ON THE HELMHOLTZ EQUATION FOR AN ACOUSTICALLY RIGID SCATTERER

by

Ergun Ar

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INTRODUCTION

The classic three-dimensional scalar scattering problem consists of determining a function ϕ^s exterior to a smooth finite boundary B which is a solution of a scalar Helmholtz equation, satisfies a Dirichlet or Neumann boundary condition on B, and obeys a radiation condition at infinity, i.e.,

$$(\nabla^2+k^2)\phi^s = 0 \tag{1}$$

$$\phi^s = -\phi^i \quad \text{or} \quad \frac{\partial\phi^s}{\partial n} = -\frac{\partial\phi^i}{\partial n} \quad \text{on B ,} \tag{2}$$

$$\lim_{r\to\infty} r\left(\frac{\partial}{\partial r}\phi^s - ik\phi^s\right) = 0 \quad , \tag{3}$$

where ϕ^i is the incident field which is known everywhere including the boundary B.

The study of the relation between this problem and potential problems (boundary value problems for the Laplace's equation, $\nabla^2\phi=0$) goes back to Lord Rayleigh[(33)]. The general problem is one of generating solutions of the Helmholtz equation (vector or scalar), which satisfy prescribed conditions on a given boundary in terms of solutions of Laplace's equation. Physically, this amounts to an attempt to infer the manner in which an obstacle perturbs the field due to a source of wave motion from a knowledge of how the same object perturbs a stationary (non-oscillatory) field, e.g. , determining an electromagnetic field from an electrostatic field. The advantage of such a procedure derives from the fact that stationary fields are physically simpler than wave phenomena and associated mathematical problems, though often still formidable, are always more easily handled.

Interest in this problem has gained new momentum in recent years (see Bouwkamp(5), Noble(31), Kleinman(21) for an extensive bibliography). The major drawback in most of the methods heretofore proposed is their intrinsic dependence on a particular geometry. That is, the techniques result from the exploitation of the geometric properties of the surface on which the boundary conditions are specified. For those shapes where the Helmholtz equation is separable, of course, the low frequency expansion may always be obtained from the series solution provided sufficient knowledge of the special functions involved is available.

Most low frequency techniques, however, have as their starting point the formulation of scattering problems as integral equations using the Helmholtz representation of the solution in terms of its properties on the boundary and the free space Green's function; i.e.,

$$\phi^s(p) = \frac{1}{4\pi} \int\!\!\int_B \left\{ \phi^s(p_B) \frac{\partial}{\partial n} u(p, p_B) - u(p, p_B) \frac{\partial}{\partial n} \phi^s(p_B) \right\} dB \tag{4}$$

where

$$u = -\frac{e^{ikR(p, p_B)}}{R(p, p_B)} ,$$

the integration is carried out over the entire scattering surface B, the normal here is taken out of B, p is the general field point, and p_B a point on B whose coordinates are the integration variables, and R is the distance between them. This formulation is also vital to the proof of existence of solutions for a general boundary by Weyl(50), Müller(30), and Leis(23). (The investigation of the solutions for the scattering problems with the help of integral equations originated by the works of Rothe(35), Sternberg(41), and Kupradse(22)). Werner(45-49) also provides different existence proofs for acoustical as well as electromagnetic scattering problems. More will be said on the existence question in the Conclusion. Noble(31) shows how the integral formulation (4) may be used to obtain a representation of the solution of a scattering problem for a general boundary as a perturbation of the solution of the corresponding potential problem. Each term in the low frequency expansion is the solution of an integral equation which differs only in its inhomogeneous part from term to term. However, this formulation does not yield an explicit representation for successive terms in general except as the formal inverse.

Long sought has been the development of a systematic procedure which will generate the solution of the Helmholtz equation, satisfying a particular boundary condition, from the solution of Laplace's equation which satisfies the same boundary condition. This goal has been achieved in a limited sense by Kleinman[21] for the Dirichlet problem on which the present work is based.

In Chapter I an integral equation for the scattered field is derived, whose kernel is the potential Neumann Green's function for the surface instead of the free space Green's function for the Helmholtz equation. Despite the fact that the integral operates over all space and the surface and it is really an integro-differential operator, it is still possible to solve the equation for the wave numbers k (assumed complex) with sufficiently small modulus. This is done in Chapter II. Also, the relation between the low frequency expansion and the Neumann-Liouville series for the solution in indicated. In Chapter III the procedure is applied to an acoustically hard sphere. Since the exact solution in this case is known , a check (for the first three terms in the low frequency expansion) is provided.

CHAPTER I

THE NEUMANN PROBLEM

1.1 A Representation Theorem

Let B denote the boundary of a smooth closed bounded surface in E^3 and V is the volume exterior to B. Erect a spherical polar coordinate system with origin interior to B and denote by p a point (r, θ, ϕ) in V and by p_B a point (r_B, θ_B, ϕ_B) on B. The distance between any two points p, $p_1 \in \bar{V} = V+B$ will be denoted by $R(p, p_1)$ and is defined as

$$R(p, p_1) = \left[r^2 + r_1^2 - 2rr_1\left(\cos\theta\cos\theta_1 + \sin\theta\sin\theta_1\cos(\phi - \phi_1)\right)\right]^{1/2} \tag{1.1}$$

Furthermore, let $c = \max r_B$ so that B is contained in a sphere of radius c and assume that the normal to B is directed inward (out of V).

Definition 1.1

A function f(p) of the coordinates of p is said to satisfy a Hölder condition at (or is Hölder continuous at) p_o if there are three positive constants, A, B, and α such that

$$\left| f(p) - f(p_o)\right| \leqslant A \cdot R^{\alpha}(p, p_o)$$

for all points p for which $R(p, p_o) \leqslant B$.

If there is a region G in which f(p) satisfies a Hölder condition at every point, with the same A, B, and α, f(p) is said to be uniformly Hölder continuous.

Definition 1.2

We shall define a surface B to be smooth (or regular) if (a) it can be described by an equation

$$r_B = g(\theta, \phi) \tag{1.2}$$

where g is a continuously differentiable function of θ and ϕ, and (b) $\hat{n}\cdot\hat{r}_B$ (where $\hat{n}$ is the unit normal, $\hat{r}_B$ is the unit radius vector) is uniformly Hölder continuous on B.

Definition 1.3

A real valued function $f: V \rightarrow E^1$ is defined to be regular (in the sense of Kellogg) at infinity if

$$rf = O(1), \quad r^2 \frac{\partial f}{\partial r} = O(1) \qquad \text{as} \qquad r \rightarrow \infty \tag{1.3}$$

uniformly in θ and ϕ.

A complex valued function is regular if both real and imaginary parts are regular.

Definition 1.4

The Neumann potential Green's function for the surface B, the existence and uniqueness of which is proven by Kellogg[20], is defined to be a function $G_o(p, p_1)$ of two points and may be written in the form

$$G_o(p, p_1) = -\frac{1}{4\pi R(p, p_1)} + u_o(p, p_1), \quad p, p_1 \in \overline{V} \tag{1.4}$$

where $u_o(p, p_1)$ has no singularities in $\overline{V}$ and

(a) $\nabla^2 u_o(p, p_1) = 0, \quad p, p_1 \in V$

(b) $\frac{\partial}{\partial n} G_o(p_B, p_1) = 0$ [This notation is used repeatedly and has the following meaning. Let ∇ be the gradient operating on coordinates of p and $\hat{n}$ the unit normal on B directed out of V. Then define

$$\nabla G_o(p_B, p_1) = \nabla G_o(p, p_1)\Big|_{p \in B} \quad \text{and} \quad \frac{\partial}{\partial n} G_o(p_B, p_1) = \hat{n} \cdot \nabla G_o(p_B, p_1).]$$

(c) u_o is regular at infinity. (1.5)

In terms of this Green's function we may state an integral representation of functions regular at infinity. This is contained in

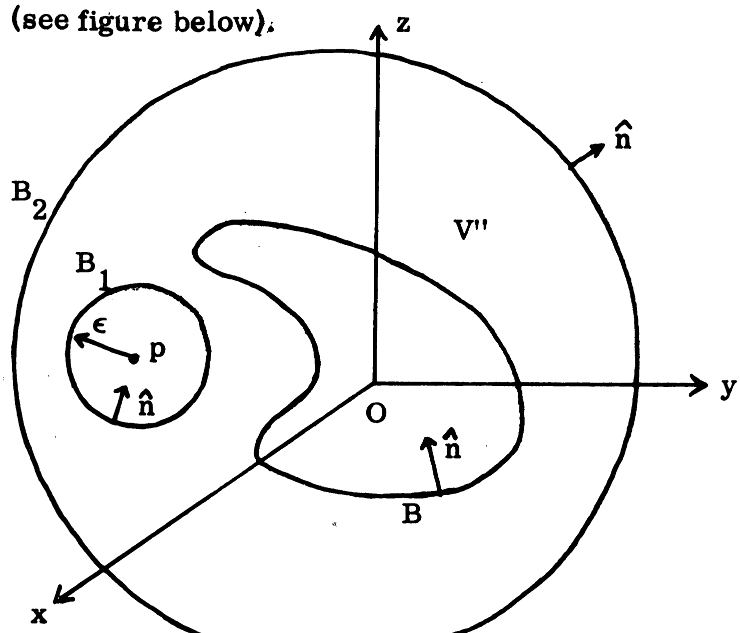
Theorem 1.1. If $\omega: \overline{V} \to E^1$ is a function which is twice differentiable in V and regular at infinity, then

$$\omega(p) = \int_V G_o(p, p_1)\nabla^2\omega(p_1)dv_1 - \int_B G_o(p, p_B)\frac{\partial}{\partial n}\omega(p_B)d\sigma_B \quad (1.6)$$

where $G_o(p, p_1)$ is the Neumann potential Green's function, dv_1 is the volume element and ∇^2 is the Laplacian both expressed in coordinates (r_1, θ_1, ϕ_1), $d\sigma_B$ is the surface element and $\partial/\partial n$ the inward normal derivative (out of V) both expressed in coordinates (r_B, θ_B, ϕ_B).

Proof.

Let $\Gamma = B + B_1 + B_2$, where B is the surface of the body assumed to be regular in the sense of the Definition 1.2, B_1 is the surface of a small sphere, with radius $\epsilon > 0$, with the center at the point p, B_2 is the surface of a large sphere containing B and B_1. Further we erect a rectangular Cartesian coordinate system with origin inside B, and let V'' denote the volume bounded by B, B_1 and B_2 (see figure below).



The functions ω, G_o and the surface Γ are sufficiently regular so that we can apply the Green's second identity; thus,

$$\int_{V''} (G_o \nabla^2 \omega - \omega \nabla^2 G_o) dv = \int_{\Gamma} (G_o \frac{\partial \omega}{\partial n} - \omega \frac{\partial G_o}{\partial n}) d\sigma \; . \qquad (1.7)$$

Here we note in passing that the usual formulation of an integral equation for a wave function involves the application of the above identity to the free space Green's function $e^{\pm ikR}/R$ (the sign ambiguity is removed with a particular choice of harmonic time factor) and the other field ω scattered by B. The integral over B_2 is then shown to vanish by virtue of the radiation condition and the integral over B_1 evaluates the scattered field. The volume integral vanishes since both functions are chosen to be solutions of the homogeneous Helmholtz equation yielding the well-known result

$$\omega(x, y, z) = \frac{1}{4\pi} \int_B \left[\frac{e^{\pm ikR}}{R} \frac{\partial \omega}{\partial n} - \omega \frac{\partial}{\partial n} \left(\frac{e^{\pm ikR}}{R} \right) \right] d\sigma \quad . \qquad (1.8)$$

Here, however, we wish to employ, not the Green's function for the Helmholtz equation but the Green's function for the potential (Laplace) equation G_o, given by (Definition 1.4)

$$G_o(p, p_1) = -\frac{1}{4\pi R(p, p_1)} + u_o(p, p_1) \; .$$

Substituting this into the Eq. (1.7) we obtain

$$\int_{V''} (-\frac{1}{4\pi R} + u_o) \nabla^2 \omega \, dv = \int_{\Gamma = B + B_1 + B_2} \left[(-\frac{1}{4\pi R} + u_o) \frac{\partial \omega}{\partial n} - \omega \frac{\partial}{\partial n} (-\frac{1}{4\pi R} + u_o) \right] d\sigma \; . \qquad (1.9)$$

<u>Integration over the surface B_1</u> :

Let $\gamma(p^*)$ denote the mean value for some $p^* \epsilon B_1$, of $u_o \frac{\partial \omega}{\partial n} - \omega \frac{\partial u_o}{\partial n}$ on B_1. Then

$$\int_{B_1} (u_o \frac{\partial \omega}{\partial n} - \omega \frac{\partial u_o}{\partial n}) d\sigma = \gamma(p^*) : 4\pi\epsilon^2 . \qquad (1.10)$$

Thus, as the radius $\epsilon \rightarrow 0$ this integral vanishes.

Also, on B_1 $\frac{\partial}{\partial n} = -\frac{\partial}{\partial R}$, and for the spherical coordinates (R, θ, ϕ) with the the center at (x, y, z), we have

$$\lim_{R \rightarrow 0} \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \, R^2 \sin\theta \Big[\frac{1}{4\pi R} \frac{\partial}{\partial R} \omega(x+R\cos\phi\sin\theta, y+R\sin\phi\sin\theta, z+R\cos\theta)$$

$$-\omega(x+R\sin\theta\cos\phi, y+R\sin\theta\sin\phi, z+R\cos\theta)\frac{\partial}{\partial R}(\frac{1}{4\pi R})\Big] = \lim_{R\rightarrow 0} \frac{1}{4\pi} \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \sin\theta \, \cdot$$

$$\cdot \Big[R\frac{\partial \omega}{\partial R} + \omega(x+R\sin\theta\cos\phi, y+R\sin\theta\sin\phi, z+R\cos\theta)\Big] = \omega(x, y, z) . \qquad (1.11)$$

Combining (1.10) and (1.11),

$$\int_{B_1} (G_o \frac{\partial \omega}{\partial n} - \omega \frac{\partial G_o}{\partial n}) d\sigma = \omega(x, y, z) \qquad (1.12)$$

Let $V' = \lim_{\epsilon \rightarrow 0} V''$. Incorporating (1.12) into (1.9) and observing that

$\left.\frac{\partial G_o}{\partial n}\right|_B = 0$, (1.9) reduces to

$$\omega(x,y,z) = \int_{V'} G_o \nabla^2 \omega dv - \int_B G_o \frac{\partial\omega}{\partial n} d\sigma - \int_{B_2} \left(G_o \frac{\partial\omega}{\partial n} - \omega \frac{\partial G_o}{\partial n}\right) d\sigma . \tag{1.13}$$

Integration over B_2:

On B_2, $\left.\frac{\partial}{\partial n}\right|_{B_2} = \frac{\partial}{\partial r_{B_2}}$.

$$\lim_{r_{B_2}\to\infty} \left| \int_{B_2} \left[G_o(p_{B_2}, p) \frac{\partial}{\partial n} \omega(p_{B_2}) - \omega(p_{B_2}) \frac{\partial}{\partial n} G_o(p_{B_2}, p) \right] d\sigma \right|$$

$$= \lim_{r_{B_2}\to\infty} \left| \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \; r_{B_2}^2 \sin\theta \left[G_o(p_{B_2}, p) \frac{\partial}{\partial r_{B_2}} \omega(p_{B_2}) - \omega(p_{B_2}) \frac{\partial}{\partial r_{B_2}} G_o(p_{B_2}, p) \right] \right|$$

$$\leqslant \lim_{r_{B_2}\to\infty} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \left\{ \left| r_{B_2}^2 G_o(p_{B_2}, p) \frac{\partial}{\partial r_{B_2}} \omega(p_{B_2}) \right| + \left| r_B^2 \omega(p_{B_2}) \frac{\partial}{\partial r_{B_2}} G_o(p_{B_2}, p) \right| \right\}$$

$$= \lim_{r_{B_2}\to\infty} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \left\{ \frac{1}{r_{B_2}} \left| r_{B_2} G_o \right| \left| r_{B_2}^2 \frac{\partial}{\partial r_{B_2}} \omega(p_{B_2}) \right| + \frac{1}{r_{B_2}} \left| r_{B_2} \omega(p_{B_2}) \right| \left| r_{B_2}^2 \frac{\partial G_o}{\partial r_{B_2}} \right| \right\} = 0, \tag{1.14}$$

since G_o and ω are both regular in the sense of Kellogg[20] (Definition 1.3).

Since $V = \lim_{r_{B_2} \to \infty} V'$, with (1.14), (1.13) becomes

$$\omega(x, y, z) = \int_V G_o(p, p_1)\nabla^2 \omega(p_1)\, dv_1 - \int_B G_o(p, p_B)\frac{\partial\omega(p_B)}{\partial n}\, d\sigma_B ,$$

proving the theorem.

1.2 A Representation of Wave Functions

A function $u: \overline{V} \to E^1$ is a scalar wave function for the volume V if

(a) u(p) is twice continuously differentiable in $\overline{V}$ (with the understanding that if $p \in B$ the limit is taken from the exterior, V),

(b) $(\nabla^2 + k^2)u(p) = 0, \quad p \in \overline{V}$, (1.15)

(c) $r\left(\frac{\partial u}{\partial r} - iku\right) = o(1)$, as $r \to \infty$, uniformly in θ and ϕ .

Other statements of the radiation condition are possible [Wilcox[53]] but this form, as given originally by Sommerfeld[39] is quite adequate for our purposes; it may be stronger than necessary but it does what we want it to do, namely, characterize radiating solutions of the Helmholtz equation.

We wish to employ Theorem 1.1 to represent scalar wave functions, but they are not regular at infinity in the sense of Kellogg[20]. In order to modify them we employ a well-known expansion theorem given with varying restriction by Atkinson[2], Sommerfeld[40], Barrar and Kay[4] and most generally by Wilcox[51].

Theorem 1.2. If u is a scalar wave function for the volume V, then

$$u = \frac{e^{ikr}}{r} \sum_{n=0}^{\infty} \frac{f_n(\theta, \phi)}{r^n} , \qquad (1.16)$$

where the series converges absolutely and uniformly for $r \geqslant c+\epsilon$, $\epsilon > 0$ and $c = \max r_B$. Furthermore, the series may be differentiated term by term with respect to r, θ and ϕ any number of times and the resulting series all converge absolutely and uniformly.

It follows from this expansion theorem that if u is a scalar wave function, then

$$\tilde{u}(p) = e^{-ikr} u(p) \tag{1.17}$$

is regular and satisfies the hypothesis of Theorem 1.1 .

Lemma 1.1 . The function $\tilde{u}(p)$ defined by (1.17) satisfies

$$\nabla^2 \tilde{u} = -\frac{2ik}{r}\frac{\partial}{\partial r}(r\tilde{u}) \quad . \tag{1.18}$$

Proof.

Since $(\nabla^2 + k^2)u = 0$, it follows that $(\nabla^2+k^2)e^{ikr}\tilde{u} = 0$.

In spherical coordinates

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \tag{1.19}$$

Since

$$\frac{\partial}{\partial r}(e^{ikr}\tilde{u}) = e^{ikr}\left\{ik\tilde{u} + \frac{\partial\tilde{u}}{\partial r}\right\} , \tag{1.20}$$

and

$$\frac{\partial^2}{\partial r^2}(e^{ikr}\tilde{u}) = e^{ikr}\left\{-k^2\tilde{u} + 2ik\frac{\partial\tilde{u}}{\partial r} + \frac{\partial^2\tilde{u}}{\partial r^2}\right\} , \tag{1.21}$$

clearly

$$\frac{\partial^2}{\partial r^2}(e^{ikr}\tilde{u}) + \frac{2}{r}\frac{\partial}{\partial r}(e^{ikr}\tilde{u}) = e^{ikr}\left\{-k^2\tilde{u} + \frac{2ik}{r}\frac{\partial}{\partial r}(r\tilde{u}) + \frac{\partial^2\tilde{u}}{\partial r^2} + \frac{2}{r}\frac{\partial\tilde{u}}{\partial r}\right\} . \tag{1.22}$$

Hence, with $(\nabla^2+k^2)e^{ikr}\tilde{u}=0$ and (1.18),

$$(\nabla^2+k^2)e^{ikr}\tilde{u}=e^{ikr}\left\{\frac{2ik}{r}\frac{\partial}{\partial r}(r\tilde{u})+\nabla^2\tilde{u}\right\}=0 \quad , \tag{1.23}$$

from which the lemma follows.

Next we note that

$$\frac{\partial}{\partial n}\tilde{u}(p_B)=\hat{n}\cdot\nabla\tilde{u}\Big|_B=\hat{n}\cdot\nabla e^{-ikr}u\Big|_B=-ik\hat{n}\cdot\hat{r}_B\tilde{u}(p_B)+e^{-ikr_B}\frac{\partial u(p_B)}{\partial n} \quad , \tag{1.24}$$

where $\hat{n}$ and $\hat{r}_B$ are unit vectors in the normal (inward) and radial directions respectively at p_B. (If B is a sphere, $\hat{n}=-\hat{r}_B$.) Incorporating these results in the representation theorem (Theorem 1.1) establishes the following.

Theorem 1.3. If

(a) u is a scalar wave function for V, the exterior of a smooth, closed, bounded surface B and

(b) $G_o(p,p_1)$ is the Neumann potential Green's function for this surface $\left(\frac{\partial}{\partial n}G_o(p_B,p_1)=0\right)$, then $\tilde{u}(=e^{-ikr}u)$ may be represented as

$$\tilde{u}(p)=-2ik\int_V\frac{G_o(p,p_1)}{r_1}\frac{\partial}{\partial r_1}\left[r_1\tilde{u}(p_1)\right]dv_1+ik\int_B G_o(p,p_B)\hat{n}\cdot\hat{r}_B\tilde{u}(p_B)\,d\sigma_B$$

$$-\int_B G_o(p,p_B)e^{-ikr_B}\frac{\partial u(p_B)}{\partial n}\,d\sigma_B \tag{1.25}$$

Included in this theorem are representations of the solutions of the two most common exterior Neumann problems for the Helmholtz equation and the surface B.

If

$$\frac{\partial u(p_B)}{\partial n} = \frac{\partial}{\partial n}\left[\frac{e^{ikR(p_B, p_o)}}{4\pi R(p_B, p_o)}\right] , \qquad (1.26)$$

then u represents the regular part of the Neumann Green's function for the Helmholtz equation. If

$$\frac{\partial u}{\partial n} = -\frac{\partial}{\partial n}\left(e^{ik\vec{r}\cdot\hat{\alpha}}\right) \text{ on } B , \qquad (1.27)$$

then u represents the field scattered when a plane acoustic wave is incident in the direction $\hat{\alpha}$ on a rigid surface B. Note that the representation (1.25) is in terms of $\tilde{u}$ but u is easily found by multiplying with the phase factor e^{ikr}.

CHAPTER II

THE SOLUTION OF THE INTEGRO-DIFFERENTIAL EQUATION FOR THE NEUMANN PROBLEM

We write the Eq. (1.25) of Chapter I in the operator form

$$\tilde{u} = L \bullet \tilde{u} + u^{(o)} \tag{2.1}$$

with

$$\omega \rightarrow L \bullet \omega = -2ik \int_V dv_1 \frac{G_o(p, p_1)}{r_1} \frac{\partial}{\partial r_1} \left[r_1 \omega(p_1) \right]$$

$$+ ik \int_B d\sigma_B G_o(p, p_B) \hat{n} \bullet \hat{r}_B \, \omega(p_B) \tag{2.2}$$

and

$$u^{(o)} = - \int_B d\sigma_B G_o(p, p_B) e^{-ikr_B} \frac{\partial u(p_B)}{\partial n} \quad . \tag{2.3}$$

An explicit solution for (2.1) may be given in the form of a Neumann-Liouville series. That is, we rewrite (2.1) in the form

$$\tilde{u} = (I - L)^{-1} \bullet u^{(o)} \tag{2.4}$$

and formally expand the inverse, obtaining

$$\tilde{u} = \sum_{n=0}^{\infty} L^n \bullet u^{(o)} \quad . \tag{2.5}$$

If we denote by $u^{(N)}$ the partial sums

$$u^{(N)} = \sum_{n=0}^{N} L^n \bullet u^{(o)} \quad , \tag{2.6}$$

it follows immediately that for $N \geqslant 1$, $u^{(N)}$ also satisfies the recursive relation

$$u^{(N)} = L \bullet u^{(N-1)} + u^{(o)} \quad . \tag{2.7}$$

Our main task is to define a proper normed linear space and show that $\tilde{u}$ (the solution we seek), $u^{(o)}$ (the known term), and all the iterates $u^{(N)}$, $N \geqslant 1$, are elements of this space and that in the sense of the norm of this space

$$\lim_{N \to \infty} u^{(N)} = \tilde{u} \quad . \tag{2.8}$$

2.1 Preliminaries

First we mention some well-known properties of spherical harmonics and expansions of the potential Green's function [e.g. Kellogg[20] (p. 143), Sommerfeld[40] (p. 123)] which will subsequently be used.

We denote by $Y_n(\theta, \phi)$ an n^{th} order spherical harmonic

$$Y_n(\theta, \phi) = \sum_{m=-n}^{n} A_{mn} P_n^m(\cos\theta) e^{im\phi} \tag{2.9}$$

and by $Y_n(\theta, \phi; \theta_1, \phi_1)$ a symmetric n^{th} order spherical harmonic

$$Y_n(\theta, \phi; \theta_1, \phi_1) = \sum_{m=0}^{n} A_{mn} P_n^m(\cos\theta) P_n^m(\cos\theta_1) \cos m(\phi - \phi_1) \quad , \tag{2.10}$$

where P_n^m are the associated Legendre functions.

These functions have the orthogonality property

$$\int_{\Omega} d\Omega_1 Y_m(\theta_1, \phi_1) Y_n(\theta, \phi; \theta_1, \phi_1) = 0, \qquad m \neq n$$
$$= Y_n(\theta, \phi), \qquad m = n , \tag{2.11}$$

where Ω is the unit sphere and $d\Omega = \sin\theta d\theta d\phi$ is the element of solid angle on it.

The static Green's function for our surface B may be written

$$G_o(p, p_1) = -\frac{1}{4\pi R(p, p_1)} + \sum_{n=0}^{\infty} \frac{Y_n(\theta, \phi)}{r^{n+1}}, \qquad r \geqslant \delta , \quad r_1 < \delta , \tag{2.12}$$

$$= -\frac{1}{4\pi R(p, p_1)} + \sum_{n=0}^{\infty} \frac{Y_n(\theta_1, \phi_1)}{r_1^{n+1}}, \qquad r_1 \geqslant \delta , \quad r < \delta , \tag{2.13}$$

$$= -\frac{1}{4\pi R(p, p_1)} + \sum_{n=0}^{\infty} \frac{Y_n(\theta, \phi; \theta_1, \phi_1)}{(r r_1)^{n+1}}, \qquad r, r_1 \geqslant \delta , \tag{2.14}$$

where the series are uniformly and absolutely convergent and may be differentiated or integrated any number of times with respect to r, θ, or ϕ; $\delta = c+\epsilon$, $\epsilon > 0$; and c the radius of the smallest sphere enclosing B.

The source term may also be expanded in spherical harmonics

$$\frac{1}{R(p, p_1)} = \sum_{n=0}^{\infty} \frac{r_<^n}{r_>^{n+1}} P_n \left[\cos\theta \cos\theta_1 + \sin\theta \sin\theta_1 \cos(\phi - \phi_1)\right] \tag{2.15}$$

where

$$r_> = \max(r, r_1), \quad r_< = \min(r, r_1) .$$

This expansion has the same convergence properties as the series in (2.12), (2.13), (2.14), provided $r \neq r_1$.

2.2 Construction of the Space

We recall from the last chapter that the function $\tilde{u} = e^{-ikr} u$, where u is a scalar wave function, is an analytic function in the complex $1/r$-plane having the expansion

$$\tilde{u} = \sum_{n=0}^{\infty} \frac{f_n(\theta, \phi)}{r^{n+1}} \quad , |r| \geqslant \delta = c + \epsilon \quad , \tag{2.16}$$

where c is the radius of the smallest sphere containing the surface B, $\epsilon > 0$. Putting $z = \frac{1}{r}$, we have

$$\tilde{u} = \sum_{n=0}^{\infty} f_n(\theta, \phi) \, z^{n+1}, \quad |z| \leqslant \frac{1}{\delta} \quad . \tag{2.17}$$

Clearly we may assume $\delta = 1$.

Next we define the following function space.

Definition 2.1.

Let W be the set of functions defined on $\bar{V}$ such that

(a) $\omega \in C^2(V)$, $\omega \in C^1(\bar{V})$,

(b) ω is analytic on the closed unit disc having the expansion

$$\omega = \sum_{n=0}^{\infty} f_n(\theta, \phi) \, z^{n+1}, \quad |z| \leqslant \frac{1}{\delta} = 1 \ ,$$

(c) $f_n(\theta, \phi) = \sum_{m=n}^{\infty} Y_m(\theta, \phi)$, where Y_m is an m^{th} order spherical

(2.18)

harmonic, i.e.

$$Y_m(\theta, \phi) = \sum_{\ell=-m}^{m} A_{\ell m} P_m^{\ell}(\cos\theta) e^{i\ell\phi} .$$

It follows immediately from this definition and the Cauchy's integral formula

$$f_n(\theta, \phi) = \frac{1}{2\pi i} \int_C \frac{\omega(\zeta)}{\zeta^{n+2}} d\zeta ,$$

where C is the unit circle around the origin, that

$$\left| f_n(\theta, \phi) \right| \leqslant \max_{\zeta \in C} \left| \omega(\zeta) \right| .$$

We define the following norm on W.

<u>Definition 2.2</u>

$$\|\omega\| = \max_{p \in \overline{V}} |\omega(p)| + \max_{\substack{|z| \leqslant 1 \\ 0 \leqslant \theta \leqslant \pi \\ 0 \leqslant \phi \leqslant 2\pi}} \left| \omega(\theta, \phi, \tfrac{1}{z}) \right| . \tag{2.19}$$

With this norm it is clear that

$$\left| f_n(\theta, \phi) \right| \leqslant \|\omega\| \tag{2.20}$$

We now proceed to solve our operator equation in the space $\left\{ W, \| \; \| \right\}$.

2.3 The Solution

Lemma 2.1

If $\frac{\partial u}{\partial n}$ is uniformly Hölder continuous on B (Eq. (2.3)) then $u^{(o)} \epsilon W$.

Proof.

The definition of $u^{(o)}$, (2.3), shows that $u^{(o)}$ consists of the potential of a single layer distribution of density $e^{-ikr_B} \frac{\partial u}{\partial n}$ plus another term (corresponding to the regular part of the Green's function) which is at least as well behaved. Therefore the differentiability of $u^{(o)}$ is essentially that of the potential. If the density $e^{-ikr_B} \frac{\partial u}{\partial n}$ is piecewise continuous then according to Kellogg[20](p. 122) the potential is infinitely differentiable in V, thus, in particular, the potential (hence $u^{(o)}$) is twice continuously differentiable in V. Furthermore, if the density is uniformly Hölder continuous then, again, according to Kellogg[20](p. 165) the potential (hence $u^{(o)}$) is continuously differentiable in the closure $\bar{V}$. Since r_B is a continuously differentiable function of θ and ϕ it follows e^{-ikr_B} is uniformly Hölder continuous, hence so is the product $e^{-ikr_B} \frac{\partial u}{\partial n}$. Therefore $u^{(o)}$ satisfies (2.18a).

$G_o(p, p_B)$ may be expanded in an absolutely and uniformly convergent series of spherical harmonics of the form [e.g. Kellogg[20](p. 143)]

$$G_o(p,p_B) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} A_{mn}(p_B) \frac{1}{r^{n+1}} P_n^m(\cos\theta) e^{im\phi} , \quad r \geqslant \delta . \tag{2.21}$$

Thus for $r \geqslant \delta$ we may rewrite $u^{(o)}$ as

$$u^{(o)} = -\sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \sum_{m=-n}^{n} P_n^m(\cos\theta) e^{im\phi} \int_B A_{mn}(p_B) e^{-ikr_B} \frac{\partial u}{\partial n} d\sigma_B \tag{2.22}$$

which is of the form

$$u^{(o)} = \sum_{n=0}^{\infty} \frac{Y_n(\theta, \phi)}{r^{n+1}} \quad . \tag{2.23}$$

and satisfies (2.18b) and (2.18c), thus, proving the lemma.

We next split the operator L defined by (2.2) as follows.

Definition 2.3

Let

$$L = kL_1 = kO + kO_1 \tag{2.24}$$

where k is the wave number. Thus L_1, O, and O_1 are all independent of k; they are given by

$$\omega \longrightarrow O \cdot \omega = -2i \int_V dv_1 \frac{G_o(p, p_1)}{r_1} \frac{\partial}{\partial r_1} \left[r_1 \omega(p_1) \right] \tag{2.25}$$

and

$$\omega \longrightarrow O_1 \cdot \omega = 1 \int_B d\sigma_B G_o(p, p_B) \hat{n} \cdot \hat{r}_B \, \omega(p_B) \quad . \tag{2.26}$$

Lemma 2.2

The operator O, defined by Eq. (2.25), maps the space W into itself.

Proof.

We have Eq. (2.25)

$$O \cdot \omega = -2i \int_V dv_1 \frac{G_o(p, p_1)}{r_1} \frac{\partial}{\partial r_1} \left[r_1 \omega(p_1) \right] \quad .$$

We separate the volume over which the integration is performed into an infinite volume, V_{ext}, where $r_1 \geqslant \delta = 1$ and the Theorem 1.2 of the last chapter holds, and a finite volume, V_{int}, between the sphere and the surface B, where the expansion theorem does not hold. We define two functions

$$\left(O\bullet\omega(p)\right)_{ext} = -2i \int_{V_{ext}} dv_1 \frac{G_o(p,p_1)}{r_1} \frac{\partial}{\partial r_1}\left[r_1\omega(p_1)\right]$$

$$\left(O\bullet\omega(p)\right)_{int} = -2i \int_{V_{int}} dv_1 \frac{G_o(p,p_1)}{r_1} \frac{\partial}{\partial r_1}\left[r_1\omega(p_1)\right] \quad . \tag{2.27}$$

If we can demonstrate that the functions defined in (2.27) are in W, then, since the space is linear, it follows that $O\bullet\omega = (O\bullet\omega)_{ext} + (O\bullet\omega)_{int}$ is also in W.

Consider $(O\bullet\omega)_{int}$ first. It is the potential of a volume distribution with the density $\frac{1}{r_1}\frac{\partial}{\partial r_1}\left[r_1\omega(p_1)\right]$ which is certainly continuous; therefore, [Kellogg[20] (p. 122)] the potential is infinitely differentiable, and hence $(O\bullet\omega)_{int} \in C^2(V)$. Also, [Kellogg[20] (p. 151, 152)] $(O\bullet\omega)_{int} \in C^1(\bar{V})$.

When $r \geqslant \delta = 1$, since $r_1 \leqslant \delta = 1$, from the formulas (2.12) and (2.15)

$$G_o(p,p_1) = -\frac{1}{4\pi}\sum_{n=0}^{\infty} \frac{r_1^n}{r^{n+1}} \cdot P_n\left[\cos\theta\cos\theta_1 + \sin\theta\sin\theta_1\cos(\phi-\phi_1)\right] + \sum_{n=0}^{\infty}\frac{Y_n(\theta,\phi)}{r^{n+1}} , \tag{2.28}$$

or, since P_n is an n^{th} order spherical harmonic,

$$G_o(p,p_1) = \sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\sum_{m=-n}^{n} A_{mn}(p_1)\,P_n^m(\cos\theta)e^{im\phi} \tag{2.29}$$

The series converges uniformly, (2.12), as does the derived series. We substitute (2.29) into (2.27); and since the integration is carried out over the finite limits, we may change the order of integration, thus

$$(O\bullet\omega)_{int} = \sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\sum_{m=-n}^{n} P_n^m(\cos\theta)e^{im\phi}\int_{V_{int}} dv_1(-2i)\frac{A_{mn}(p_1)}{r_1}\frac{\partial}{\partial r_1}\left[r_1\omega(p_1)\right], \; r \geqslant \delta = 1, \tag{2.30}$$

which is of the form

$$(O\bullet\omega)_{int} = \sum_{n=0}^{\infty} \frac{Y_n(\theta,\phi)}{r^{n+1}} \quad , \qquad r \geqslant \delta = 1 \; , \tag{2.31}$$

and satisfies (2.18b) and (2.18c). Therefore,

$$(O\bullet\omega)_{int} \in W \; . \tag{2.32}$$

Now we consider $(O\bullet\omega)_{ext}$. If V_{ext} is replaced by a large but finite volume then again it follows [Kellogg[20] (p. 122, 152)] that, in the volume considered, ω is twice continuously differentiable, and it is once continuously differentiable in its closure.

Next we study the exterior integral as volume extends to infinity. Explicitly,

$$(O\bullet\omega)_{ext} = -2i \lim_{\mu\to\infty} \int_1^{\mu} dr_1 \int_0^{2\pi} d\phi_1 \int_0^{\pi} d\theta_1 \, r_1^2 \sin\theta_1 \, \frac{G_o(p,p_1)}{r_1} \frac{\partial}{\partial r_1}\left[r_1\omega(p_1)\right] \quad , \tag{2.33}$$

and it is sufficient to show that the integrand is of $O(\frac{1}{r_1^2})$ for large r_1. Since $\omega(p_1) \in W$, it follows that

$$\omega(p_1) = \sum_{n=0}^{\infty} \frac{f_n(\theta_1,\phi_1)}{r_1^{n+1}} \quad , \qquad r_1 \geqslant \delta = 1$$

and, therefore, that

$$\frac{\partial}{\partial r_1}\left[r_1\omega(p_1)\right] = -\sum_{n=1}^{\infty} \frac{n f_n(\theta_1,\phi_1)}{r_1^{n+1}} \quad . \tag{2.34}$$

Thus for large r_1, $\frac{\partial}{\partial r_1}\left[r_1\omega(p_1)\right] = O(\frac{1}{r_1^2})$. Furthermore, the expansions of $G_o(p,p_1)$ given in (2.13) and (2.15) show that for large r_1, $\frac{G_o(p,p_1)}{r_1} = O(\frac{1}{r_1^2})$.

Thus, as $r_1 \longrightarrow \infty$ the integrand is of $O(\frac{1}{r_1^2})$ and $(O\bullet\omega)_{ext}$ exists.

Now we pursue this calculation carefully to show that $(O\bullet\omega)_{ext}$ satisfies the expansion properties required of elements of W. Thus, we rewrite (2.33) for $r, r_1 \geqslant \delta = 1$ (see expansion (2.14) for G_o) as

$$(O\bullet\omega)_{ext} = \int_1^{\infty} dr_1 \int_0^{2\pi} d\phi_1 \int_0^{\pi} d\theta_1 r_1 \sin\theta_1 \left\{ -\frac{1}{4\pi R(p, p_1)} + \sum_{m=0}^{\infty} \frac{Y_m(\theta, \phi; \theta_1 \phi_1)}{(r r_1)^{m+1}} \right\} \cdot \sum_{n=0}^{\infty} \frac{f_n(\theta, \phi)}{r_1^{n+1}} \tag{2.35}$$

where we have absorbed the factor 2ikn in the functions $f_n(\theta_1, \phi_1)$. Now consider separately the integrals involving the regular and singular parts of the static Green's function, treating the regular part, $(O\bullet\omega)^{reg}_{ext}$, first. In this case both series are uniformly convergent, and the integral has been shown to exist; thus we may interchange the order of integration and summation and perform the integration using the orthogonality properties of spherical harmonics (2.11) and the definition of $f_n(\theta, \phi)$ (2.18 c) to obtain

$$(O\bullet\omega)^{reg}_{ext} = \int_1^{\infty} dr_1 \int_0^{2\pi} d\phi_1 \int_0^{\pi} d\theta_1 \sin\theta_1 \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{Y_m(\theta, \phi; \theta_1, \phi_1) f_n(\theta_1, \phi_1)}{r^{m+1} r_1^{m+n+1}}$$

$$= \sum_{n=1}^{\infty} \sum_{m=n}^{\infty} \frac{Y_m(\theta, \phi)}{(m+n) r^{m+1}} \quad . \tag{2.36}$$

Absorbing the constant factors in the spherical harmonics and shifting index, we have

$$(O\bullet\omega)^{reg}_{ext} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{Y_{m+n+1}(\theta,\phi)}{r^{m+n+2}} \quad . \tag{2.37}$$

Using the identity

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} Q(m,n) \equiv \sum_{n=0}^{\infty}\sum_{m=0}^{n} Q(m,\ n-m)$$

and since the series involved are absolutely convergent, we obtain

$$(O\bullet\omega)^{reg}_{ext} = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{Y_{n+1}(\theta,\phi)}{r^{n+2}} \quad . \tag{2.38}$$

The coefficients in $Y_{n+1}(\theta,\phi)$ may depend on m, but the summation over m is still a spherical harmonic of order $n+1$, hence (2.38) is of the form

$$(O\bullet\omega)^{reg}_{ext} = \sum_{n=0}^{\infty}\frac{Y_{n+1}(\theta,\phi)}{r^{n+2}} \quad , \tag{2.39}$$

and satisfies (2.18b) and (2.18c).

We now pursue the analysis involving the singular part of the Green's function. The expansion of $1/R$, (2.15), is not convergent at $r=r_1$. From (2.35) we see that

$$(O\bullet\omega)^{sing}_{ext} = -\frac{1}{4\pi}\int_1^{\infty} dr_1 \int_0^{2\pi} d\phi_1 \int_0^{\pi} d\theta_1 \frac{\sin\theta_1}{R(p,p_1)}\sum_{n=1}^{\infty}\frac{f_n(\theta_1,\phi_1)}{r_1^{n}} \quad . \tag{2.40}$$

Since the series in (2.40) is uniformly convergent and the infinite integral has been shown to exist, we may interchange the order of summation and integration, and absorb the factor $(-\frac{1}{4\pi})$ into f_n, obtaining

$$(O\bullet\omega)^{sing}_{ext} = \sum_{n=1}^{\infty} \int_1^{\infty} dr_1 \int_0^{2\pi} d\phi_1 \int_0^{\pi} \frac{\sin\theta_1}{R(p,p_1)} \frac{f_n(\theta_1,\phi_1)}{r_1^n} . \tag{2.41}$$

We now employ the expansion for $1/R$, (2.15), to obtain

$$(O\bullet\omega)^{sing}_{ext} = \sum_{n=1}^{\infty} \left\{ \int_1^{r} dr_1 \int_0^{2\pi} d\phi_1 \int_0^{\pi} d\theta_1 \frac{\sin\theta_1}{r_1^n} f_n(\theta_1,\phi_1) \sum_{m=0}^{\infty} \frac{r_1^m}{r^{m+1}} Y_m(\theta,\phi;\theta_1,\phi_1) + \right.$$

$$\left. + \int_r^{\infty} dr_1 \int_0^{2\pi} d\phi_1 \int_0^{\pi} d\theta_1 \frac{\sin\theta_1}{r_1^n} f_n(\theta_1,\phi_1) \sum_{m=0}^{\infty} \frac{r^m}{r_1^{m+1}} Y_m(\theta,\phi;\theta_1,\phi_1) \right\} . \tag{2.42}$$

Although the inner summation is singular at $r=r_1$, $\theta=\theta_1$, $\phi=\phi_1$, it is a straightforward matter to exclude a small neighborhood of $p=(r,\theta,\phi)$ from the integral in which case the interchange of summation and integration is legitimate and then show that the integral over the excluded neighborhood may be made arbitrarily small by taking the neighborhood sufficiently small [e.g. Kellogg[20] (p. 148)] . Thus we find, again using the orthogonality properties

$$(O\bullet\omega)^{sing}_{ext} = \sum_{n=1}^{\infty} \sum_{m=n}^{\infty} \left\{ \frac{Y_m(\theta,\phi)(r^{m-n+1}-1)}{r^{m+1}(m-n+1)} + \frac{Y_m(\theta,\phi)}{(n+m)r^n} \right\} . \tag{2.43}$$

We note that in obtaining (2.43) the condition (2.18c) is necessary, since without this property, terms involving $\log r$ would occur. Again absorbing the constants in the spherical harmonics from (2.43) we obtain

$$(O\bullet\omega)^{sing}_{ext} = \sum_{n=1}^{\infty} \frac{1}{r^n} \sum_{m=n}^{\infty} Y_n(\theta,\phi) + \sum_{n=1}^{\infty} \sum_{m=n}^{\infty} \frac{Y_m(\theta,\phi)}{r^{m+1}} \quad . \tag{2.44}$$

Using the identity following Eq. (2.37) on the second series of (2.44), and using the similar argument, we have

$$(O\bullet\omega)^{sing}_{ext} = \sum_{n=1}^{\infty} \frac{1}{r^n} \sum_{m=n}^{\infty} Y_m(\theta,\phi) + \sum_{n=0}^{\infty} \frac{Y_{n+1}(\theta,\phi)}{r^{n+2}} \quad . \tag{2.45}$$

Now we combine the Eqs. (2.39) and (2.45) to obtain, for $r \geqslant \delta = 1$,

$$(O\bullet\omega)_{ext} = (O\bullet\omega)^{reg}_{ext} + (O\bullet\omega)^{sing}_{ext} = \sum_{n=1}^{\infty} \frac{1}{r^n} \sum_{m=n}^{\infty} Y_m(\theta,\phi) + \sum_{n=0}^{\infty} \frac{Y_{n+1}(\theta,\phi)}{r^{n+2}} \quad , \tag{2.46}$$

or, by shifting index,

$$(O\bullet\omega)_{ext} = \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \sum_{m=n+1}^{\infty} Y_m(\theta,\phi) + \sum_{n=1}^{\infty} \frac{Y_n(\theta,\phi)}{r^{n+1}} \quad . \tag{2.47}$$

This is of the form

$$(O\bullet\omega)_{ext} = \sum_{n=0}^{\infty} \frac{f_n(\theta,\phi)}{r^{n+1}} \quad , \qquad r \geqslant \delta = 1 \quad , \tag{2.48}$$

satisfying (2.18b) and (2.18c),

with

$$f_o = \sum_{m=1}^{\infty} Y_m(\theta, \phi) \tag{2.49}$$

$$f_n = \sum_{m=n}^{\infty} Y_m(\theta, \phi) \ , \qquad n \geqslant 1 \ . \tag{2.50}$$

Hence,

$$(O \circ \omega)_{ext} \in W \ . \tag{2.51}$$

Finally, the Lemma 2.2 follows when we combine (2.51) with (2.32).

Lemma 2.3 The operator O_1, defined by Eq. (2.26), maps the space W into itself.

Proof.

We have

$$O_1 \circ \omega = i \int_B d\sigma_B \, G_o(p, p_B) \, \hat{n} \circ \hat{r}_B \, \omega(p_B)$$

$$= i \int_B d\sigma_B \left[-\frac{1}{4\pi R(p, p_B)} + u_o(p, p_B) \right] \hat{n} \circ \hat{r}_B \, \omega(p_B) \ , \tag{2.26}$$

where u_o is a regular potential function at all points $p \in \overline{V}$. The Eq. (2.26) shows that $O_1 \circ \omega$ is the potential of a single surface layer distribution of density $i \hat{n} \cdot \hat{r}_B \omega(p_B)$ plus another term (corresponding to the regular part u_o) which is at least as well behaved. If the density $i \hat{n} \cdot \hat{r}_B \omega(p_B)$ is piecewise continuous then [Kellogg[20] (p. 122)] the potential is infinitely differentiable in V, thus, in particular, the potential $O_1 \circ \omega$ is twice continuously differentiable in V.

Furthermore, if density is uniformly Hölder continuous then [Kellogg[20] (p. 165)] the potential is continuously differentiable in the closure $V\left(O_1 \cdot \omega \in C^1(\overline{V})\right)$. But $\omega \in W$ ensures that $\omega \in C^1(\overline{V})$. With the fact that the surface is closed and finite, it follows that $\omega(p_B)$ is uniformly Hölder continuous. Thus the density $i\hat{n} \cdot \hat{r}_B \omega(p_B)$ will be uniformly Hölder continuous if $\hat{n} \cdot \hat{r}_B$ is. This however, is one of the smoothness requirements on B. (See the definition of smoothness for the surface B given in Chapter I. Indeed, the above need promoted us to give this particular smoothness definition.)

For $r \geqslant \delta = 1$, using the previous similar argument, we have

$$G_o(p, p_B) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} A_{mn}(p_B) \frac{1}{r^{n+1}} P_n^m(\cos\theta) e^{im\phi} \quad , \quad r \geqslant \delta = 1 \ . \tag{2.52}$$

Substituting into (2.26), we obtain

$$O_1 \cdot \omega = \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \sum_{m=-n}^{n} P_n^m(\cos\theta) e^{im\phi} \int_B A_{mn}(p_B) \left[i\hat{n} \cdot \hat{r}_B \omega(p_B) \right] d\sigma_B \tag{2.53}$$

which is of the form

$$O_1 \cdot \omega = \sum_{n=0}^{\infty} \frac{Y_n(\theta, \phi)}{r^{n+1}} \quad , \quad r \geqslant \delta = 1 \ , \tag{2.54}$$

and satisfies (2.18b) and (2.18c). With this, we conclude that

$$O_1 \cdot \omega \in W$$

proving the lemma.

From the Lemmas 2.2 and 2.3 and from the Definition 2.3 (2.4) it immediately follows that

Corollary 2.1 The operators L_1 and $L = kL_1 = kO + kO_1$ map the space W into itself.

Our next task is to show that operator L (2.24) is bounded. However, to do this we shall need the following estimate.

Lemma 2.4

For every $\omega \epsilon W$ the estimate

$$|\omega - \frac{f_o}{r}| \leqslant \frac{M}{r^2} \|\omega\| , \qquad p \epsilon \overline{V} , \qquad |z| = \frac{1}{|r|} \leqslant \delta ,$$

is valid. M is a constant independent of ω; f_o, the so-called "radiation pattern", is the first coefficient in the expansion of ω .

Proof

Recall we may assume $\delta = c + \epsilon = 1$.

For $r \leqslant 1$, since $|f_o| \leqslant \|\omega\|$ (Eq. (2.20)) ,

$$\left|\omega - \frac{f_o}{r}\right| \leqslant |\omega| + \frac{|f_o|}{r} \leqslant \|\omega\| + \frac{\|\omega\|}{r} \leqslant \frac{\|\omega\|}{r^2} + \frac{\|\omega\|}{r^2} = \frac{2}{r^2} \|\omega\| . \tag{2.55}$$

For $r \geqslant \delta = 1$, since $\sum_{n=1}^{\infty} \frac{f_n}{r^{n-1}}$ is analytic its maximum is achieved on the circle $|z| = \frac{1}{|r|} = 1$; therefore,

$$\left| \omega - \frac{f_o}{r} \right| = \frac{1}{r^2} \left| \sum_{n=1}^{\infty} \frac{f_n}{r^{n-1}} \right| \leqslant \frac{1}{r^2} \left| \sum_{n=1}^{\infty} f_n \right| . \tag{2.56}$$

On the other hand

$$\left| \sum_{n=1}^{\infty} \frac{f_n}{r^{n+1}} \right| = \left| \omega - \frac{f_o}{r} \right| \leqslant |\omega| + \frac{|f_o|}{r} , \qquad r \geqslant \delta = 1 . \tag{2.57}$$

Since $|f_o| \leqslant \|\omega\|$ and $1/r \leqslant 1$, it follows that

$$\left| \sum_{n=1}^{\infty} \frac{f_n}{r^{n+1}} \right| \leqslant 2 \|\omega\| .$$

In particular, at $r = \delta = 1$

$$\left| \sum_{n=1}^{\infty} f_n \right| \leqslant 2 \, \|\omega\| \, . \qquad (2.58)$$

Substituting this in (2.56) yields

$$\left| \omega - \frac{f_o}{r} \right| \leqslant \frac{2}{r^2} \, \|\omega\| \quad , \qquad r \geqslant \delta = 1 \, . \qquad (2.59)$$

From (2.59) and (2.55) the lemma follows

<u>Lemma 2.5</u> Operator L_1 (2.24) is bounded.

<u>Proof.</u>

We want to show that there exists a constant $M < \infty$ such that if $\omega \epsilon W$ then $\|L_1 \bullet \omega\| \leqslant M \, \|\omega\|$. Since

$$L_1 = O + O_1 \qquad (2.60)$$

and

$$\|L_1 \bullet \omega\| = \|O \bullet \omega + O_1 \bullet \omega\| \leqslant \|O \bullet \omega\| + \|O_1 \bullet \omega\| \; , \qquad (2.61)$$

it is sufficient to show that

$$\|O \bullet \omega\| \leqslant M_1 \, \|\omega\| \; , \qquad M_1 < \infty \qquad (2.62)$$

and

$$\|O_1 \bullet \omega\| \leqslant M_2 \, \|\omega\| \; , \qquad M_2 < \infty \; . \qquad (2.63)$$

Consider (2.62) first. We integrate (2.25) by parts with respect to r_1 once to obtain

$$O \bullet \omega = -2i \int_V dv_1 \, \frac{G_o(p, p_1)}{r_1} \, \frac{\partial}{\partial r_1} \left[r_1 \omega(p_1) \right] = -2i \int_V dv_1 \, \frac{G_o(p, p_1)}{r_1} \, \frac{\partial}{\partial r_1} \left[r_1 \omega(p_1) - f_o(\theta_1, \phi_1) \right]$$

$$= 2i \int_B d\sigma_B \; G_o(p, p_B) \left[\omega(p_B) - \frac{f_o}{r_B} \right] + 2i \int_V dv_1 \, \frac{1}{r_1} \, \frac{\partial}{\partial r_1} \left[r_1 G_o(p, p_1) \right] \left[\omega(p_1) - \frac{f_o}{r_1} \right] ,$$

(2.64)

where $f_o(\theta, \phi)$ is the "radiation pattern" for ω.

The integrated term vanishes at the upper limit, $r_1 \longrightarrow \infty$, since (see Eqs. (2.13), (2.15) and (2.16))

$$r_1 G_o(p, p_1)\left[r_1 \omega(p_1) - f_o(\theta_1, \phi_1)\right] = O(\frac{1}{r_1}) \text{ as } r_1 \longrightarrow \infty ,$$

and gives rise to the surface integral at the lower limit.

Making use of the estimate

$$\left|\omega - \frac{f_o}{r}\right| \leqslant \frac{M}{r^2} \|\omega\| , \qquad p \in \overline{V} , \qquad |z| = \frac{1}{|r|} \leqslant 1 ,$$

of Lemma 2.4 in Eq. (2.64) we see that for $p \in \overline{V}$

$$\left|O \circ \omega\right| \leqslant 2M \left[\int_B \left|G_o(p, p_B)\right| \frac{1}{r_B} d\sigma_B + \int_V \frac{1}{r_1^2}\left|\frac{\partial}{\partial r_1}(r_1 G_o)\right| dv_1\right] \frac{\|\omega\|}{r} . \tag{2.65}$$

The surface integral may be rewritten, separating out the singular part of the Green's function, as

$$\int_B \left|G_o(p, p_B)\right| \frac{1}{r_B} d\sigma_B \leqslant \int_B \frac{1}{4\pi r_B R(p, p_B)} d\sigma_B + \int_B \left|u_o(p, p_B)\right| \frac{1}{r_B} d\sigma_B .$$

The first term on the right is the potential of a single layer distribution of density $1/4\pi r_B$. Since $r_B \neq 0$ (the origin was taken within B) and the surface is smooth, closed and finite, this density is uniformly Hölder continuous which means [Kellogg[20] (p. 165)] that the potential is continuously differentiable for all points $p \in \overline{V}$. The second term on the right hand side of (2.66) is the integral of a bounded function over a finite surface and hence is also bounded. Thus for some $N < \infty$,

$$\int_B \left|G_o(p, p_B)\right| \frac{1}{r_B} d\sigma_B < N , \qquad p \in \overline{V} . \tag{2.66}$$

The volume integral in (2.65) is also bounded since the integrand is sufficiently well behaved. At the singularity of G_o ,

$$\frac{1}{r_1^2} \left| \frac{\partial}{\partial r_1} (r_1 G_o) \right| = O(1/R^2) \quad \text{as} \quad R \longrightarrow 0 ,$$

and is therefore integrable over any finite volume containing the singularity [Kellogg[20] (p. 148)] . Furthermore,

$$\frac{1}{r_1^2} \left| \frac{\partial}{\partial r_1} (r_1 G_o) \right| = O(r/r_1^4) \quad \text{as} \quad r_1 \longrightarrow \infty ,$$

thus for some $N_1 < \infty$

$$\int_V \frac{1}{r_1^2} \left| \frac{\partial}{\partial r_1} \left[r_1 G_o(p, p_1) \right] \right| dv_1 < N_1 \cdot r , \; p \in \overline{V} . \tag{2.67}$$

With (2.65), (2.66) and (2.67) we have for some constant N_3

$$\left| O \bullet \omega \right| \leqslant N_3 \cdot \left\| \omega \right\| , \qquad p \in \overline{V} . \tag{2.68}$$

In particular this is true for the maximum value of $|O \bullet \omega|$, therefore, renaming the constants,

$$\max_{p \in \overline{V}} \left| O \bullet \omega \right| \leqslant M_1 \left\| \omega \right\| . \tag{2.69}$$

Also, since $G_o(p, p_1)$ has its singularity only for real values of r at $p=p_1$ (integration is carried out over the real range $1 \leqslant r_1 \leqslant \infty$), it follows from (2.65) and the expansion of G_o that for a positive constant M_2 independent of ω, we have

$$\max_{|z| \leqslant 1} |O \bullet \omega| \leqslant M_2 \, \|\omega\| \quad . \tag{2.70}$$

Hence with (2.70) and (2.69),

$$\|O \bullet \omega\| \leqslant M \, \|\omega\| \tag{2.71}$$

for an appropriate constant M.

Next we establish (2.63), thus proving the lemma. With the definition of O_1 (2.26) we see that

$$|O_1 \bullet \omega| \leqslant \int_B |G_o(p, p_B)| \, |\hat{n} \bullet \hat{r}_B| \, |\omega(p_B)| \, d\sigma_B \quad . \tag{2.72}$$

By definition,

$$|\omega(p_B)| \leqslant \|\omega\| \quad . \tag{2.73}$$

Also $\hat{n}$ and $\hat{r}_B$ are unit vectors,

$$|\hat{n} \bullet \hat{r}_B| \leqslant 1 \ . \tag{2.74}$$

Thus,

$$|O_1 \bullet \omega| \leqslant \|\omega\| \int_B |G_o(p, p_B)| \, d\sigma_B \quad . \tag{2.75}$$

By arguments similar to those above, we have

$$\max_{p \in \bar{V}} |O_1 \bullet \omega| \leqslant N_1 \|\omega\| , \tag{2.76}$$

and

$$\max_{|z| \leqslant 1} |O_1 \bullet \omega| \leqslant N_2 \|\omega\| ,$$

for some constants N_1, N_2 .

Hence for an appropriate constant $N > 0$.

$$\|O_1 \bullet \omega\| \leqslant N \|\omega\| . \tag{2.77}$$

With this the lemma is proven.

Since $L = kL_1$, and L_1 has been proven to be bounded, we immediately have the following

Corollary 2.2

$\|L\| < 1$ for sufficiently small $|k|$.

Lemma 2.6 If u is a scalar wave function (see (1.15)) then $\tilde{u} = e^{-ikr} u$ is an element of W.

Proof

u is a scalar wave function, and as such (see Chapter I, Section 1.2) $u \in C^2(\bar{V})$. Since e^{-ikr} is analytic in r and r is continuously differentiable on B (see Chapter I, Definition 1.2), $\tilde{u} = e^{-ikr} u$ satisfies the requirements that $\tilde{u} \in C^2(V)$ and $\tilde{u} \in C^1(\bar{V})$.

Furthermore, scalar wave functions may be expanded in spherical harmonics [e.g. Sommerfeld[40] (p. 143)] in the following well-known manner

$$u(p) = \sum_{n=0}^{\infty} h_n(kr)\, Y_n(\theta, \phi) , \qquad r \geqslant \delta = 1 \tag{2.78}$$

where $h_n(kr)$ are spherical Hankel functions of the first kind ,

$$h_n(kr) = \frac{e^{ikr} i^{-n-1}}{r} \sum_{m=0}^{n} \frac{(n+m)!}{(n-m)!\, m!} \left(\frac{-1}{2\, ikr}\right)^m . \tag{2.79}$$

Equations (2.78) and (2.79) yield

$$\tilde{u} = e^{-ikr} u = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{Y_n(\theta, \phi) \cdot i^{-n-1} (n+m)!}{r^{m+1} (n-m)!\, m!} \quad \frac{1}{(2ik)^m} . \tag{2.80}$$

Using the following identity

$$\sum_{n=0}^{\infty} \sum_{m=0}^{n} D(m, n) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} D(m, n+m) \quad ,$$

and absorbing the constants into spherical harmonics, (2.80) reduces to

$$\tilde{u} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{Y_{n+m}(\theta, \phi)}{r^{m+1}} \quad , \tag{2.81}$$

or

$$\tilde{u} = \sum_{n=0}^{\infty} \frac{f_n(\theta, \phi)}{r^{n+1}} \quad , \tag{2.82}$$

where

$$f_n(\theta, \phi) = \sum_{\ell=0}^{\infty} Y_{n+\ell}(\theta, \phi) \equiv \sum_{m=n}^{\infty} Y_m(\theta, \phi) \quad ,$$

proving the lemma.

We are finally in a position to state and prove our main result.

Theorem 2.1.

Let B be a closed, finite surface imbedded in E^3 and let it be described by the equation $r_B = g(\theta, \phi)$ where g is continuously differentiable for $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \phi \leqslant 2\pi$. Let $\hat{n} \cdot \hat{r}_B$ be uniformly Hölder continuous. If u(p) is a scalar wave function for V, the exterior of B, then there exists a disc around the origin in the complex wave number plane $|k| < |k_o|$ such that for k inside this disc u(p) is given explicitly by the convergent expansion

$$u(p) = e^{ikr} \sum_{n=0}^{\infty} L^n \bullet u^{(o)} \quad , \tag{2.83}$$

where

$$L \bullet u^{(o)} = -2ik \int_V dv_1 \frac{G_o(p, p_1)}{r_1} \frac{\partial}{\partial r_1} \left[r_1 u^{(o)}(p_1) \right] + ik \int_B d\sigma_B G_o(p, p_B) \hat{n} \cdot \hat{r}_B u^{(o)}(p_B) , \tag{2.84}$$

$$u^o(p) = - \int_B G_o(p, p_B) e^{-ikr_B} \frac{\partial}{\partial n} u(p_B) \, d\sigma_B \quad , \tag{2.85}$$

$G_o(p, p_B)$ is the static Neumann Green's function $\left(\frac{\partial}{\partial n} G_o(p, p_B) = 0 \right)$, and the normal is taken out of V.

Proof.

Multiplying both sides of (2.83) by e^{-ikr} we obtain

$$\tilde{u} = \sum_{n=0}^{\infty} L^n \bullet u^{(o)} \quad . \tag{2.86}$$

We proceed to prove (2.86), since this would be equivalent to proving (2.83). $\tilde{u}$, $u^{(o)}$, and the partial sums $u^{(N)}$ are all in W (Lemmas 2.1, 2.6, Corollary 2.1). Thus $\|\tilde{u} - u^{(N)}\|$ is meaningful for any $N \geqslant 0$. We shall prove the theorem by showing that for any $\epsilon > 0$, there exists a positive integer $N_o(\epsilon)$ such that

$$\left\| \tilde{u} - u^{(N)} \right\| < \epsilon \quad \text{provided } N > N_o . \tag{2.87}$$

From our operator equation $\tilde{u} = L \bullet \tilde{u} + u^{(o)}$ and from the definition of partial Neumann-Liouville sums

$$u^{(N)} = \sum_{n=0}^{N} L^n \bullet u^{(o)} \left(\text{which implies } u^{(N)} = L \bullet u^{(N-1)} + u^{(o)} \right)$$

we have, by induction,

$$\tilde{u} - u^{(N)} = L^{N+1} \bullet \tilde{u} \quad , \tag{2.88}$$

and hence

$$\left\| \tilde{u} - u^{(N)} \right\| \leqslant \|L\|^{N+1} \, \|\tilde{u}\| \quad . \tag{2.89}$$

Since $|k| < |k_o|$, we have $\|L\| < 1$ (Corollary 2.2), and since $\|\tilde{u}\|$ is bounded, for a given $\epsilon > 0$

$$\|L\|^{N+1} \, \|\tilde{u}\| < \epsilon \tag{2.90}$$

provided

$$N+1 > \frac{\log \frac{\epsilon}{\|\tilde{u}\|}}{\log \|L\|} \quad . \tag{2.91}$$

This proves the theorem. Representations of wave functions in two important special cases follow immediately.

Corollary 2.3 The Green's function of the second kind for the Helmholtz equation and surface B $\left(\frac{\partial G_k}{\partial n} = 0 \text{ on } B\right)$ is

$$G_k(p, p_o) = -\frac{e^{ikR(p, p_o)}}{4\pi R(p, p_o)} + u(p, p_o) \tag{2.92}$$

where $u(p, p_o)$ is given explicitly by (2.86) with

$$u^{(o)}(p, p_o) = -\frac{1}{4\pi} \int_B G_o(p, p_B)\, e^{-ikr_B} \frac{\partial}{\partial n} \left[\frac{e^{ikR(p_B, p_o)}}{R(p_B, p_o)} \right] d\sigma_B \quad . \tag{2.93}$$

This, of course, follows immediately by taking the normal derivatives of both sides of (2.92) and observing that $\left.\frac{\partial G_k}{\partial n}\right|_B = 0$, then by substituting the result

$$\frac{\partial}{\partial n} u(p_B, p_o) = \frac{\partial}{\partial n} \left[\frac{e^{ikR(p_B, p_o)}}{4\pi R(p_B, p_o)} \right] \tag{2.94}$$

into the Eq. (2.85) .

Corollary 2.4 The velocity potential u^t when a plane acoustic wave is incident in a direction $\hat{\alpha}$ on a rigid surface B ($\frac{\partial u^t}{\partial n} = 0$ on B) is

$$u^t = e^{ik\vec{r} \cdot \hat{\alpha}} + u(p) \quad , \tag{2.95}$$

where $u(p)$ is defined explicitly in (2.83) with

$$u^{(o)}(p) = \int_B G_o(p, p_B)\, e^{-ikr_B} \frac{\partial}{\partial n} \left(e^{ik\vec{r}_B \cdot \hat{\alpha}} \right) d\sigma_B \quad . \tag{2.96}$$

Similarly this follows from the fact that normal derivative of the velocity potential vanishes on the surface.

Theorem 2.2 (Uniqueness)

If u_1 and u_2 are scalar wave functions for V and

$$\frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} \quad \text{on B}$$

then $u_1 \equiv u_2$, $p \in \bar{V}$.

Proof.

Suppose $u_1 \not\equiv u_2$. Then subtracting the equations

$$u_1 = L\bullet u_1 + u^{(o)}$$
$$u_2 = L\bullet u_2 + u^{(o)} \quad , \tag{2.97}$$

(where $u^{(o)}$ is the same in both equations since the normal derivatives are equal) we obtain

$$u_1 - u_2 = L\bullet (u_1 - u_2)$$

and

$$||u_1 - u_2|| \leqslant ||L||\,||u_1 - u_2|| \quad . \tag{2.98}$$

Since $||u_1 - u_2|| \neq 0$, we may divide obtaining $\|L\| > 1$. This violates Corollary 2.2, the boundedness of the operator, and theorem is proved.

2.4 Remarks on the Low Frequency Expansion

We have shown that the solution to the equation

$$\tilde{u} - kL_1 \bullet \tilde{u} = u^{(o)}$$

is given by

$$\tilde{u} = e^{-ikr} u = \sum_{n=0}^{\infty} k^n L_1^n \bullet u^{(o)} \quad , \tag{2.99}$$

where the operator L_1 is independent of k. If the boundary data is analytic in k, as is the case in Corollaries 2.3 and 2.4, then $u^{(o)}$ has the expansion

$$u^{(o)}(p) = \sum_{n=0}^{\infty} a_n(p)\, k^n \quad . \tag{2.100}$$

Substituting (2.100) into (2.99) and observing that both series are absolutely convergent, we obtain

$$\tilde{u} = \sum_{m=0}^{\infty} k^m \sum_{n=0}^{m} L_1^n \bullet a_{m-n}(p) \quad , \tag{2.101}$$

or

$$\tilde{u} = \sum_{m=0}^{\infty} u_m k^m \tag{2.102}$$

where

$$u_m = \sum_{n=0}^{m} L_1^n \bullet a_{m-n}(p) \quad . \tag{2.103}$$

From (2.103) it follows that

$$u_o = a_o$$
$$u_m = a_m + L_1 \bullet u_{m-1} \quad . \tag{2.104}$$

Equations (2.102) and (2.103) represent a low frequency expansion of $\tilde{u}$. If we had assumed the expansion (2.102), substituted it, together with the expansion (2.100) for $u^{(o)}$ into the equation $\tilde{u} = kL_1 \bullet \tilde{u} + u^{(o)}$ and equated the coefficients of k, then we would have obtained the u_m exactly as given by (2.103).

For the scalar wave function u(p) we have the corresponding expansion

$$u(p) = e^{ikr}\tilde{u} = \sum_{n=0}^{\infty} \frac{(ikr)^n}{n!} \sum_{m=0}^{\infty} u_m k^m = \sum_{n=0}^{\infty} k^n \sum_{m=0}^{n} \frac{(ir)^{n-m}}{(n-m)!} u_m \quad .$$

$$= \sum_{n=0}^{\infty} k^n \sum_{m=0}^{n} \frac{(ir)^{n-m}}{(n-m)!} \sum_{\nu=0}^{m} L_1^{\nu} \bullet a_{m-\nu}(p) \quad . \tag{2.105}$$

For these expansions the radius of convergence in the k-plane is $|k_o| > 0$ (Theorem 2.1).

Next we note the relation between the low frequency expansion and the Neumann-Liouville expansion. Specifically, partial sums $u^{(N)}$ in the Neumann-Liouville series are (with (2.100))

$$u^{(N)} = \sum_{n=0}^{N} L^n \cdot u^{(o)} = \sum_{n=0}^{N} k^n L_1^{\,n} \cdot \sum_{m=0}^{\infty} a_m k^m \quad . \tag{2.106}$$

Adding and substracting the same quantity, we have

$$u^{(N)} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} k^{n+m} L_1^{\,n} \cdot a_m - \sum_{n=N+1}^{\infty} \sum_{m=0}^{\infty} k^{n+m} L_1^{\,n} \cdot a_m \quad . \tag{2.107}$$

Using Cauchy's form of the product of two series to rewrite the first sum and shifting the index in the second enable us to write

$$u^{(N)} = \sum_{m=0}^{\infty} \sum_{n=0}^{m} k^m L_1^{\,n} \cdot a_{m-n} - \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} k^{n+m+N+1} L_1^{\,n+N+1} \cdot a_m \quad .$$

Splitting the first sum and again adjusting the index of the second yields three blocks of terms

$$u^{(N)} = \sum_{m=0}^{N} \sum_{n=0}^{m} k^m L_1^{\,n} \cdot a_{m-n} + \sum_{m=N+1}^{\infty} \sum_{n=0}^{m} k^m L_1^{\,n} \cdot a_{m-n}$$

$$- \sum_{m=0}^{\infty} \sum_{n=0}^{m} k^{m+N+1} L_1^{\,n+N+1} \cdot a_{m-n} \quad .$$

The first block is seen to be, with (2.102) and (2.103), the sum of the first N terms of the low frequency expansion while the remaining blocks may be combined by further reordering to yield

$$u^{(N)} = \sum_{m=0}^{N} k^n u_m + \sum_{m=N+1}^{\infty} \sum_{n=0}^{m} k^m L_1^n \circ a_{m-n} - \sum_{m=N+1}^{\infty} \sum_{n=0}^{m-N-1} k^m L_1^{n+N+1} \circ a_{m-N-1-n}$$

$$= \sum_{m=0}^{N} k^m u_m + \sum_{m=N+1}^{\infty} \left\{ \sum_{n=0}^{m} k^m L_1^n \circ a_{m-n} - \sum_{n=N+1}^{m} k^m L_1^n \circ a_{m-n} \right\} ,$$

and finally ,

$$u^{(N)} = \sum_{m=0}^{N} u_m k^m + \sum_{m=N+1}^{\infty} k^m \sum_{n=0}^{N} L_1^n \circ a_{m-n} \quad . \tag{2.108}$$

With (2.103) we see that the first sum on the right represents the first N terms of the series (2.102). Thus the N^{th} term of the Neumann-Liouville series is seen to contain terms of all order in k, the first N of which correspond exactly to the first N terms of the low frequency expansion.

CHAPTER III

AN APPLICATION: SCATTERING OF A PLANE WAVE OF SOUND BY AN ACOUSTICALLY RIGID SPHERE

Both as a check and an illustration we apply the techniques described in the previous chapters to a specific problem--one which is indicated by the title of this chapter. In this case, the exact result is known and we are able to show, not only that the iteration produces the correct result, but how the N^{th} iterate approximates the exact result.

The surface B is now a sphere of radius a whose center is taken as the origin of the coordinate system. The static Green's function of the second kind for this sphere is (e.g. Morse and Feshbach[29])

$$G_o(p, p_1) = -\frac{1}{4\pi}\sum_{n=0}^{\infty}\left(\frac{r_<^n}{r_>^{n+1}} + \frac{n}{n+1}\,\frac{a^{2n+1}}{(rr_1)^{n+1}}\right) P_n(\cos\gamma) \tag{3.1}$$

where $r_< = \min(r, r_1)$, $r_> = \max(r, r_1)$

and $\cos\gamma = \hat{r}\cdot\hat{r}_1 = \cos\theta\cos\theta_1 + \sin\theta\sin\theta_1\cos(\phi-\phi_1)$.

The incident field is a plane wave which, without loss of generality, is chosen as propagating down the z-axis, i.e.,

$$u^i = e^{-ik\vec{r}\cdot\hat{i}_z} = e^{-ikz} = e^{-ikr\cos\theta} \tag{3.2}$$

The boundary values of interest are (with the well-known plane wave expansion)

$$\frac{\partial u^i}{\partial n} = -\frac{\partial}{\partial r}\, e^{-ikr\cos\theta}\Big|_{r=a} = -k\sum_{n=0}^{\infty}(-i)^n(2n+1)\, j_n'(ka)\, P_n(\cos\theta) \tag{3.3}$$

where j_n is the spherical Bessel function and the prime denotes differentiation with respect to ka, i.e.,

$$j_n'(ka) = \frac{d}{d(ka)}\left[\left(\frac{\pi}{2ka}\right)^{1/2} J_{n+1/2}(ka)\right] \tag{3.4}$$

The scattered field, u (where $\partial u/\partial n = -\partial u^i/\partial n$ on B), is given by the methods described previously as

$$u = \lim_{N\to\infty} e^{ikr} u^{(N)} \tag{3.5}$$

where

$$u^{(N)} = \sum_{n=0}^{N} L^n \circ u^{(o)} \quad , \tag{3.6}$$

$$L\circ u^{(o)}(p) = \frac{ik}{2\pi}\int_V \sum_{n=0}^{\infty}\left\{\frac{r_<^n}{r_>^{n+1}} + \frac{n}{n+1}\frac{a^{2n+1}}{(rr_1)^{n+1}}\right\} P_n(\cos\gamma)\frac{1}{r_1}\frac{\partial}{\partial r_1}\left[r_1 u^{(o)}(p_1)\right] dv_1$$

$$+\frac{ik}{4\pi}\int_B \sum_{n=0}^{\infty}\frac{2n+1}{n+1}\frac{a^n}{r^{n+1}} P_n(\cos\gamma)\, u^{(o)}(p_B)\, d\sigma_B \tag{3.7}$$

(cos γ always involves coordinates of p and the integration variables) and

$$u^{(o)}(p) = \frac{ke^{-ika}}{4\pi}\int_B \sum_{n=0}^{\infty}\frac{2n+1}{n+1}\frac{a^n}{r^{n+1}} P_n(\cos\gamma)\sum_{m=0}^{\infty}(-i)^m(2m+1)j_m'(ka)P_m(\cos\theta_B)d\sigma_B \tag{3.8}$$

The orthogonality of the Legendre functions enables us to evaluate $u^{(o)}$ and the first three iterates. Omitting the details, these are found to be

$$u^{(o)}(p) = ka\, e^{-ika} \sum_{n=0}^{\infty} \frac{(2n+1)}{(n+1)} (-i)^n \frac{a^{n+1}}{r^{n+1}} j_n'(ka) P_n(\cos\theta) \tag{3.9}$$

$$u^{(1)}(p) = ka\, e^{-ika} \frac{a}{r} j_o'(ka)(1+ika)$$

$$+ ka\, e^{-ika} \sum_{n=1}^{\infty} \frac{2n+1}{n+1} (-i)^n \left(\frac{a}{r}\right)^{n+1} j_n'(ka) P_n(\cos\theta) \left[1 + ika - ikr\right] \tag{3.10}$$

$$u^{(2)}(p) = ka\, e^{-ika} \frac{a}{r} j_o'(ka) \left[1 + ika - (ka)^2\right] + ka\, e^{-ika} \sum_{n=1}^{\infty} \frac{2n+1}{n+1} (-i)^n \left(\frac{a}{r}\right)^{n+1} \quad x$$

$$x \quad j_n'(ka) P_n(\cos\theta) \left[1 + ik(a-r) + k^2 ra - \frac{(kr)^2(n-1)}{2n-1} - \frac{(ka)^2(n^2+n-1)}{(n+1)(2n-1)}\right] . \tag{3.11}$$

The exact expression for u is (e.g. Morse and Feshbach[29])

$$u(p) = - \sum_{n=0}^{\infty} (-i)^n (2n+1) j_n'(ka)\, P_n(\cos\theta) \frac{h_n(kr)}{h_n'(ka)} , \tag{3.12}$$

where the prime again denotes differentiation with respect to ka (see (3.4)) and h_n is a spherical Hankel function of the first kind. Expliticly

$$h_n(z) = \frac{e^{iz}}{z} i^{-n-1} \sum_{m=0}^{n} \frac{(n+m)!}{(n-m)!\, m!} \frac{1}{(-2iz)^m} . \tag{3.13}$$

With this definition we find that

$$\frac{h_n(kr)}{h'_n(ka)} = -ka\, e^{ik(r-a)} \left(\frac{a}{r}\right)^{n+1} \frac{\displaystyle\sum_{m=0}^{n} \frac{(2n-m)!}{(n-m)!\, m!} (-2ikr)^m}{\displaystyle\sum_{m=0}^{n} \frac{(2n-m)!}{(n-m)!\, m!} (-2ika)^m (n+1-m-ika)} \tag{3.14}$$

The ratio of the two polynomials, of degree n in kr in the numerator and $n+1$ in ka in the denominator, may be re-expanded, for ka sufficiently small, in ascending powers of k. Thus

$$\frac{h_n(kr)}{h'_n(ka)} = -ka \frac{e^{ik(r-a)}}{n+1} \left(\frac{a}{r}\right)^{n+1} \sum_{\ell=0}^{\infty} \alpha_\ell k^\ell \quad , \quad n \geqslant 0 \tag{3.15}$$

where the coefficients α_ℓ are functions of r, a, and n. The first three are found to be

$$\begin{aligned}
\alpha_o &= 1 \\
\alpha_1 &= ia \, , \qquad n = 0 \\
& i(a-r) \, , \; n > 0 \\
\alpha_2 &= -a^2 , \qquad n = 0 \\
&= ar - \frac{n-1}{2n-1} r^2 - \frac{n^2+n-1}{(n+1)(2n-1)} a^2 \quad , \qquad n > 0
\end{aligned} \tag{3.16}$$

In terms of these expansions the exact result for u, Eq. (3.12), may be rewritten as

$$u(p) = ka\, e^{ik(r-a)} \sum_{n=0}^{\infty} (-i)^n \frac{2n+1}{n+1} \left(\frac{a}{r}\right)^{n+1} j'_n(ka)\, P_n(\cos\theta) \sum_{\ell=0}^{\infty} \alpha_\ell k^\ell \, . \tag{3.17}$$

If we denote by $u_N(p)$ the expression resulting from taking only the first N terms in the expansions in k in (3.17), that is,

$$u_N = ka\, e^{ik(r-a)} \sum_{n=0}^{\infty} (-i)^n \frac{2n+1}{n+1} \left(\frac{a}{r}\right)^{n+1} j_n'(ka)\, P_n(\cos\theta) \sum_{\ell=0}^{N} \alpha_\ell k^\ell \tag{3.18}$$

then we see that, for the values of N computed,

$$u_N = e^{ikr} u^{(N)} \tag{3.19}$$

where the first three iterates, $u^{(N)}$, are given in (3.9) through (3.11).

CHAPTER IV

CONCLUSION

The main result of this work consists of (a) the derivation of an integro-differential equation for the exterior Neumann problem whose kernel is the potential Green's function of the second kind, and (b) an effective approximation method for it.

We shall now indicate some of the areas which are the natural extensions of this (and the Dirichlet) problem, and which are not included in the present work.

4.1 Functional Analytic Aspects

It is immediately evident from its definition that the space in which the perturbation is performed is not complete. For this reason, it was necessary to show that the solution to the problem was an element of this space and that in the sense of our norm iterates converge to this function. However, let the norm be given as follows:

$$\|u\| = \max_{p\in\overline{V}} |u(p)| + \max_{p\in\overline{V}} |D^1 u(p)| + \max_{|z|\leqslant 1} |u(\theta, \phi, \tfrac{1}{z})| , \quad 0\leqslant \theta \leqslant \pi , \quad 0\leqslant\phi\leqslant 2\pi ,$$

where D^1 denotes the first derivatives with respect to any one of the variables θ, ϕ, or r, and $|z| = \frac{1}{|r|} \leqslant 1$ is the unit disc in the complex z plane defined in Chapter II. Then it should be possible to show (the work on this is being completed) that this new space is a Banach space, and that the operator $L = kL_1$, defined in Chapter II, is compact. With these facts, we may appeal to the Banach fixed point theorem to conclude the existence (and uniqueness) of the solution to the equation $\tilde{u} + L\cdot\tilde{u} = u^{(o)}$.

Another norm with which the present problem may, with somewhat more tediousness, be solved is

$$\|u\| = \max\left\{ \max_{p\in\overline{V}} |u(p)| \,,\ \max_{\substack{0\leqslant\theta\leqslant\pi \\ 0\leqslant\phi\leqslant 2\pi \\ r\geqslant b}} \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} \frac{\left|Y_m^{\,n}\right|}{r^{n+1}} \right\} ,$$

where b some constant greater then 1, and Y_m^n are the spherical harmonics for u, specifically,

$$u = \sum_{n=0}^{\infty} \frac{f_n(\theta,\phi)}{r^{n+1}} \,, \quad r \geqslant 1$$

$$f_n(\theta,\phi) = \sum_{m=n}^{\infty} Y_m^{\,n}(\theta,\phi)$$

$$Y_m^{(n)} \sum_{\ell=-m}^{m} A_{\ell m}^{n} P_m^{\ell} (\cos\theta)\, e^{i\ell\phi} \quad .$$

Again, we note, with this norm the space W is not complete.

4.2 Non-Separable Surfaces. Bodies with Edges.

It is now possible to "solve" the Neumann (and Dirichlet) problem for the scalar Helmholtz equation in the regions exterior to a non-separable body, provided k, the complex wave number, is sufficiently small in modulus, and the solution of the Laplace's equation can be obtained for the body in question. This is done for one such body, an ogive (see Ar[(1)]). It should be noted in this connection that the explicit representation of the solution of the Neumann (and Dirichlet) problem has been proven only for smooth bodies (see Chapter I for the definition of smoothness). However, preliminary calculations for the circular disc support the hypothesis that the representation remains valid for bodies with edges (e.g. an ogive). To prove this, however, will require a different definition of norm, since it is known (e.g. Bouwkamp[5], Meixner[28]) that wave functions associated with bodies with edges have singular derivatives. Thus, convergence of the iterates in these cases will have to be established in some other norm.

Another non-separable body is the torus. The solution for this problem is also presently under consideration. Since, in this case, the smoothness requirements are satisfied, the present norm is sufficient to justify the iteration.

4.3 The Radius of Convergence of the Low Frequency Expansion

While we have proven that the series convergences for $|k|$ sufficiently small, that is, there exists some number $|k_o| > 0$ such that the series converges for $|k| < |k_o|$, no indication was given as to how large $|k_o|$ may be. If the boundary data are analytic in k, this problem of estimation is equivalent to finding the radius of convergence of the low frequency expansion. Such estimates are available only for special surfaces (e.g. Darling and Senior[38]) and the general problem remains unsolved. If the exact radius of convergence is found, the analytic continuation into the complex k-plane is then possible. Restriction of this continuation to the real line would in effect "solve" the persistent problem of the "resonance region".

4.4 Extension to Vector (Electromagnetic) Problems

Here the goal is an explicit iterative solution, as opposed to Stevenson's technique[42] in which each successive term in the series solution can be found only by solving a new problem.

4.5 Two-Dimensional Low Frequency Scattering Problems.

The success in three-dimensional problems is due in part to the existence of the expansion (see Chapter I) for the wave functions. While the comparable expansion for two-dimensional wave functions is more complicated (Karp[18]), its very existence offers some hope that an iterative method analogous to that for three-dimensional problems could be found. In one respect, success in this area would be more far reaching than in the three-dimensional case since, in contrast to that case, two-dimensional potential problems are all essentially solvable using conformal mapping; thus, all two-dimensional scattering problems would also be solvable.

REFERENCES

(1) Ar, Ergun, "Low Frequency Scattering from an Ogive," The University of Michigan, Radiation Laboratory Report No. 7030-3-T; 1965. (Has been submitted to Quart. Appl. Math. for publication.)

(2) Atkinson, F.V., "On Sommerfeld's Radiation Condition," Philos. Mag., XL, Series 7, No. 305, pp. 645-651; 1949.

(3) Baker, B.B. and E.T. Copson, The Mathematical Theory of Huygens' Principle, Oxford: Clarendon Press; 1950.

(4) Barrar, R.B. and A.F. Kay, "A Series Development of a Solution of the Wave Equation in Powers of 1/r," Internal Memorandum, Tech. Res. Group, Inc., NYC.

(5) Bouwkamp, C.J., "Diffraction Theory," Rep. Progr. Phys., 17, pp. 35-100; 1954.

(6) Collins, W.D., "Some Scalar Diffraction Problems for a Spherical Cap," Arch. Rat. Mech. Anal., 10, pp. 249-266; 1962.

(7) Darling, D.A., "Some Relations Between Potential Theory and the Wave Equation," The University of Michigan, Radiation Laboratory Report No. 2871-5-T; 1960.

(8) Dolph, C.L. and R.B. Barrar, "On a Three-Dimensional Transmission Problem of Electromagnetic Theory," Arch. Rat. Mech. Anal., 3, No. 6, pp. 725-743; 1954.

(9) Ehrmann, H., "Iterationsverfahren mit veränderlichen Operatoren," Arch. Rat. Mech. Anal., 4, No. 1, pp. 45-46; 1959-60.

(10) Ehrman, H., "On Implicit Function Theorems and the Existence of Solutions of Non-linear Equations," University of Wisconsin Technical Report No. 343; 1962.

(11) Garabedian, P.R., "An Integral Equation Governing Electromagnetic Waves," Quart. Appl. Math., XII, No. 4, pp. 428-433; 1955.

(12) Hartman, P. and C. Wilcox, "On Solutions of the Helmholtz Equation in Exterior Domains," Math. Zeitschr, 75, pp. 228-255; 1961.

(13) Heins, A.E., "Function-Theoretic Aspects of Diffraction Theory," Electromagnetic Waves (R. Langer, ed.). Madison: The University of Wisconsin Press; 1962.

(14) Heins, A.E., "Axially-Symmetric Boundary Value Problems," Bull. Am. Math. Soc., 71, No. 6; 1965.

(15) deHoop, A.T., "On the Scalar Diffraction by a Circular Aperture in an Infinite Plane Screen," Appl. Sci. Res., B4, pp. 151-160; 1954.

(16) Hörmander, L., Linear Differential Operators. Academic Press, Inc.; 1963.

(17) Ikebe, T., "Eigenfunction Expansions Associated with Schroedinger Operators and Their Applications to Scattering Theory," Arch. Rat. Mech. Anal., 5, No. 1, pp. 1-34; 1960.

(18) Karp, S.N., "A Convergent 'Far Field' Expansion for Two-Dimensional Radiation Functions," Comm. Pure Appl. Math., XIV, pp. 427-434; 1961.

(19) Kato, T., "Growth Properties of Solutions of the Reduced Wave Equation with a Variable Coefficient," Comm. Pure Appl. Math., XII, pp. 403-425; 1959.

(20) Kellogg, O.D., Foundations of Potential Theory. New York: Dover Publications; 1953.

(21) Kleinman, R.E., "The Dirichlet Problem for the Helmholtz Equation." Arch. Rat. Mech. Anal., 18, pp. 205-229; 1965.

(22) Kupradse, W.D., Randwertaufgaben der Schwingungstheorie und Integralgleichungen. Veb Deutscher Verlag Der Wissenschaften: Berlin;

(23) Leis, R., "Über das Neumannsche Randwertproblem für die Helmholtzsche Schwingungsgleichung," Arch. Rat. Mech. Anal., 2, pp. 101-113; 1958.

(24) Levine, L.M., "A Uniqueness Theorem for the Reduced Wave Equation," Comm. Pure Appl. Math., XVII, No. 2; 1964.

(25) Magnus, W., "Über Eindeutigkeitsfragen bei einer Randwertaufgabe," Jahresber. Deut. Math., Verein 52, pp. 177-188; 1942.

(26) Magnus, W., "Fragen der Eindeutigkeit und des Verhaltens in Unendlichen für Lösungen von $\Delta u + k^2 u = 0$," Math. Seminar der Univ. Hamburg, 16, p. 77; 1949.

(27) Magnus, W. and F. Oberhettinger, Formulas and Theorems for the Special Functions of Mathematical Physics. New York: Chelsea Publ. Co.; 1949.

(28) Meixner, J., "Die Kantenbedingung in der Theorie der Beugung elektromagnetischer Wellen an vollkommen leitenden ebenen Schirmen," Ann. Phys., 6, pp. 2-9; 1949.

(29) Morse, P.M. and H. Feshbach, Methods of Theoretical Physics. New York: McGraw-Hill; 1953.

(30) Müller, C., "Zur Methoden der Strahlungskapazität von H. Weyl," Math. Z., 56, pp. 80-83; 1952.

(31) Noble, B., "Integral Equation Perturbation Methods in Low Frequency Diffraction," Electromagnetic Waves (R. Langer, ed.). Madison: The University of Wisconson Press; 1962.

(32) Odeh, F. and C. Zemach, "Uniqueness of Radiative Solutions to the Schroedinger Wave Equation," Arch. Rat. Mech. Anal., 5, No. 3, pp. 226-237; 1960.

(33) Lord Rayleigh, "On the Incidence of Aerial and Electric Waves upon Small Obstacles in the Form of Ellipsoids or Elliptic Cylinders and on the Passage of Electric Waves through a Circular Aperture in a Conducting Screen," Philos. Mag., XLIV, pp. 28-52; 1897.

(34) Rellich, F., "Über das asymptotische Verhalten der Lösungen von $\Delta u + \lambda u = 0$ in unendlichen Gebieten," Jahresber. Deut. Math., Verein 53, pp. 57-64; 1943.

(35) Rothe, E., "Über die Integralgleichungen des Skineffekts," J. für Reine und Angew. Math., 170, No. 4, pp. 218-230; 1934.

(36) Saunders, W.K., "On Solutions of Maxwell's Equations in an Exterior Region," Proc. National Academy of Sciences, 38, No. 4, pp. 342-348; 1952.

(37) Senior, T.B.A., "Scalar Diffraction by a Prolate Spheroid at Low Frequencies," Can. J. Phys., 38, pp. 1632-1641; 1960.

(38) Senior, T.B.A. and D.A. Darling, "Low Frequency Expansions for Scattering by Separable and Non-Separable Bodies," J. Acoust. Soc. Amer., 37, No. 2, pp. 228-234; 1965.

(39) Sommerfeld, A., "Der Greensche Funktion der Schwingungsgleichung," Jahresber. Deut. Math., Verein. 21, pp. 309-353; 1912.

(40) Sommerfeld, A., Partial Differential Equations in Physics. New York: Academic Press; 1949.

(41) Sternberg, W., "Anwendung der Integralgleichungen in der elektromagnetichen Lichttheory," Zeit. für Physik, 64, pp. 638-649; 1930.

(42) Stevenson, A.F., "Solution of Electromagnetic Scattering Problems as Power Series in the Ratio (Dimension of Scatterer/Wavelength)," J. Appl. Phys., 24, pp. 1134-1142; 1953).

(43) Stratton, J.A., Electromagnetic Theory. New York: McGraw-Hill; 1941.

(44) Thomas, D.P., "Diffraction by a Spherical Cap," Proc. Cambridge Philos. Soc., 59, pp. 197-209; 1963.

(45) Werner, P., "Zur mathematischen Theorie akustischer Wellenfelder," Arch. Rat. Mech. Anal., 6-7, pp. 231-260; 1960-61.

(46) Werner, P., "Boundary Value Problems in the Theory of Acoustic Wave Fields," (Two Parts) University of Wisconsin Technical Report No. 233 and 234; 1961.

(47) Werner, P., "On the Exterior Boundary Value Problem of Perfect Reflection for Stationary Electromagnetic Wave Fields," University of Wisconsin Technical Report No. 286; 1962.

(48) Werner, P., "Randwertprobleme der mathematischen Akustik," Arch. Rat. Mech. Anal., 10, pp. 29-66; 1962.

(49) Werner, P., "Beugungsprobleme der mathematischen Akustik," Arch. Rat. Mech. Anal., 12, pp. 155-184; 1963.

(50) Weyl, H., "Kapazität von Strahlungsfeldern," Math. Z., 55, pp. 187-198; 1952.

(51) Wilcox, C.H., "A Generalization of Theorems of Rellich and Atkinson," Proc. Amer. Math. Soc., 7, pp. 271-276; 1956.

(52) Wilcox, C.H., "An Expansion Theorem for Electromagnetic Fields," Comm. Pure Appl. Math., IX, pp. 115-134; 1956.

(53) Wilcox, C.H., "Spherical Means and Radiation Conditions," Arch. Rat. Mech. Anal., 3, pp. 133-148; 1959.

(54) Williams, W.E., "Diffraction by a Disk," Proc. Roy. Soc. London. Ser. A 267, pp. 77-87; 1962.

(55) Williams, W.E., "The Reduction of Boundary Value Problems to Fredholm Integral Equations of the Second Kind," Z. Angew. Math. Phys., 13, pp. 133-152; 1962.