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On a Topology for Invariant Subspaces

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This paper is concerned with a topology on the collection of invariant subspaces for a given operator on Hilbert space. Let $\mathscr{H}$ be a complex Hilbert space, and let $\mathscr{L}(\mathscr{H})$ denote the algebra of all (bounded, linear) operators on $\mathscr{H}$. For every fixed element T of $\mathscr{L}(\mathscr{H})$, we denote by $\mathscr{I}_T$ the collection of all invariant subspaces of T; i.e., $\mathscr{M}$ is an element of $\mathscr{I}_T$ if $\mathscr{M}$ is a (closed) subspace of $\mathscr{H}$ for which $T\mathscr{M} \subset \mathscr{M}$. The set $\mathscr{I}_T$ is never empty since (0) and $\mathscr{H}$ are in it. Consider the metric Θ defined on $\mathscr{I}_T$ by the equation $\Theta(\mathscr{M}, \mathscr{N}) = \| P_{\mathscr{M}} - P_{\mathscr{N}} \|$, where $P_{\mathscr{K}}$ denotes the (orthogonal) projection onto a subspace $\mathscr{K}$ of $\mathscr{H}$. This metric, defined on the collection of *all* subspaces of $\mathscr{H}$, has been studied previously. In particular, it has been shown [9] that if $\mathscr{M}$ and $\mathscr{N}$ are subspaces of $\mathscr{H}$ such that $\Theta(\mathscr{M}, \mathscr{N}) < 1$, then $\dim \mathscr{M} = \dim \mathscr{N}$ and $\dim \mathscr{M}^\perp = \dim \mathscr{N}^\perp$. (An alternate proof of this fact appears in the proof of Theorem 3.) Several properties of the metric space $\mathscr{I}_T = (\mathscr{I}_T, \Theta)$ are immediately clear. First, since $\| P - Q \| \leqslant 1$ for any projections P and Q, Θ is bounded by 1. Second, if $\{\mathscr{M}_n\}$ is a Cauchy sequence in $\mathscr{I}_T$, then the sequence $\{P_{\mathscr{M}_n}\}$ is Cauchy in $\mathscr{L}(\mathscr{H})$ and hence converges uniformly to a projection P. Since $P_{\mathscr{M}_n} T P_{\mathscr{M}_n} = T P_{\mathscr{M}_n}$ for all n, it follows that $PTP = TP$ and hence that $\mathscr{M} = P\mathscr{H}$ is an invariant subspace for T. Clearly the sequence $\{\mathscr{M}_n\}$ converges to $\mathscr{M}$ in $\mathscr{I}_T$, and thus $\mathscr{I}_T$ is a *complete* metric space. Finally, if $\mathscr{I}_T^{m,n}$ is defined (for every pair of cardinal numbers m and n satisfying $m + n = \dim \mathscr{H}$) to be the collection of all subspaces $\mathscr{M} \in \mathscr{I}_T$ such that $\dim \mathscr{M} = m$ and $\dim \mathscr{M}^\perp = n$, then each component of $\mathscr{I}_T$ is contained in some one $\mathscr{I}_T^{m,n}$.

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Before proceeding to consideration of the topology induced on $\mathscr{I}_T$ by Θ, we prove the following rather technical lemma:

LEMMA 1.1. *For $i = 1, 2$, let $\mathscr{M}_i$ be a subspace of $\mathscr{H}$ and let $C_i \in \mathscr{L}(\mathscr{H})$ satisfy $\| C_i x \| \geqslant \epsilon_i \| x \|$ ($\epsilon_i > 0$) for all $x \in \mathscr{M}_i$. If $\mathscr{N}_i = C_i \mathscr{M}_i$, $i = 1, 2$, then $\mathscr{N}_i$ is closed and*

$$\| P_{\mathscr{N}_1} - P_{\mathscr{N}_2} \| \leqslant \left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} \right) \| C_1 - C_2 \| + \left(\frac{1}{\epsilon_1} \| C_2 \| + \frac{1}{\epsilon_2} \| C_1 \| \right) \| P_{\mathscr{M}_1} - P_{\mathscr{M}_2} \|.$$

Proof. It is clear that $\mathscr{N}_1$ and $\mathscr{N}_2$ are closed. If we denote the identity operator on $\mathscr{H}$ by 1, then

$$\begin{aligned} \| P_{\mathscr{N}_1} - P_{\mathscr{N}_2} \| &\leqslant \| P_{\mathscr{N}_1} (1 - P_{\mathscr{N}_2}) \| + \| (1 - P_{\mathscr{N}_1}) P_{\mathscr{N}_2} \| \\ &= \| (1 - P_{\mathscr{N}_2}) P_{\mathscr{N}_1} \| + \| (1 - P_{\mathscr{N}_1}) P_{\mathscr{N}_2} \|. \end{aligned}$$

Thus, by symmetry, it suffices to prove that

$$\| (1 - P_{\mathscr{N}_2}) P_{\mathscr{N}_1} \| \leqslant \frac{1}{\epsilon_1} \| C_1 - C_2 \| + \frac{1}{\epsilon_1} \| C_2 \| \| P_{\mathscr{M}_1} - P_{\mathscr{M}_2} \|,$$

and this goes as follows: Let x be a unit vector in $\mathscr{H}$, and write $x = y_1 + z_1$ where $y_1 \in \mathscr{N}_1$ and $z_1 \in \mathscr{N}_1^{\perp}$. Let w_1 be the unique vector in $\mathscr{M}_1$ satisfying $C_1 w_1 = y_1$, let $w_2 = P_{\mathscr{M}_2} w_1$, and let $y_2 = C_2 w_2$. Then $y_2 \in \mathscr{N}_2$, and we have

$$\begin{aligned} \| (1 - P_{\mathscr{N}_2}) P_{\mathscr{N}_1} x \| &= \| (1 - P_{\mathscr{N}_2}) y_1 \| \\ &\leqslant \| (1 - P_{\mathscr{N}_2}) y_2 \| + \| (1 - P_{\mathscr{N}_2})(y_1 - y_2) \| \\ &= \| (1 - P_{\mathscr{N}_2})(C_1 w_1 - C_2 w_2) \| \\ &\leqslant \| C_1 w_1 - C_2 w_1 \| + \| C_2 w_1 - C_2 w_2 \| \\ &\leqslant \| C_1 - C_2 \| \| w_1 \| + \| C_2 \| \| (P_{\mathscr{M}_1} - P_{\mathscr{M}_2}) w_1 \| \\ &\leqslant \| C_1 - C_2 \| \frac{1}{\epsilon_1} \| y_1 \| + \| C_2 \| \| P_{\mathscr{M}_1} - P_{\mathscr{M}_2} \| \frac{1}{\epsilon_1} \| y_1 \| \\ &\leqslant \frac{1}{\epsilon_1} \| C_1 - C_2 \| + \frac{1}{\epsilon_1} \| C_2 \| \| P_{\mathscr{M}_1} - P_{\mathscr{M}_2} \|, \end{aligned}$$

as desired.

Our interest in the topology of the metric space $\mathscr{I}_T$ was generated by its connection with the following idea: If $T \in \mathscr{L}(\mathscr{H})$, denote by $\mathfrak{A}_T$

the weakly-closed algebra generated by T and 1, and denote by $\mathfrak{A}'_T$ the algebra of all operators that commute with $\mathfrak{A}_T$. It is clear that $\mathfrak{A}_T$ consists of the weak closure of the set of all polynomials $p(T)$, and that $\mathfrak{A}'_T$ consists of exactly those operators that commute with T. A subspace $\mathscr{M}$ of $\mathscr{H}$ is said to be *hyperinvariant for* T if it is invariant under the algebra $\mathfrak{A}'_T$; i.e., if $T'\mathscr{M} \subset \mathscr{M}$ for every operator T' such that $T'T = TT'$. Clearly the hyperinvariant subspaces for T form a closed subset of $\mathscr{I}_T$. We also say that $\mathscr{M} \in \mathscr{I}_T$ is *inaccessible* if the only continuous mapping ϕ of the interval $[0, 1]$ into $\mathscr{I}_T$ with $\phi(0) = \mathscr{M}$ is the constant map $\phi(t) \equiv \mathscr{M}$.

THEOREM 1. *If $T \in \mathscr{L}(\mathscr{H})$ and $\mathscr{M}$ is an inaccessible invariant subspace for T, then $\mathscr{M}$ is hyperinvariant for T.*

Proof. Let $S \in \mathfrak{A}'_T$, and let Δ be the subset $\Delta = \{\lambda : 0 \leqslant \lambda < 1/\| S \|\}$ of the real line. Then for each $\lambda \in \Delta$, the operator $1 - \lambda S$ is invertible, and hence $\mathscr{M}_\lambda = (1 - \lambda S)\mathscr{M}$ is a subspace of $\mathscr{H}$. Moreover, $\mathscr{M}_\lambda \in \mathscr{I}_T$ for each $\lambda \in \Delta$, since $T\mathscr{M}_\lambda = T(1 - \lambda S)\mathscr{M} = (1 - \lambda S)T\mathscr{M} \subset (1 - \lambda S)\mathscr{M} = \mathscr{M}_\lambda$. It follows from Lemma 1.1 that the map $\lambda \to \mathscr{M}_\lambda$ from Δ into $\mathscr{I}_T$ is continuous, since

$$\| P_{\mathscr{M}_\alpha} - P_{\mathscr{M}_\beta} \| \leqslant \{\|(1 - \alpha S)^{-1}\| + \|(1 - \beta S)^{-1}\|\} | \alpha - \beta | \, \| S \|,$$

and the map $A \to A^{-1}$ is continuous in the uniform topology. Since $\mathscr{M} = \mathscr{M}_\lambda$ for $\lambda = 0$ and $\mathscr{M}$ is inaccessible, we must have $\mathscr{M} = \mathscr{M}_\lambda = (1 - \lambda S)\mathscr{M}$ for each $\lambda \in \Delta$, and it follows that $S\mathscr{M} \subset \mathscr{M}$, completing the argument.

COROLLARY 1.2. *If $\mathscr{M}$ is an isolated point of $\mathscr{I}_T$, then $\mathscr{M}$ is hyperinvariant for T.*

We say that a subspace $\mathscr{M} \in \mathscr{I}_T$ *commutes with* $\mathscr{I}_T$ if $P_{\mathscr{M}}$ commutes with $P_{\mathscr{N}}$ for every $\mathscr{N} \in \mathscr{I}_T$, and that $\mathscr{M}$ is a *pinch point* of $\mathscr{I}_T$ if for every $\mathscr{N} \in \mathscr{I}_T$, either $\mathscr{N} \subset \mathscr{M}$ or $\mathscr{M} \subset \mathscr{N}$.

COROLLARY 1.3. *If $\mathscr{M}$ is an invariant subspace for T which commutes with $\mathscr{I}_T$, then $\mathscr{M}$ is hyperinvariant for T.*

Proof. $\mathscr{M}$ is an isolated point of $\mathscr{I}_T$.

The following corollary is a recent result of P. Rosenthal [*10*]:

COROLLARY 1.4. *If $\mathscr{M} \in \mathscr{I}_T$ is a pinch point of $\mathscr{I}_T$, then $\mathscr{M}$ is hyperinvariant for T.*

Proof. Such an invariant subspace $\mathscr{M}$ commutes with $\mathscr{I}_T$.

We next give some examples which illustrate various possible properties of spaces $\mathscr{I}_T$, and also furnish applications of our results.

EXAMPLE 1.5. Let $\mathscr{H}$ be an n-dimensional Hilbert space ($n < \infty$), and let T be an operator on $\mathscr{H}$ having n distinct eigenvalues. It is easy to see that every invariant subspace of T is spanned by eigenvectors, and hence corresponds to a subset of the set of eigenvalues for T. Thus $\mathscr{I}_T$ is a discrete metric space consisting of 2^n points. If T is normal, then $\mathscr{I}_T$ is commutative, and the distance between any two distinct points in $\mathscr{I}_T$ is 1.

EXAMPLE 1.6. Let $\mathscr{H}$ be a 2-dimensional Hilbert space with orthonormal basis $\{e_0, e_1\}$. Let $\mathscr{N} \subset \mathscr{H}$ be the 1-dimensional subspace spanned by e_1, and for $\lambda \geqslant 0$, let $\mathscr{M}_\lambda \subset \mathscr{H}$ be the 1-dimensional subspace spanned by the unit vector

$$f_\lambda = \frac{\lambda e_1 + e_2}{(1 + \lambda^2)^{1/2}}.$$

A straightforward computation shows that for every $\lambda \geqslant 0$,

$$\| P_{\mathscr{M}_\lambda} - P_{\mathscr{N}} \| = \frac{1}{(1 + \lambda^2)^{1/2}}.$$

Let T_λ be the operator on $\mathscr{H}$ defined by $Te_1 = e_1$, and $Tf_\lambda = 2f_\lambda$. Then the space $\mathscr{I}_{T_\lambda}$ is discrete and consists of the four points (0), $\mathscr{N}$, $\mathscr{M}_\lambda$, and $\mathscr{H}$. Furthermore, as noted above, $\Theta(\mathscr{N}, \mathscr{M}_\lambda) = 1/(1 + \lambda^2)^{1/2}$. This example shows that a point (e.g. $\mathscr{M}_\lambda$) in $\mathscr{I}_T$ can be isolated without commuting with $\mathscr{I}_T$, and it also shows that the distance between an isolated point and its complement can be any number in the half-open interval (0, 1].

EXAMPLE 1.7. Let $\mathscr{H}$ be the Hilbert space consisting of all functions f that are square integrable with respect to Lebesgue measure on the interval [0, 1], and let T be the Volterra operator on $\mathscr{H}$ defined by $(Tf)(x) = \int_0^x f(t)\,dt$. It is known [*6*] that the invariant subspaces of T are exactly the subspaces $\mathscr{M}_\lambda (0 \leqslant \lambda \leqslant 1)$ defined by $\mathscr{M}_\lambda = \{f \mid f(x) = 0 \text{ for } 0 \leqslant x \leqslant \lambda\}$. Since $\mathscr{I}_T$ is linearly ordered, the distance between any two distinct points of $\mathscr{I}_T$ is 1, so that $\mathscr{I}_T$ is a discrete space consisting of a continuum of points, and hence is nonseparable. Note that it follows from Corollary 1.4 that each $\mathscr{M}_\lambda$ is hyperinvariant for T. (Sarason [*11*] has recently obtained the stronger result that $\mathfrak{A}_T = \mathfrak{A}_T'$.)

EXAMPLE 1.8. Let $\mathscr{H}$ and $\mathscr{K}$ be separable, infinite-dimensional Hilbert spaces. We shall now construct an operator $T \in \mathscr{L}(\mathscr{H} \oplus \mathscr{K})$ with the following property:

> (P) The subspace $\mathscr{H} \oplus (0)$ is a pinch point of the space $\mathscr{I}_T$, and the only other pinch points of $\mathscr{I}_T$ are (0) and $\mathscr{H} \oplus \mathscr{K}$.

As far as we know, no example of an operator having property (P) has previously been given. The present construction employs Corollary 1.4 and Theorem 4 (Section 3). To begin the construction, let $\mathscr{B}$ be a fixed two-dimensional Hilbert space, and let $\mathscr{H}$ be the Hilbert space of all sequences $(x_0, x_1, \ldots)$ where each $x_i \in \mathscr{B}$ and $\Sigma \| x_i \|^2 < \infty$. Consider the operator D on $\mathscr{H}$ defined by the equation

$$D(x_0, x_1, x_2, \ldots) = (\alpha_0 x_1, \alpha_1 x_2, \alpha_2 x_3, \ldots),$$

where for each n, $\alpha_n = 1/2^n$. The following lemma concerning D seems interesting in its own right:

LEMMA 1.9. *There exists an infinite-dimensional subspace $\mathscr{V}$ of $\mathscr{H}$ with the property that every vector of the form $w = Dx + y$, where $x \in \mathscr{H}$ and $0 \neq y \in \mathscr{V}$, is cyclic for D* (i.e., $\bigvee\{w, Dw, D^2w, \ldots\} = \mathscr{H}$).

Proof. Since $\mathscr{B}$ is separable, there exists a countable set $S = \{b_i\}_{i=1}^{\infty}$ of unit vectors that is dense in the unit sphere of $\mathscr{B}$. The desired subspace $\mathscr{V}$ will be constructed by exhibiting a suitable orthonormal basis for $\mathscr{V}$, and this basis is best understood by considering the following array:

$$\begin{bmatrix} b_1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2}b_2 & 0 & 0 & 0 & \cdots \\ 0 & b_1 & 0 & 0 & \cdots \\ \frac{1}{3}b_3 & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{2}b_2 & 0 & 0 & \cdots \\ 0 & 0 & b_1 & 0 & \cdots \\ \frac{1}{4}b_4 & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{3}b_3 & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{2}b_2 & 0 & \cdots \\ 0 & 0 & 0 & b_1 & \cdots \\ \frac{1}{5}b_5 & \cdots & \cdots & \cdots & \cdots \\ \vdots & \ddots & & & \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \end{bmatrix}$$

If the nth column of this array is regarded as a vector $t_n \in \mathscr{H}$, $0 \leqslant n < \infty$, then t_n is orthogonal to t_m for $n \neq m$ and $\| t_n \|^2 = \pi^2/6$ for all n. Define $\mathscr{V}$ to be the subspace spanned by the orthogonal family $\{t_n\}_{n=0}^{\infty}$. Let $w = Dx + y$, where $x \in \mathscr{H}$, $0 \neq y \in \mathscr{V}$, and let $\mathscr{M} \subset \mathscr{H}$ be the subspace $\mathscr{M} = \bigvee\{w, Dw, D^2w,...\}$. We wish to prove that $\mathscr{M} = \mathscr{H}$, and this will be done as follows: Let z_0 be an arbitrary unit vector in $\mathscr{B}$. We show that the vector $(z_0, 0, 0,\cdots)$ lies in $\mathscr{M}$, and an induction argument shows that $\mathscr{M}$ contains every vector of the form $(z_0, z_1,..., z_n, 0, 0,...)$ where the z_i are arbitrary, and thus is equal to $\mathscr{H}$. (This is a standard sort of argument, the prototype of which is to be found in [7, p. 303].) The vector $y \in \mathscr{V}$ can be written as $y = \sum_{i=0}^{\infty} \gamma_i t_i$ where $\Sigma | \gamma_i |^2 < \infty$, and if x is the vector $x = (x_0, x_1,...)$ in $\mathscr{H}$, then $w = Dx + y$ is the vector

$$w = (\alpha_0 x_1, \alpha_1 x_2,...) + (\gamma_0 b_1, \tfrac{1}{2}\gamma_0 b_2, \gamma_1 b_1, \tfrac{1}{3}\gamma_0 b_3, \tfrac{1}{2}\gamma_1 b_2, \gamma_2 b_1,...)$$

Let $\epsilon > 0$ be given, and note that to prove that $(z_0, 0, 0,...) \in \mathscr{M}$, it suffices to show that $\mathscr{M}$ contains a vector whose distance from the given vector is less than ϵ. Since $\sum_{n=0}^{\infty} | \gamma_n |^2$ converges, we can choose j so that $| \gamma_j | \geqslant | \gamma_n |$ for all $0 \leqslant n < \infty$, and since $y \neq 0$, $| \gamma_j | > 0$. Furthermore, since z_0 is a unit vector and S is dense in the unit sphere of $\mathscr{B}$, we can choose $k > 3$ so that $\| b_k - z_0 \| < \epsilon/3$ and also so that

$$\frac{k \| x \|}{2^k | \gamma_j |} < \epsilon/3 \qquad \text{and} \qquad \frac{k}{2^k} < \epsilon/3.$$

Examination shows that the vector y has for one of its components (say the mth component) the vector $(1/k)\gamma_j b_k$, and since $k > 3$, it is clear that $m > k$. Now consider the vector

$$\begin{aligned}
&\frac{k}{| \gamma_j |} \frac{1}{\alpha_0 \cdots \alpha_{m-1}} D^m w - (z_0, 0, 0,...) \\
&= \frac{k}{| \gamma_j |} \frac{1}{\alpha_0 \cdots \alpha_{m-1}} D^{m+1} x + \left\{ \frac{k}{| \gamma_j |} \frac{1}{\alpha_0 \cdots \alpha_{m-1}} D^m y - (z_0, 0, 0,...) \right\} \\
&= \frac{k}{| \gamma_j |} \left(\frac{\alpha_m}{\alpha_0} x_{m+1}, \frac{\alpha_m \alpha_{m+1}}{\alpha_0 \alpha_1} x_{m+2}, \frac{\alpha_m \alpha_{m+1} \alpha_{m+2}}{\alpha_0 \alpha_1 \alpha_2} x_{m+3},... \right) + (b_k - z_0, 0, 0,...) \\
&\quad + \left(0, \frac{\alpha_m}{\alpha_0} \frac{k}{k-1} \frac{\gamma_{j+1}}{| \gamma_j |} b_{k-1}, \frac{\alpha_m \alpha_{m+1}}{\alpha_0 \alpha_1} \frac{k}{k-2} \frac{\gamma_{j+2}}{| \gamma_j |} b_{k-2},... \right).
\end{aligned}$$

Easy arithmetic now shows that each of the three summands on the right-hand side of the last equation is a vector of norm less than $\epsilon/3$, and hence we have established that $(z_0, 0, 0,...) \in \mathscr{M}$. An induction argument, whose details we leave to the interested reader, shows that $\mathscr{M}$ contains every vector of the form $(z_0, z_1, \ldots, z_n, 0, 0, \ldots)$ where the z_i are arbitrary in $\mathscr{B}$, and hence that $\mathscr{M} = \mathscr{H}$. This completes the proof of the lemma.

To continue with the construction of the proposed operator T on $\mathscr{H} \oplus \mathscr{K}$, let V be any bounded operator from $\mathscr{K}$ to $\mathscr{H}$ having no null space and satisfying $V\mathscr{K} \subset \mathscr{V}$. (Since $\mathscr{K}$ and $\mathscr{V}$ are infinite-dimensional, such operators exist in abundance.) The required operator T is then given matricially by

$$T = \begin{pmatrix} D & V \\ 0 & 0 \end{pmatrix},$$

where this matrix is understood to act on $\mathscr{H} \oplus \mathscr{K}$ in the usual fashion. To see that T has the properties claimed for it, note first that $\mathscr{H} \oplus (0) \in \mathscr{I}_T$. To show that $\mathscr{H} \oplus (0)$ is a pinch point of $\mathscr{I}_T$, it suffices to show that if $\mathscr{M} \subset \mathscr{H} \oplus \mathscr{K}$ is an invariant subspace for T and $\mathscr{M}$ contains a vector $(x, y) \in \mathscr{H} \oplus \mathscr{K}$ with $y \neq 0$, then $\mathscr{H} \oplus (0) \subset \mathscr{M}$. But this is clear, since $\mathscr{M}$ contains $T(x, y) = (Dx + Vy, 0) = (w, 0)$, and hence by Lemma 1.9, $\mathscr{M}$ contains $\mathscr{H} \oplus (0)$. Suppose now that $\mathscr{P} \in \mathscr{I}_T$ is some other pinch point of $\mathscr{I}_T$. Since $\mathscr{H} \oplus (0)$ is a pinch point, it must be that $\mathscr{P} \subset \mathscr{H} \oplus (0)$ or $\mathscr{H} \oplus (0) \subset \mathscr{P}$. In the latter case, $\mathscr{P} = \mathscr{H} \oplus \mathscr{N}$ for some subspace $\mathscr{N} \subset \mathscr{K}$, and since every subspace of the form $\mathscr{H} \oplus \mathscr{T}$ where $\mathscr{T} \subset \mathscr{K}$ lies in $\mathscr{I}_T$, it is clear that $\mathscr{P}$ cannot be a pinch point unless $\mathscr{N} = \mathscr{K}$. On the other hand, if $\mathscr{P} \subset \mathscr{H} \oplus (0)$, then $\mathscr{P}$ must be a pinch point of the space $\mathscr{I}_D$. Thus the proof will be completed by showing that $\mathscr{I}_D$ has no pinch points other than (0) and $\mathscr{H}$. To see this, note first that D is unitarily equivalent to the operator

$$D_1 = \begin{pmatrix} D_2 & 0 \\ 0 & D_2 \end{pmatrix},$$

where D_2 is the operator on ℓ_2 defined by

$$D_2(\eta_0, \eta_1, \eta_2, \ldots) = (\eta_1, \tfrac{1}{2}\eta_2, \tfrac{1}{4}\eta_3, \ldots).$$

Thus it suffices to show that $\mathscr{I}_{D_1}$ has no pinch points other than (0) and $\ell_2 \oplus \ell_2$. Let $\mathscr{Q}$ be a pinch point of $\mathscr{I}_{D_1}$ and observe that by Corollary 1.4, $\mathscr{Q}$ is hyperinvariant for D_1. Furthermore, by Theorem 4 (Section 3), $\mathscr{Q}$ is of the form $\mathscr{Q} = \mathscr{T} \oplus \mathscr{T}$ where $\mathscr{T}$ is an invariant subspace for D_2. Since every subspace $\mathscr{S} \oplus (0) \subset \ell_2 \oplus \ell_2$, where $\mathscr{S}$ is invariant for D_2, lies in $\mathscr{I}_{D_1}$, and the invariant subspaces for D_2 are known to be linearly ordered [6], it is clear that either $\mathcal{Q} = (0) \oplus (0)$ or $\mathcal{Q} = \ell_2 \oplus \ell_2$, completing the proof.

This concludes our list of examples. We end this section with the following theorem:

THEOREM 2. *If T_1 and T_2 are similar operators, then $\mathcal{I}_{T_1}$ and $\mathcal{I}_{T_2}$ are homeomorphic topological spaces.*

Proof. Let S be an invertible operator such that $ST_1S^{-1} = T_2$. The mapping $\mathcal{M} \to S\mathcal{M}$ is clearly a 1–1 mapping of $\mathcal{I}_{T_1}$ onto $\mathcal{I}_{T_2}$, and from Lemma 1.1 we know that

$$\| P_{S\mathcal{M}} - P_{S\mathcal{N}} \| \leqslant 2 \| S \| \| S^{-1} \| \| P_{\mathcal{M}} - P_{\mathcal{N}} \|,$$

and similarly that

$$\| P_{\mathcal{M}} - P_{\mathcal{N}} \| \leqslant 2 \| S \| \| S^{-1} \| \| P_{S\mathcal{M}} - P_{S\mathcal{N}} \|,$$

whence the result.

2. Isolated Invariant Subspaces of Normal Operators

A natural problem that arises is that of characterizing the isolated invariant subspaces and the hyperinvariant subspaces of a given operator or a given class of operators. In this section we consider and completely solve this problem for the class of normal operators. We begin by reminding the reader that if $T \in \mathcal{L}(\mathcal{H})$, then the *von Neumann algebra generated by* T, denoted hereafter by $\mathcal{V}_T$, is the smallest selfadjoint, weakly-closed operator algebra that contains T and $1_{\mathcal{H}}$. The *commutant* of a given von Neumann algebra $\mathcal{V}_T$ will be denoted, as usual, by $\mathcal{V}_T'$. If T is a normal operator, then a projection $E \in \mathcal{L}(\mathcal{H})$ is a *spectral projection* for T if $E \in \mathcal{V}_T$. (It is well known that if $\mathcal{H}$ is separable, then the spectral projections for T are exactly the values of the unique spectral measure associated with T. On nonseparable spaces, this need not be the case.)

THEOREM 3. *If T is a normal operator, the following statements are equivalent for a subspace $\mathcal{M} \in \mathcal{I}_T$:*

(a) $\Theta(\mathcal{M}, \mathcal{N}) = 1$ *for all $\mathcal{N} \in \mathcal{I}_T$ such that $\mathcal{N} \neq \mathcal{M}$,*
(b) *$\mathcal{M}$ is an isolated point of $\mathcal{I}_T$,*
(c) *$\mathcal{M}$ is hyperinvariant for T,*
(d) *$P_{\mathcal{M}}$ is a spectral projection for T.*

Proof. We shall prove (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a). That (a) implies (b) is trivial, and that (b) implies (c) is Corollary 1.3. To prove that (c) implies (d) it suffices, in view of the well-known fact that $\mathscr{V}_T'' = \mathscr{V}_T$ [5], to show that $P_{\mathscr{M}}$ commutes with every element of $\mathscr{V}_T'$; or, what is the same thing, that $\mathscr{M}$ is reducing for every operator $S \in \mathscr{V}_T'$. Since $\mathscr{V}_T'$ is selfadjoint, it suffices to prove that $\mathscr{M}$ is invariant under every operator $S \in \mathscr{V}_T'$. But this is immediate, since by Fuglede's theorem, $\mathscr{V}_T' = \mathfrak{A}_T'$, and $\mathscr{M}$ is given to be hyperinvariant.

We turn now to the proof that (d) implies (a). We first establish the following fact: if $\mathscr{M}$ and $\mathscr{N}$ are subspaces of $\mathscr{H}$ such that $\| P_{\mathscr{M}} - P_{\mathscr{N}} \| = \alpha < 1$, then $\mathscr{N}$ and $\mathscr{M}^\perp$ are complementary (i.e., $\mathscr{N} \cap \mathscr{M}^\perp = (0)$ and $\mathscr{N} + \mathscr{M}^\perp = \mathscr{H}$). Note first that if $f \in \mathscr{N}$, then

$$\|(1 - P_{\mathscr{M}})f\| = \|(P_{\mathscr{N}} - P_{\mathscr{M}})f\| \leqslant \alpha \|f\|,$$

so that

$$\| P_{\mathscr{M}} f\|^2 = \|f\|^2 - \|(1 - P_{\mathscr{M}})f\|^2 \geqslant (1 - \alpha^2)\|f\|^2.$$

Thus $P_{\mathscr{M}}$ is bounded below on $\mathscr{N}$ and it follows that $P_{\mathscr{M}}\mathscr{N}$ is a closed subspace of $\mathscr{M}$. To show that $P_{\mathscr{M}}\mathscr{N} = \mathscr{M}$, suppose that $x \in \mathscr{M}$ is such that $x \perp P_{\mathscr{M}}\mathscr{N}$. Then for every $y \in \mathscr{N}$, $0 = (x, P_{\mathscr{M}} y) = (P_{\mathscr{M}}x, y) = (x, y)$, and thus $x \in \mathscr{N}^\perp$. It follows that $\|(P_{\mathscr{M}} - P_{\mathscr{N}})x \| = \| x \|$, which implies that $x = 0$, and hence that $P_{\mathscr{M}}\mathscr{N} = \mathscr{M}$. (This proves that if $\| P_{\mathscr{M}} - P_{\mathscr{N}} \| < 1$, then $\dim \mathscr{M} = \dim \mathscr{N}$.) We note next that $\mathscr{N} \cap \mathscr{M}^\perp = (0)$, for otherwise $\| P_{\mathscr{M}} - P_{\mathscr{N}} \| = 1$. Also, for every $x \in \mathscr{H}$, $P_{\mathscr{M}}x \in \mathscr{M} = P_{\mathscr{M}}\mathscr{N}$, and thus there is a unique vector $y \in \mathscr{N}$ such that $P_{\mathscr{M}}x = P_{\mathscr{M}} y$. Hence $z = x - y \in \mathscr{M}^\perp$, and we have written $x = y + z$ with $y \in \mathscr{N}$ and $z \in \mathscr{M}^\perp$. Thus $\mathscr{N} + \mathscr{M}^\perp = \mathscr{H}$, and we have shown that $\mathscr{M}^\perp$ and $\mathscr{N}$ are complementary.

We complete the proof that (d) implies (a) as follows: If $P_{\mathscr{M}}$ is a spectral projection for T, and $\mathscr{N} \in \mathscr{I}_T$ is such that $\| P_{\mathscr{M}} - P_{\mathscr{N}} \| < 1$, then $\mathscr{N}$ and $\mathscr{M}^\perp$ are complementary invariant subspaces for T. Let Q be the bounded idempotent on $\mathscr{H}$ having range $\mathscr{N}$ and null space $\mathscr{M}^\perp$. Since $\mathscr{N}, \mathscr{M}^\perp \in \mathscr{I}_T$, we have $QT = TQ$ and, by Fuglede's theorem, $QT^* = T^*Q$. It follows that $\mathscr{N} = \text{range } Q$ is invariant for T^*, and hence that $\mathscr{N}$ is reducing for T. Thus $P_{\mathscr{N}} \in \mathscr{V}_T'$ so that $P_{\mathscr{M}}P_{\mathscr{N}} = P_{\mathscr{N}}P_{\mathscr{M}}$, and since $\| P_{\mathscr{N}} - P_{\mathscr{M}} \| < 1$, we have $\mathscr{M} = \mathscr{N}$.

3. Tensor Products and Unilateral Shifts

In this section we consider the problem of determining the isolated invariant subspaces and the hyperinvariant subspaces for certain tensor products and also for unilaterial shifts. For information concerning tensor products of operators, the reader is referred to ([5], p. 22).

THEOREM 4. *Let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert spaces. A subspace $\mathscr{N} \subset \mathscr{H} \otimes \mathscr{K}$ is hyperinvariant for the operator $T \otimes 1_{\mathscr{K}} \in \mathscr{L}(\mathscr{H} \otimes \mathscr{K})$ if and only if $\mathscr{N}$ is of the form $\mathscr{N} = \mathscr{M} \otimes \mathscr{K}$, where $\mathscr{M}$ is a hyperinvariant subspace for T.*

Proof. If we choose a basis $\{x_\alpha\}_{\alpha \in A}$ for $\mathscr{K}$, then $\mathscr{H} \otimes \mathscr{K}$ can be regarded as the direct sum $\mathscr{H} \otimes \mathscr{K} = \sum_{\alpha \in A} \oplus \mathscr{H}_\alpha$, where each $\mathscr{H}_\alpha$ is a copy of $\mathscr{H}$. The operators on $\mathscr{H} \otimes \mathscr{K}$ can then be regarded as those $n \times n$ matrices with entries from $\mathscr{L}(\mathscr{H})$ that act on $\mathscr{H} \otimes \mathscr{K}$ as bounded operators, where $n = \text{card } A = \dim \mathscr{K}$. In particular, the operator $T \otimes 1$ becomes the $n \times n$ diagonal matrix with T in every position on the diagonal. From this vantage point it is easy to verify that $(\mathfrak{A}_{T \otimes 1})'$ consists exactly of all those $n \times n$ matrices (that act as bounded operators on $\sum_\alpha \oplus \mathscr{H}_\alpha$) with entries from $\mathfrak{A}_T'$; i.e., $(\mathfrak{A}_{T \otimes 1})' = \mathfrak{A}_T' \otimes \mathscr{L}(\mathscr{K})$. This enables us to decide exactly which subspaces $\mathscr{N}$ of $\mathscr{H} \otimes \mathscr{K}$ are invariant under $\mathfrak{A}_T' \otimes \mathscr{L}(\mathscr{K})$ (i.e., hyperinvariant for $T \otimes 1_{\mathscr{K}}$) as follows: Every diagonal $n \times n$ matrix having one $1_{\mathscr{H}}$ on the diagonal and all other diagonal entries equal to zero lies in $(\mathfrak{A}_{T \otimes 1})'$, and $\mathscr{N}$ must be reducing for such an operator. Thus $\mathscr{N} = \sum_{\alpha \in A} \oplus \mathscr{M}_\alpha$ where $\mathscr{M}_\alpha \subset \mathscr{H}_\alpha$. Furthermore, every partially-isometric matrix having a $1_{\mathscr{H}}$ in one off-diagonal position and 0's elsewhere, lies in $(\mathfrak{A}_{T \otimes 1})'$ and it follows that there is a fixed subspace $\mathscr{M}$ of $\mathscr{H}$ such that for each $\alpha \in A$, $\mathscr{M}_\alpha = \mathscr{M}$. Finally, since every operator $T' \otimes 1$ with $T' \in \mathfrak{A}_T'$ also lies in $(\mathfrak{A}_{T \otimes 1})'$, it is clear that $\mathscr{M}$ must be invariant under $\mathfrak{A}_T'$. Thus $\mathscr{N}$ must be of the form $\mathscr{M} \otimes \mathscr{K}$, where $\mathscr{M}$ is hyperinvariant for T. Since every subspace $\mathscr{N} = \mathscr{M} \otimes \mathscr{K}$ of this form is obviously invariant under $(\mathfrak{A}_{T \otimes 1})'$, the proof is complete.

COROLLARY 3.1. *If $\mathscr{N}$ is an inaccessible [isolated] invariant subspace for the operator $T \otimes 1_{\mathscr{K}}$ on $\mathscr{H} \otimes \mathscr{K}$, then $\mathscr{N} = \mathscr{M} \otimes \mathscr{K}$, where $\mathscr{M}$ is an inaccessible [isolated] invariant subspace for T.*

Proof. Since $\mathscr{N}$ is inaccessible, it follows from Theorems 1 and 4 that $\mathscr{N} = \mathscr{M} \otimes \mathscr{K}$ where $\mathscr{M}$ is a hyperinvariant subspace for T.

The result now follows easily from the fact that if $\mathscr{P}$ is any invariant subspace for T, then

$$\| P_{\mathscr{P}\otimes\mathscr{K}} - P_{\mathscr{M}\otimes\mathscr{K}} \| = \| P_{\mathscr{P}} - P_{\mathscr{M}} \|.$$

These results on tensor products enable us to describe the hyperinvariant and isolated invariant subspaces of all unilateral shift operators. We remind the reader that a *unilateral shift of multiplicity* 1 is an operator U defined on a separable Hilbert space $\mathscr{H}$ with orthonormal basis $\{x_n\}_{n=0}^{\infty}$ by the equation

$$Ux_n = x_{n+1}, \qquad n = 0, 1, 2, \dots .$$

A unilateral shift of multiplicity n (where n is any cardinal number) is an operator of the form $U \otimes 1_{\mathscr{K}_n}$ where U is a unilateral shift of multiplicity 1 and $\mathscr{K}_n$ is an n-dimensional Hilbert space.

THEOREM 5. *Let U be a unilateral shift of multiplicity* 1 *on the Hilbert space $\mathscr{H}$. Then every subspace $\mathscr{M} \in \mathscr{I}_U$ is hyperinvariant for U and the only inaccessible points of $\mathscr{I}_U$ are* (0) *and $\mathscr{H}$. Furthermore, if n is any cardinal number and $\mathscr{K}_n$ is an n-dimensional Hilbert space, then the hyperinvariant subspaces of $U \otimes 1_{\mathscr{K}_n}$ are exactly the subspaces $\mathscr{M} \otimes \mathscr{K}_n$ where $\mathscr{M} \in \mathscr{I}_U$, and the only inaccessible invariant subspaces of $U \otimes 1_{\mathscr{K}_n}$ are* (0) *and $\mathscr{H} \otimes \mathscr{K}_n$.*

Proof. It is known [3] that every operator on $\mathscr{H}$ that commutes with U is a weak limit of polynomials in U; it follows that the hyperinvariant subspaces of U are exactly the invariant subspaces of U. Now let $\mathscr{M} \in \mathscr{I}_U$ be such that $(0) \neq \mathscr{M} \neq \mathscr{H}$. It is known [1] that there exists an isometry $V \in \mathscr{L}(\mathscr{H})$ that commutes with U such that $\mathscr{M} = V\mathscr{H}$. For $0 \leqslant \lambda < 1$, the operator $V - \lambda 1$ is bounded below on $\mathscr{H}$ and thus $\mathscr{M}_\lambda = (V - \lambda 1)\mathscr{H}$ is closed and in $\mathscr{I}_U$. By Lemma 1.1,

$$\| P_{\mathscr{M}_\lambda} - P_{\mathscr{M}} \| \leqslant \frac{2\lambda - \lambda^2}{1 - \lambda},$$

so that $\mathscr{M}$ is accessible provided only that $\mathscr{M}_\lambda \neq \mathscr{M}$ for every $0 < \lambda < 1$. To see that this is indeed the case, suppose that for some $0 < \lambda_0 < 1$, $(V - \lambda_0 1)\mathscr{H} = \mathscr{M}_{\lambda_0} = \mathscr{M}$. Then for every $x \in \mathscr{H}$, $Vx - \lambda_0 x = y \in \mathscr{M}$, and hence $x = (1/\lambda_0)(Vx - y) \in \mathscr{M}$. Thus $\mathscr{M} = \mathscr{H}$, which is a contradiction.

The results for $U \otimes 1_{\mathscr{K}_n}$ follow immediately from the above theorem and Corollary 3.1.

4. Direct Sums and Finite-Dimensional Operators

Let S and T be operators on Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$, respectively, and consider the operator $S \oplus T$ on $\mathscr{H} \oplus \mathscr{K}$. In general there is no simple relationship between the invariant subspaces of S and T and those of $S \oplus T$. However, it was recently shown by Crimmins and Rosenthal [4] that if the spectra $\sigma(S)$ and $\sigma(T)$ of S and T are related in a certain way, then every invariant subspace of $S \oplus T$ is a direct sum of an invariant subspace of S with one of T. To be specific, if σ is any compact subset of the plane, let $\eta[\sigma]$ denote the complement of the unbounded component of the complement of σ. The above-mentioned result of Crimmins and Rosenthal can be stated thus: If $\eta[\sigma(S)]$ and $\eta[\sigma(T)]$ are disjoint, then every $\mathscr{P} \in \mathscr{I}_{S \oplus T}$ is of the form $\mathscr{P} = \mathscr{M} \oplus \mathscr{N}$, where $\mathscr{M} \in \mathscr{I}_S$ and $\mathscr{N} \in \mathscr{I}_T$. This leads easily to the following theorem:

Theorem 6. *Let $S \in \mathscr{L}(\mathscr{H})$ and $T \in \mathscr{L}(\mathscr{K})$ be such that $\eta[\sigma(S)]$ and $\eta[\sigma(T)]$ are disjoint. Then the topological space $\mathscr{I}_{S \oplus T}$ is homeomorphic to the space $\mathscr{I}_S \times \mathscr{I}_T$ with the product topology.*

Proof. If $\mathscr{P}$ is an invariant subspace of $S \oplus T$, then $\mathscr{P} = \mathscr{M} \oplus \mathscr{N}$, where $\mathscr{M} \in \mathscr{I}_S$ and $\mathscr{N} \in \mathscr{I}_T$, and it follows that the map $\mathscr{P} \to (\mathscr{M}, \mathscr{N})$ is a 1–1 map of $\mathscr{I}_{S \oplus T}$ onto the Cartesian product $\mathscr{I}_S \times \mathscr{I}_T$. That this map is a homeomorphism when $\mathscr{I}_S \times \mathscr{I}_T$ is given the product topology is immediate from the equation

$$\| P_{\mathscr{M}_1 \oplus \mathscr{N}_1} - P_{\mathscr{M}_2 \oplus \mathscr{N}_2} \| = \sup\{\| P_{\mathscr{M}_1} - P_{\mathscr{M}_2} \|, \| P_{\mathscr{N}_1} - P_{\mathscr{N}_2} \|\}.$$

This theorem can be used to reduce the study of the space $\mathscr{I}_T$ for an operator T on a finite-dimensional space to the case that T is nilpotent. In particular, if $T \in \mathscr{L}(\mathscr{H})$ where $\mathscr{H}$ is finite-dimensional, then T is similar to an operator T_1 on $\mathscr{H}$ with the property that there is an orthonormal basis for $\mathscr{H}$ relative to which the matrix for T_1 is in Jordan canonical form. By Theorem 2, the spaces $\mathscr{I}_T$ and $\mathscr{I}_{T_1}$ are homeomorphic. Furthermore, T_1 is a direct sum of operators each having singleton spectrum, and by Theorem 6, $\mathscr{I}_{T_1}$ is homeomorphic to the corresponding product space. Finally, a finite-dimensional operator S with singleton spectrum $\{\lambda\}$ has exactly the same invariant subspaces as the operator $S - \lambda$, and $S - \lambda$ is nilpotent. This proves the following theorem:

Theorem 7. *If T is an operator on a finite-dimensional Hilbert space, then there exists an integer $k \geqslant 1$ and nilpotent operators $N_1, \ldots, N_k$*

on finite-dimensional spaces such that $\mathscr{I}_T$ *is homeomorphic to the product space*

$$\mathscr{I}_{N_1} \times \mathscr{I}_{N_2} \times \cdots \times \mathscr{I}_{N_k}.$$

The next step in our program is to describe a somewhat novel way of looking at nilpotent operators on finite-dimensional spaces. This approach will then be used to prove the following theorems:

THEOREM 8. *If* $T \in \mathscr{L}(\mathscr{H})$ *where* $\mathscr{H}$ *is n-dimensional* $(1 \leqslant n < \infty)$ *and the spectrum of* T *consists of a single point, then the components of* $\mathscr{I}_T$ *are arcwise connected and are exactly the sets* $\mathscr{I}_T^{k,n-k}$, $k = 0, 1, \ldots, n$.

THEOREM 9. *If* $T \in \mathscr{L}(\mathscr{H})$ *where* $\mathscr{H}$ *is n-dimensional* $(1 < n < \infty)$, *then* $\mathscr{I}_T$ *contains isolated points different from* (0) *and* $\mathscr{H}$ *if and only if* $\sigma(T)$ *contains more than one point, or* $\mathscr{I}_T$ *is linearly ordered (by inclusion). Furthermore, if* T *is not a scalar, then* T *always has hyperinvariant subspaces different from* (0) *and* $\mathscr{H}$.

Let N be a positive integer and let $\mathscr{D}_i$ $(1 \leqslant i \leqslant N)$ be finite-dimensional Hilbert spaces such that

$$(0) \neq \mathscr{D}_N \subset \mathscr{D}_{N-1} \subset \cdots \subset \mathscr{D}_2 \subset \mathscr{D}_1.$$

Let $\mathscr{H}$ be the Hilbert space $\mathscr{H} = \mathscr{D}_N \oplus \mathscr{D}_{N-1} \oplus \cdots \oplus \mathscr{D}_2 \oplus \mathscr{D}_1$, and note that an arbitrary vector x in $\mathscr{H}$ can be written as an N-tuple

$$x = (x_N, x_{N-1}, \ldots, x_1), \qquad x_i \in \mathscr{D}_i.$$

Consider the operator T on $\mathscr{H}$ defined by

$$T(x_N, x_{N-1}, \ldots, x_1) = (0, x_N, x_{N-1}, \ldots, x_2).$$

It is clear that T is a nilpotent operator (of index N). What is more important is that T is a universal model (up to similarity) for nilpotent operators on a finite-dimensional space.

LEMMA 4.1. *If* N *is any nilpotent operator on a finite-dimensional Hilbert space, then* N *is similar to a nilpotent operator* T *of the type described above.*

This lemma is more or less obvious, and instead of giving its proof we content ourselves with treating an example. Suppose that N is a nilpotent operator on an 8-dimensional Hilbert space $\mathscr{H}$. Then N is similar to an operator N_1 whose matrix relative to an orthonormal basis $\{w_1, w_2, w_3, x_1, x_2, y_1, y_2, z_1\}$ for $\mathscr{H}$ is in Jordan canonical form, say

$$N_1 = \left(\begin{array}{ccc|cc|cc|c} 0 & 1 & & & & & & \\ & 0 & 1 & & & & & \\ & & 0 & & & & & \\ \hline & & & 0 & 1 & & & \\ & & & & 0 & & & \\ \hline & & & & & 0 & 1 & \\ & & & & & & 0 & \\ \hline & & & & & & & 0 \end{array}\right)$$

Let $\mathscr{D}_1 = \bigvee \{w_1, x_1, y_1, z_1\}$, $\mathscr{D}_2 = \bigvee \{w_2, x_2, y_2\}$, and $\mathscr{D}_3 = \bigvee \{w_3\}$. Identify $\mathscr{D}_2$ with a subspace of $\mathscr{D}_1$ via the mapping $w_2 \to w_1$, $x_2 \to x_1$, $y_2 \to y_1$, and identify $\mathscr{D}_3$ with a subspace of $\mathscr{D}_2$ via the mapping $w_3 \to w_2$. Then $\mathscr{D}_3 \subset \mathscr{D}_2 \subset \mathscr{D}_1$, and $\mathscr{H} = \mathscr{D}_3 \oplus \mathscr{D}_2 \oplus \mathscr{D}_1$. Furthermore, the operator T that N_1 becomes under these identifications satisfies

$$T(t_3, t_2, t) = (0, t_3, t_2)$$

for an arbitrary vector $(t_3, t_2, t_1) \in \mathscr{D}_3 \oplus \mathscr{D}_2 \oplus \mathscr{D}_1 = \mathscr{H}$.

The proof of Theorem 8 depends upon Lemma 4.1 and the following additional lemmas:

LEMMA 4.2. *Let $\mathscr{K}$ be a Hilbert space and suppose that $\mathscr{M}_\lambda$ is a function from the interval* $[0, 1]$ *to subspaces of $\mathscr{K}$ that is continuous in the metric Θ (defined on all subspaces of $\mathscr{K}$). Suppose also that $f(\lambda)$ is a strongly-continuous function from* $[0, 1]$ *to $\mathscr{K}$ such that for every λ, $f(\lambda) \notin \mathscr{M}_\lambda$. Then $\mathscr{N}_\lambda = \bigvee \{f(\lambda), \mathscr{M}_\lambda\}$ is continuous on* $[0, 1]$ *in the metric Θ.*

Proof. The function $h(\lambda) = P_{\mathscr{M}_\lambda} f(\lambda)$ is clearly strongly continuous, so that $g(\lambda) = f(\lambda) - h(\lambda)$ is also. Since $g(\lambda)$ never vanishes, the function $k(\lambda) = g(\lambda)/\| g(\lambda)\|$ is also strongly continuous on $[0, 1]$. Since $k(\lambda)$ is always a unit vector orthogonal to $\mathscr{M}_\lambda$, an easy computation shows that for every $x \in \mathscr{K}$, $P_{\mathscr{N}_\lambda} x = P_{\mathscr{M}_\lambda} x + (x, k(\lambda))\, k(\lambda)$. Thus,

$$\begin{aligned} \| P_{\mathscr{N}_\lambda} x - P_{\mathscr{N}_\mu} x \| &\leqslant \| P_{\mathscr{M}_\lambda} x - P_{\mathscr{M}_\mu} x \| + \|(x, k(\lambda))\, k(\lambda) - (x, k(\mu))\, k(\mu)\| \\ &\leqslant \| P_{\mathscr{M}_\lambda} - P_{\mathscr{M}_\mu} \| \, \| x \| + 2 \| x \| \, \| k(\lambda) - k(\mu)\|, \end{aligned}$$

which proves that $P_{\mathscr{N}_\lambda}$ is continuous in the metric Θ.

COROLLARY 4.3. *Let* $\{f_i(\lambda)\}_{i=1}^{n}$ *be a collection of strongly-continuous functions from* $[0, 1]$ *to a Hilbert space* $\mathscr{K}$ *such that for every* λ, *the set* $\{f_1(\lambda),..., f_n(\lambda)\}$ *is linearly independent. Then the function* $\mathscr{M}_\lambda = \bigvee \{f_1(\lambda),..., f_n(\lambda)\}$ *is continuous on* $[0, 1]$ *in the metric* $\mathbf{\Theta}$.

Proof. This follows immediately by induction from Lemma 4.2.

Proof of Theorem 8. We may suppose that T is nilpotent, and by Lemma 4.1 we may assume that $\mathscr{H}$ is a Hilbert space of the form $\mathscr{H} = \mathscr{D}_N \oplus \cdots \oplus \mathscr{D}_1$ where $\mathscr{D}_N \subset \cdots \subset \mathscr{D}_1$, and that T is the operator on $\mathscr{H}$ defined by

$$T(x_N, x_{N-1},..., x_1) = (0, x_N, x_{N-1},..., x_2).$$

The proof now proceeds by induction on the dimension of $\mathscr{H}$. If $\mathscr{H}$ is either one or two dimensional, then it is obvious that the theorem is true. Thus we may suppose that $\mathscr{H}$ is n-dimensional and that the theorem has been proved for all nilpotent operators on Hilbert spaces of dimension less than n. We know that each component of $\mathscr{I}_T$ is contained in some $\mathscr{I}_T^{k,n-k}$, so our task is to show that each of the sets $\mathscr{I}_T^{k,n-k}$ $(0 \leqslant k \leqslant n)$ is arcwise connected. For brevity we shall denote the subspace $(0) \oplus \cdots \oplus (0) \oplus \mathscr{D}_1$ of $\mathscr{H}$ by $\tilde{\mathscr{D}}_1$, and we define $\tilde{\mathscr{D}}_2,..., \tilde{\mathscr{D}}_N$ similarly. The crux of what is to be proved is contained in the following lemma:

LEMMA 4.4. *If* $\mathscr{M} \in \mathscr{I}_T$ *is such thar* $\mathscr{M} \cap \tilde{\mathscr{D}}_1 \neq \mathscr{M}$ *and* $\mathscr{M} \cap \tilde{\mathscr{D}}_1 \neq \tilde{\mathscr{D}}_1$, *then* $\mathscr{M}$ *can be joined by an arc in* $\mathscr{I}_T$ *to a subspace* $\mathscr{N} \in \mathscr{I}_T$ *such that* $\dim(\mathscr{N} \cap \tilde{\mathscr{D}}_1) > \dim(\mathscr{M} \cap \tilde{\mathscr{D}}_1)$.

Proof. Since $\mathscr{M} \cap \tilde{\mathscr{D}}_1 \neq \tilde{\mathscr{D}}_1$, there exists a vector $x_0 \in \tilde{\mathscr{D}}_1$ such that $x_0 \notin \mathscr{M} \cap \tilde{\mathscr{D}}_1$. Let j be the largest integer such that $P_{\tilde{\mathscr{D}}_j}\mathscr{M} \neq 0$, and note that since $\mathscr{M} \cap \tilde{\mathscr{D}}_1 \neq \mathscr{M}$, $j > 1$. Let $y_1,..., y_r$ be a basis for $P_{\tilde{\mathscr{D}}_j}\mathscr{M}$, and choose $x_1,..., x_r \in \mathscr{M}$ so that $P_{\tilde{\mathscr{D}}_j}x_i = y_i$ $(1 \leqslant i \leqslant r)$. Let $\mathscr{P}$ denote the subspace

$$\mathscr{P} = \{x \in \mathscr{M} \mid P_{\tilde{\mathscr{D}}_j} x = 0\},$$

and observe that $\mathscr{M} \cap \tilde{\mathscr{D}}_1 \subset \mathscr{P}$ and that $\mathscr{M}$ is the disjoint sum $\mathscr{M} = \bigvee \{x_1,..., x_r\} + \mathscr{P}$. For $0 \leqslant \lambda \leqslant 1$, let $x(\lambda) = (1 - \lambda) x_1 + \lambda x_0$, and let $\mathscr{M}_\lambda$ be the subspace

$$\mathscr{M}_\lambda = \bigvee \{x(\lambda), x_2,..., x_r\} + \mathscr{P}.$$

It is clear that for $0 \leqslant \lambda \leqslant 1$, $x(\lambda)$ is linearly independent of the subspace $\bigvee \{x_2,..., x_r\} + \mathscr{P}$, since $x(0) = x_1$ and for $\lambda > 0$ the contrary would imply that $x_0 \in \mathscr{M}$. Furthermore, since $T\mathscr{M}_\lambda \subset \mathscr{P} \subset \mathscr{M}_\lambda$, we have $\mathscr{M}_\lambda \in \mathscr{I}_T$ $(0 \leqslant \lambda \leqslant 1)$. Thus by Lemma 4.2, $\mathscr{M}_\lambda$ is a continuous function from [0, 1] into $\mathscr{I}_T$, and of course $\mathscr{M}_0 = \mathscr{M}$. Since $x_0 \in \mathscr{M}_1 \cap \tilde{\mathscr{D}}_1$, $x_0 \notin \mathscr{M} \cap \tilde{\mathscr{D}}_1$ and $\mathscr{M}_1 \supset \mathscr{P} \supset \mathscr{M} \cap \tilde{\mathscr{D}}_1$, it follows that $\dim(\mathscr{M}_1 \cap \tilde{\mathscr{D}}_1) > \dim(\mathscr{M} \cap \tilde{\mathscr{D}}_1)$, so that the proof is complete with $\mathscr{N} = \mathscr{M}_1$.

Returning to the proof of the theorem, it is clear that the sets $\mathscr{I}_T^{0,n} = (0)$ and $\mathscr{I}_T^{n,0} = \{\mathscr{H}\}$ are arcwise connected, so that it suffices to consider $\mathscr{I}_T^{k,n-k}$ where $0 < k < n$. The proof now splits into cases depending on whether $k < \dim \tilde{\mathscr{D}}_1$ or $k \geqslant \dim \tilde{\mathscr{D}}_1$. Suppose first that $k < \dim \tilde{\mathscr{D}}_1$. Then every k-dimensional subspace of $\tilde{\mathscr{D}}_1$ lies in $\mathscr{I}_T^{k,n-k}$, and if we define $\mathscr{J}$ to be the collection of all such subspaces, then $\mathscr{J}$ is arcwise connected by [*8*, Theorem 7]. Thus it suffices to show that any point $\mathscr{M} \in \mathscr{I}_T^{k,n-k}$ such that $\mathscr{M} \not\subset \tilde{\mathscr{D}}_1$ can be joined to $\mathscr{J}$ by an arc lying in $\mathscr{I}_T^{k,n-k}$. Since $\mathscr{M} \cap \tilde{\mathscr{D}}_1 \neq \mathscr{M}$ and $\mathscr{M} \cap \tilde{\mathscr{D}}_1 \neq \tilde{\mathscr{D}}_1$ $(k < \dim \tilde{\mathscr{D}}_1)$, we can apply Lemma 4.4 a finite number of times to obtain a subspace $\mathscr{N} \in \mathscr{I}_T^{k,n-k}$ such that $\mathscr{M}$ is joined to $\mathscr{N}$ by an arc in $\mathscr{I}_T^{k,n-k}$ and such that $\dim(\mathscr{N} \cap \tilde{\mathscr{D}}_1)$ is maximal. Since $\mathscr{N} \cap \tilde{\mathscr{D}}_1 = \tilde{\mathscr{D}}_1$ is impossible, it must be that $\mathscr{N} \cap \tilde{\mathscr{D}}_1 = \mathscr{N}$, so that $\mathscr{N} \in \mathscr{J}$, and the proof of this case is completed.

Turning now to the case that $k \geqslant \dim \tilde{\mathscr{D}}_1$, we can apply Lemma 4.4 again to see that every $\mathscr{M} \in \mathscr{I}_T^{k,n-k}$ can be joined by an arc lying in $\mathscr{I}_T^{k,n-k}$ to a subspace $\mathscr{N}$ such that $\mathscr{N} \supset \tilde{\mathscr{D}}_1$. Thus it suffices to show that the collection $\mathscr{L}$ of all subspaces in $\mathscr{I}_T^{k,n-k}$ that contain $\tilde{\mathscr{D}}_1$ is arcwise connected. This is done via the induction hypothesis on the dimension of $\mathscr{H}$. Consider the operator T_1 given by the product $(P_{\mathscr{H} \ominus \tilde{\mathscr{D}}_1}) T$ restricted to the Hilbert space $\mathscr{H} \ominus \tilde{\mathscr{D}}_1$. This operator is nilpotent, and hence by the induction hypothesis, the set $\mathscr{I}_{T_1}^{m,p}$ is arcwise connected, where $m = k - (\dim \tilde{\mathscr{D}}_1)$ and $p = n - (\dim \tilde{\mathscr{D}}_1 + m)$. It is easy to see that a subspace $\mathscr{M} \subset \mathscr{H} \ominus \tilde{\mathscr{D}}_1$ lies in $\mathscr{I}_{T_1}^{m,p}$ if and only if $\mathscr{M} \oplus \tilde{\mathscr{D}}_1$ lies in $\mathscr{L}$, and since

$$\| P_{\mathscr{M}_1 \oplus \tilde{\mathscr{D}}_1} - P_{\mathscr{M}_2 \oplus \tilde{\mathscr{D}}_1} \| = \| P_{\mathscr{M}_1} - P_{\mathscr{M}_2} \|,$$

this proves that the set $\mathscr{L}$ is arcwise connected, completing the proof.

We proceed immediately to the proof of Theorem 9.

Proof of Theorem 9. It suffices to prove the theorem for any operator similar to T, so we may assume that there is an orthonormal basis for $\mathscr{H}$ relative to which the matrix for T is in Jordan canonical form. If $\mathscr{I}_T$ is linearly ordered, then by Corollary 1.4 every point of $\mathscr{I}_T$ is isolated. Furthermore, if $\sigma(T)$ contains more than one point, then by Theorem 7, T can be written as $T = T_1 \oplus T_2$ corresponding to a decomposition of $\mathscr{H}$ as $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$, and $\mathscr{I}_T = \mathscr{I}_{T_1} \times \mathscr{I}_{T_2}$. It is clear in this case that the subspaces $(0) \oplus \mathscr{H}_2$ and $\mathscr{H}_1 \oplus (0)$ are isolated points of $\mathscr{I}_T$. Turning to the proof of the implication going the other way, it suffices to show (in view of Theorem 8) that if $\sigma(T)$ consists of a single point and $\mathscr{I}_T$ is not linearly ordered, then every set $\mathscr{I}_T^{k,n-k}$, where $0 < k < n$, contains at least two distinct points. We may suppose that T is nilpotent, and by an application of Lemma 4.1 we may assume that $\mathscr{H} = \mathscr{D}_N \oplus \cdots \oplus \mathscr{D}_1$ where $\mathscr{D}_N \subset \cdots \subset \mathscr{D}_1$, and that T is defined by the equation

$$T(x_N, \dots, x_1) = (0, x_N, \dots, x_2).$$

We may also suppose, by induction, that if S is any nilpotent operator on an $n-1$ dimensional space and $\mathscr{I}_S$ is not linearly ordered, then there exist at least two distinct points in $\mathscr{I}_S^{k,n-1-k}$ for every $0 < k < n-1$. To see that $\mathscr{I}_T^{n-1.1}$ contains at least two distinct $n-1$ dimensional subspaces $\mathscr{M}$ and $\mathscr{N}$, note that otherwise T^* has only one 1-dimensional invariant subspace, and thus the Jordan matrix for T^* contains only one Jordan block. This says that the invariant subspaces for T^* are linearly ordered, which implies that those for T are linearly ordered also, which is a contradiction. To show that $\mathscr{I}_T^{k,n-k}$ contains more than one point for $0 < k < n-1$, consider $T \mid \mathscr{M}$, the restriction of T to the $n-1$ dimensional invariant subspace $\mathscr{M}$. By induction the argument is complete unless the invariant subspaces of $T \mid \mathscr{M}$ are linearly ordered. If this is the case, note that $\mathscr{M}$ contains a vector x such that $T^{n-2}x \neq 0$. This implies that in the decomposition $\mathscr{H} = \mathscr{D}_N \oplus \cdots \oplus \mathscr{D}_1$, we must have

$$N = n-1, \dim \mathscr{D}_{n-1} = \dim \mathscr{D}_{n-2} = \cdots = \dim \mathscr{D}_2 = 1,$$

and $\dim \mathscr{D}_1 = 2$. (Since $\mathscr{I}_T$ is not linearly ordered, $\dim \mathscr{D}_1 > 1$.) That $\mathscr{I}_T^{k,n-k}$ contains at least two distinct points for $0 < k < n-1$ now follows by applying induction to the operator $T \mid \mathscr{D}_{n-2} \oplus \cdots \oplus \mathscr{D}_1$. To prove the final statement of the theorem, note that if T is not a scalar operator, then T has an eigenspace different from (0) and $\mathscr{H}$, and any such eigenspace is hyperinvariant for T.

Corollary 4.5. *If T is any operator on a finite-dimensional Hilbert space, then the isolated points of $\mathscr{I}_T$ can be specified exactly.*

Proof. By Theorem 7 there exists an integer $k \geqslant 1$ and nilpotent operators $N_1, \dots, N_k$ on finite-dimensional spaces such that $\mathscr{I}_T$ is homeomorphic to the product space $\mathscr{I}_{N_1} \times \cdots \times \mathscr{I}_{N_k}$. By Theorem 9 the isolated invariant subspaces of the operators N_i ($1 \leqslant i \leqslant k$) are completely determined, and clearly a subspace $\mathscr{M} = (\mathscr{M}_1, ..., \mathscr{M}_k) \in \mathscr{I}_T$ will be isolated if and only if each of the subspaces $\mathscr{M}_i \in \mathscr{I}_{N_i}$ is an isolated point in $\mathscr{I}_{N_i}$.

We close this section by remarking that Brickman and Fillmore [2] recently described the invariant subspace lattice of an arbitrary operator T on a finite-dimensional space. Their results, however, seem inapplicable to the study of the metric properties of $\mathscr{I}_T$.

5. Concluding Remarks

(1) The problem of specifying exactly the hyperinvariant subspaces of a finite-dimensional operator T does not seem to have a simple solution.

(2) In a metric space an isolated point is necessarily inaccessible. On the other hand, if T is a normal operator or a finite-dimensional operator, then a point in $\mathscr{I}_T$ is inaccessible if and only if it is isolated. Is this true for every operator T?

(3) Is $\mathscr{I}_T$ always locally connected? Are the components of $\mathscr{I}_T$ always arcwise connected?

(4) The reader will have noted that every nonscalar operator discussed in this paper has hyperinvariant subspaces different from (0) and $\mathscr{H}$. Whether this is true for every nonscalar operator is a very difficult open question. An affirmative answer would solve the invariant subspace problem, and a negative answer would settle a long-standing question as to whether every irreducible operator algebra is strongly dense in $\mathscr{L}(\mathscr{H})$.

(5) The question of whether every nonzero compact operator has a hyperinvariant subspace is also open. An affirmative answer to this question would solve several hard open questions in the theory of compact operators.

(6) If T is an operator that satisfies $\mathfrak{A}_T = \mathfrak{A}_T'$, then every invariant subspace for T is hyperinvariant for T. If T is either a normal or a finite-dimensional operator, then the converse is true. Is the converse always true? Note that even if T is a normal operator, it is not necessarily true that $\mathfrak{A}_T = \mathfrak{A}_T''$.

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