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Some Remarks on Critical Point Theory in Hilbert Space (Continuation)

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This paper is a continuation of the paper of the same title in the Proceedings of the Symposium on Nonlinear Problems, held at the Mathematical Research Center, U.S. Army, at the University of Wisconsin April 30 to May 2, 1962 (University of Wisconsin Press, 1963, pp. 233-256). References to definitions, lemmas etc. always refer to that papers as do bibliographical references (numbers in brackets). The particular subject matter of the present paper is indicated in the title of the following Section 7.

7. Localization and addition theorems.

THEOREM 7.1. *Let the scalar* $i = i(x)$ *satisfy the assumptions* (A)-(E) *of Section 4. Let c be a critical level of i, and let a, b be two numbers such that* $a < c < b$ *and such that c is the only critical level in the closed interval* $[a, b]$ *while* $\bar{\imath}_a$ *is not empty. Let* σ *be the critical set at level c. Then every open neighborhood U of* σ *contains a neighborhood W of* σ *such that the critical group of dimension q at level c (see Definition 4.4) is isomorphic to the homology group of the same dimension of the couple* $(W, W \cap i_c)$.

(See [6, Theorem 8.2] for the finite dimensional case).

For the proof we need some lemmas.

LEMMA 7.1. *For any* ϵ *with* $0 < \epsilon < b - c$ *the critical group of dimension q at the level c is isomorphic to*

$$H_q(\bar{\imath}_{c+\epsilon}, i_c). \tag{7.1}$$

PROOF. By Definition 4.4 and Theorem 4.2 the critical group in question is isomorphic to

$$H_q(\bar{\imath}_{c+\epsilon}, \bar{\imath}_a). \tag{7.2}$$

Now by Lemma 5.6 the pair $(i_c, \bar{\imath}_a)$ is homotopically trivial, and since $\bar{\imath}_{c+\epsilon} \supset i_c \supset \bar{\imath}_a$ the isomorphism of the groups (7.1), (7.2) follows from Lemma 2.4.

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LEMMA 7.2. (α). *There exists an open neighborhood* $U_2 \subset S_{a,b}$ *of* σ *where* $S_{a,b}$ *is defined as in Lemma 4.8.*

(β). *For* $0 \leqslant t \leqslant T$ *let* $\Delta(x_0, t)$ *be the deformation defined in Definition 4.3, and for any set* $S \subset V$ *let* $\Delta_t S$ *denote the image of* S *under the* map $\Delta(\cdot, t)$. *If then* U *is an open neighborhood of* σ *there exists an open neighborhood* $U_1 \subset U$ *of* σ *such that*

$$\Delta_t U_1 \subset U, \qquad 0 \leqslant t \leqslant T. \tag{7.3}$$

PROOF OF (α). The set Γ of all critical points of i in V is compact (Lemma 4.3) and has no points in common with the boundary $\dot{V}$ of V (by assumption D). As a closed subset of Γ, the set σ has also these properties; moreover σ has no points in common with the closure $\bar{S}'_{a,b}$ of the complement $S'_{a,b}$ of $S_{a,b}$. Therefore the compact set σ has a positive distance from $\dot{V} \cup \bar{S}'_{a,b}$ which proves the assertion (α).

PROOF OF (β). By Definition 4.3

$$\Delta(\xi, t) = \xi \qquad \text{for} \qquad \xi \in \sigma. \tag{7.4}$$

Now $\Delta(x_0, t)$ depends continuously on x_0 uniformly in t (see e.g. [9, Lemma 4.4]). Therefore (7.4) implies the existence of a positive $\rho = \rho(\xi)$ such that $\Delta(x, t) \in U$ for $0 \leqslant t \leqslant T$ and for x_0 in the ball $V_\rho(\xi)$ of center ξ and radius ρ. Since σ is contained in the open set U we may, in addition, require ρ to be such that $V_\rho(\xi) \subset U$. But the compactness of σ assures the existence of a finite number of points $\xi_1, \ldots, \xi_r$ in σ such that $\sigma \subset \cup V_{\rho_i}(\xi_i) \subset U$ where $\rho_i = \rho(\xi_i)$. Then $U_1 = \cup V_{\rho_i}(\xi_i)$ satisfies the requirements of our assertion (β).

LEMMA 7.3. *Let* U *be an open neighborhood of* σ. *Then there exist positive numbers* ϵ *and* e *such that*

$$\Delta(x_0, T) \in U \cup i_{c-e} \qquad \text{for} \qquad x_0 \in \bar{i}_{c+\epsilon}. \tag{7.5}$$

PROOF. Let first ϵ be any number satisfying

$$0 < \epsilon < b - c. \tag{7.6}$$

Now $\bar{i}_{c+\epsilon} = S_{a,c+\epsilon} \cup i_a$. If $x_0 \in i_a$ then (7.5) is true for any e satisfying

$$0 < e < c - a, \tag{7.7}$$

since i is not increasing under Δ (Lemma 4.11). Let now $x_0 \in S_{a,c+\epsilon}$, and let U_1 be as in the (β)-part of Lemma 7.2. Then (7.5) is true for $x_0 \in S_{a,c+\epsilon} \cap U_1$. It remains to prove (7.5) for $x_0 \in S_{a,c+\epsilon} \cap U_1'$ where U_1' is the complement of U_1. We note that by (7.6)

$$S_{a,c+\epsilon} \cap U_1' \subset \tilde{S}, \qquad \text{where} \qquad \tilde{S} = S_{a,b} \cap U_1'. \tag{7.8}$$

Now

$$\tilde{S} \cap \Gamma = \Phi \qquad (\Phi \text{ the empty set}). \tag{7.9}$$

Indeed: $\sigma \cap U_1' = \Phi$ since $\sigma \in U_1$. On the other hand $(\Gamma - \sigma) \cap S_{a,b} = \Phi$ since c is the only critical level in $[a, b]$. Equation (7.9), together with the compactness of Γ and the closedness of $\tilde{S}$, implies that these two sets have a positive distance. If we denote this distance by 3ρ then the ρ-neighborhood $\tilde{S}_\rho$ of $\tilde{S}$ has a distance greater than ρ from Γ. Consequently if M is a constant satisfying

$$M > \begin{cases} \| g(x) \| \\ \rho/T \end{cases} \qquad \text{for} \qquad x \in V, \tag{7.10}$$

if $\mu = \rho M$, and if Γ_μ is as defined in Lemma 4.5, then, noting that $\tilde{S} \subset \tilde{S}_\rho \subset \Gamma_\mu$ we see from Lemma 4.6 that there exists a constant m such that

$$\| g(x) \| > m > 0 \qquad \text{for} \qquad x \in \tilde{S}_\rho . \tag{7.11}$$

Let now $x_0 \in \tilde{S}$. Then, because of (7.9), $\Delta(x_0, t)$ is by its Definition 4.3 the solution $x(t)$ of the problem $dx/dt = -g(x)$, $x(0) = x_0$. From this and from (7.10) we see that

$$\| \Delta(x_0, t) - x_0 \| = \left\| \int_0^t g(x(\tau))\, d\tau \right\| \leqslant Mt \leqslant \rho \qquad \text{for} \qquad 0 \leqslant t \leqslant \frac{\rho}{M}.$$

This shows that $\Delta(x_0, t) \in \tilde{S}_\rho$ for the t-values indicated such that for these t, (7.11) holds with $x = x(t) = \Delta(x_0, t)$. We see therefore from (4.15) that

$$i(\Delta x_0, \rho M^{-1}) = i(x_0) - \int_0^{\rho M^{-1}} \| g(x(\tau)) \|^2\, d\epsilon < i(x_0) - \frac{m^2 \rho}{M} .$$

But, by (7.10), $T > \rho/M$, and since $i(\Delta(x_0, t)$ is not increasing in t we see that

$$i(\Delta(x_0, T) < i(x_0) - \frac{m^2 \rho}{M} \qquad \text{for} \qquad x_0 \in \tilde{S}. \tag{7.12}$$

Because of (7.8) this inequality holds for $x_0 \in S_{a,c+\epsilon} \cap U_1'$. For such x_0, $i(x_0) \leqslant c + \epsilon$. Therefore by (7.12)

$$i(\Delta(x_0, T)) < c - \left(\frac{m^2 \rho}{M} - \epsilon \right) .$$

But this implies $\Delta(x_0, t) \in i_{c-e}$ if we choose

$$e = \epsilon = \min \left(b - c, c - a, \frac{m^2 \rho}{2M} \right) .$$

This finishes the proof of (7.5).

LEMMA 7.4. *In the notation of Lemma 7.3 the couples* $(\bar{i}_{c+\epsilon}, i_c)$ *and* $(\Delta_T \bar{i}_{c+\epsilon}, i_c \cap \Delta_T \bar{i}_{c+\epsilon})$ *are homotopically equivalent, and therefore (Lemma 2.2)*

$$H_q(\bar{i}_{c+\epsilon}, i_c) \approx H_q(\Delta_T \bar{i}_{c+\epsilon}, i_c \cap \Delta_T \bar{i}_{c+\epsilon}). \tag{7.13}$$

PROOF. It is easily verified from the properties of the deformation $\Delta_t (0 \leqslant t \leqslant T)$ that the assumptions of Lemma 2.1 are satisfied with

$$B = \bar{i}_{c+\epsilon}, \qquad A = i_c, \qquad D = \Delta_T \bar{i}_{c+\epsilon}, \qquad C = i_c \cap D. \tag{7.14}$$

PROOF OF THEOREM 7.1. Let ϵ and e be as in Lemma 7.3. Using the notation of (7.14) we set

$$W = U \cap D, \qquad W_1 = U' \cap D, \tag{7.15}$$

where U' is the complement of U. The proof consists in excising W_1 from the couple (D, C). Obviously

$$D - W_1 = W. \tag{7.16}$$

We assert moreover that

$$C - W_1 = i_c \cap W. \tag{7.17}$$

Indeed we see from (7.5) and (7.14) that $D \subset U \cup i_{c-e}$, and therefore from (7.15)

$$W_1 \subset U' \cap (U \cup i_{c-e}) = U' \cap i_{c-e} \subset i_{c-e} \subset i_c. \tag{7.18}$$

Thus $W_1 = i_c \cap W_1$, and the left member of (7.17) may be written as $i_c \cap (D - W_1)$, and (7.17) now follows from (7.16).

In order to apply the excision theorem for singular homology theory ([3, VII, Theorem 9.1]) we have to verify that in the relative topology of D the interior of C contains the closure of W_1. Now in this topology $C = i_c \cap D$ is open since i_c is (absolutely) open. Thus we have to prove $\bar{W}_1 \subset C$. But we see from (7.18) that $\bar{W}_1 \subset \bar{i}_{c-e} \subset i_c$ and therefore from (7.14), (7.15)

$$\bar{W}_1 = i_c \cap \bar{W}_1 = i_c \cap (\overline{U' \cap D}) = i_c \cap \bar{U}' \cap D \subset i_c \cap D = C.$$

We thus may apply the excision theorem which by (7.16), (7.17) yields the isomorphism $H_q(D, C) \approx H_q(W, i_c \cap W)$. Combining this with (7.13) we see (using (7.14)) that $H_q(\bar{i}_{c+\epsilon}, i_c) \approx H_q(W, i_c \cap W)$. By Lemma 7.1 this isomorphism proves Theorem 7.1.

THEOREM 7.2. *Let* σ *be as in Theorem 7.1. Assume* $\sigma = \sigma_1 \cup \sigma_2$, *where* σ_1 *and* σ_2 *have a positive distance from each other, and let* U *be an open neighborhood of* σ. *Then there exist disjoint neighborhoods* W_j *of* σ_j $(j = 1, 2)$ *such that* $W = W_1 \cup W_2 \subset U$ *and such that the* qth *critical group of* i *at level* c *is isomorphic to the direct sum*

$$H_q(W_1, i_c \cap W_1) \dotplus H_q(W_2, i_c \cap W_2). \tag{7.19}$$

PROOF. Since σ is compact and σ_1 and σ_2 have a positive distance it follows that σ_1 and σ_2 are compact. Therefore there exist open neighborhoods U_1 and U_2 of σ_1 and σ_2 respectively with a positive distance from each other. Obviously we may also require that these neighborhoods are subsets of U. Then their union is contained in U, and for the purpose of our proof it is no loss of generality to assume that $U = U_1 \cup U_2$. Let now $W \subset U$ be a neighborhood of σ which satisfies the assertions of Theorem 7.1, and let $W_j = W \cap U_j$. Then $W = W_1 \cup W_2$, and by the direct sum theorem ([3; p. 33]) the direct sum (7.19) is isomorphic to $H_q(W, i_c \cap W)$. Theorem 7.2 now follows from Theorem 7.1.

THEOREM 7.3. *In addition to our previous assumptions let σ consist of a finite number of distinct critical points $\sigma_1, \ldots, \sigma_p$. Then there exist (arbitrarily small) disjoint neighborhoods Z_j of σ_j such that the* qth *critical group of i at level c is isomorphic to the direct sum of the groups*

$$H_q(\sigma_j \cup Z_j, Z_j), \qquad j = 1, \ldots, p. \tag{7.20}$$

PROOF. By Theorem 5.2 our critical group is isomorphic to $H_q(\sigma \cup i_c, i_c)$. We will prove that every open neighborhood U of σ contains a neighborhood Z of σ such that

$$H_q(\sigma \cup i_c, i_c) \approx H_q(\sigma \cup Z, Z). \tag{7.21}$$

Let us show first that this assertion implies our theorem. For $j = 1, \ldots, p$, let $U_j \subset U$ be disjoint open neighborhoods of σ_j. It is then no loss of generality to assume that $U = \cup_1^p U_j$. If now $Z \subset U$ satisfies (7.21) and if $Z_j = Z \cap U_j$ such that $Z = \cup_1^p Z_j$, then by the direct sum theorem the right member of (7.21) is isomorphic to the direct sum of the groups (7.20).

Now the proof of (7.21) follows closely that of Theorem 7.1 if instead of (7.14) we introduce the notation

$$B = \sigma \cup i_c, \qquad A = i_c, \qquad D = \Delta_T B, \qquad C = i_c \cap D. \tag{7.22}$$

Noting that

$$\Delta_T \sigma = \sigma, \qquad i_c \cap \sigma = \Phi, \qquad \Delta_T i_c \subset i_c, \tag{7.23}$$

we see that

$$C = \Delta_T A. \tag{7.24}$$

This together with (7.22) shows that the couple (D, C) is obtained from the couple (B, A) by the deformation Δ_t. Therefore

$$H_q(\sigma \cup i_c, i_c) = H_q(B, A) \approx H_q(D, C). \tag{7.25}$$

If we define again W and W_1 by (7.15) we see that (7.16) and (7.17) hold. Consequently by excising W_1 we obtain from (2.25)

$$H_q(\sigma \cup i_c, i_c) \approx H_q(W, i_c \cap W). \tag{7.26}$$

We now set $Z = U \cap \Delta_T i_c$. We then see from (7.15), (7.22) and (7.23) that $W = \sigma \cup Z$ and $i_c \cap W = Z$. Therefore (7.21) follows from (7.26).