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# CIVIL ENGINEERING.

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## *Minimum Material in a Trussed Girder. (A Thesis.)* By WILLIAM BOUTON, (University of Michigan.)

*I propose to find the inclination of the braces (or ties), and the relation between the length and depth of a truss with horizontal chords, so as to demand the least amount of material in the construction, when the transverse section of the several parts are made proportional to the strains to which they are subjected.*

An exact analytical solution, I consider, from the nature of the case, impossible; but I desire to make an exact solution upon the following hypotheses, which may be more or less nearly realized in practice:

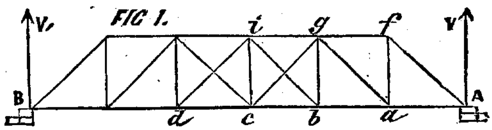
1st. Assume that the weight of the bridge is a uniform permanent load.

2d. Consider the surcharge—or moving load—as uniform as far as it extends.

3d. Assume that the joints at the ends of the braces or ties are perfectly flexible.

These assumed, and I can use the formulas which have been developed in the course of lectures on constructions, and which have been reproduced in this *Journal* by Professor Wood in his articles on "The General Problem of Trussed Girders," to which I shall hereafter refer. I will first consider the panel system, in which the inclined pieces are braces, and the vertical ones ties; in other words, I will consider a Howe's truss,—letting the load be on the lower chord.

FIG 1.




*Notation.*

Let $L = AB =$ the span,

$D = af =$ the depth,

$l = Aa = ab = bc =$ the length of each bay,

$N =$ number of bays, $\therefore l = L \div N$,

$W_1 =$ the weight of the frame,

$w_1 = W_1 \div N =$ the weight of one panel,

$p =$ the weight of the surcharge on one panel,

$r =$ the weight of the surcharge per foot of length,

$c = \frac{w_1}{p}$, and $w_1 = \frac{crL}{N}, p = \frac{rL}{N}$,

$$\therefore p + w_1 = (1 + c)\frac{rL}{N}, \quad . \quad . \quad (1.)$$

$W =$ the total load $= (p + w_1)N$,

$\theta = gaf$ the inclination of a brace.

We have

$$D = \frac{l}{\tan\theta} = \frac{L}{N\tan\theta}, \quad . \quad . \quad (2.)$$

The length of a brace $= D\sec\theta = \frac{L\sec\theta}{N\tan\theta}$, . . (3.)

We will take up the problem by parts, and suppose, first, that the braces only are varied in size, and that the load is uniform over the whole length.

The strain on the $n$th brace is, (see the second of equations (45) p. 381, vol. xlviii,) $\frac{1}{2}(N - 2n + 1)(p + w_1)\sec\theta$, . . (4.)

By making $n = \frac{1}{2}N$ in the 3d of the same equations, we find for the strain at the middle of the lower chord,

$$\frac{1}{8}\frac{N^2 l(p + w_1)}{D} = \frac{WL}{8D} = \frac{WN}{8}, \quad . \quad . \quad (5,)$$

and, by the hypothesis, the chord is to be made large enough throughout to resist this strain.

Making $\sec\theta = 1$ in (4), and we have

$$\frac{1}{2}(N - 2n + 1)(p + w_1), \quad . \quad . \quad (6,)$$

for the strain on the $n$th vertical tie; but as they are to be of uniform size, they must all be as large as the first one, the strain on which is found by making $n = 1$ in (6). Hence the strain is

$$\frac{1}{2}(N - 1)(p + w_1), \quad . \quad . \quad (7.)$$

Let $K =$ the section of a brace,

$S =$ the strain on a unit of section.

Hence (4) and (1) give

$$\text{K} = \frac{(\text{N}-2n+1)\,(1+\text{c})}{\text{S}}\,\frac{r\,\text{L}}{2\text{N}}\sec\theta, \quad . \qquad (8.)$$

In (8) make $n = 1, 2, 3$, &c., . . . . . to $\frac{1}{2}$N, and we shall form a series, the successive terms of which give the sections of the braces in their order. Summing the series, and we have for the total sections of the braces on half the bridge,

$$\Sigma\text{K} = \frac{\text{N}\,(1+c)\,r\,\text{L}}{8\text{s}}\sec\theta,$$

which, multiplied by the length of a brace, equation (3), gives for the volume of half the braces,

$$\frac{(1+c)\,r\,\text{L}^2}{8\text{s}}, \frac{1}{\cos\theta\,\sin\theta}, \quad . \quad . \qquad (9,)$$

which is a minimum, for $\theta = 45°$. Hence, the minimum amount of material in the braces is

$$\frac{(1+c)\,r\,\text{L}^2}{2\text{s}} = \frac{p+w_1}{2\text{s}} = \frac{\text{NL}}{\text{N}} \quad . \qquad (10.)$$

This shows that the total amount of material in the braces is the same as if the truss were a king-post with braces inclined at an angle of 45 degrees. The aggregate length of the ties will be the same whether the truss be composed of two or many braces; hence, equation (7) shows that the amount of material in the truss is least when N is least, or equal 2; hence the king-post truss requires *less* material for the ties and braces. Equation (5) shows that the strain on the chord varies with N; hence, the section of the chord is least in the king-post truss. When the inclination of the braces is varied, we shall still find an advantage in the king-post truss, but not as great as in the present case.

Next, *suppose that both the ties and braces are proportioned according to the strains to which they are subjected*, the load being uniform throughout.

By summing the series found by equation (6), and multiplying by the length of a tie, equation (2), we find for the volume of half the ties,

$$\frac{(1+c)\,r\,\text{L}^2}{8\text{s}}, \frac{1}{\tan\theta} \quad . \quad . \qquad (11.)$$

Twice the sum of (9) and (11) gives the total volume of the ties and braces, which is

$$\frac{(1+c)\,r\,\text{L}^2}{4\text{s}}\left(\frac{1}{\cos\theta\,\sin\theta} + \frac{1}{\tan\theta}\right) \quad . \qquad (12,)$$

which is a minimum for $\theta = 54°\ 45'$.

In the following table, the second column shows the relative amount of material in the braces only, for different inclinations; and the fourth column shows the relative amount in both ties and braces:

| $\theta$ | $\frac{1}{\cos\theta\sin\theta}$ | $\theta$ | $\frac{1}{\cos\theta\sin\theta}+\frac{1}{\tan\theta}$ |
|---|---|---|---|
| 45° | 100·00 | 54° 45′ | 100· |
| 40° or 50° | 101·50 | 50° | 101·5 |
| 35° or 55° | 106·40 | 45° | 106·4 |
| 30° or 60° | 115·50 | 40° | 113·7 |
| 25° or 65° | 130·55 | 35° | 125·7 |
| 20° or 70° | 195·85 | 30° | 142·9 |
| 15° or 75° | 200·00 | 25° | 168·1 |
| 10° or 80° | 368·09 | 20° | 207·1 |
| 5° or 85° | 515·00 | 10° | 407·3 |
| 0° or 90° | $\infty$ | | |
| | | 60° | 102·1 |
| | | 65° | 108·8 |

If we make the strains those due to a moving load, the braces (in the form of counter braces) will pass beyond the centre, and the *coefficient* of $\frac{1}{\tan\theta}$ will be less than that of $\frac{1}{\cos\theta\sin\theta}$; hence, will reduce the value of $\theta$ for the function a minimum toward 45°, a limit it can never reach. Indeed, it cannot reach 48°; for using the first two terms of equation (20) and make $w_1 = 0$, and $n_1 = \text{N}-1$, and solve for a minimum, and we find

$$(\text{N}^2-1)\tan^2\theta = \frac{31}{24}\text{N}^2 - \frac{28}{24}.$$

If N be very large, we have approximately

$$\tan^2\theta = \frac{31}{24} \therefore \theta = 48° \, 40',$$

which is the limiting value, but the practical value is nearer 54°.

*General Problem.*

The amount of material in the chords diminishes as the depth of the truss increases—N and L being constant—but this diminishes $\theta$, hence increases the material in the ties and braces when $\theta$ is less than 54° 45′. Hence, there must be some value of $\theta$ which will give the amount of material required for *all* the parts a minimum.

When the load is uniform throughout and permanent, we know that no counter braces are necessary, and that all the braces incline towards the centre. But the surcharge which moves on or off is uniform over only a portion of the length, and the most unfavorable case is when it extends from one end of the bridge to the point considered. This case necessitates counter-braces. To find the number of braces which, in this case, must incline from either end, we place the second of (46) page 381, vol. xlviii, equal zero and solve for $n$. Call the rational value $n_0$, and we have

$$n_0 = \text{N}\frac{w}{p} + \text{N} + \tfrac{1}{2} - \sqrt{\text{N}^2\left(\frac{w^2}{p^2}+\frac{w}{p}\right)+\tfrac{1}{4}}, \quad . \qquad (13.)$$

Then will the *number* of braces required be the integer part of $n_0$; or, if $n_0$ is an integer, it will be $n_0-1$. Call it $n_1$.

From the second of (46) above referred to, we have for the section of the $n$th brace,

$$\text{K} = \Big[(\text{N}-n)\,(\text{N}-n+1)p + (\text{N}-2n+1)\,\text{N}\,w_1\Big]\frac{\sec\theta}{2\,\text{S}\,\text{N}},$$

which, multiplied by the length, equation (3), gives for the volume of the $n$th brace,

$$\Big[(\text{N}-n)\,(\text{N}-n+1)\,p + (\text{N}-2n+1)\,\text{N}\,w_1\Big]\frac{\text{L}}{2\text{S}\,\text{N}^2\sin\theta\cos\theta} \quad (14.)$$

To find the volume of half the braces, form a series by making $\text{N}=1$, 2, 3, &c., to $n_1$, and sum the series. Two series will be formed, one the coefficients of $p$; the other of $w_1$.

1. The coefficient of $p$.

By the method of the orders of differences, the sum of the series,

$$\text{S} = n\,a + \frac{n\,(n-1)}{2}d_1 + \frac{n\,(n-1)\,(n-2)}{2\cdot 3}d_2 + \&c.,$$

in which $n =$ the number of terms,

$a =$ the first term,

$d_1, d_2$, &c., $=$ the first terms of the several orders of differences.

The terms of the series are

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | $\text{N}\,(\text{N}-1)$, | $(\text{N}-1)$, | $(\text{N}-2)$, | $(\text{N}-2)$, | $(\text{N}-3)$, | $(\text{N}-3)$, | $(\text{N}-4)$. |
| 1st dif. | $-2\text{N}+2$ | | $-2\text{N}+4$ | | $-2n+6$. | | |
| 2d dif. | | $+2$ | | $+2$ | | $+2$. | |
| 3d dif. | | | $0$ | | $0$. | | |

Substituting in the formula, and we have

$\Big[n_1\,\text{N}\,(\text{N}-n_1) + \tfrac{1}{3}\,n_1\,(n_1^2-1\Big]$ to be multiplied by $\dfrac{p\,\text{L}}{2\text{S}\,\text{N}^2\sin\theta\cos\theta}$.

2. Coefficient of $w_1$.

This becomes negative when we pass the centre, and each term cancels a corresponding one before we arrive at the centre; hence we need sum only from $n=1$ to $n=\text{N}-n_1$.

The terms of these series are

| | | | | | | |
|---|---|---|---|---|---|---|
| | $\text{N}\,(\text{N}-1)$ | | $\text{N}\,(\text{N}-3)$ | | $\text{N}\,(\text{N}-5)$ | $\text{N}\,(\text{N}-7)$. |
| 1st dif. | | $-2\text{N}$ | | $-2\text{N}$ | $-2\text{N}$. | |
| 2d dif. | | | $0$ | $0$, | | |

and the sum for $(\text{N}-n_1)$ terms is

$n_1\,\text{N}\,(\text{N}-n_1)$ which is to be multiplied by $\dfrac{\text{N}\,w_1\,\text{L}}{2\text{S}\,\text{N}^2\sin\theta\cos\theta}$.

Adding the results, and using equation (1), and we have for *the total volume in half the braces*,

$$\Big[\text{N}\,n_1\,(\text{N}-n_1)\,(1+e) + \tfrac{1}{3}\,n_1\,(n_1^2-1)\Big]\frac{r\,\text{L}^2}{2\text{S}\,\text{N}^3}\,\frac{1}{\sin\theta\cos\theta} \quad (15.)$$

*Material in the Vertical Bars.*

The equation for the strain is the same as for the braces, except that sec $\theta = 1$. The length is $\frac{L}{N \tan \theta}$. It will make a difference, whether N be odd or even. If N be odd, there will be $\frac{1}{2}(N-1)$ ties in half the bridge; and hence, $\frac{1}{2}(N-1)$ terms in the series. Hence the volume is

$$\left[\tfrac{1}{4} N (N^2-1)(1+c) + \tfrac{1}{6}(N-1)\left(\tfrac{1}{4}(N-1)^2-1\right)\right] \frac{r L^2}{2S N^3}, \frac{1}{\tan \theta}.^* \quad (16.)$$

If N be even, there will be a middle tie, and after summing the series for $n_1 = \frac{1}{2}N$ terms, we must deduct one-half the volume of the middle tie. The volume of the middle tie is found by making $n = \frac{1}{2}N$, and sec $\theta = 1$ in the first term of (14), and using tan θ for $\frac{1}{\sin}$ θ. Hence the half volume is

$$\tfrac{1}{8} N (N+2) \frac{r L^2}{2S N^3} \frac{1}{\tan \theta}.$$

Summing $\frac{1}{2}N$ terms of the series, and deducting the preceding expression, and we have for the volume of one-half the ties,

$$\left[\tfrac{1}{4} N^3 (1+c) + \tfrac{1}{6} N (\tfrac{1}{4} N^2 - 1) - \tfrac{1}{8} N (N+2)\right] \frac{r L^2}{2S N}, \frac{1}{\tan \theta} \quad (17.)$$

*Material in the Chords.*

The equation of the strain is (see the third of (45), page 381, vol. xlviii),

$$\tfrac{1}{2}(N-n)\, n\, (p + w_1) \tan \theta, \quad (18.)$$

For the half span the limits for $n$ are, when N is even, $n = 1$ to $n = \frac{1}{2}(N-2)$ in the upper chord, and $n = \frac{1}{2}N$ in the lower. If N is odd, the limits are $n = 1$ to $n = \frac{1}{2}(N-1)$ in both chords, by omitting half the central bay in the lower, and including it in the upper chord, since their sections are the same. In either case the summation will be

$$\tfrac{1}{6} N (N^2-1)(1+c) \frac{r L^2}{2S N^2} \quad . \quad . \quad (19.)$$

Adding (15), (16), or (17) as the case may be, and (19), and we have for half the volume of the truss,

$$\left\{ \begin{array}{l} \quad \left[ N n_1 (N - n_1)(1+c) + \tfrac{1}{3} n_1 (n_1^2 - 1) \right] \\ + \left\{ \begin{array}{l} (\text{for } n \text{ odd}) \tfrac{1}{4} N (N^2-1)(1+c) + \tfrac{1}{6}(N-1) \\ (\text{for } n \text{ even}) \tfrac{1}{4} N^3 (1+c) + \tfrac{1}{6} N (\tfrac{1}{4} N^2 - 1) \end{array} \right\} \\ + \quad \tfrac{1}{6} N^2 (N^2-1)(1+c) \tan \theta \end{array} \right\}$$

$$\left\{ \begin{array}{l} \dfrac{1}{\cos \theta \sin \theta} \\ \left. \begin{array}{l} \left(\tfrac{1}{4}(N-1)^2 - 1\right) \\ -\tfrac{1}{8} N (N+2)(1+c) \end{array} \right\} \dfrac{1}{\tan \theta} \end{array} \right\} \frac{r L^2}{2S N^3} \quad . \quad (20.)$$

* We have made no distinction between the resistance to tension and compression.

which is to be a minimum.

$$\therefore \left\{ \begin{array}{l} \left[ \text{N}\, n_1(\text{N}-n_1)(1+c) + \frac{1}{3} n_1(n_1{}^2-1) \right] \\ -\left\{ \begin{array}{l} (n \text{ odd})\ \frac{1}{4}\text{N}(\text{N}^2-1)(1+c) + \frac{1}{6}(\text{N}-1) \\ (n \text{ even})\ \frac{1}{4}\text{N}^3(1+c) + \frac{1}{6}\text{N}(\frac{1}{4}\text{N}^2-1) \end{array} \right\} \\ \quad + \frac{1}{8}\text{N}^2(\text{N}^2-1)(1+c)\tan^2\theta \end{array} \right\}$$

$$\left\{ \begin{array}{l} (\tan^2\theta - 1) \\ \left\{ \begin{array}{l} \left(\frac{1}{4}(\text{N}-1)^2 - 1\right) \\ -\frac{1}{8}\text{N}(\text{N}+2)(1+c) \end{array} \right\} \end{array} \right\} = 0, \quad \cdot \quad \cdot \quad (21).$$

We see that the inclination is dependent upon the number of bays. To solve (21) assume N and $c$, and find $n_1$ by means of (13), and then we may find $\theta$. In this way the following table has been computed.

TABLE *showing the inclination of the braces and the ratio of the length to the depth of a* HOWE'S TRUSS, *for a minimum amount of material, load on the lower chord:*

| Value of N. | Uniform and permanent load. $p=0$ and $c=\infty$ | | $p=w_1 \therefore c=1$ | | $p=2w_1 \therefore c=\frac{1}{2}$ | | | $w=0 \therefore c=0$ | |
|---|---|---|---|---|---|---|---|---|---|
| | $\theta$ | $\frac{\text{L}}{\text{D}}$ | $\theta$ | $\frac{\text{L}}{\text{D}}$ | $\theta$ | $\frac{\text{L}}{\text{D}}$ | Rel. value of $\frac{r\text{L}^2}{\text{S}}$ Eq. (20.) | $\theta$ | $\frac{\text{L}}{\text{D}}$ |
| | ° ′ | | ° ′ | | ° ′ | | | ° ′ | |
| 2 | 40 53 | 1·73 | 40 53 | 1·73 | 40 53 | 1·73 | 100 | 40 53 | 1·73 |
| 3 | 39 14 | 2·45 | 39 36 | 2·48 | 39 43 | 2·49 | 135 | 39 55 | 2·51 |
| 4 | 35 47 | 2·88 | 36 25 | 2·95 | 36 35 | 2·96 | | 37 23 | 3·05 |
| 5 | 34 12 | 3·40 | 34 51 | 3·48 | 35 5 | 3·51 | | 35 39 | 3·59 |
| 6 | 31 51 | 3·72 | 32 29 | 3·81 | 32 30 | 3·82 | | 33 21 | 3·95 |
| 7 | 30 26 | 4·11 | 31 28 | 4·28 | 31 46 | 4·34 | | 32 28 | 4·45 |
| 8 | 29 1 | 4·46 | 29 48 | 4·58 | 30 13 | 4·66 | 217 | 31 7 | 4·82 |
| 9 | 28 7 | 4·81 | 28 54 | 4·97 | 29 11 | 5·03 | | 30 0 | 5·20 |
| 10 | 26 48 | 5·05 | 27 41 | 5·24 | 27 48 | 5·27 | | 28 6 | 5·34 |
| 12 | 25 2 | 5·60 | 25 54 | 5·82 | 26 13 | 5·91 | | 27 28 | 6·23 |
| 15 | 23 5 | 6·39 | 23 52 | 6·64 | 24 10 | 6·73 | | 25 0 | 6·99 |
| 20 | 20 31 | 7·49 | 21 21 | 7·81 | 21 39 | 7·93 | 274 | 22 32 | 8·36 |
| 25 | 18 36 | 8·41 | 18 41 | 8·45 | 18 57 | 8·58 | | 20 22 | 9·23 |
| 30 | 17 21 | 9·37 | 17 51 | 9·66 | 18 8 | 9·84 | | 18 56 | 10·30 |
| 40 | 15 23 | 11·00 | 15 40 | 11·22 | 15 54 | 13·39 | | 16 41 | 12·00 |
| 50 | 13 34 | 12·87 | 14 7 | 12·57 | 14 23 | 18·52 | 506 | 15 3 | 13·44 |
| 60 | 12 26 | 13·23 | | | | | | 13 49 | 14·80 |

For $p = 2w_1$, I have added a third column of the relative value of the coefficients of $\frac{r\,L^2}{S}$ in equation (20). By this it appears that to support a *moving* surcharge, $r$ L, plus a *uniform permanent* load $\frac{1}{2}\,r$ L, will require 5·06 times the material when the span is divided into fifty bays that it would if divided into two.

Although we find that the material does not increase with the number of bays so fast as it would if $\theta$ remained constant, still the increase is so great as to forbid our increasing N until there is no danger of rupture by flexure in the braces. But N must be increased until there is no danger of breaking in the loaded chord from transverse strain.

Let P be the greatest concentrated load which can come on any bay *e.g.*, on the driving wheels of a locomotive, $\therefore \frac{1}{4}\,\mathrm{P}l = \frac{1}{6}\,\mathrm{R}\,bd^2$, but $bd = \mathrm{K} \;\therefore\; \mathrm{K} = \frac{3}{2}\,\frac{\mathrm{P}l}{\mathrm{R}b}$, which section must be less than that required to satisfy the formula $\frac{1}{2}(\mathrm{N}-n)\,n\,(p + w_1)$ tang $\theta = \mathrm{S\,K}$. See equation (18.) This may be effected by making $l$, the length of the bay, small, or making $d$ larger in proportion to $b$ throughout the span. On the whole, I think it would be better to increase the section of the loaded chord so as to prevent danger from transverse strain, and not alter the disposition of the other parts, for such an alteration would increase the material in all the other parts, to avoid a danger which only requires the increase of one part.

## *Triangular Systems.*

When the load is on the upper chord we find from equations (43) and (44), page 379, vol. xlviii, that

$$\left[(\mathrm{N}-n)^2 p + (\mathrm{N}^2 - 2n\mathrm{N})\,w_1\right]\frac{\sec\theta}{2\,\mathrm{N}} = \text{the maximum strain on the pair}$$

of braces at the end of the $n$th bay.

$(\mathrm{N}-n)\,n\,(p + w_1)\tan\theta$ = strain on $n$th bay of upper chord.

$\left[\mathrm{N}\left(n-\frac{1}{2}\right)-(n-1)\,n\right](p + w_1)\tan\theta$ = strain on $n$th bay of lower chord.

Sum these equations for all integral values of $n$ from the end to the centre of the truss.

1°. For the braces WHEN N IS EVEN and the first bar is a brace,

$n = 0$ to $\frac{1}{2}(\mathrm{N}-2)$ for the main braces,
$n = 1$ to $\frac{1}{2}\,\mathrm{N}$ for the ties.

Expanding the equation for the braces and ties with these limits, and we get for the total section of half the braces and ties when N is even,

$$\Sigma\mathrm{K} = \frac{7\,\mathrm{N}^3 p + 6\,\mathrm{N}^3 w_1 + 2\,\mathrm{N}p}{12} \cdot \frac{\sec\theta}{2\,\mathrm{N\,S}}$$

The length of a brace is $\frac{\mathrm{L}\sec\theta}{2\,\mathrm{N}\tan\theta}$ ; hence, Volume of half the braces and ties,

$$= \frac{(7+6c)\,N^2+12}{4} \cdot \frac{r\,L^2}{12\,N^2\,s} \cdot \frac{1}{\cos\theta \sin\theta}$$

WHEN N IS ODD.

The limits for the main braces are, $n = 0$ and $n = \frac{1}{2}(N-1)$.
for the ties $n = 1$ and $n = \frac{1}{2}(N-1)$.

$$\therefore \text{ the volume} = \frac{(7+6c)\,N^2+5+6c}{4} \cdot \frac{r\,L^2}{12\,N^2 s} \cdot \frac{1}{\cos\theta \sin\theta}$$

*The Chords.*

*For* N *even* the limits are $n = 1$ and $n = \frac{1}{2}N$, by subtracting $\frac{1}{2}(N-\frac{1}{2}N)\frac{1}{2}N = \frac{1}{8}N^2$ for half the central bay in the upper chord.

$$\therefore \text{ the volume} = (2\,N^3 + N)(1+c)\frac{r\,L^2}{2\,N^2 s} \tan\theta.$$

*For* N *odd*, the limits are $n = 1$ and $n = \frac{1}{2}(N-1)$ for the upper chord, and $n = 1$ and $n = \frac{1}{2}(N+1)$ for the lower chord, minus half the central bay of the lower chord, viz: $\frac{1}{8}(N+1)$,

$$\therefore \text{ volume} = (2N^3 + N)(1+c)\frac{r\,L^2}{2s\,N^2} \tan\theta, \text{ the same result as before.}$$

Hence the total material in half the truss is

$$\left[\left\{\begin{array}{l} \frac{1}{4}\Big((7+6c)\,N^2+2\Big)(\text{for } N \text{ even}) \\ \frac{1}{4}\Big((7+6c)\,N^2+5+6c\Big)\text{for } N \text{ odd}) \end{array}\right\}\frac{1}{\cos\theta\sin\theta} + (2\,N^3+N)(1+c)\tan\theta\right]\frac{r\,L^2}{12\,s\,N^2}, \quad . \quad . \quad . \quad (22.)$$

which for a minimum gives

$$\begin{array}{l} N \text{ even}, \\ N \text{ odd}, \end{array}\left\{\begin{array}{l} (7+6c)N+2 \\ (7+6c)\,N^2+5+6c \end{array}\right\}(\tan^2\theta - 1) + (8\,N^3+4\,N)(1+c)\tan^2\theta = 0, \quad . \quad . \quad . \quad (23.)$$

It will be observed that I have considered the compressive and tensive resistances equal; otherwise the final equation would be slightly modified.

From equation (23) and the equation $\frac{L}{2\,N\tan\theta} = D$, I have made the following table:

| Values of | | | |
|---|---|---|---|
| N | $c$ | $\theta$ | $\frac{L}{D}$ |
| | | ° ′ | |
| 6 | $\frac{1}{2}$ | 19 11 | 4·17 |
| 9 | $\frac{1}{2}$ | 16 14 | 5·03 |
| 40 | $\frac{1}{2}$ | 8 8 | 11·43 |
| 50 | 0 | 7 28 | 13·11 |
| 12 | 0 | 14 37 | 6·25 |

I have chosen these values of N and $c$ wholly at random; and it will be perceived that the ratio of L to D corresponds very nearly with the results for the panel system for the same value of N.

*Approximation.*

When N is large we may drop all the terms below $N^2$, and we shall have

$$(7 + 6\,c)\,(\tan^2\theta - 1) + 8\,N\,(1 + c)\,\tan^2\theta = 0. \qquad (24.)$$

In this, when $N = 9$ and $c = \frac{1}{2}$, the formula differs from the exact one, less than a minute in the value of $\theta$. For larger values of N the difference will be inappreciable, as shown from the following table:

| N = | $c$ = | $\theta$ | $\frac{L}{D}$ |
|---|---|---|---|
| | | ° ′ | |
| 12 | 0 | 14 37 | 6·25 |
| 40 | $\frac{1}{2}$ | 8 8 | 11·43 |
| 50 | 0 | 7 28 | 13·11 |

This approximation may be made for the panel system to get the relation of L to D, since the relation is nearly the same in both systems. The peculiar value of the factor $n_1$ in the panel system forbade such an approximation in the general equation.

In like manner, we may determine the relation between the length and depth, and the proper inclination of the braces, in any truss in which the strains may be expressed by a continuous function.

The relation of $p$ to $w_1$ in any practical case cannot be accurately known beforehand, but as all possible inclinations are found from the equation between that resulting when $p = 0$, and that when $w_1 = 0$, and as these values do not differ more than two degrees, a very close approximation may be very readily arrived at. The function varies slowly about its minimum so that a *degree* even will make little difference.

We should diminish the number of bays as much as possible consistently with the liability to rupture by transverse strains and flexure by too great length, and having fixed upon N, determine the relation of $p$ to $w_t$ as nearly as possible, and then make $\frac{L}{D}$ correspond to the equation for minimum material.

It will be seen that the results which I have obtained for the minimum material, all approximate the value of $\theta$ required for the minimum depression of a single triangle composed of bars of uniform section, and sustaining a weight at its apex; that is, they are nearer it, (30°,) than they would be if inclined so that the material in the ties and braces above should be a minimum. Let me suggest to the patient reader that a triangle sustaining compression in both its braces, is not in the same condition as one sustaining compression in one and extension in the other, as in the case of all bridges. Let me also inquire what effect the compression in the upper chord has upon the rigidity of the whole? Is 30° then an approximation to the inclination for the greatest rigidity in a trussed girder?

### *Multiple Systems or Lattice Trusses.*

In adhering to the simple systems I have shown that the minimum material in the *whole* is obtained by increasing that in the braces above the minimum amount, and so increasing the depth of the truss. Cannot both advantages be combined by retaining the depth of the truss while increasing the inclination of the braces by passing them across two or more panels? This course, while it increases the length of the braces, so much reduces their section as to require less material if we neglect flexure. But both these modifications leave them much more liable to flexure; moreover, the strains cannot be rigidly analyzed, so that much more will have to be added for safety, as well as to prevent the increased liability to flexure, so that we should have little gain. This criticism does not apply to Whipple's truss, or to any other in which the inclined bars are ties. In them I think the multiple system must be advantageous, although it cannot be rigidly calculated.

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## *On Uniform Stress in Girder Work; illustrated by reference to two bridges recently built.* By Mr. CALLCOTT REILLY, Assoc. Inst. Civ. Eng.

From Newton's London Journal of Arts, June, 1865.

This communication was suggested by a previous discussion at the Institution, when Mr. Phipps (M. Inst., C.E.,) condemned the trough-shaped section commonly adopted for the top and bottom members of truss girders, because the intensity per square unit of the stress upon any vertical cross section was necessarily variable when the connexion of the vertical web with the trough was made in the usual manner. In the construction of the iron work of the two bridges under consideration, attention was invited chiefly to those details which were de-