

SHELL-MODEL CALCULATION IN ^{24}Mg

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Abstract: A shell-model calculation is performed for ^{24}Mg assuming wave functions with maximum symmetry in the spacial coordinates. Several interactions have been used including the Kallio-Kolltveit hard-core potential. A reasonable agreement is obtained for the energy levels, while the E2 branching ratios are in many cases different from the experimental values. The eigenfunctions have large overlap with the predicted SU_3 wave functions. The influence of particle-hole excitations in the spectrum is discussed.

1. Introduction

The shell-model approach including configuration mixing has been applied with considerable success in the beginning of the sd shell. Such calculations are based on the assumption that the ^{16}O nucleus constitutes a tightly bound system. In nuclei, which have only a few particles in addition to this core system, the low-energy part of the spectra should therefore be well described by the degrees of freedom of the extra particles. The available single-particle levels in this region are known ¹⁾ to be $1d_{5/2}$, $2s_{1/2}$ and $1d_{3/2}$ which appear in the one-particle nuclei ^{17}O and ^{17}F .

Many calculations ²⁻⁸⁾ have been carried out along these lines in order to analyse the low-energy spectrum of the nuclei ^{18}O , ^{18}F , ^{19}O , ^{19}F and ^{20}Ne . The results confirm the basic shell-model ideas. The level separations and the properties of the corresponding wave functions seem to be well described by this method. However, there are some cases where the discrepancy between the experiments and the theoretical predictions are still too large to be explained as due to the uncertainties in the two-body interaction and the single-particle wave functions. This has especially been found ⁹⁾ for some specific states in ^{18}O , where the core system is important even for the low-energy excitations. Still, the present conclusion is that these nuclei are reasonably well described in the shell-model picture, and the main component of the wave functions are due to the additional particles outside ^{16}O .

In this situation it is very interesting to go one step further and try to analyse the nuclei beyond ^{20}Ne in the shell-model scheme. How far away from the closed shell

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is the method adequate, and where in this region should a different point of view be chosen in order to reproduce the energy spectra? We know that in the middle of the sd shell the rotational model introduced by Bohr and Mottelson^{10, 11)} has been applied with good results. What will the shell model give here?

Unfortunately, standard shell-model calculation is very complicated as soon as one exceeds three- or four-particle systems. In the sd region the multiplicity of states in the shell-model basis for the two particle nuclei ^{18}F and ^{18}O does not exceed five for a given spin J and isospin T . Already in the ^{20}Ne case, this multiplicity has reached 75. Thus, computation of an enormous number of matrix elements is required, followed by a complicated diagonalization procedure. Even with high speed computers, it seems impracticable with the present mathematical methods to perform such shell-model calculations for nuclei beyond ^{20}Ne .

However, an interesting feature of the shell-model wave functions, which seems to simplify the situation, has been discovered in the beginning of the sd shell. Due to the fact that the whole second oscillator shell is taken into account in the calculations, the single-particle oscillator Hamiltonian is invariant under SU_3 transformations. Elliott¹²⁾ has shown that the calculated eigenstates for nuclei in this region to a large extent preserve the SU_3 symmetry, even when one includes symmetry breaking terms as the single-particle $l \cdot s$ term and realistic two-body forces.

This symmetry property of the eigenstates explains the close connection between the rotational-model approach and standard shell-model calculations. In typical shell-model nuclei as ^{19}F and ^{20}Ne , a collective-model interpretation based on Nilsson's deformed single-particle level scheme gives an equally good agreement^{7, 14-16)}.

Higher in the sd shell especially around $A = 25$, the energy spectra show typical band structure according to the collective-rotational model¹⁷⁾. It is interesting to study such nuclei from a conventional shell-model point of view in order to test to what extent such wave functions can explain this collective behaviour.

Based on the SU_3 symmetry of the eigenstates, Elliott and Harvey¹³⁾ have proposed an explanation of the level schemes for all nuclei throughout the shell. The interpretation is based on the SU_3 coupling scheme assuming that the eigenstates have a large overlap with SU_3 wave functions. This idea has been applied to the nucleus ^{24}Mg with a reasonable good agreement for the ground state rotational band^{13, 18)}.

In this work we have analysed the same nucleus from a more conventional shell-model point of view. In light nuclei it appears that the α -particle nuclei with even $N = Z$ are well described in the LS coupling scheme. We have therefore assumed that the eigenstates of ^{24}Mg can be described by the set of eight-particle wave functions which have maximum space symmetry characterized by the partition $[f] = [44]$, while the first 16 particles constitute an inert and spherical core. Such a model contains enough shell-model configurations to give a reasonable test of the SU_3 coupling scheme.

The assumption of maximum space symmetry is based on the fact that the two-particle interaction in nuclei is of short range and attractive and therefore gives the

largest matrix elements between configurations which are spatially symmetrical. Such configurations should, consequently, be the most important ones in the low-energy part of the spectrum. This idea has lately been tested by Inoue *et al.* ⁸⁾ in the case of the ^{20}Ne , another α -particle nucleus. They found that the next highest symmetry contributed with only 10 % to the eigenstates in the low-energy part of the spectrum. The other symmetries gave nearly no contribution at all.

2. The model

According to the individual-particle model, we assume that the 16 first particles occupy the two first oscillator shells and generate an inert and spherical core. The effect of the core is represented in the calculation by a single-particle, shell-model potential. The eight additional particles move in this well, restricted by the Pauli principle. As the possible orbits we choose the single-particle levels with positive parity which appear in the low-energy spectrum of ^{17}O ; see fig. 1. This restricts our model to positive-parity states only.

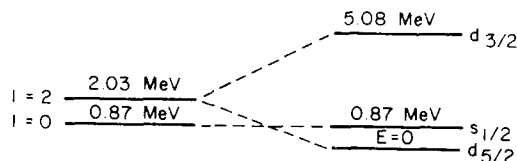


Fig. 1. Single-particle orbitals in the 2s-1d shell from ^{17}O .

The spectrum of ^{24}Mg is then assumed to be due to a residual two-body interaction between the additional particles in these orbits.

Our model Hamiltonian then has the form

$$H = H_0 + H_s + V,$$

where

$$H_s = \sum_i \left\{ -\frac{\hbar^2}{2m} \nabla_i^2 + \delta \mathbf{l}_i \cdot \mathbf{s}_i + U(r_i) \right\},$$

$$V = \sum_{i < j} V_{ij}, \quad (1)$$

H_0 is the Hamiltonian of the core. The sum in eq. (1) is over the external particles, $U(r_i)$ being the field of the core. The parameter $\delta = 2.03$ MeV is the strength of the $\mathbf{l} \cdot \mathbf{s}$ term adjusted to reproduce the splitting between $1d_{\frac{5}{2}}$ and $1d_{\frac{3}{2}}$ in the spectrum of ^{17}O ; see fig. 1.

From the arguments in sect. 1, we further assume that our wave functions, which represent the eight particles outside the core, have maximum symmetry in the spatial coordinates. This selects the LS -coupling scheme as the natural one. The spatial part of the wave functions, therefore, transforms according to the $[f] = [44]$ representa-

tion of the symmetry group S_8 , giving the conjugate symmetry $[\tilde{f}] = [22222]$ for the spin-isospin part to account for the Pauli principle. Thus all wave functions have $T = 0$ and $S = 0$.

We can now write our wave functions as

$$\Psi(s^k d^{8-k}[44]\alpha L = J) = \frac{1}{\sqrt{h[44]}} \sum_{\{r_8\}} \Phi(s^k d^{8-k}[44]\alpha L; \{r_8\}) X(\gamma^8[44]00; \{\tilde{r}_8\}), \quad (2)$$

TABLE I
The number of basic states with space symmetry $[f] = [44]$

	L-values												
$s^k d^{8-k}$	0	1	2	3	4	5	6	7	8	9	10	11	12
d^8	4	1	6	4	8	4	7	3	4	2	2		1
sd^7	1	4	7	7	8	8	6	5	4	2	1	1	
$s^2 d^6$	3	2	7	5	8	5	6	3	3	1	1		
$s^3 d^5$	1	2	3	4	4	3	3	2	1	1			
$s^4 d^4$	1		2		2	1	1		1				
total	10	9	25	20	30	21	23	13	13	6	4	1	1

with $k = 0, 1, 2, 3$ and 4 . It is a sum over products of a space part and a spin-isospin part where the summation runs through all the different Yamanouchi symbols $\{r_8\}$ according to the standard Young tableau $[f] = [44]$ for S_8 , the n -particle spin-isospin wave functions being given by $X(\gamma^n[\tilde{f}]TS; \{r_n\})$ (ref. ¹⁹). In our notation $\{r_8\}$ represents the eight characteristic integers $(r_8 r_7 \dots r_1)$ for the Yamanouchi symbols. The factor $h[44]$ is the dimension of the representation of the symmetric group S_8 with partition $[f] = [44]$.

In table I we list the number of basic states. We see that the problem will be to generate matrix elements and perform diagonalization of matrices with dimension up to 30.

The matrix elements of the single-particle Hamiltonian in eq. (1) are easily calculated to

$$\langle s^k d^{8-k} \alpha L = J | H_s | s^{k'} d^{8-k'} \alpha' L = J \rangle = \delta_{\alpha\alpha'} \delta_{kk'} \{8\varepsilon_2 + k(\varepsilon_0 - \varepsilon_2)\}. \quad (3)$$

The values of ε_l are estimated from the spectrum of ^{17}O (see fig. 1) to be

$$\varepsilon_0 = 0.871 \text{ MeV}, \quad \varepsilon_2 = 2.03 \text{ MeV}. \quad (4)$$

The final spectrum, however, depends only on the difference between these parameters as long as we do not try to fit the absolute binding energy of the system. Due to the

structure of the wave functions (2), the $I \cdot s$ term in H_s does not contribute to the matrix elements in eq. (3).

The standard approach in order to calculate the matrix elements of the two-particle interaction in the Hamiltonian (1) is to use the technique of fractional parentage coefficients. This seems, however, to be too complicated in our case. Another method which we have chosen is to reduce all the matrix elements to integrals over space coordinates only and then apply an operator technique due to Moshinsky²⁰). This is a very powerful method in cases with high symmetry in the spatial coordinates. The same approach has been applied in a similar calculation⁶) of the spectrum of ^{20}Ne .

The most general two-body interaction which contribute to the matrix elements with wave functions of the type given in eq. (2) can be written as

$$V_{ij} = [V_s^+(r_{ij})P^1 + V_t^+(r_{ij})P^3]_{\frac{1}{2}}[1 + P_{ij}^M] + [V_s^-(r_{ij})P^1 + V_t^-(r_{ij})P^3]_{\frac{1}{2}}[1 - P_{ij}^M], \quad (5)$$

where P_{ij}^M is the Majorana exchange operator, P^1 and P^3 the singlet and triplet projection operators, respectively, and $V_s^\pm(r_{ij})$ and $V_t^\pm(r_{ij})$ are radial forms of the interaction in the different channels. All other terms in the two-body interaction are not scalars in the spin-isospin space and consequently give zero contributions to the matrix elements.

The structure of the interaction in eq. (5) is of the form $T_{ij}(r_{ij})U_{ij}(\sigma_i \cdot \sigma_j, \tau_i \cdot \tau_j)$. For such an interaction operator, we have

$$\begin{aligned} \langle s^k d^{8-k}[44]\alpha L = J | \sum_{i < j} T_{ij} U_{ij} | s^{k'} d^{8-k'}[44]\alpha' L = J \rangle \\ = \frac{28}{h[44]} \sum_{\{r_8\} \{r'_8\}} \langle s^k d^{8-k}[44]\alpha L; \{r_8\} | T_{78} | s^{k'} d^{8-k'}[44]\alpha' L; \{r'_8\} \rangle \\ \times \langle \gamma^8[\tilde{44}]00; \{\tilde{r}_8\} | U_{78} | \gamma^8[\tilde{44}]00; \{\tilde{r}'_8\} \rangle. \end{aligned} \quad (6)$$

From the knowledge of the fractional parentage coefficients in the spin-isospin space¹⁹), we can easily calculate the contribution from the last term in eq. (6) and reduce the whole sum to a space integral for T_{ij} . If we then apply this method for each term in the two-body interaction (5), we get the following formula:

$$\begin{aligned} \langle s^k d^{8-k}[44]\alpha L = J | \sum_{i < j} V_{ij} | s^{k'} d^{8-k'}[44]\alpha' L = J \rangle \\ = \langle s^k d^{8-k}[44]\alpha L | \sum_{i < j} \frac{1}{2}(V_s^+ + V_t^+)P_{ij}^+ + \frac{1}{10}(V_s^- + 9V_t^-)P_{ij}^- | s^{k'} d^{8-k'}[44]\alpha' L \rangle. \end{aligned} \quad (7)$$

where the operator P_{ij}^+ (P_{ij}^-) has the eigenvalue $+1$ (-1) when acting on a two-particle state which is spatially symmetric (antisymmetric), and the eigenvalue zero otherwise.

This expression of the two-body matrix elements gives a convenient starting point for using the operator technique introduced by Moshinsky²⁰) to obtain the space part of the wave functions and the corresponding matrix elements. In order to establish the necessary equations we shall in the following give a short review of the technique.

We define the single-particle wave functions as a creation operator acting on a "vacuum state" by the equations

$$\begin{aligned} |lm\rangle_s &\equiv a_{lms}^+|0\rangle, & l = 0, 2; m = -l, \dots, +l, \\ a_{lms}|0\rangle &\equiv 0, & s = 1, 2, \dots, 8 \text{ (particle index)}, \end{aligned} \quad (8)$$

with commutation relations

$$[a_{lms}, a_{l'm's'}^+] = \delta_{ss'}\delta_{ll'}\delta_{mm'}.$$

Our single-particle wave functions are restricted to the second oscillator shell. They are therefore completely characterized by the two quantum numbers l and m .

The eight-particle wave functions are then given as homogeneous polynomials in the creation operators acting on the vacuum state. These polynomials must have the same symmetry property under the symmetry group as our original wave functions. We also want them to behave properly under rotation in the three-dimensional space. These two properties are characterized by the quantum numbers $[f]$ and L . An additional quantum number α is introduced in order to distinguish wave functions with the same $[f]$ and L .

From table 1 we know the multiplicity of the basic states for each L -value. The problem is to determine the same number of polynomials which satisfy the following set of equations:

$$C_{rr}P_\alpha(L)|0\rangle = 4P_\alpha(L)|0\rangle, \quad r = 1, 2, \quad (9)$$

$$C_{ss}P_\alpha(L)|0\rangle = 0, \quad s = 3, 4, \dots, 8, \quad (10)$$

$$C_{uv}P_\alpha(L)|0\rangle = 0, \quad u < v = 2, 3, \dots, 8, \quad (11)$$

$$L_0P_\alpha(L)|0\rangle = LP_\alpha(L), \quad (12)$$

$$L_{+1}P_\alpha(L)|0\rangle = 0, \quad (13)$$

where

$$C_{rs} = \sum_{lm} a_{lms}^+ a_{lms},$$

$$L_q = \sum_{lmm'} \sqrt{l(l+1)} \langle lm1q|lm'\rangle \sum_s a_{lms}^+ a_{lms}.$$

Here $\langle lm1q|lm'\rangle$ is the usual Clebsch-Gordan coupling coefficient.

The two eqs. (12) and (13) represent the condition that the polynomials have correct behaviour under three-dimensional rotations with the z -component of the angular momentum equal to L . The three first equations (9)–(11) restrict the polynomials to the partition $[f] = [44]$.

In this operator formalism, the two-body interaction in eq. (7) is given by

$$V = \sum_{\substack{l_1 l_2 \\ l'_1 l'_2}} \sum_A G(l_1 l_2 l'_1 l'_2; A) P(l_1 l_2 l'_1 l'_2; A), \quad (14)$$

where

$$P(l_1 l_2 l'_1 l'_2; A) = \frac{1}{2} \sum_{M=-A}^{+A} \sum_{s, t=1}^8 [a_{l_1 s}^+ a_{l_2 t}^+]_{AM} [a_{l'_1 s} a_{l'_2 t}]_{AM},$$

$$G(l_1 l_2 l'_1 l'_2; A) = \langle (l_1 l_2) A | \frac{1}{2} (V_s^+ + V_t^+) P_{12}^+ + \frac{1}{10} (V_s^- + 9V_t^-) P_{12}^- | (l'_1 l'_2) A \rangle.$$

The symbols $[\dots]_M$ stands for the vector coupling of the two operators. For the matrix elements in eq. (7), we get

$$\langle (\text{sd})^8 [44] \alpha L = J | \sum_{i < j} V_{ij} | (\text{sd})^8 [44] \alpha' L = J \rangle = \langle 0 | P_\alpha^+(L) V P_\alpha(L) | 0 \rangle$$

$$= \sum_{\substack{l_1 l_2 l'_1 l'_2 \\ A}} C_{\alpha\alpha'}^L(l_1 l_2 l'_1 l'_2; A) G(l_1 l_2 l'_1 l'_2; A), \quad (15)$$

with the coefficients $C_{\alpha\alpha'}^L(l_1 l_2 l'_1 l'_2; A)$ given by the operator product

$$C_{\alpha\alpha'}^L(l_1 l_2 l'_1 l'_2; A) = \frac{1}{2} \sum_{M=-A}^{+A} \sum_{s, t=1}^8 \langle 0 | P_\alpha^+(L) [a_{l_1 s}^+ a_{l_2 t}^+]_{AM} [a_{l'_1 s} a_{l'_2 t}]_{AM} P_\alpha^L(L) | 0 \rangle. \quad (16)$$

The matrix elements are given as a sum over products of a coefficient $C_{\alpha\alpha'}^L(l_1 l_2 l'_1 l'_2; A)$ and a two-particle matrix element $G(l_1 l_2 l'_1 l'_2; A)$. The last factor contains all the physical information in the problem, the structure of the two-body interaction and the form of the single-particle wave functions. This formula is very convenient for repeated applications under different physical conditions. Once the coefficients $C_{\alpha\alpha'}^L(l_1 l_2 l'_1 l'_2; A)$ are calculated, it is a straightforward task to generate the matrices and perform the diagonalization.

In order to generate the eight-particle polynomials which satisfy eqs. (9)–(13), we follow a procedure suggested by Moshinsky²⁰⁾. We start by defining homogeneous antisymmetric two-particle polynomials of the type

$$A_{lm}^{1,2} = \sum_{m_1 m_2} \langle l_1 m_1 l_2 m_2 | lm \rangle (a_{l_1 m_1}^+ a_{l_2 m_2}^+ - a_{l_1 m_1, 2}^+ a_{l_2 m_2, 1}^+), \quad (17)$$

which have definite angular momentum $l = 1, 2, 3$ with projection m along the z -axis.

It is easy to verify that any homogeneous polynomial of degree four in these $A_m^{1,2}$ is a solution of eqs. (9)–(11). In order to have a solution of the whole set of eqs. (9)–(13), we must choose a combination of the $A_m^{1,2}$ which also have a total angular momentum L and $M = L$ along the z -axis. This is easily done by suitable angular momentum coupling of the two-particle polynomials.

From eq. (17) we generate four-particle polynomials

$$B_{l_1 l_2 l m}^{1,2} = \sum_{m_1 m_2} \langle l_1 m_1 l_2 m_2 | lm \rangle A_{l_1 m_1}^{1,2} A_{l_2 m_2}^{1,2}. \quad (18)$$

Clearly the $B_{l_1 l_2 l m}^{1,2}$ represent four particles with the symmetry $[f] = [22]$.

The last step in constructing the eight-particle polynomials is then similarly given by

$$P_\alpha(L) = \sum_{M_1 M_2} \langle L_1 M_1 L_2 M_2 | LL \rangle B_{l_1 l_2 L_1 M_1}^{1,2} B_{l_3 l_4 L_2 M_2}^{1,2}. \quad (19)$$

This is a systematic and convenient procedure for machine computation. However, from the straightforward construction of all $B_{l_1 l_2 l m}^{1, 2}$ in eq. (18), we get a number of polynomials which are linear combinations of the other ones and have to be thrown out before constructing the $P_\alpha(L)$. Similarly there will be extra ones $P_\alpha(L)$, which are picked out before applying Schmidt's orthogonalization process to them.

The number of linearly independent polynomials for $[f] = [22]$ and $[f] = [44]$ symmetry are thus checked against Elliott's table (ref. ¹²), p. 136).

Because of the orthogonalization the eight-particle polynomials are no longer characterized by the quantum numbers l of two and four particles but are numbered successively by α .

The solutions in eq. (19) show one important feature. They contain operators with particle indices $s = 1$ and 2 only. This simplifies the structure and makes the whole calculation relatively easy. The reason is that we have restricted our wave functions to maximum symmetry in the spatial coordinates.

Once we have found the necessary number of independent and orthogonal solutions of eqs. (9)–(13), all the coefficients $C_{\alpha\alpha'}^L(l_1 l_2 l_1' l_2'; A)$ can be calculated from eq. (16).

3. Calculation of the energy spectrum

In the calculation previously on ^{24}Mg by Elliott and Harvey ¹³), a two-body central interaction is used both with Serber and Rosenfeld exchange mixture. The energy spectrum was determined from the maximum SU_3 state $(\lambda\mu) = (84)$ with space symmetry $[f] = [44]$. In one case also $(\lambda\mu) = (46)$ and (08) were included. The results show typical rotational behaviour with a $K = 0$ ground state band followed by an excited $K = 2$ band. This structure is also seen in the experimental spectrum ²²). However, one major difficulty arises in this approach. The calculated $K = 2$ band starts far below the experimental one.

In the present work all the basic states with the maximum space symmetry are included; see table 1. This corresponds to the following SU_3 representations:

$$\begin{aligned}
 [f] = [44] (\lambda\mu) = & (84), (73), (46), (81), (54), \\
 & (08), (62)^2, (35), (43), (51)^2, \\
 & (24), (32), (40)^2, (13), (02).
 \end{aligned}$$

To construct the energy matrices we use the method described earlier. All the polynomials needed are calculated from eq. (19). The only difficulties arise from the fact that the technique gives too many polynomials which in general are linearly dependent. However, they can be separated into four groups according to the distribution of s- and d-particles; see table 1, and the orthogonalization procedure is only necessary inside each of these groups. This simplifies the calculation considerably. The coefficients $C_{\alpha\alpha'}^L(l_1 l_2 l_1' l_2'; A)$ are generated from eq. (16), and the energy matrices are constructed as linear combinations of two-body matrix elements; see eq. (14). The

TABLE 2
The eigenvalues and percentage intensity of SU_3 components for the eight lowest states

J^π	Force	Calculated energy	$(\lambda\mu)$ values						
			84	73	46	81	08/54	62	other ($\lambda\mu$)
0 ⁺	A	0	87.7		4.8		2.4	4.4	0.7
	B	0	69.8		14.4		7.0	6.6	2.2
	C	0	73.5		9.8		6.7	7.4	2.6
	D	0	86.0		0.1		2.0	9.3	2.6
2 ⁺	A	1.35	91.4		3.8		0.9	3.6	0.3
	B	1.55	77.1	0.1	12.5		4.1	5.4	0.8
	C	1.56	82.2		7.9		3.4	5.7	0.8
	D	1.63	91.1		0.2		0.9	6.9	0.9
4 ⁺	A	3.62	90.9	0.4	5.1		0.4	2.5	0.7
	B	3.28	74.2		21.3	0.1	0.5	3.3	0.6
	C	3.46	78.8	0.1	16.3		0.7	3.6	0.5
	D	3.70	87.6	2.9	0.2	0.1	0.7	6.7	1.8
2 ⁺	A	2.45	91.0	0.4	4.8		1.4	2.2	0.2
	B	2.81	76.8	0.8	14.0		4.0	3.6	0.8
	C	2.72	82.2	0.9	7.9		3.4	5.7	0
	D	3.06	92.4	1.2	0.4		0.9	4.3	0.8
3 ⁺	A	3.43	92.4	1.3	3.5			2.5	0.3
	B	3.90	83.3	1.9	9.7			4.2	0.9
	C	3.65	86.0	2.7	6.0			4.1	1.2
	D	3.91	90.0	3.1	0.5		0.2	4.8	1.4
4 ⁺	A	4.27	84.6	2.0	10.3		0.3	2.0	0.8
	B	3.85	69.8	3.5	16.5		2.4	4.6	3.2
	C	3.69	69.4	4.7	15.6	0.1	1.6	4.5	4.1
	D	5.16	82.6	3.3	7.5	0.1	0.4	3.3	2.8
0 ⁺	A	7.17	3.6		56.7		25.2	11.0	3.5
	B	6.90	5.0		32.0		27.1	23.0	12.9
	C	5.24	3.8		38.9		22.6	22.5	12.2
	D	7.36	0.1		72.7		11.7	10.8	4.7
2 ⁺	A	8.82	5.8	2.8	58.7	0.7	25.6	4.5	1.9
	B	8.75	18.2	12.6	26.0	2.7	26.2	5.5	8.8
	C	7.54	14.4	5.1	41.5	2.1	24.3	6.8	5.8
	D	8.66	1.1	12.6	63.8	2.3	8.8	3.3	8.1

A - Gaussian Serber, B - Gaussian Rosenfeld, C - Yukawa, D - Kallio-Kolltveit interaction.

calculations are carried out for various interactions which have been applied in the sd shell with good agreement. This is done in order to eliminate the uncertainty of the interaction as much as possible. In all cases we have used the harmonic oscillator wave functions with a size parameter

$$b = \sqrt{\frac{\hbar}{m\omega}} = 1.8 \text{ fm.}$$

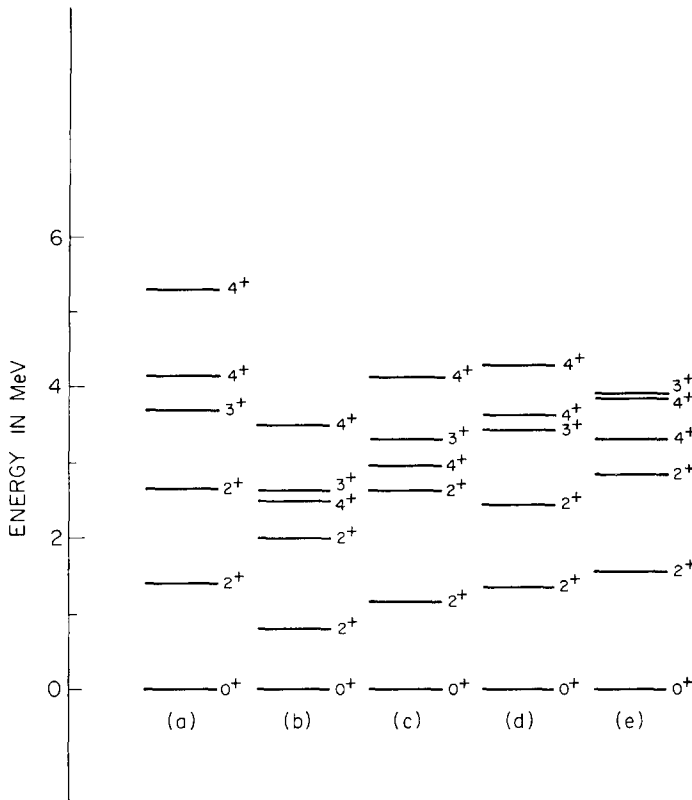


Fig. 2. Energy levels with Gaussian potential. Calculation by Elliott and Harvey¹⁹; (a) - Serber mixture ($\lambda\mu = (84)$), (b) - Rosenfeld mixture ($\lambda\mu = (84)$), (c) - Rosenfeld mixture ($\lambda\mu = (84) + (46) + (62)$). Our calculation with all ($\lambda\mu$); (d) - Serber mixture, (e) - Rosenfeld mixture.

To make the connection with the SU_3 model of Elliott and Harvey, a calculation is performed with the same interaction. It has the following radial dependence:

$$V(r_{ij}) = V_0 \exp [-(r_{ij}/r_0)^2],$$

with $V_0 = -60$ MeV and $r_0 = 1.8$ fm. The result for the six lowest levels is given in fig. 2. It shows the influence of the number of basic states used in the calculation.

agreement with the experimental spectrum. It has the form

$$V_{s,t}^{\pm}(r_{ij}) = V_{s,t}^{\pm} \frac{1}{r_{ij}/a} \exp[-r_{ij}/a],$$

where the strengths of the different components are

$$\begin{aligned} V_s^+ &= -27 \text{ MeV}, & V_t^+ &= -35 \text{ MeV}, \\ V_s^- &= 0, & V_t^- &= +13.5 \text{ MeV}, \end{aligned}$$

with

$$a = \frac{2}{3}b\sqrt{2}.$$

Here a is the range of the interaction and b the size parameter of the oscillator wave functions.

To study the effect of more realistic forces with hard core, we also used the Kallio-Kolltveit (KK) interaction applied in this region by many authors⁶⁾. The 16 lowest levels for all these interactions are shown in fig. 3 with the experimental spectrum.

The general feature of the various level schemes shows more or less the same characteristics. We have a $K = 0$ ground state rotational band with approximately an $L(L+1)$ dependence. Some distortion is observed for higher L -values. This effect is, however, too large in comparison with the experimental situation and also compared to the simpler SU_3 model. Some improvement has been obtained for the $K = 2$ band which now starts at approximately 3 MeV. In all cases the odd-state force increases the energy of the $K = 2, J = 2$ state. To analyse this effect further we increased the odd-state component in the case of Gaussian interaction to far above the Rosenfeld mixture, but there the rotational structure began to break down, thus giving no further improvement.

Of all different interactions which have been used, the KK force with hard core gives the best all-over fit. The level density in the low-energy region is less than in all other cases. However, the deviations from the experimental spectrum are still too large, and it seems to indicate that important degrees of freedom are present in the spectrum which has not yet been included in the model.

The splitting between the 2s and 1d single-particle levels has been taken from the ^{17}O spectrum to be 1.16 MeV; see fig. 1, and it is assumed that this is the same throughout the sd shell. However, a comparison with the ^{39}Ca spectrum indicates that the ordering of these levels has changed, while the strength of the $l \cdot s$ force is more or less the same. In a Hartree-Fock calculation Muthukrishnan and Baranger²³⁾ found the same effect for the s- and d-levels, and the crossing occurred around $A = 30$. We have here tested the influence of this parameter in the spectrum, but we found rather small changes which did not give any significant improvement.

4. The structure of the wave functions and electromagnetic transitions

The SU_3 classification is known to be a good physical scheme in the beginning of the sd shell. We therefore make a transformation in our basic states from the arbitrary

TABLE 3

The (84) component of the wave functions for the same four cases as in table 2

State	Force	(84) normalized to 1			Intensities of other ($\lambda\mu$)
		$K = 0$	$K = 2$	$K = 4$	(%)
Ground	A	1			12.3
	B	1			30.2
	C	1			24.5
	D	1			14.0
First 2 ⁺	A	-0.9728	-0.1640		8.6
	B	-0.9460	-0.2946		22.9
	C	-0.9544	-0.2674		17.8
	D	-0.9629	-0.2376		8.9
First 4 ⁺	A	-0.9304	-0.2491	-0.0036	9.1
	B	-0.9620	-0.1643	-0.0292	25.8
	C	-1.0000	-0.0043	0.0204	21.4
	D	-0.6888	-0.6364	0.0121	12.6
Second 2 ⁺	A	-0.2149	0.9834		9.0
	B	-0.4141	0.9226		23.2
	C	0.3629	-0.9442		18.6
	D	0.2727	-0.9718		7.6
First 3 ⁺	A		1		7.6
	B		1		16.7
	C		1		12.5
	D		1		10.0
Second 4 ⁺	A	0.3770	-0.9811	0.0311	15.4
	B	0.1071	-0.9805	0.0071	30.2
	C	0.0513	-1.0063	0.0262	30.6
	D	0.7810	-0.7429	0.0252	17.4

A - Gaussian Serber, B - Gaussian Rosenfeld, C - KK, D - Yukawa odd-state.

classification α , see eq. (19), to the physically more significant quantum numbers ($\lambda\mu$). This is obtained by diagonalizing the quadrupole-quadrupole interaction Q_{ij}^2 (ref. ²⁰), whose eigenvalues are known to be

$$E_{(\lambda\mu)L} = \frac{2}{3}(\lambda^2 + \mu^2 + \lambda\mu + 3\lambda + 3\mu) - \frac{1}{2}L(L+1).$$

Our basic states are then chosen to be eigenstates of this interaction. We are not able to distinguish between SU_3 states which are degenerate for Q_{ij}^2 . This is the case for $(\lambda\mu) = (08)$ and (54) . Also some $(\lambda\mu)$ values occur twice. However, this does not give rise to any major difficulties since the most important ones are easily separable.

In table 3 we have given the eigenstates for the eight lowest levels in the spectrum. This shows that (84) is the dominant component in the two lowest rotational band. The Gaussian-Serber and KK interaction are in particularly good agreement with the SU_3 model. In all cases a repulsive odd-state component gives rise to larger admixture of other $(\lambda\mu)$ values which also shows up in the spectrum as a distortion of the rotational structure. The rest of the strength in the eigenstates is distributed over very few additional $(\lambda\mu)$ states; (46), (08)/(54) and (62). One remarkable feature of the KK force is the dominant influence of (62). This may be interpreted as a resistance against transition from prolate to oblate deformation, caused by the repulsive hard core.

TABLE 4
Calculated branching ratios for E2 transitions

Transition	Experimental	Pure (84)	Gaussian Serber	Gaussian Rosenfeld	KK	Yukawa odd
$2_2 \rightarrow 0$	75	84	25	1	7	2
$2_2 \rightarrow 2_1$	25	16	75	99	93	98
$0_2 \rightarrow 2_1$	83	0	97	72	94	84
$0_2 \rightarrow 2_2$	17	0	3	28	6	16
$4_2 \rightarrow 2_1$	91	96	29	99	98	98
$4_2 \rightarrow 2_2$	4.5	3	33	0.5	1.5	1
$4_2 \rightarrow 4_1$	4.5	1	38	0.5	0.5	1
$2_3 \rightarrow 0$	55	0	72	17	95	16
$2_3 \rightarrow 2_1$	45	0	28	83	5	84
$3_1 \rightarrow 2_1$	100	99.1	95	66	80	66
$3_1 \rightarrow 2_2$	$\ll 1$	0.8	3	33	9	15
$3_1 \rightarrow 4_1$	$\ll 1$	0.1	2	1	11	19

The next band is in all cases predominantly (46) and starts with $J = 0$ and 2. Here the mixing of $(\lambda\mu)$ values are considerably higher, but still rather few are of importance.

In table 4 we have given the structure of the (84) component in the wave function distributed over the different K -values in the SU_3 scheme. The physical interpretation of this quantum number is the projection of the angular momentum along the intrinsic axis. A considerable admixture of K -values is obtained in the wave function, especially for the KK force. However, the experimental classification of these states as belonging to a $K = 0$ and $K = 2$ band seems to survive an extensive shell-model calculation.

The experimental decay scheme for the low-lying states of ^{24}Mg is well known²²). A small admixture of M1 transition has been reported in some cases²⁴), but the main contribution is due to the E2 mode. This is in good agreement with our model which predicts zero M1 transitions.

The reduced electric quadrupole transition probability can be written as ²¹⁾

$$\begin{aligned} B(E2; J_i \rightarrow J_f) &= \sum_{qM_f} |\langle J_i M_i | M(E2; q) | J_f M_f \rangle|^2 \\ &= \frac{1}{2J_i + 1} |\langle J_i || M(E2) || J_f \rangle|^2, \end{aligned}$$

where the operator $M(E2; q)$ between $T = 0$ states has the form

$$M(E2; q) = \frac{1}{2} e \sum_i r_i^2 Y_{2q}(\Omega_i).$$

The summation goes over all particles. The most sensitive test of the wave functions is the branching ratios. For the absolute transition rates such factors as the radial form of the single-particle wave functions and the effective charge due to the distortion of the core, have major influence. We therefore concentrate on branching ratios where these effects are of minor importance.

In table 5 we have given all branching ratios for the eight lowest levels. As a comparison we have listed the result from the pure SU_3 model assuming no K -mixing in the levels. For the two lowest rotational bands no improvement has been obtained over the simple model and in nearly all cases serious disagreements are found. The major reason for the large change from the (84) configuration is due to the K -mixing in the eigenstates; see table 4. The transition operator $M(E2; q)$ is diagonal in the SU_3 scheme. Hence the other $(\lambda\mu)$ values play a minor role in the transition strength. Reasonable results have been obtained for transitions from the second 0^+ and third 2^+ states where the pure model does not give any contribution. This agreement might not be fully significant, however, since the competing transitions are highly depressed by the energy factor in the formula for the lifetime.

The experimental transition rates for ^{24}Mg follow very closely the prediction by the SU_3 model. This is also known to be the case for other nuclei in this region ²⁵⁾. However, it seems clear that an extensive shell model calculation has difficulties to reproduce this result. The configuration mixing does not improve the picture, as one would have expected.

5. Discussion and conclusion

This calculation has been based on the assumption that the eight-particle wave function for ^{24}Mg can be represented by the maximum space symmetry states $[f] = [44]$ only. From similar calculations in the sd shell it seems to be well established that the LS coupling scheme is a good approximation. The calculation reproduces the well-known rotational structure with a $K = 0$ and a $K = 2$ band. However, our results also show some disagreements, especially in the case of transition rates. This indicates that additional degrees of freedom which have been omitted, are of importance. Two sources of correction to our model can be thought of as the major reasons.

First of all, states with higher space symmetry may have some influence. Also core excitation seems possible from the experience in similar cases.

From the calculation of ^{20}Ne by Inoue *et al.*⁸⁾, it seems probable that only the next higher symmetry $[f] = [431]$ should be of importance, at least for interactions without tensor components. Such states can mix with the $[f] = [44]$ configurations only through the $l \cdot s$ term in the Hamiltonian (1). In a calculation of the second-order correction to the energy we only need to take into account the (84) component of the eigenstates. This gives the following expression:

$$E[(84)\alpha J] = \delta^2 \sum_{(\lambda\mu)\beta} \frac{|\langle [431](\lambda\mu)\beta JT | \sum_i \mathbf{l}_i \cdot \mathbf{s}_i | [44](84)\alpha JT \rangle|^2}{E_{(84)\alpha J} - E_{(\lambda\mu)\beta J}} \\ \approx (\delta^2/\Delta) \sum_{(\lambda\mu)\beta} |\langle [431](\lambda\mu)\beta JT | \sum_i \mathbf{l}_i \cdot \mathbf{s}_i | [44]\alpha(84)JT \rangle|^2,$$

where Δ is the average excitation energy. There are rather few terms in the sum over $(\lambda\mu)$; (92), (65), $(73)^2$ and (46). We have calculated the sum for (92) to be approximately 0.1, and it is nearly independent of α and J . A reasonable estimate of the factor δ^2/Δ gives ≈ 0.5 MeV. With this result no change in the energy levels is obtained.

Both due to the energy denominator and the matrix element, the (92) term in the sum should be the dominant one. It is therefore unlikely that including all $(\lambda\mu)$ values will change this result significantly. Banerjee *et al.*¹⁸⁾ estimated the same term in a variational approach to give a large contribution to the energy spectrum. However, this seems not to be confirmed by a more detailed calculation.

A similar estimate has been performed for the core excitation effect. The lowest-order correction of this type is obtained by promoting two particles from the p-shell to the sd shell. Aligning the particles to the maximum total $(\lambda\mu)$, we get a state of the form

$$|\Psi\rangle = |(\text{sd})^{10}; p^{-2}(104) KLS = T = 0\rangle.$$

The off-diagonal matrix element for the interaction between this state and the eight-particle states can be calculated in the SU_3 scheme^{9, 26)}. An estimate of the unperturbed positions of these configurations is obtained from the binding energy of ^{26}Al , ^{24}Mg and ^{14}N . This gives approximately 10 MeV. The second-order energy correction gives ≈ -0.4 MeV for the lowest $K = 0$ states and ≈ -0.1 MeV for the $K = 2$ states. This result has the right trend, but the effect is still too small. However, there are many close-lying particle-hole configurations which could improve this result. A further study on this point is in progress.

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References

- 1) F. Ajzenberg-Selove and T. Lauritsen, *Nuclear Physics* **11** (1959) 1
- 2) J. P. Elliott and B. H. Flowers, *Proc. Roy. Soc.* **A229** (1955) 536
- 3) M. G. Redlich, *Phys. Rev.* **110** (1958) 468
- 4) I. Talmi and I. Unna, *Nuclear Physics* **30** (1962) 280
- 5) J. F. Dawson, I. Talmi and J. D. Walecka, *Ann. of Phys.* **18** (1962) 339
- 6) T. Engeland and A. Kallio, *Nuclear Physics* **59** (1964) 211
- 7) J. D. Talman, *Can. J. Phys.* **40** (1962) 1730; *Phys. Rev.* **135** (1964) B1302
- 8) T. Inoue, T. Sebe, H. Hagiwara and A. Arima, *Nuclear Physics* **59** (1964) 1
- 9) T. England, *Nuclear Physics* **72** (1965) 68;
P. Federman and I. Talmi, *Phys. Lett.* **19** (1965) 490
- 10) A. Bohr, *Mat. Fys. Medd. Dan. Vid. Selsk* **26**, No. 14 (1952)
- 11) A. Bohr and B. Mottelson, *Mat. Fys. Medd. Dan. Vid. Selsk.* **27**, No. 16 (1953)
- 12) J. P. Elliott, *Proc. Roy. Soc.* **A245** (1958) 128, 562
- 13) J. P. Elliott and M. Harvey, *Proc. Roy. Soc.* **A272** (1963) 557
- 14) A. E. Litherland *et al.*, *Phys. Rev. Lett.* **7** (1961) 98
- 15) A. E. Litherland, M. J. L. Yates, B. M. Hinds and D. Eccleshall, *Nuclear Physics* **44** (1963) 220
- 16) E. B. Paul, *Phil. Mag.* **15** (1957) 311
- 17) A. E. Litherland *et al.*, *Can. J. Phys.* **36** (1958) 378
- 18) M. K. Banerjee and C. A. Levinson, *Phys. Rev.* **130** (1963) 1036;
M. K. Banerjee, C. A. Levinson and S. Meskow, *Phys. Rev.* **130** (1963) 1064
- 19) H. A. Jahn and H. van Wieringen, *Proc. Roy. Soc.* **A209** (1951) 502
- 20) V. Bargmann and M. Moshinsky, *Nuclear Physics* **18** (1960) 697;
M. Moshinsky, *Nuclear Physics* **31** (1962) 384;
M. Moshinsky, *Lecture notes, Latin American School of Physics* (1962)
- 21) J. M. Blatt and V. F. Weisskopf, *Theoretical nuclear physics* (J. Wiley and Sons, New York, 1952)
- 22) P. M. Endt and van der Leun, *Nuclear Physics* **34** (1962) 1;
R. W. Ollerhead, J. A. Kuehner, R. J. A. Levesque and E. W. Blackmore, *Bull. Am. Phys. Soc.* **11** (1966) 64
- 23) R. Muthukrishnan and M. Baranger, *Phys. Lett.* **18** (1965) 160
- 24) R. Batchelor, A. J. Ferguson, H. E. Gove and A. E. Litherland, *Nuclear Physics* **16** (1960) 38
- 25) A. E. Litherland, P. J. M. Smulders and T. K. Alexander, *Bull. Am. Phys. Soc.* **11** (1966) 65
- 26) K. T. Hecht, *Nuclear Physics* **62** (1965) 1