M^*_p - P)\|_{xy}}.$$

<u>**Case 2.2**</u>: If $0 \le \|R^T(M^*_p - P)\|_{xy} \le D_2$, then $d(M^*_p, W^*_p) = (R^T(M^*_p - P))^2_z$ $V^*_{px} = (R^T(M^*_p - P))_x$, and $V^*_{py} = (R^T(M^*_p - P))_y$.

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