

NOTE

POLYNOMIAL APPROXIMATIONS OF EXPONENTIAL INTEGRALS

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Abstract—A method is described to calculate polynomial approximations for exponential integrals. Tables of coefficients are presented for exponential integrals of the second, third and fourth order, which allow their rapid evaluation to an absolute accuracy of about 1×10^{-7} .

IN RADIATIVE transfer problems angular integration of the transfer equations frequently produces terms involving exponential integrals. The n -th order exponential integral $E_n(x)$ is defined by the relation

$$E_n(x) = \int_1^{\infty} e^{-xt/t^n} dt.$$

Here $E_1(x)$ may be evaluated in a number of ways, including series and asymptotic expansions, or less precisely using polynomial or rational approximations.⁽¹⁾ A recurrence relation may be used to obtain higher order integrals from $E_1(x)$.

The rational approximations are suitable for use in a computer subroutine, but they involve the evaluation of either the exponential function or natural logarithm, or both (depending on the value of the argument x), a factor which considerably increases computing time. Moreover, values obtained by the recurrence relation for $x > 5$ are subject to considerable error. For some applications it is important to have an accurate fast method of evaluating the integrals, and for this reason a series of polynomial approximations has been devised.

Suppose a polynomial of degree m

$$p_m(x) = \sum_{i=0}^m A_i x^i$$

is used to approximate $E_n(x)$ on a subinterval (x_1, x_2) of the positive x -axis, using a least squares technique to determine the coefficients. The condition to be satisfied is that

$$\int_{x_1}^{x_2} [p_m(x) - E_n(x)]^2 dx$$

is a minimum. In practice $p_m(x)$ is expressed in terms of Legendre polynomials $P_i(x)$, orthogonal on (x_1, x_2) , viz.

$$p_m(x) = \sum_{i=0}^m B_i P_i(x).$$

The least squares condition gives

$$B_i = \frac{\int_{x_1}^{x_2} P_i(x) E_n(x) dx}{\int_{x_1}^{x_2} (P_i(x))^2 dx} \quad i = 0, 1, \dots, m.$$

The integral in the numerator may be expressed as a linear sum of integrals of the form

$$d_j = \int_{x_1}^{x_2} x^j E_n(x) dx \quad j = 0, 1, \dots, i.$$

By successive integrations by parts, or by induction on j , it can be shown that

$$d_j = \sum_{k=0}^j \frac{(j)!}{(j-k)!} [x_1^{j-k} E_{n+1+k}(x_1) - x_2^{j-k} E_{n+1+k}(x_2)].$$

This expression was used for values of x_1 greater than about 1.0. Values of $E_1(x)$ were obtained from highly accurate tables,^(2,3) and the recurrence relation employed for higher order integrals, using double precision arithmetic.

For small values of x_1 and x_2 the numerical values of the terms in the sum are many orders of magnitude greater than d_j , resulting in severe loss of accuracy. For x_1 and x_2 between 0.0 and 1.0, the values of d_j were obtained by Gaussian quadrature of the integral. The exponential integrals were accurately obtained from the series expansion⁽¹⁾ of $E_1(x)$, which converges rapidly on this interval, followed by the recurrence relation.

Radiative transfer applications generally require a high absolute accuracy for the exponential integral. Approximations were obtained for E_4 , E_3 and E_2 giving an absolute accuracy of about 1×10^{-7} . The intervals and polynomial coefficients are given in the accompanying tables. For $x > 14.0$ the value of the exponential integrals is taken as zero. A computer capable of representing about eight significant decimal digits is required to maintain full accuracy.

Near $x = 0.0$, the slope of $E_2(x)$ becomes very large, making polynomial approximations less accurate in this region. In the interval (0.0, 0.005) the function was evaluated by the expression

$$E_2(x) \sim 1 + (\gamma - 1 + \ln x)x - \frac{1}{2}x^2 + \frac{1}{12}x^3$$

where $\gamma = 0.577215665$ is Euler's constant.

TABLE 1. POLYNOMIAL COEFFICIENTS FOR THE APPROXIMATION OF $E_2(x)$

X1	X2	M	A(0)	A(1)	A(2)	A(3)	A(4)	A(5)	A(6)
.005	.01	5	.99863856E+00	-.64849862E+01	.28699864E+03	-.20230211E+05	.92987275E+06	-.18864517E+08	
.01	.02	5	.99719845E+00	-.57667626E+01	.14015630E+03	-.48733759E+04	.11095948E+06	-.111211459E-07	
.02	.03	5	.99512047E+00	-.52178244E+01	.81215706E+02	-.16637561E+04	.22506892E+05	-.13650376E+06	
.03	.05	5	.99227478E+00	-.47562499E+01	.50897054E+02	-.65548040E+03	.55339225E+04	-.20871535E+05	
.05	.08	5	.98743435E+00	-.42707709E+01	.31161615E+02	-.24918857E+03	.13013803E+04	-.30413637E+04	
.08	.14	5	.97890829E+00	-.37515615E+01	.18314077E+02	-.87710226E+02	.27097516E+03	-.37351991E+03	
.14	.25	6	.96617789E+00	-.32935617E+01	.11163358E+02	-.38542755E+02	.90594739E+02	-.12506467E+03	-.75047464E+02
.25	.40	6	.94232903E+00	-.27602169E+01	.66185588E+01	-.13205705E+02	.18236745E+02	-.14695374E+02	-.51126570E+01
.40	.70	6	.91002540E+00	-.23189262E+01	.40972991E+01	-.55123669E+01	.50594481E+01	-.27377738E+01	-.65312535E+00
.70	1.00	5	.83257151E+00	-.14824257E+01	.18822436E+01	-.13316793E+01	.54534096E+00	-.97555565E-01	
1.00	1.50	6	.78964619E+00	-.14714204E+01	.14733368E+01	-.95133859E+00	.39167846E+00	-.93110982E-01	-.97039862E-02
1.50	2.50	6	.67880797E+00	-.10419146E+01	.78456730E+00	-.35571203E+00	.98812997E-01	-.15521811E-01	.10573839E-02
2.50	4.00	6	.48261635E+00	-.57616474E+00	.31201874E+00	-.96201227E-01	.17518586E-01	-.17638563E-02	.75988859E-04
4.00	6.00	6	.25318620E+00	-.22816343E+00	.89305974E-01	-.19262687E-01	.23975807E-02	-.16236047E-03	.46529558E-05
6.00	8.00	5	.61520729E-01	-.36981621E-01	.90654130E-02	-.11287099E-02	.71170586E-04	-.18139326E-05	
8.00	10.00	4	.76228002E-02	-.30017878E-02	.44776459E-03	-.29936337E-04	.75588086E-06		
10.00	12.00	3	.50963609E-03	-.12675641E-03	.10549960E-04	-.29525857E-06			
12.00	14.00	2	.18470219E-04	-.26366002E-05	.9441851E-07				

TABLE 2. POLYNOMIAL COEFFICIENTS FOR THE APPROXIMATION OF $E_1(v)$

X1	X2	M	A(0)	A(1)	A(2)	A(3)	A(4)	A(5)	A(6)
.00	.01	5	.5000000E+00	-.00027661E+00	.35030680E+01	-.14451565E+03	.91145339E+04	-.27510454E+06	
.01	.05	5	.40000486E+00	-.09367752E+00	.24542231E+01	-.18666420E+02	.16957034E+03	-.78233485E+03	
.05	.10	5	.40987973E+00	-.09250148E+00	.19531425E+01	-.68473836E+01	.24141927E+02	-.43633200E+02	
.10	.20	5	.40650246E+00	-.09644750E+00	.14000413E+01	-.33014884E+01	.50086080E+01	-.53301442E+01	
.20	.40	5	.49797209E+00	-.02834808E+00	.12496552E+01	-.15556700E+01	.14440030E+01	-.65387457E+00	
.40	.70	5	.49201603E+00	-.06644115E+00	.03952389E+00	-.76213922E+00	.40763092E+00	-.10229208E+00	
.70	1.00	5	.49203747E+00	-.78785009E+00	.70970913E+00	-.42183647E+00	.15271436E+00	-.25082541E-01	
1.00	1.50	5	.46228186E+00	-.69286540E+00	.52553268E+00	-.24177842E+00	.63959788E-01	-.74386331E-02	
1.50	2.50	6	.43684018E+00	-.61053170E+00	.42011457E+00	-.17665829E+00	.46264005E-01	-.69480434E-02	.45746719E-03
2.50	4.00	6	.34656643E+00	-.39789156E+00	.20831318E+00	-.62459659E-01	.11121848E-01	-.10999885E-02	.46719667E-04
4.00	6.00	6	.20000240E+00	-.17767995E+00	.68695208E-01	-.14662192E-01	.18086938E-02	-.12154629E-03	.36603193E-05
6.00	8.00	5	.52636082E-01	-.31518925E-01	.77010713E-02	-.95616821E-03	.60148400E-04	-.15299014E-05	
8.00	10.00	4	.68132071E-02	-.26793015E-02	.39918259E-03	-.26660540E-04	.67255760E-06		
10.00	12.00	3	.46746949E-03	-.11619393E-03	.06840454E-05	-.27038491E-06			
12.00	14.00	2	.17225127E-04	-.24581282E-05	.88003757E-07				

TABLE 3. POLYNOMIAL COEFFICIENTS FOR THE APPROXIMATION OF $E_4(x)$

X1	X2	M	A(0)	A(1)	A(2)	A(3)	A(4)	A(5)	A(6)
.00	.02	3	.33333321E+00	-.49991784E+00	.48723621E+00	-.60065675E+00			
.02	.10	4	.33333145E+00	-.49976349E+00	.48623636E+00	-.548877654E+00	.67427701E+00		
.10	.20	4	.33329039E+00	-.49820931E+00	.46314345E+00	-.38876234E+00	.73942719E+00		
.20	.40	4	.33300495E+00	-.49306872E+00	.47766140E+00	-.27717782E+00	.10435406E+00		
.40	.70	5	.33253226E+00	-.48847193E+00	.41209314E+00	-.26007762E+00	.11597315E+00	-.26091905E-01	
.70	1.00	4	.32618509E+00	-.44764367E+00	.30568652E+00	-.11924852E+00	.21083163E-01		
1.00	1.50	5	.32369356E+00	-.44028710E+00	.30183790E+00	-.12713362E+00	.31436526E-01	-.34847974E-02	
1.50	2.50	6	.31416286E+00	-.40980641E+00	.26366584E+00	-.10465057E+00	.26191553E-01	-.38024567E-02	.24418347E-03
2.50	4.00	6	.26476052E+00	-.29454781E+00	.14995841E+00	-.43917471E-01	.76716851E-02	-.74718008E-03	.31347324E-04
4.00	6.00	6	.16347761E+00	-.14351490E+00	.54917252E-01	-.11618356E-01	.14224296E-02	-.94971422E-04	.26886469E-05
6.00	8.00	5	.45805090E-01	-.277338178E-01	.66609363E-02	-.82506626E-03	.51796873E-04	-.13152097E-05	
8.00	10.00	4	.61485433E-02	-.24149872E-02	.35942825E-03	-.23983739E-04	.60455421E-06		
10.00	12.00	3	.43135189E-03	-.10716388E-03	.89272471E-05	-.24915099E-06			
12.00	14.00	2	.16129989E-04	-.23012048E-05	.82165827E-07				

The coefficients were used in a subroutine run on an IBM 7090 computer. With the exception of the region near $x = 0.0$ for $E_2(x)$, the average time for evaluation was about 0.24 m sec, almost half the average time of the system subroutine for the exponential function. A Fortran IV listing of the subroutines may be obtained from the author.

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REFERENCES

1. M. ABRAMOWITZ and I. A. STEGUN, *Handbook of Mathematical Functions*, U.S. Government Printing Office, Washington (1964).
2. F. E. HARRIS, *Math. Tables Aids Comp.* **11**, 9 (1957).
3. J. MILLER and P. HURST, *Math. Tables Aids Comp.* **12**, 187 (1958).