

A General Theory of Polynomial Conjoint Measurement^{1,2}

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The present theory generalizes conjoint measurement in five major respects. (a) It is formulated in terms of partially rather than fully ordered data. (b) It applies to both ordinal and numerical data. (c) It is applicable to finite as well as infinite data structures. (d) It provides a necessary and sufficient condition for measurement. (e) This condition applies to any polynomial measurement model; that is, any model where each data element is expressed as a specified real-valued, order-preserving polynomial function of its components.

Examples of polynomial measurement models include Savage's subjective expected utility model, Hull's and Spence's performance models, Luce's choice model, and multidimensional scaling models.

It is shown that a data structure D satisfies a given polynomial measurement M if and only if D satisfies an abstract irreflexivity axiom with respect to M . The interpretation of the result and its implications to measurement theory are discussed.

1. INTRODUCTION

The decomposition of complex phenomena into sets of basic factors according to specifiable rules of combination may be regarded as one of the goals of scientific investigation. Performance, for instance, is decomposed, according to some behavior theories, into its learning, incentive, and drive components. Whenever the factors involved can be independently measured, the problem is to account for their joint effects by the appropriate combination rule. In many applications, however, no adequate independent measurement of the separate factors is available, and only the order of their joint effects is given. Consequently, one would like to solve the decomposition and the measurement problems simultaneously by obtaining measurements of the basic factors such that when they are combined according to the hypothesized composition rule they account for the given order of the joint effects. The problem of obtaining measurement scales for both the dependent and the independent variables based on the order of their joint effects according to some specifiable com-

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position rule, is the conjoint measurement problem. The composition rule is called a conjoint measurement model.

Consider data obtained from a three-dimensional factorial experiment $A \times B \times C$ where each cell corresponds to a given treatment combination (a, b, c) , and each cell entry represents the effects of this treatment combination. For example, each cell may correspond to a commodity bundle consisting of given amounts of bread, wine, and beer; and each cell entry may be the maximal amount of money the consumer is willing to pay for it. If the composition rule is additive, one seeks real-valued utility functions for the commodities involved such that the utility of any commodity bundle equals the sum of the utilities of its components, and the order of these utilities corresponds to the consumer's ordering of the commodity bundles. If, however, the contributions of some of the components, e.g., wine and beer, are not independent, a more complicated measurement model or composition rule is called for.

Any (partially) ordered set of data, such as the above, where each datum can be regarded as the effect of treatment combination (a, b, \dots, k) of the factors A, B, \dots, K is called a *data structure*, denoted D , and each separate datum in the structure is referred to as a *data element*. A *polynomial measurement model* is defined as a composition rule which represents each data element as a specified polynomial function of its components.

A data structure D is said to satisfy a polynomial measurement model M whenever there exists a real-valued function f defined on D and real-valued functions f_A, f_B, \dots, f_K defined on the factors A, B, \dots, K such that, for any data element (a, b, \dots, k) :

$$(i) f(a, b, \dots, k) = M(f_A(a), f_B(b), \dots, f_K(k))$$

where M is a polynomial function of its arguments, that is, a specified combination of sums, differences and products of the functions f_A, f_B, \dots, f_K :

$$(ii) \text{ for all } x = (a, b, \dots, k), x' = (a', b', \dots, k'),$$

$$x >_0 x' \text{ implies } f(x) > f(x'),$$

$$x =_0 x' \text{ implies } f(x) = f(x'),$$

where $>_0$ and $=_0$ denote the order observed in the data. (1.1)

Thus, a data structure satisfies a polynomial measurement model M whenever it is possible to scale each of its components or treatments, such that every data element is represented as a specified polynomial of the scale value of its components, and such that the representation preserves the order of the data.

The previous example of preference order on commodity bundles is regarded as an *ordinal* data structure since the consumer's prices are used only to determine the

preference order. If these prices, however, are viewed as an absolute rather than an ordinal preference scale, the data structure is called *numerical*. A numerical data structure D_g is a data structure D together with a real valued function g defined for all x in D . Thus, the consumer's buying prices may constitute a numerical data structure where g is the function assigning the corresponding price to any commodity bundle. In this case, one may seek utility functions for the commodities involved such that the actual price attached to a given commodity bundle equals the sum of the utilities of its components.

A numerical data structure D_g is said to satisfy a polynomial measurement model M whenever D satisfies M in the sense of (1.1) with the specific function g used in place of f . (1.2)

Measurement models are referred to as ordinal or numerical when they are applied to ordinal or numerical data structures, respectively. Clearly, whenever the data satisfy a numerical model they also satisfy the corresponding ordinal model, but not conversely. The problem of determining whether a data structure satisfies a given measurement model is equivalent to that of determining whether the corresponding system of polynomial equations and inequalities is solvable. The systems generated by ordinal data structures are homogeneous, whereas those generated by numerical data structures are nonhomogeneous, i.e., they contain numerical constants. From a measurement-theory viewpoint, ordinal models are regarded as fundamental measurement in the sense that numbers are introduced only via the measurement model, whereas numerical models are regarded as derived measurement in the sense that they are based on some prior numerical assignment.

Classical measurement models (Campbell, 1920; Hölder, 1901) were constructed for structures which include a full ordering of the object set and an empirical concatenation operation, such as the juxtaposition of objects in a balance pan. The absence of a natural concatenation operation in the behavioral sciences has led to the development of measurement models of a different kind. During the last decade it has been shown (Debreu, 1960; Krantz, 1964; Luce and Tukey, 1964; Luce, 1965; Pfanzagl, 1959; Suppes and Winet, 1955) that if the data structure is rich enough (dense or continuous), then an axiomatization in terms of the ordering of the joint effects of two factors yields an interval scale measurement of the additive type. Necessary and sufficient conditions for additivity for finite data have been established by Scott (1964) and by Tversky (1964). The relationships between extensive (classical) and conjoint measurement have been explored by Luce (1966). Nonadditive measurement models have also been investigated. Marley (1964) and Roskies (1965) studied multiplicative models and Fishburn (1965) and Krantz and Tversky (1966) investigated some simple polynomial forms.

These results, however, apply to only a portion of the measurement models used or proposed in psychological literature. Moreover, many measurement models have

been stated in terms of the existence of numerical representations of a specified form, rather than in terms of an axiomatic structure from which the desired representation is derived.

The present development extends earlier results in five major respects. First, it is formulated in terms of partially, rather than fully, ordered data; hence it applies to instances where some observations are unavailable (missing data) or some alternatives are not comparable. Second, it applies to both finite and infinite data, thus avoiding the distinction between them in the present context. Third, the theory can be applied to both ordinal and numerical data structures. Fourth, it provides conditions that are both necessary and sufficient for measurement by a polynomial measurement model. Fifth, and most important, these conditions apply to any polynomial measurement model, thus providing a complete characterization of all data that are measurable by such models. The scope of the present theory is illustrated by few examples of polynomial measurement models which have not been previously encompassed within the conjoint measurement framework.

A. *The subjective expected utility model* (Savage, 1954). According to this model, choices among gambles maximize subjective expected utility which equals the sum of the utilities of the outcomes weighted by their subjective probabilities of occurrence. Formally, let x be a gamble with outcomes o_1, \dots, o_n obtained contingent upon events e_1, \dots, e_n ; and let x' be a gamble with outcomes o'_1, \dots, o'_n obtained contingent upon events e'_1, \dots, e'_n . The subjective expected utility model states that

$$x >_s x' \quad \text{iff} \quad \sum_{i=1}^n u(o_i) s(e_i) > \sum_{i=1}^n u(o'_i) s(e'_i), \quad (1.3)$$

where u and s are the utility and subjective probability functions associated with the outcomes and the events respectively.

B. *Hull's and Spence's performance models* (Hilgard, 1956). Consider a factorial experiment, $H \times D \times K$, in which some performance measure, such as latency, is observed under different combinations of learning (H), drive (D) and incentive (K). Let $x = (h, d, k)$ and $x' = (h', d', k')$ be any pair of such treatment combinations. The Hullian model asserts that

$$x >_0 x' \quad \text{iff} \quad f_H(h)f_D(d)f_K(k) > f_H(h')f_D(d')f_K(k'), \quad (1.4)$$

where f_H , f_D , and f_K are the learning, drive, and incentive scales. Note that this model is equivalent to the additive model since the order is invariant under a logarithmic transformation, provided all values are positive.

Spence's model, on the other hand, states that

$$x >_0 x' \quad \text{iff} \quad f_H(h)(f_D(d) + f_K(k)) > f_H(h')(f_D(d') + f_K(k')). \quad (1.5)$$

Note that for a fixed level of any of the factors the two models yield identical predictions. Consequently, in order to compare the models one has to vary all three factors simultaneously. A detailed conjoint measurement analysis of these models is given in Krantz and Tversky (1966).

The above equations express the ordinal versions of Hull's and Spence's models. If the response measure employed is treated as an absolute rather than an ordinal performance scale, the models would have to satisfy the additional constraints imposed by (1.2).

C. *The Bradley-Terry-Luce (B.T.L.) model* (Luce, 1959). This model is applicable to numerical data structures and specifies the relation of pairwise choice probabilities between alternatives to their scale values. Let $p(x, y)$ be the probability with which x is chosen over y ; the B.T.L. model states that

$$p(x, y) = \frac{v(x)}{v(x) + v(y)}; \quad \text{hence} \quad \frac{p(x, y)}{p(y, x)} = \frac{v(x)}{v(y)} \quad (1.6)$$

(for all pair-wise probabilities different from 0 or 1).

Although the above is not a polynomial measurement model it can be expressed as one since (1.6) is equivalent to:

$$\log p(x, y) - \log p(y, x) = v'(x) - v'(y) \quad \text{where} \quad v' = \log v.$$

The ordinal version of the B.T.L. model is given by:

$$p(x, y) > p(w, z) \quad \text{iff} \quad u(x) - u(y) > u(w) - u(z), \quad (1.7)$$

which is equivalent to Marschak's (1960) strong (or Fechnerian) utility model.

D. *Multidimensional scaling models* (Coombs, 1964; Shepard, 1966). These models differ markedly from those studied in the conjoint measurement framework. The data consist of proximities (e.g., similarities or distances) between pairs of objects. The objects are viewed as points in some n -dimensional metric space and the order of the proximities between points is accounted for by the order of their distances (according to Coombs's distance model) or by the order of their scalar products (according to nonmetric factor analysis). Let $d(x, y)$ denote the proximity or the psychological distance between objects x and y . Nonmetric factor analysis is defined by

$$d(x, y) >_0 d(w, z) \quad \text{iff} \quad \sum_{i=1}^n x_i y_i > \sum_{i=1}^n w_i z_i, \quad (1.8)$$

where the x_i 's are the unknown coordinates of point x which may be regarded as the loadings of the point on the n hypothesized dimensions or factors. Note that (1.8)

is the ordinal form of the classical factor analytic model in which $d(x, y)$ corresponds to the product moment correlation between x and y .

Coombs' (Euclidean) distance model is given by

$$d(x, y) >_0 d(w, z) \quad \text{iff} \quad \sum_{i=1}^n (x_i - y_i)^2 > \sum_{i=1}^n (w_i - z_i)^2. \quad (1.9)$$

The multidimensional unfolding model is a special case of (1.9) where $x = w$ is an individual and y, z are alternatives or stimuli. This model represents individuals and stimuli as points in a joint space. An individual x prefers alternative y over z if and only if y is "closer" to him than z .

An essential difference between the last two models and those discussed earlier is that the latter yield scales of specifiable objects or factors, whereas the former yield sets of coordinates chosen to minimize the dimensionality of the resulting space.

The conditions under which a data structure satisfies a given polynomial measurement model M are investigated in the next section. First, each data element is represented by a unique polynomial determined by M . The order relation induced by the data on these polynomials is extended in a non-unique way, and it is shown that if the irreflexivity axiom is satisfied then the data are embeddable in a nonzero fully-ordered Archimedean ring which, in turn, can be embedded uniquely in the real-number field. The interpretation of the result and its implications to measurement theory are discussed in the last section.

2. THE BASIC THEORY

*A data structure*⁴. $D = \langle \mathcal{D}, \geq_0 \rangle$ is a system where

\mathcal{D} is a subset of the product set $A \times B \times \cdots \times K$ of some finite number of disjoint sets A, B, \dots, K ; (2.1)

\mathcal{D} is partially ordered under \geq_0 . That is, \geq_0 is a binary relation defined on \mathcal{D} which satisfies the following conditions for all x, y, z in \mathcal{D} :

- (i) Reflexivity: $x \geq_0 x$;
- (ii) Transitivity: $x \geq_0 y$ and $y \geq_0 z$ imply $x \geq_0 z$. (2.2)

$x =_0 y$ is defined as $x \geq_0 y$ and $y \geq_0 x$. $x >_0 y$ is defined as $x \geq_0 y$ and not $y \geq_0 x$. A data structure, thus defined, is a partially ordered set of tuples of the form (a, b, \dots, k) where a is in A, b in B, \dots, k in K , called *data elements*. Their coordinates a, b, c, \dots are

⁴ The theory is formulated in terms of ordinal data structures. The application to numerical data is outlined later.

called *components*, and the set of all components is denoted by C which is a subset of $A \cup B \cup \dots \cup K$.

A factorial experiment provides an example of a data structure where the sets A, B, \dots, K are the independent variables or the factors, the components correspond to their levels, and the ordered data elements represent the ordered cell entries or treatment conditions.

A data structure D is said to satisfy a polynomial measurement model M whenever there exists a real-valued function f defined on D and real-valued functions f_A, f_B, \dots, f_K defined on A, B, \dots, K , respectively, such that:

- (i) $f(a, b, \dots, k) = M(f_A(a), f_B(b), \dots, f_K(k))$,
- where M is any polynomial function of its arguments;
- (ii) for any $x = (a, b, \dots, k), x' = (a', b', \dots, k')$ in D ,
- $$x >_0 x' \text{ implies } f(x) > f(x'),$$
- $$x =_0 x' \text{ implies } f(x) = f(x'). \tag{2.3}$$

In order to investigate the conditions under which measurement models are satisfied by data structures, certain constructions are introduced. Let $\Delta = R[C]$ be the ring of polynomials in C with real coefficients. That is, the elements of Δ are polynomials whose variables are elements of C and whose coefficients are real numbers. The elements of Δ , denoted by $\alpha, \beta, \gamma, \dots$, are called polynomials. In particular, 0 denotes the zero polynomial. For a discussion of polynomial rings, see Van der Waerden (1948).

Next, every data element is represented by a polynomial according to the measurement model M . Thus, M assigns to each $x = (a, b, \dots, k)$ in \mathcal{D} a unique $M(x) = M(a, b, \dots, k)$ in Δ . Note, however, that these images of the data elements are polynomials with the elements of C as their indeterminates, and not real numbers.

To illustrate the mapping of \mathcal{D} into Δ let $D = \langle \mathcal{D}, \geq_0 \rangle$ be a data structure, where $\mathcal{D} = A \times B \times C$ with a typical data element (a, b, c) . According to an additive measurement model, denoted M_1 , (a, b, c) is mapped into $M_1(a, b, c) = a + b + c$. Another measurement model, denoted M_2 , represents (a, b, c) by

$$M_2(a, b, c) = a(b + c) = ab + ac.$$

This representation induces a natural partial order on the images of \mathcal{D} defined by⁵

- (i) $M(x) -_1 M(y)$ iff $x =_0 y$,
 - (ii) $M(x) >_1 M(y)$ iff $x >_0 y$.
- $$\tag{2.4}$$

⁵ Throughout the paper \geq_i denotes some binary relation, not necessarily an order, where $>_i$ and $=_i$ are its symmetric and its non-symmetric parts respectively.

Note that different measurement models assign different formal elements to the same data element. Thus, although a data structure generates a unique polynomial ring \mathcal{A} , different measurement models define different order relations on \mathcal{A} . Hence the relations $=_1$ and $>_1$ depend both on the data structure considered and on the measurement model applied.

In order to embed \mathcal{A} in the real number field we first embed it in an ordered ring. A system $\langle R, \geq_i \rangle$ is an *ordered ring* whenever it satisfies the following three conditions:

R is a ring, i.e., it is a group under addition and a semi-group under multiplication where the latter is distributive over the former. (2.5)

\geq_i is a full order on R , i.e., all x, y, z in R satisfy:

- (i) Reflexivity: $x \geq_i x$
- (ii) Transitivity: $x \geq_i y$ and $y \geq_i z$ imply $x \geq_i z$
- (iii) Antisymmetry: $x \geq_i y$ and $y \geq_i x$ imply $x = y$
- (iv) Connectivity: $x \geq_i y$ or $y \geq_i x$. (2.6)

The order is compatible with the ring structure, i.e., all x, y, z in R satisfy:

- (i) Additivity: $x \geq_i y$ implies $x + z \geq_i y + z$
- (ii) Multiplicativity: $x \geq_i y$ and $z \geq_i 0$ imply $xz \geq_i yz$. (2.7)

First, the partial order on \mathcal{A} is extended in the following manner. A pair of binary relations, denoted $=_2$ and $>_2$, is called a *regular extension* of $=_1$ and $>_1$, respectively, if they satisfy the following two conditions.

$\alpha =_2 \beta$ whenever one of the following holds:

- (i) $\alpha =_1 \beta$ or $\alpha = \beta$ (inclusion and reflexive closure);
- (ii) $\alpha = \alpha' + \alpha'', \beta = \beta' + \beta''$ and $\alpha' =_2 \beta', \alpha'' =_2 \beta''$ (additive closure);
- (iii) $\alpha = \alpha'\alpha'', \beta = \beta'\beta''$ and $\alpha' =_2 \beta', \alpha'' =_2 \beta''$ (multiplicative closure). (2.8)

$\alpha >_2 \beta$ whenever one of the following holds:

- (i) $\alpha >_1 \beta$ (inclusion);
- (ii) $\alpha = \alpha' + \alpha'', \beta = \beta' + \beta''$ and $\alpha' >_2 \beta', \alpha'' \geq_2 \beta''$ (additive closure), where $\alpha \geq_2 \beta$ is defined as either $\alpha =_2 \beta$ or $\alpha >_2 \beta$;
- (iii) $\alpha = \alpha'\alpha'', \beta = \beta'\beta''$ and either:
 $\alpha' >_2 \beta' \geq_2 0, \alpha'' \geq_2 \beta'' >_2 0$ or $0 \geq_2 \beta' >_2 \alpha', 0 >_2 \beta'' \geq_2 \alpha''$
(multiplicative closure).

In addition, \succ_2 has to satisfy the following connectivity and Archimedean properties:

- (iv) $\alpha \succ_2 \beta$ or $\beta \succcurlyeq_2 \alpha$
- (v) For any $\alpha \neq_2 0, \beta$ there exists an integer n such that $n \alpha \succ_2 \beta$. (2.9)

The set of all regular extensions of a given data structure is denoted by E_{Δ} . Thus, E_{Δ} is the set of all pairs of binary relations of the form $(=_2, \succ_2)$ satisfying (2.8) and (2.9). A single binary relation, say \succ_2' , satisfying the conditions in (2.9) is said to be in E_{Δ} whenever there exist some $=_2'$ satisfying the conditions in (2.8) such that the pair $(=_2', \succ_2')$ is a regular extension.

To demonstrate that $=_1$ and \succ_1 always have a regular extension let both $=_2$ and \succ_2 be the universal relation. That is, let both $\alpha =_2 \beta$ and $\alpha \succ_2 \beta$ hold for all α, β in Δ . It is easy to verify that both sets of conditions in (2.8) and (2.9) are satisfied in this case and hence the existence of a regular extension is ensured. Furthermore, as a direct consequence of our definition we obtain:

LEMMA 2.10. $=_2$ is an equivalence relation.

Proof. (i) Reflexivity follows immediately from Part (i) of (2.8).

(ii) Transitivity: Assume $\alpha =_2 \beta$ and $\beta =_2 \gamma$, hence, by Part (ii) of (2.8), $\alpha + \beta =_2 \gamma + \beta$. Adding $-\beta$ to both sides and applying Part (ii) again yields $\alpha =_2 \gamma$ as required.

(iii) Symmetry follows from the fact that $=_1$ is symmetric, by Definitions 2.2 and 2.4, together with the symmetry of Definition 2.8.

The fact that the universal relation on Δ is a regular extension of any partial order indicates that although \succ_1 is a strict order relation, i.e., it is an irreflexive, asymmetric and transitive binary relation, \succ_2 need not be. In order to obtain a regular extension in which \succ_2 is a strict order relation, the following axiom is introduced.

AXIOM 2.11 (irreflexivity). *There exists at least one \succ_2 in E_{Δ} such that $\alpha \succ_2 \alpha$ does not hold for at least one α .*

Before discussing the irreflexivity axiom we prove:

LEMMA 2.12. *For any given \succ_2 in E_{Δ} , $\alpha \succ_2 \alpha$ for one α in Δ implies $\beta \succ_2 \beta$ for any β in Δ .*

Proof. Since $\alpha \succ_2 \alpha$, by hypothesis, and $\beta - \alpha =_2 \beta - \alpha$, by Part (i) of (2.8), then $\alpha + (\beta - \alpha) \succ_2 \alpha + (\beta - \alpha)$, by Part (ii) of (2.9), and hence $\beta \succ_2 \beta$.

Therefore if for any given \succ_2 there exists a single α in Δ such that $\alpha \succ_2 \alpha$ then \succ_2 is reflexive. Consequently, the irreflexivity axiom (2.11) is equivalent to the condition that not all \succ_2 are reflexive. A regular extension $(=_2, \succ_2)$ is called irreflexive if \succ_2 is irreflexive. An alternative formulation of this condition is given by the following:

AXIOM 2.13 (trichotomy). *There exists a $>_2$ in E_Δ and some α, β in Δ for which only one of the following holds:*

$$(i) \alpha >_2 \beta; \quad (ii) \alpha =_2 \beta; \quad (iii) \beta >_2 \alpha.$$

The equivalence of the two axioms can be readily established. First, assume irreflexivity is violated, hence, $\alpha >_2 \alpha$ for at least one α and by Lemma 2.12, $\alpha >_2 \alpha$ for all α and for all $>_2$. But since $\alpha =_2 \alpha$, by Part (i) of (2.8), the trichotomy axiom is violated. Conversely, if the trichotomy axiom does not hold, there exist some α, β in Δ such that $\alpha >_2 \beta$ and $\beta \geq_2 \alpha$. Consequently, by Part (ii) of (2.9), $\alpha + \beta >_2 \alpha + \beta$ for some α, β , and hence for all α, β in Δ . Therefore, a particular regular extension satisfies the irreflexivity axiom if and only if it satisfies the trichotomy axiom which implies the equivalence of the two axioms.

Although the relations $=_2$ and $>_2$ are not uniquely defined, the minimal binary relations satisfying (2.8) and Parts (i), (ii), and (iii) of (2.9) are uniquely determined by the data. They are referred to as the *polynomial closure* of the observed order since they include the observed order of the data and are included in any regular extension. Consequently, if $\alpha >_2 \alpha$ can be obtained in a given data structure without using parts (iv) and (v) of (2.9), the irreflexivity axiom must be violated. This follows readily from the fact that $\alpha >_2 \alpha$ holds in all regular extensions for at least one α , and hence—by Lemma 2.12—for all α , violating the irreflexivity axiom.

The converse, unfortunately, does not hold in general. That is, the fact that the polynomial closure of the observed order does not yield $\alpha >_2 \alpha$ is not sufficient to guarantee that the irreflexivity axiom is satisfied.

The axiom is illustrated by a concrete example. Let $D = \langle A \times B, \geq_0 \rangle$ with components a_1, a_2 in A and b_1, b_2 in B . Consider the following rank order of their joint effects.

	b_1	b_2
a_1	1	3
a_2	4	2

The order of the data elements is given by the following chain of inequalities:

$$(a_2, b_1) >_0 (a_1, b_2) >_0 (a_2, b_2) >_0 (a_1, b_1).$$

If the measurement model considered is additive, the above is represented by the following chain of polynomial inequalities:

$$a_2 + b_1 >_1 a_1 + b_2 >_1 a_2 + b_2 >_1 a_1 + b_1.$$

Summing the first two and the last two polynomials, respectively, and applying Part (ii) of Definition 2.9 yields

$$\begin{aligned} \alpha &= (a_2 + b_1) + (a_1 + b_2) = a_1 + a_2 + b_1 + b_2 >_2 \\ &(a_2 + b_2) + (a_1 + b_1) = a_1 + a_2 + b_1 + b_2 = \alpha, \end{aligned}$$

contrary to the irreflexivity axiom. It can be easily shown that this data structure does not satisfy the additive model. If, on the other hand, the measurement model is multiplicative, it is easy to verify that the scale values $f_A(a_1) = 2, f_A(a_2) = -3, f_B(b_1) = -2, f_B(b_2) = 1$ constitute a multiplicative representation of D and it can be further shown that the irreflexivity axiom is satisfied in D relative to the multiplicative measurement model.

In the following discussion it is shown that the irreflexivity axiom is both necessary and sufficient for some \geq_2 to be a weak order. This order, however, is not necessarily antisymmetric as required by (2.6). The difficulty can be resolved naturally by introducing equivalence classes.

Since, by Lemma 2.10, $=_2$ is an equivalence relation on \mathcal{A} , let $\mathcal{A}/=_2$ denote the set of equivalence classes of \mathcal{A} with respect to $=_2$, and let $\bar{\alpha}$ denote the equivalence class containing α . Thus,

$$\bar{\alpha} = \{\beta \text{ in } \mathcal{A}: \beta =_2 \alpha\}. \tag{2.14}$$

Addition and multiplication in $\mathcal{A}/=_2$ are defined by

$$\begin{aligned} \text{(i)} \quad \bar{\alpha} + \bar{\beta} &= \overline{\alpha + \beta}; \\ \text{(ii)} \quad \bar{\alpha}\bar{\beta} &= \overline{\alpha\beta}. \end{aligned} \tag{2.15}$$

To show that the operations are well defined, let α', β' be elements of $\bar{\alpha}$ and $\bar{\beta}$, respectively. Hence, by (2.14), $\alpha' =_2 \alpha$ and $\beta' =_2 \beta$ and, by (2.8) Part (ii), $\alpha' + \beta' =_2 \alpha + \beta$ and hence $\overline{\alpha' + \beta'} = \overline{\alpha + \beta}$. Similarly, by (2.8) Part (iii), $\alpha'\beta' =_2 \alpha\beta$ and hence $\overline{\alpha'\beta'} = \overline{\alpha\beta}$, which shows that both operations are independent of the representative elements which were chosen to define the equivalence classes. Next, for any irreflexive regular extension on \mathcal{A} define, its counterpart on $\mathcal{A}/=_2$ by

$$\bar{\alpha} \geq_3 \bar{\beta} \text{ iff } \alpha \geq_2 \beta. \tag{2.16}$$

To show that the definition is independent of the choice of α and β , consider any α' in $\bar{\alpha}$ and β' in $\bar{\beta}$. Assuming $\alpha \geq_2 \beta$ and applying Parts (ii) of (2.8) and (2.9) repeatedly yields

$$\alpha' + \beta =_2 \alpha + \beta';$$

consequently,

$$\alpha + (\alpha' + \beta) \geq_2 \beta + (\alpha + \beta'),$$

and

$$\alpha' + (\alpha + \beta) \geq_2 \beta' + (\alpha + \beta);$$

hence,

$$\alpha' \geq_2 \beta'$$

for all α' in $\bar{\alpha}$, β' in $\bar{\beta}$, which ensures that \geq_3 is indeed well defined.

A ring is called a zero-ring whenever the multiplication is trivial, i.e., all products are zero.

LEMMA 2.17. Δ/\equiv_2 is a nonzero ring, provided $>_0$ is nonempty.⁶

Proof. Since Δ is, by construction, a ring, and the mapping from Δ onto Δ/\equiv_2 is, by Definitions 2.15, a homomorphism, Δ/\equiv_2 is also a ring because a homomorphic image of a ring is itself a ring. See, for example, Van der Waerden (1948, p. 38). Since $>_0$ is, by assumption, nonempty there exist some α, β in Δ such that $\alpha >_2 0$ and $\beta >_2 0$. Hence, by Part (iii) of (2.9), $\alpha\beta >_2 0$. If, on the other hand, Δ/\equiv_2 is a zero ring we obtain $\bar{\alpha}\bar{\beta} = \bar{0}$ for all $\bar{\alpha}, \bar{\beta}$ and hence, by Definition 2.16, $\alpha\beta \equiv_2 0$. Applying Part (ii) of (2.8) to the above yields $\alpha\beta >_2 \alpha\beta$, contrary to the irreflexivity axiom (2.11).

LEMMA 2.18. \geq_3 is a full order (2.6) on Δ/\equiv_2 .

Proof. (i) *Reflexivity:* Since $\alpha \equiv_2 \alpha$, by (2.8) Part (i), we obtain $\alpha \geq_2 \alpha$ and hence $\bar{\alpha} \geq_3 \bar{\alpha}$.

(ii) *Transitivity:* Assume $\bar{\alpha} \geq_3 \bar{\beta}$ and $\bar{\beta} \geq_3 \bar{\gamma}$; hence, $\alpha \geq_2 \beta$ and $\beta \geq_2 \gamma$. Adding the inequalities and applying Parts (ii) of (2.8) and (2.9) yields

$$\alpha + \beta \geq_2 \gamma + \beta; \text{ hence } \alpha \geq_2 \gamma, \text{ and } \bar{\alpha} \geq_3 \bar{\gamma}.$$

(iii) *Antisymmetry:* Assume $\bar{\alpha} \geq_3 \bar{\beta}$ and $\bar{\beta} \geq_3 \bar{\alpha}$; hence $\alpha \geq_2 \beta$ and $\beta \geq_2 \alpha$ which, by the irreflexivity axiom, imply $\alpha \equiv_2 \beta$ and hence $\bar{\alpha} = \bar{\beta}$.

(iv) *Connectivity:* Since, by Part (iv) of (2.9), either $\alpha \geq_2 \beta$ or $\beta >_2 \alpha$, we obtain either $\bar{\alpha} \geq_3 \bar{\beta}$ or $\bar{\beta} \geq_3 \bar{\alpha}$.

LEMMA 2.19. \geq_3 is compatible with the ring structure.

Proof. (i) *Additivity:* Given $\bar{\alpha} \geq_3 \bar{\beta}$ we have $\alpha \geq_2 \beta$, and hence by Parts (ii) of (2.8) and (2.9), $\alpha + \gamma \geq_2 \beta + \gamma$, and consequently

$$\bar{\alpha} + \bar{\gamma} = \overline{\alpha + \gamma} \geq_3 \overline{\beta + \gamma} = \bar{\beta} + \bar{\gamma}$$

for all γ in Δ .

(ii) *Multiplicativity:* Given $\bar{\alpha} \geq_3 \bar{\beta}$ and $\bar{\gamma} \geq_3 \bar{0}$, we have $\alpha \geq_2 \beta$ and $\gamma \geq_2 0$; hence, by Parts (iii) of (2.8) and (2.9), $\alpha\gamma \geq_2 \beta\gamma$, and consequently

$$\bar{\alpha}\bar{\gamma} = \overline{\alpha\gamma} \geq_3 \overline{\beta\gamma} = \bar{\beta}\bar{\gamma},$$

which establishes Lemma 2.19.

A fully ordered ring $\langle R, \geq_i \rangle$ is called *Archimedean* if for any $x \neq 0, y$ in R there exists an integer n such that $nx \geq_i y$. To verify that $\langle \Delta/\equiv_2, \geq_3 \rangle$ is an Archimedean ring, note that for any $\alpha \neq 0, \beta$ in Δ there exists, by Part (v) of (2.9), an integer n such that $n\alpha >_2 \beta$ and hence $n\bar{\alpha} \geq_3 \bar{\beta}$ as required. As a consequence of this fact and the three preceding lemmas we obtain

⁶ The case where $>_0$ is empty can be excluded with no loss of generality since if $>_0$ is empty then the trivial solution $f(x) = 0$ for any x in D satisfies all polynomial measurement models.

THEOREM 2.20. $\langle \Delta / \equiv_2, \geq_3 \rangle$ is a nonzero fully ordered Archimedean ring.

The relationships between such rings and the real number field is given by the following result due to Hion; see Fuchs (1963, p. 126) for the proof.

THEOREM 2.21 (Hion). A nonzero fully ordered Archimedean ring is order-isomorphic to a unique subring of the real number field, taken with its usual ordering.

The theorem states that there exists a unique order-preserving isomorphism f from a nonzero fully ordered Archimedean ring onto a subring of the reals. Our next theorem relies heavily on this result.

THEOREM 2.22 (representation). For a data structure D to satisfy a polynomial measurement model M it is necessary and sufficient that it satisfies the irreflexivity axiom (2.11). Furthermore, for any choice of an irreflexive $>_2$, the resulting numerical assignment is uniquely determined.

Proof. Sufficiency: Let $x = (a, b, \dots, k)$ be any data element; hence, by construction $M(x)$ is in Δ and $\overline{M(x)}$ is an element of $\langle \Delta / \equiv_2, \geq_3 \rangle$ which, by Theorems 2.20 and 2.21, is isomorphic to a unique real-number subring. Thus, there exists a real-valued function f such that

$$f(\overline{M(x)}) = f(\overline{M(a, b, \dots, k)}) = M(f(a), f(b), \dots, f(k)),$$

since M is by definition a polynomial function of its arguments and f is a ring-isomorphism. To show that f reflects the order on D assume $x >_0 y$ for some x, y in D .

Hence:

$$\begin{aligned} x >_0 y & \text{ implies } M(x) >_1 M(y) & \text{ [by (2.4)]} \\ & \text{ implies } M(x) >_2 M(y) & \text{ [by (2.9), part (i)]} \\ & \text{ implies } \overline{M(x)} >_3 \overline{M(y)} & \text{ [by (2.16)].} \end{aligned}$$

Therefore, by (2.20) and (2.21) there exists an order-preserving real-valued function f such that:

$$f(\overline{M(x)}) > f(\overline{M(y)}) \quad \text{if} \quad x >_0 y.$$

Similarly,

$$\begin{aligned} x =_0 y & \text{ implies } M(x) =_1 M(y) & \text{ [by Definition (2.4)]} \\ & \text{ implies } M(x) =_2 M(y) & \text{ [by (2.8), Part (i)]} \\ & \text{ implies } \overline{M(x)} = \overline{M(y)} & \text{ [by (2.14)],} \end{aligned}$$

and hence by the same argument,

$$f(\overline{M(x)}) = f(\overline{M(y)}) \quad \text{if} \quad x =_0 y.$$

Consequently, f preserves the order, and D satisfies M .

Necessity: Assume M is satisfied by a given D ; hence there exists a real-valued function f such that

$$\begin{aligned} x >_0 y & \text{ implies } f(M(x)) > f(M(y)) \text{ and} \\ x =_0 y & \text{ implies } f(M(x)) = f(M(y)). \end{aligned}$$

It can be easily verified that the relations $=$ and $>$ on the reals satisfy Definitions 2.8 and 2.9. Consequently, the pair $(=, >)$, defined by

$$\begin{aligned} f(\alpha) > f(\beta) & \text{ iff } \alpha >_2 \beta \text{ and} \\ f(\alpha) = f(\beta) & \text{ iff } \alpha =_2 \beta, \end{aligned}$$

is in E_d . But since $>$ is clearly irreflexive, E_d contains at least one irreflexive regular extension. This establishes the necessity of the irreflexivity axiom and completes the proof of the theorem, since the uniqueness follows at once from Theorem 2.21.

The theory was formulated in terms of ordinal data structures. In order to apply the result to numerical data structures, we extend the definition of a data structure to include the real numbers with their usual ordering.

An extended data structure $D^* = \langle \mathcal{D}^*, \geq_0 \rangle$ is a system where

(i) \mathcal{D}^* is a subset of $D \cup R$ where $\mathcal{D} = A \times B \times \cdots \times K$ for some finite number of disjoint sets A, B, \dots, K and R , where R is the set of real numbers.

(ii) \geq_0 is a partial order on \mathcal{D}^* whose restriction to R coincides with the usual full order of R . (2.23)

According to this definition, a data element is either a tuple of the form (a, b, \dots, k) or a real number. The mapping of \mathcal{D}^* into its corresponding polynomial ring Δ is defined as follows:

For any polynomial measurement model M and any x in \mathcal{D}^* ,

$$\begin{aligned} \text{(i)} \quad M(x) &= M(a, b, \dots, k) \text{ if } x \text{ is in } \mathcal{D} \\ \text{(ii)} \quad M(x) &= x \text{ if } x \text{ is in } R. \end{aligned} \tag{2.24}$$

The order on Δ is defined as in (2.4) and it can be easily shown that the result obtained for ordinal data structures is applicable to numerical data structures as well.

3. DISCUSSION

A logical analysis or an axiomatic treatment of measurement models may serve several interrelated purposes. The present development explores the mathematical structure of polynomial measurement theory, interrelates various measurement models within a unified conceptual framework, and leads to the formulation of some mathe-

mathematical problems whose solution may illuminate the structures under study. The present theory, however, does not provide any simple set of empirically testable conditions which can be easily interpreted as a substantive theory. Furthermore, the general theory does not provide any constructive procedure for obtaining the desired numerical representation.

The theory is summarized by the last theorem which asserts that a data structure satisfies a polynomial measurement model if and only if it satisfies the irreflexivity axiom. Characterizations in terms of necessary and sufficient conditions have been established only for the finite linear case (Scott, 1964; Tversky, 1964). Axiomatizations applied to the infinite case, such as Luce and Tukey (1964), Pfanzagl (1959) or Suppes and Winet (1955) impose conditions such as solvability or continuity which simplify the axioms and their interpretation considerably, but restrict the applicability of the system to dense or continuous infinite data structures. The present theory applies equally well to the finite and to the infinite cases, indicating that the structure of the measurement model remains unchanged whether one regards a data structure as finite or infinite. Similarly, the theory encompasses both ordinal and numerical data structures indicating the existence of a close relation between the corresponding fundamental and derived measurement models.

In contrast to most measurement models which lead to at most an interval or a ratio scale, the numerical assignment obtained by Theorem 2.22 is unique. This uniqueness, however, is relative to the choice of a particular regular extension and there are, in general, infinitely many such regular extensions. Hence the problem of obtaining a unique numerical solution is equivalent to that of choosing a particular irreflexive regular extension.

Another point of departure from previous work in the field is the omission of the connectedness or the completeness axiom according to which all data elements are comparable. In discussing the completeness axiom in the context of utility theory, Aumann writes:

“Of all the axioms of utility theory, the completeness axiom is perhaps the most questionable. Like others of the axioms, it is inaccurate as a description of real life; but unlike them, we find it hard to accept even from the normative viewpoint. Does ‘rationality’ demand that an individual make definite preference comparisons between *all* possible lotteries (even on a limited set of basic alternatives)? For example, certain decisions that our individual is asked to make might involve highly hypothetical situations, which he will never face in real life; he might feel that he cannot reach an ‘honest’ decision in such cases. Other decision problems might be extremely complex, too complex for intuitive ‘insight,’ and our individual might prefer to make no decision at all in these problems.... Is it ‘rational’ to force decisions in such cases?” (1962, p. 446)

The present theory is developed in terms of partially rather than fully ordered data since in many cases of interest the data are only partially ordered. For example, con-

sider a data structure where each one of n voters rank orders each one of m candidates. The order is only partial since there is no natural way to determine whether voter i likes candidate x more than voter j likes candidate y . In the absence of a satisfactory solution to the problem of inter-personal comparability of utility, the use of partially ordered measurement models is unavoidable.

The above example, as well as (1.8) and (1.9), serves to illustrate another feature of the present theory. Unlike previous conjoint measurement results which are restricted to data structures of the factorial type, the present approach also applies to the type of data usually analyzed by multidimensional scaling methods. In these models, the observed order on pairs of points, such as voter i prefers candidate x to y , is interpreted as an ordering of the distances between them. If the distance function can be expressed as a polynomial function of the coordinates, the present theory provides a necessary and sufficient condition for embedding a data structure in a real n -space with a fixed dimensionality.

The generality of the present theory, however, stems not only from the fact that most measurement models proposed can be expressed as polynomial functions but also from the well-known result that any continuous real-valued function on a closed bounded region can be uniformly approximated arbitrarily closely by a polynomial function. Hence the type of measurement model considered provides a very close uniform approximation to the data even when the data generating process or the decomposition function cannot be expressed as a polynomial function.

Since the theory applies to all (polynomial) measurement models, the exact form of the restrictions imposed on the data depends on the particular measurement model considered. However, once the measurement model is specified, the axiom may be used to derive testable conditions. To illustrate, the cancellation law used by Luce and Tukey (1964) is derived from the irreflexivity axiom for the additive measurement model. Consider a data structure $D = \langle A \times P, \geq_0 \rangle$ with a, b, c in A and p, q, r in P , where \geq_0 is a weak order of $A \times P$. The cancellation law states that

$$(a, q) \geq_0 (b, p) \quad \text{and} \quad (b, r) \geq_0 (c, q)$$

imply

$$(a, r) \geq_0 (c, p). \tag{3.1}$$

Assume the irreflexivity axiom holds in D relative to the additive model but the above cancellation is not satisfied. Hence we obtain $(a, q) \geq_0 (b, q)$, $(b, r) \geq_0 (c, q)$ and $(c, p) >_0 (a, r)$ since \geq_0 is a weak order. Applying the additive model and Part (i) of (2.9) yields

$$M(a, q) = a + q \geq_2 b + p = M(b, p),$$

$$M(b, r) = b + r \geq_2 c + q = M(c, q),$$

$$M(c, p) = c + p >_2 a + r = M(a, r).$$

Adding the inequalities and applying Part (ii) of (2.9) we obtain

$$\alpha = a + b + c + p + q + r \succ_2 a + b + c + p + q + r = \alpha.$$

But since $\alpha \succ_2 \alpha$ is obtained by using Parts (i) and (ii) of (2.9), it must hold in any regular extension, and hence the irreflexivity axiom is contradicted, which completes the derivation of the cancellation law.

In investigating this model Krantz has shown that:

“Merely by assuming an equivalence relation, together with the Luce-Tukey axioms specialized for it, one can introduce by definition a ‘concatenation’ operation in the object set. The resulting structure is shown to be a commutative group. The order relation is then introduced, and the measurement theorems follow standard theorems on ordered groups.” (1964, p. 249)

Thus, Krantz has shown that under the Luce-Tukey axioms, a fully-ordered data structure can be embedded in the additive group of reals using the above concatenation as a group operation. The role of such operation in the present system is filled by (formal) addition of polynomials.

Another consequence of the irreflexivity axiom for a closely related measurement model is the bisymmetry axiom applied by Pfanzagl (1959) to utility theory and to psychophysics. The axiom is stated in terms of a concatenation operation defined for all pairs of data elements, where $x * y$ may be interpreted as a (fixed) probability mixture of some alternatives x, y , or as a bisection of the interval $[x, y]$. The bisymmetry axiom states that

$$(x * y) * (z * w) = (x * z) * (y * w). \tag{3.2}$$

If we let $M(x * y) = px + (1 - p)y$ and apply the asymmetry axiom, (3.2) follows at once, since, under this interpretation, it reduces to an identity in Δ .

In a paper entitled “Measurement Structures and Linear Inequalities,” Scott (1964) applied a general criterion for the solvability of finite systems of linear inequalities to obtain a necessary and sufficient condition for finite fully ordered data structures to satisfy a linear measurement model. In order to relate Scott’s condition to the irreflexivity axiom let us reformulate his Theorem 1.4.

THEOREM 3.3. (Scott). *Let D be a finite fully ordered data structure and D^+ the free commutative semigroup generated by D . D satisfies some linear measurement model if and only if there exists a binary relation \succsim_i on D^+ which includes \succsim_0 and satisfies the following conditions for all x, x', y, y' in D^+ :*

- (i) $x \succsim_i y$ or $y \succsim_i x$.
- (ii) $x \succsim_i y$ and $x' \succsim_i y'$ imply $x + x' \succsim_i y + y'$
- (iii) $x + x' = y + y'$ and $x \succsim_i y$ imply $y' \succsim_i x'$.

Since, by Theorem 2.20, the system $\langle \mathcal{A}/=_{2}, \geq_{3} \rangle$ is a fully ordered ring, where \geq_{3} includes \geq_{0} , the relation \geq_{3} clearly satisfies (i) and (ii). Suppose (iii) does not hold; then we obtain both $x \geq_{3} y$ and $x' >_{3} y'$, by the completeness of the order, and $x + x' >_{3} y + y'$ contrary to our hypothesis that $x + x' = y + y'$.

In a closely related development, Tversky (1964) showed that the following generalized cancellation law is both necessary and sufficient for additivity. Let D and D^{+} be defined as in Theorem 3.3 and let $>_{i}$ and $=_{i}$ be the additive closures of $>_{0}$ and $=_{0}$, respectively (see Parts (i) and (ii) of (2.8) and (2.9) for a complete definition). The generalized cancellation law is given by

$$\alpha + x >_{i} \alpha + y \quad \text{implies} \quad x >_{0} y \quad (3.4)$$

for all x, y in D and α in D^{+} .

A minor difference between these approaches, in addition to the form of the conditions, is that Scott embedded a finite subset of an ordered group in the additive group of reals, whereas Tversky embedded a finitely generated, partially ordered semi-group, in the additive semi-group of reals. Both approaches may be regarded as special cases of the present theory, where the data structure is finite, the order is complete, and the measurement model is linear.

It is interesting to note that, under the above definition of the order, the irreflexivity axiom is equivalent to (3.4) and hence to Scott's condition for fully ordered data structures. Consequently, it can be shown that, provided D^{+} is Archimedean, D need not be finite and hence the conditions given in Theorem 3.3 or in (3.4) are necessary and sufficient for additivity for infinite data structures as well.

To prove that the irreflexivity axiom is equivalent to the generalized cancellation law (3.4), assume first that the latter does not hold. Then there exist some x, y in D and α in D^{+} such that:

$$\alpha + x >_{i} \alpha + y, \text{ but } y >_{0} x \text{ by the completeness of } \geq_{0}.$$

Hence, by Part (ii) of (2.9), $\alpha + y >_{i} \alpha + x$ contrary to the irreflexivity axiom. Next, assume irreflexivity does not hold; then for any α in D^{+} and x in D , $\alpha + x >_{i} \alpha + x$ since $>_{i}$ is included in $>_{2}$. Therefore, by the generalized cancellation law, $x >_{0} x$ contrary to the hypothesis that $>_{0}$ is irreflexive; hence (3.4) is violated.

One essential difference between the results obtained for the finite linear case and the present theory is that in the former the irreflexivity of the minimal binary relation satisfying Parts (i) and (ii) of (2.8) and (2.9) is both necessary and sufficient for measurement, whereas in the latter the irreflexivity of the minimal binary relation satisfying Parts (i), (ii) and (iii) of (2.8) and (2.9) is necessary but not sufficient for measurement. This raises an interesting mathematical problem, namely the characterization of those measurement models for which the above condition is both necessary and sufficient. For these models, the theory can be both simplified and strengthened by omitting Parts (iv) and (v) of (2.9) and replacing the present irreflexivity condition by the

irreflexivity of the minimal binary relation satisfying (2.9), i.e., the polynomial closure of the observed order.

It can be readily shown, (see Tversky, 1964), that the irreflexivity of the corresponding minimal binary relation is both necessary and sufficient for measurement by a linear model provided the order is finitely generated. The order of a data structure is finitely generated whenever it is obtainable upon repeated applications of transitivity and the group (or the ring) operations from a finite number of inequalities. Clearly, the order of any finite data structure is finitely generated, but the converse is not true. One is tempted to conjecture that when the order is finitely generated, the same result holds for nonlinear models as well. Stated differently, the conjecture is that any partially ordered commutative ring whose order is finitely generated can be embedded in the real-number field.

Additional open problems include: (a) The formulation of simple testable axiomatic structures for specific measurement models such as those mentioned in the introduction; (b) The development of appropriate error theories together with a statistical analysis of the problem of goodness-of-fit of the data to models; (c) The construction of algorithms or scaling procedures for obtaining numerical solutions.

Finally, and most important, is the discovery of the appropriate measurement model or decomposition function that characterizes the data. This problem, however, involves extramathematical considerations since it lies in the juncture of substantive and measurement theories.

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