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Commutators and Certain II_1-Factors

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1. INTRODUCTION

Recently some progress has been made in the structure theory of commutators in von Neumann algebras [*1*]–[*3*]. The theory is far from complete, however, and one of the most intractable of the unsolved problems, is that of determining the commutators in a finite von Neumann algebra. A commutator in a finite von Neumann algebra must, of course, have central trace zero, and is not unreasonable to hope that the commutators in such an algebra are exactly the operators with central trace zero. However, despite considerable effort, this has been proved only in case the algebra is a finite direct sum of algebras of type I_n [*4*].

In this note, we consider a certain class of factors of type II_1 discovered by Wright [*13*], and we show that every *Hermitian* operator with trace zero in such a factor is a commutator in the factor. This is accomplished by first proving that every Hermitian operator with central trace zero in an arbitrary finite von Neumann algebra of type I is a commutator in the algebra.

Finally, we turn our attention to the problem of characterizing the linear manifold $[\mathfrak{A}, \mathfrak{A}]$ spanned by the commutators in an arbitrary von Neumann algebra $\mathfrak{A}$ of type II_1. We give three characterizations; in particular we show that $[\mathfrak{A}, \mathfrak{A}]$ coincides with the set of all linear combinations $\sum_{i=1}^{n} \alpha_i E_i$ where $\sum_{i=1}^{n} \alpha_i = 0$ and each E_i is equivalent in $\mathfrak{A}$ to $I - E_i$.

2. FINITE ALGEBRAS OF TYPE I

A *von Neumann algebra* is a weakly closed, self-adjoint algebra of operators that contains the identity operator on its underlying Hilbert space. Our standard reference for the general theory of von Neumann algebras is Dixmier's book [6]. An operator A in a von Neumann algebra $\mathcal{A}$ is a *commutator* in $\mathcal{A}$ if there exist operators B and C in $\mathcal{A}$ with $A = BC - CB$. If $\mathcal{H}$ is any (complex) Hilbert space, the von Neumann algebra consisting of all (bounded, linear) operators on $\mathcal{H}$ will be denoted by $\mathcal{L}(\mathcal{H})$.

In this section, we show that every Hermitian operator with central trace zero in an arbitrary finite von Neumann algebra $\mathcal{A}$ of type I is a commutator in $\mathcal{A}$. For this purpose, the following lemmas are needed.

LEMMA 2.1. *If $\{\lambda_i\}_{i=1}^n$ is any sequence of real numbers such that $|\lambda_i| \leqslant 1$ $(1 \leqslant i \leqslant n)$ and $\sum_{i=1}^n \lambda_i = 0$, then there exists a permutation π on the set of the first n positive integers such that for all k with $1 \leqslant k \leqslant n$,*

$$\left| \sum_{i=1}^{k} \lambda_{\pi(i)} \right| \leqslant 1.$$

The proof of this lemma is an easy exercise and is omitted.

LEMMA 2.2. *Suppose that A is the $n \times n$ diagonal matrix*

$$A = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix},$$

where $\sum_{i=1}^n \lambda_i = 0$. Then A is the commutator $A = BC - CB$, where

$$B = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 1 & & & & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & & & & \\ \sigma_1 & 0 & & & \\ & \sigma_2 & \ddots & & \\ & & \ddots & \ddots & \\ & & & \sigma_{n-1} & 0 \end{pmatrix},$$

and the numbers σ_k are defined as $\sigma_k = \sum_{i=1}^k \lambda_i$.

Proof. Compute.

LEMMA 2.3. *Suppose that A is a Hermitian contraction with central trace zero in an n-homogeneous, finite von Neumann algebra of type* I. *Then A is the commutator $A = BC - CB$ of two operators B and C in $\mathcal{A}$ satisfying $\| B \|, \| C \| \leqslant 2$.*

Proof. One knows that the maximal ideal space $\mathfrak{X}$ of the center of $\mathfrak{A}$ is an extremally disconnected, compact Hausdorff space [*11*], and that $\mathfrak{A}$ is $*$-isomorphic to the C^*-algebra $M_n(\mathfrak{X})$ of all continuous functions from $\mathfrak{X}$ to the full ring M_n of $n \times n$ complex matrices [*9*]. (The operations in $M_n(\mathfrak{X})$ are defined pointwise, and the norm is the supremum norm). Thus the problem can be transferred to $M_n(\mathfrak{X})$. If A_1 is the image in $M_n(\mathfrak{X})$ of the operator A, then of course A_1 is a Hermitian contraction, and furthermore trace $(A_1(x)) = 0$ for all $x \in \mathfrak{X}$ ([*9*], p. 1409). According to [*5*], Corollary 3.3, A_1 is unitarily equivalent in $M_n(\mathfrak{X})$ to an element A_2 such that $A_2(x)$ is diagonal for each $x \in \mathfrak{X}$. It thus suffices to show that A_2 is the commutator in $M_n(\mathfrak{X})$ of two elements B and C satisfying $\| B \|, \| C \| \leqslant 2$, and this goes as follows. Let A_2 be the function

$$A_2(x) = \begin{pmatrix} \lambda_1(x) & & & \\ & \lambda_2(x) & & \\ & & \ddots & \\ & & & \lambda_n(x) \end{pmatrix}$$

and let x_0 be an arbitrary point of $\mathfrak{X}$. Since for $1 \leqslant i \leqslant n$,

$$|\lambda_i(x_0)| \leqslant \sup\{\| A_2(x)\| : x \in \mathfrak{X}\} = \| A_2 \| = \| A \| \leqslant 1,$$

it follows from Lemma 2.1 that there exists a permutation π of the first n positive integers such that

$$\left| \sum_{i=1}^{k} \lambda_{\pi(i)}(x_0) \right| \leqslant 1 \qquad (1 \leqslant k \leqslant n).$$

Since the functions λ_i are continuous and $\mathfrak{X}$ is totally disconnected, there exists a compact open neighborhood $\mathscr{U}_{x_0}$ of x_0 such that for every $x \in \mathscr{U}_{x_0}$,

$$\left| \sum_{i=1}^{k} \lambda_{\pi(i)}(x) \right| \leqslant 2 \qquad (1 \leqslant k \leqslant n).$$

Furthermore there exists a unitary permutation matrix U_{x_0} in M_n such that

$$U_{x_0} A_2 U_{x_0}^* = \begin{pmatrix} \lambda_{\pi(1)} & & & \\ & \lambda_{\pi(2)} & & \\ & & \ddots & \\ & & & \lambda_{\pi(n)} \end{pmatrix}.$$

Now the collection $\{\mathscr{U}_{x_0}\}_{x_0 \in \mathfrak{X}}$ is an open covering of $\mathfrak{X}$, and since $\mathfrak{X}$ is compact, there is a finite subcovering $\{\mathscr{U}_{x_1}, \mathscr{U}_{x_2}, \ldots, \mathscr{U}_{x_m}\}$. Since each $\mathscr{U}_{x_i}$ is compact as well as open, we may suppose, by shrinking the $\mathscr{U}_{x_i}$ if necessary, that the sets $\mathscr{U}_{x_i}$ are pairwise disjoint. A unitary element U in $M_n(\mathfrak{X})$ can now be defined by setting $U(x) = U_{x_0}$ for each $x \in \mathscr{U}_{x_0}$, and the function

$$UA_2U^*(x) = \begin{pmatrix} \beta_1(x) & & & \\ & \beta_2(x) & & \\ & & \ddots & \\ & & & \beta_n(x) \end{pmatrix}$$

satisfies

$$\left| \sum_{i=1}^{k} \beta_i(x) \right| \leqslant 2 \qquad (1 \leqslant k \leqslant n,\ x \in \mathfrak{X}).$$

The proof is completed by applying the construction of Lemma 2.2 to UA_2U^* to yield elements B and C in $M_n(\mathfrak{X})$ such that $UA_2U^* = BC - CB$ and $\| B \|, \| C \| \leqslant 2$.

THEOREM 1. *A Hermitian operator with central trace zero in an arbitrary finite von Neumann algebra of type* I *is a commutator in the algebra.*

Proof. If $\mathfrak{A}$ is an arbitrary finite von Neumann algebra of type I, then there is an increasing sequence N (finite or infinite) of positive integers and a corresponding sequence $\{\mathfrak{A}_n\}_{n \in N}$ of n-homogeneous algebras such that $\mathfrak{A} = \sum_{n \in N} \oplus \mathfrak{A}_n$. Since $\mathfrak{A}$ consists of all operators $A = \sum_{n \in N} \oplus A_n$, where $A_n \in \mathfrak{A}_n$ and the sequence $\{\| A_n \|\}_{n \in N}$ is bounded, the result follows immediately from Lemma 2.3.

3. THE II_1-FACTORS OF WRIGHT

We begin this section by reviewing the construction of a special class of II_1-factors that were discovered by Wright [*13*]. Throughout this section N will denote the set of all integers greater than 1. For each $n \in N$, let $\mathscr{H}_n$ be n-dimensional complex Hilbert space, and let $\mathfrak{A} = \sum_{n \in N} \oplus \mathscr{L}(\mathscr{H}_n)$. Then $\mathfrak{A}$ is a finite von Neumann algebra of type I whose center $\mathscr{Z}$ can be (and is hereafter) identified with the algebra of all continuous complex-valued functions on $\beta(N)$, the Stone-Čech compactification of the discrete space N. Let the unique central trace on the algebra $\mathfrak{A}$ be the mapping denoted by $A \rightarrow A^\natural$, and recall that if $A = \sum \oplus A_n \in \mathfrak{A}$, then $A^\natural$ is the continuous function on $\beta(N)$ whose value at an integer n is $A^\natural(n) = (1/n)t_n(A_n)$, where t_n denotes the unnormalized trace on the algebra $\mathscr{L}(\mathscr{H}_n)$. It is known ([13], Theorems 2.6 and 3.1) that there is a one-to-one correspondence between the maximal (two-sided) ideals $\mathscr{M}$ of $\mathfrak{A}$ and the points p of $\beta(N)$ defined by

$$\mathscr{M} = \{T \in \mathfrak{A} : (T^*T)^\natural (p) = 0\}.$$

Wright showed in [13] that, if $\mathscr{M}$ is a maximal ideal corresponding to a point p in $\beta(N) - N$, then the quotient algebra $\mathscr{W} = \mathfrak{A}/\mathscr{M}$ is an AW^*-factor of type II_1. Subsequently, Feldman showed in [7] that $\mathscr{W}$ has a faithful weakly closed representation, and hence is a von Neumann factor of type II_1. (He also showed that every such faithful representation of $\mathscr{W}$ necessarily acts on a nonseparable Hilbert space.) Throughout the remainder of this section, $\mathscr{W}$ will denote the II_1 von Neumann factor $\mathscr{W} = \mathfrak{A}/\mathscr{M}$ corresponding to a fixed point p of $\beta(N) - N$. It is easy to identify the unique numerical trace on the factor $\mathscr{W}$; its value at an element $A + \mathscr{M}$ of $\mathscr{W}$ is $A^\natural(p)$ ([6] Corollaire, p. 272).

We proceed now to the central lemma of this section, which shows that any element of $\mathscr{W}$ whose numerical trace is zero can be "lifted" to an operator in $\mathfrak{A}$ having central trace zero.

LEMMA 3.1. *Let $A + \mathscr{M}$ be an element of $\mathscr{W}$ such that $A^\natural(p) = 0$. Then there exists an operator B in $\mathfrak{A}$ such that $B + \mathscr{M} = A + \mathscr{M}$ and such that $B^\natural = 0$. Furthermore, if $A + \mathscr{M}$ is a Hermitian element of $\mathscr{W}$, then B can be taken to be Hermitian.*

Proof. To prove the first assertion of the theorem, it suffices to exhibit an operator C in $\mathscr{M}$ such that $C^\natural = A^\natural$. (Then define $B = A - C$.) For this purpose, define $C = A^\natural$. Then at least $C^\natural = A^\natural$, and we proceed to show that $A^\natural$ lies in $\mathscr{M}$, i.e., that $[(A^\natural)^*A^\natural]^\natural(p) = 0$. Note that $[(A^\natural)^*A^\natural]^\natural = (A^\natural)^*A^\natural$, and that the value of this continuous function [on $\beta(N)$] at an arbitrary integer n is $|A^\natural(n)|^2$. Thus for every $n \in N$ we have $(C^*C)^\natural(n) = |A^\natural(n)|^2$, and since N is dense in $\beta(N)$, we have $(C^*C)^\natural(p) = |A^\natural(p)|^2 = 0$, so that $C \in \mathscr{M}$ as desired.

Observe that in the argument just completed, if A is a Hermitian operator, then $B = A - C = A - A^\natural$ is also Hermitian. Thus the second assertion of the lemma follows by noting that if $A + \mathscr{M}$ is Hermitian, then $A - A^* \in \mathscr{M}$ and $A + \mathscr{M} = \frac{1}{2}(A + A^*) + \mathscr{M}$.

We now turn to the main result of our investigation.

THEOREM 2. *Every Hermitian operator with trace zero in the* II_1*-factor* $\mathcal{W} = \mathfrak{A}/\mathcal{M}$ *is a commutator in* $\mathcal{W}$. *Thus, every operator in* $\mathcal{W}$ *with trace zero is the sum of two commutators in* $\mathcal{W}$.

Proof. If $A + \mathcal{M}$ is a Hermitian element of $\mathcal{W}$ with $A^\natural(p) = 0$, then by Lemma 3.1 there exists a Hermitian operator $B \in \mathfrak{A}$ such that $A - B \in \mathcal{M}$ and $B^\natural = 0$. By Theorem 1, B is a commutator in $\mathfrak{A}$, and hence $A + \mathcal{M} = B + \mathcal{M}$ is a commutator in $\mathcal{W}$.

The last assertion of the theorem follows by decomposing an arbitrary operator in $\mathcal{W}$ with trace zero into its real and imaginary parts, and observing that each part also has trace zero.

4. COMMUTATORS, NILPOTENTS AND PROJECTIONS

In this section we shall give three characterizations of the linear span $[\mathfrak{A}, \mathfrak{A}]$ of the commutators in a von Neumann algebra $\mathfrak{A}$ of type II_1.

LEMMA 4.1. *Every commutator in a von Neumann algebra* $\mathfrak{A}$ *of type* II_1 *is the sum of ten operators in* $\mathfrak{A}$ *each having square zero.*

Proof. By the "halving lemma" ([*6*], Corollaire 3, p. 229), there is a projection $E \in \mathfrak{A}$ such that $\mathfrak{A}$ is spatially isomorphic to the algebra $M_2(E\mathfrak{A}E)$ of all 2×2 matrices over the algebra $E\mathfrak{A}E$.

Thus we may regard a commutator $T \in \mathfrak{A}$ as the commutator of two 2×2 matrices in $M_2(E\mathfrak{A}E)$. By so doing, we see that the resulting matrix for T has the property that the sum of its diagonal entries is the sum of four commutators in $E\mathfrak{A}E$. Thus T may be written as the sum of two 2×2 matrices in $M_2(E\mathfrak{A}E)$ each having the property that the sum of its diagonal entries is the sum of two commutators in $E\mathfrak{A}E$. The construction of [*10*], Theorem 5 can now be applied to show that each of these two matrices is the sum of five nilpotents in $E\mathfrak{A}E$ of index two.

LEMMA 4.2. *Let* $\mathfrak{A}$ *be a von Neumann algebra of type* II_1, *and let* T *be an operator in* $\mathfrak{A}$ *such that* $T^2 = 0$. *Then there is a projection* E *in* $\mathfrak{A}$ *with* $ET = T$, $TE = 0$ *and* $E \sim I - E$.

Proof. The partial isometry V appearing in the canonical polar decomposition of T maps the orthogonal complement of the null space of T isometrically onto the closure of the range of T. The projections $F = V^*V$ and $G = VV^*$ are orthogonal and $TG = 0$

since the null space of T contains its range. Clearly $GT = T$ and the product of $I - F - G$ with T in either order is zero. Now apply the "halving lemma" ([6], Corollaire 3, p. 229) to write $I - F - G$ as the orthogonal sum of two equivalent projections. Either of the latter added to G gives a projection E having the desired properties.

LEMMA 4.3. *If T is an operator satisfying $T^2 = 0$ in a von Neumann algebra $\mathcal{A}$ of type* II_1, *then T can be written as a linear combination*

$$T = \sum_{i=1}^{16} \alpha_i E_i ,$$

where $\sum_{i=1}^{16} \alpha_i = 0$ and each E_i is a projection in $\mathcal{A}$ equivalent to $I - E_i$.

Proof. According to Lemma 4.2, there is a projection $E \in \mathcal{A}$ satisfying $ET = T$, $TE = 0$ and $E \sim I - E$. The projections E and $I - E$ together with a partial isometry implementing their equivalence can be used to establish a spatial isomorphism between $\mathcal{A}$ and $M_2(E\mathcal{A}E)$ which carries T onto a matrix in $M_2(E\mathcal{A}E)$ of the form

$$\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$$

Since A can be written as a linear combination of four unitary operators in $E\mathcal{A}E$ ([6], Proposition 3, p. 4), it suffices to deal with operators in $M_2(E\mathcal{A}E)$ of the form

$$N = \begin{pmatrix} 0 & U \\ 0 & 0 \end{pmatrix}$$

where U is unitary.

Now define

$$P = \frac{1}{2}\begin{pmatrix} I & U \\ U^* & I \end{pmatrix}, \quad Q = \frac{1}{2}\begin{pmatrix} I & -iU \\ iU^* & I \end{pmatrix},$$

$$G = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}, \quad S = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix},$$

$$X = 2P - I, \quad \text{and} \quad Y = 2Q - I.$$

Then

$$N = P + iQ - \left(\frac{1+i}{2}\right) G - \left(\frac{1+i}{2}\right) H$$

expresses N as a linear combination of four projections so that the coefficients sum to zero.

Finally, note that S, X, and Y are symmetries (Hermitian unitary operators) in $M_2(E\mathfrak{A}E)$ satisfying $SGS = H$, $XQX = I - Q$, and $YPY = I - P$. This shows that each projection in question is equivalent to its orthogonal complement.

The preceeding lemmas enable us to give several characterizations of $[\mathfrak{A}, \mathfrak{A}]$.

THEOREM 3. *In a von Neumann algebra $\mathfrak{A}$ of type II_1, the following four linear manifolds coincide:*

(1) *The set $[\mathfrak{A}, \mathfrak{A}]$ of all finite sums of commutators in $\mathfrak{A}$.*

(2) *The set $\mathscr{N}_a$ of all finite sums of nilpotent operators of index two in $\mathfrak{A}$.*

(3) *The set $\mathscr{P}_a$ of all linear combinations $\sum_{i=1}^{n} \alpha_i E_i$, where $\sum_{i=1}^{n} \alpha_i = 0$ and each E_i is a projection in $\mathfrak{A}$ equivalent to $I - E_i$.*

(4) *The set $\mathscr{S}_a$ of all linear combinations of symmetries $S \in \mathfrak{A}$ such such that the projections $1/2(I \pm S)$ are equivalent in $\mathfrak{A}$.*

Proof. We prove the inclusions

$$[\mathfrak{A}\,\mathfrak{A}] \subset \mathscr{N}_a \subset \mathscr{P}_a \subset \mathscr{S}_a \subset [\mathfrak{A}, \mathfrak{A}].$$

That $[\mathfrak{A}, \mathfrak{A}] \subset \mathscr{N}_a$ is Lemma 4.1, and that $\mathscr{N}_a \subset \mathscr{P}_a$ is Lemma 4.4. To see that $\mathscr{P}_a \subset \mathscr{S}_a$, it suffices to note that if E is a projection in $\mathfrak{A}$ equivalent to $I - E$, then $S = 2E - I$ is a symmetry in $\mathfrak{A}$, and $\frac{1}{2}(I + S) = E \sim I - E = \frac{1}{2}(I - S)$.

Finally, to show that $\mathscr{S}_a \subset [\mathfrak{A}, \mathfrak{A}]$, let S be a symmetry in $\mathfrak{A}$ and suppose that $E \sim I - E$, where $E = \frac{1}{2}(I + S)$. If U is a partial isometry in $\mathfrak{A}$ implementing the equivalence of E and $I - E$, then $H = U + U^*$ is a symmetry in $\mathfrak{A}$, and an easy calculation shows that $SH = -HS$. It follows that

$$S = \tfrac{1}{2}[(SH)H - H(SH)],$$

so that S is a commutator in $\mathfrak{A}$, and the proof is complete.

The following corollary is an immediate consequence of Theorems 2 and 3.

COROLLARY 4.4. *If $\mathscr{W}$ is one of the II_1-factors of Wright, then each of the four linear manifolds $[\mathscr{W}, \mathscr{W}]$, $\mathscr{N}_w$, $\mathscr{P}_w$, and $\mathscr{S}_w$ coincides with the subspace of all operators in $\mathscr{W}$ with trace zero.*

We remark that Theorem 3 is valid if $\mathfrak{A}$ is any properly infinite von Neumann algebra. In that case, all four of the linear manifolds mentioned coincide with $\mathfrak{A}$ itself ([*10*], Theorem 5, [*12*], Theorem 2, p. 295). Furthermore, it is an easy exercise (using [*4*], Remark 1, p. 866 and Lemma 4.1 of the present paper) to verify that if $\mathfrak{A}$ is a finite von Neumann algebra of type I, then $[\mathfrak{A}, \mathfrak{A}] = \mathcal{N}_a$. Thus the equation $[\mathfrak{A}, \mathfrak{A}] = \mathcal{N}_a$ is valid in an arbitrary von Neumann algebra $\mathfrak{A}$.

5. Concluding Remarks

(1) A more difficult version of Lemma 2.1 in which the λ_i are allowed to be complex and it is required only to find a permutation π such that for all $1 \leqslant k \leqslant n$,

$$\left| \sum_{i=1}^{k} \lambda_{\pi(i)} \right| \leqslant 2,$$

was recently proved by John Dyer (unpublished). This has the immediate corollary that every normal operator with central trace zero in a finite von Neumann algebra $\mathfrak{A}$ of type I is a commutator in $\mathfrak{A}$. (For the proof, just copy the proofs of Lemmas 2.2 and 2.3 and Theorem 1.)

(2) Let $\mathfrak{A}$ be a von Neumann algebra, and denote by $\mathscr{E}_a$ the linear manifold of all linear combinations in $\mathfrak{A}$ of the form $\sum \alpha_i E_i$ where for each i, $E_i \sim I - E_i$. It is clear that $\mathscr{P}_a$ is a hyperplane in $\mathscr{E}_a$, and it follows immediately from Corollary 4.4 that if $\mathfrak{A}$ is a II_1-factor of Wright, then $\mathscr{E}_a = \mathfrak{A}$. Conversely, one can ask which algebras $\mathfrak{A}$ of type II_1 satisfy $\mathscr{E}_a = \mathfrak{A}$. This question seems difficult, but Theorem 3 shows at least that such an algebra $\mathfrak{A}$ must be a factor in which $[\mathfrak{A}, \mathfrak{A}]$ is norm closed and consists exactly of those operators in $\mathfrak{A}$ with trace zero.

(3) We conjecture that $\mathscr{E}_a = \mathfrak{A}$ in every II_1-factor $\mathfrak{A}$. If indeed this turns out to be the case, and if this could be established, along with the fact that $I \notin [\mathfrak{A}, \mathfrak{A}]$, without *a priori* use of the trace, a new proof of the existence of the trace would be available. For then by Remark 2, $[\mathfrak{A}, \mathfrak{A}]$ would be a proper norm closed hyperplane in $\mathfrak{A}$, and would therefore coincide with the null space of the numerical trace. Thus one would have a norm continuous linear functional vanishing on all commutators and taking the value one at I. By Dixmier's Approximation Theorem ([*6*], Théorème 1, p. 272) this functional would necessarily be the trace.

(4) Note that if $\mathfrak{A}$ is a II_1-algebra which is *not* a factor, then $\mathfrak{A} \neq \mathscr{E}_a$. However, a method employed by Fillmore and Topping [8] can be used to show that the set $\mathscr{Q}$ of all projections $E \in \mathfrak{A}$ with $E \sim I - E$ generates $\mathfrak{A}$ as an algebra.

Furthermore, if $\mathfrak{A}$ is spanned by $\mathscr{Q}$ *over its center*, then the above arguments can easily be adapted to conclude that $[\mathfrak{A}, \mathfrak{A}]$ is equal to the null space of the central trace of $\mathfrak{A}$. (We are indebted to Professor Irving Kaplansky for this observation.)

(5) Sunouchi [12], using the theorem of Dixmier just mentioned, showed that if $\mathfrak{A}$ is any finite von Neumann algebra, then $[\mathfrak{A}, \mathfrak{A}]$ is norm-dense in the null space of the central trace on $\mathfrak{A}$.

References

1. Brown, A. and Pearcy, C., Structure of commutators of operators. *Ann. Math.* **82** (1965), 112–127.
2. Brown, A. and Pearcy, C., Commutators in factors of type III. *Canadian J. Math.* **18** (1966), 1152–1160.
3. Brown, A., Pearcy, C. and Topping, D., Commutators and the strong radical. *Duke J. Math.* (to be published).
4. Deckard, D. and Pearcy, C., On continuous matrix-valued functions on a Stonian space. *Pacific J. Math.* **14** (1964), 857–869.
5. Deckard, D. and Pearcy, C., On matrices over the ring of continuous complex-valued functions on a Stonian space. *Proc. Am. Math. Soc.* **14** (1963), 322–328.
6. Dixmier, J., Les algèbres d'opérateurs dans l'espace hilbertien, Gauthier-Villars, Paris, 1957.
7. Feldman, J., Nonseparability of certain finite factors. *Proc. Am. Math. Soc.* **7** (1956), 23–26.
8. Fillmore, P. and Topping, D., Operator algebras generated by projections. *Duke Math. J.* **34** (1967), 333–336.
9. Pearcy, C., A complete set of unitary invariants for operators generating finite W^*-algebras of type I. *Pacific J. Math.* **12** (1962), 1405–1416.
10. Pearcy, C. and Topping, D., Sums of small numbers of idempotents. *Michigan Math. J.* **14** (1967), 453–465.
11. Stone, M., Boundedness properties in function lattices. *Canadian J. Math.* **1** (1949), 176–186.
12. Sunouchi, H., Infinite Lie rings. *Tôhoku Math. J.* **8** (1956), 291–307.
13. Wright, F., A reduction for algebras of finite type. *Ann. Math.* **60** (1954), 560–570.