facilitate the solution of the partial differential equation characterizing field problems. These field simulation techniques can be conveniently classified in terms of the types of variables which are discretized and the types of variables which are maintained in continuous form.

In a method described by Little [1] and Soudack a hybrid Monte Carlo method is employed to treat parabolic equations in one and two space dimensions without performing a difference approximation. Miura [2] introduced a hybrid approach for minimizing the amount of analog equipment required in simulating partial differential equations by discretizing the space variable and maintaining the time variable in continuous form (DSCT). In the continuous-space discrete-time (CSDT) method for treating one-dimensional parabolic equations as described by Witsenhausen [3] and others a boundary-value problem is solved iteratively for each time level of a diffusion process. Finally, the author [4] has described a method of handling a large class of field problems by discretizing both the space and time variables (DSDT) and employing a passive analog network as a matrix inversion subroutine in a digital program. The present paper is concerned with the analysis of errors arising in the CSDT and the DSDT method. Sensitivity analysis and sensitivity techniques are employed to this end.

Sensitivity functions are partial derivatives expressing the transient perturbation in solution variables resulting from small perturbations in parameters. Sensitivity methods have been applied successfully in various control system problems (see Tomovic [5]), and the extension of these techniques to the analysis of hybrid computer systems when solving systems or ordinary differential equations has been described by the author [6]. In the latter application sensitivity functions can be employed to specify tolerances for units comprising hybrid loops, so as to maintain the solution within prescribed error bounds; they can be employed to estimate the accuracy of hybrid solutions; and they can be utilized to devise compensation procedures to minimize the effect of accumulated errors. From the point of view of mechanization, sensivity techniques have the advantage that the entire dynamic range of a computer can be employed to study the effect of small perturbations. If these perturbations were to be applied in the original compter setup, their effect would be masked by other errors and by noise. An additional advantage of the sensitivity approach is that the mechanization of the sensitivity equations is usually very similar to that employed in solving the actual problem, so that little additional equipment and little additional programming is required.

In the present paper application of sensitivity techniques to CSDT simulations are made to study the effect of :

1) Failure to match perfectly the second boundary condition at the end of each series of iterations.

2) Sampling the continuous (with respect to space) solution at each time level.

3) Discretization of the time domain.

4) Analog inaccuracies.

In applying sensitivity techniques to the DSDT method, sensitivity to the following sources of errors are considered :

1) Failure to converge completely to the correct solution at each time step.

2) Discretization of the time domain.

3) Component errors in the analog network.

It is demonstrated that sensitivity techniques constitute an effective tool for the characterization and specification of hybrid simulations of physical fields, and that a plot of the appropriate sensitivity functions provides a direct insight into per step and accumulated error problems and the relative advantages and disadvantages of alternative mechanization techniques.

REFERENCES

[1] Little, W.D. : « Hybrid Comupter Solutions of Partial Differential Equations by Monte Carlo Methods », Ph.D. thesis (EE), Univ. of British Columbia, Vancouver, October 1965.

[2] MIURA, T. and J. IWATA : « Time-sharing Computation Utilizing Analog Memory », Annals, International Association for Analog Computation, Volume V, July 1963, pp. 141-149.

[3] WITSENHAUSEN, H. . « Hybrid Solutions of Initial Value Problems for Partial Differential Equations », Proceedings, IFIPS Congress, New York, May 1965.

[4] KARPLUS, W.J. : « A Hybrid Computer Technique for Treating Nonlinear Partial Differential Equations », IEEE, Transactions on Electronic Computers, Volume EC-13, October 1964, pp. 597-605.

[5] TOMOVIC, R. : « Sensitivity Analysis of Dynamic Systems », McGraw-Hill Book Company, New York, 1964.

[6] KARPLUS, W.J. : « Error Analysis of Hybrid Computer Systems », Simulation, Volume 6, February 1966, pp. 121-136.

ERROR-ANALYSIS OF HYBRID COMPUTER LOOPS

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The introduction of sampling into an analog computation loop has become quite common, e.g., through the generation of nonlinear functions by table-look-up in a digital computer. It is well recognized that dynamic errors (analogous to truncation errors produced in the numerical solution of differential equations) are introduced by this sampling and various analytic and experimental investigations of the errors have been made. Most analytic investigations which have appeared have been carried out in the frequency domain and because of their inherent complexity have been limited to linear time-invariant systems of first and second order. While these limited analyses are very useful because they give a « feel » for the nature of dynamic errors produced in more complex problems, they do not provide a quantitative evaluation of errors in everyday problems. In this presentation a new method for analyzing dynamic errors in hybrid computing loops is described. It overcomes the above mentioned disadvantages in that it is readily applied to linear time-invariant systems of any order, avoids cumbersome frequency-domain derivations, presents the error information in a highly usable form, and is well suited to numerical calculations.

To illustrate the form which the analysis takes, consider the following example. It is desired to obtain a solution of the differential equations.

$$\dot{x} = Ax, \quad x(0) = \text{initial conditions}, \qquad (1)$$

where $x = {}^{t}(x^1, ..., x^n)$ is a n-vector and A is an $n \times n$ matrix. Suppose the vector Ax is generated in two parts : $(A - B)x_1$ by direct analog computation; and Bx by sampling, delaying the samples by one sample period T, and holding through the sample periods. This situation effectively models a hybrid computing loop where Bx is generated by table-look-up in a digital computer. The equation actually solved by the hybrid computer is

$$\dot{y} = (A - B)y + By\ (kT - T), \qquad (2)$$
$$kT \leqslant t < kT + T, \quad k = 0, 1, 2, \ldots$$

If the solution of (2) is evaluated at

$$t = kT\ [y_k = y\ (Tk)]$$

it can be shown that

$$y_k = e^{\tilde{A}Tk}\, \tilde{x}_0 + R^k r \qquad (3)$$

To have no error at the sample instants it is necessary that

$y_k = x\ (Tk) = e^{ATk}\, x\ (0)$, i.e., $\tilde{A} = A$, $\tilde{x}_0 = x\ (0)$ and $R^k r = 0$. The error analysis shows that

$$\tilde{A} = A - \frac{3}{2}\ TBA + \frac{1}{12}\ T^2\ (ABA + 13\ BA^2 + 26\ B^2A) + \ldots \qquad (4)$$

$$R = -\ TB + \frac{1}{2}\ T^2\ (AB + B^2) + \ldots \qquad (5)$$

$$\tilde{x}_0 = x\ (0) + 2\ T^2BA\ x\ (0) + \ldots \qquad (6)$$

$$r = -\ 2\ T^2BA\ x\ (0) + \ldots \qquad (7)$$

where terms of order T^3 and higher have been omitted and where $y_0 = x\ (0)$, $y_{-1} = x\ (0)$. The equations show that $\tilde{A}/A$ is the principal source of error when T is sufficiently small. Thus the elements of $\tilde{A}$ may be interpreted as the parameters of the equation actually solved by the hybrid computer. When applied to specific examples, the above formulas confirm results obtained earlier by z-transform methods.

Formulas similar to those above are derived for other types of hybrid computer loops and form the basis for a discussion of several methods data extrapolation which improve computation accuracy.

SPECTRAL ERROR ANALYSIS OF HYBRID SUBROUTINES

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In hybrid programming two types of digital subroutines are encountered : functional and non-functional. Functional subroutines are executed at run time and provide the main program with analog data in the same way as an analog subprogram. An example of a functional subroutine is the integration of force equations. Other types of subroutines such as logical calculations or iteration procedures are called non-functional.

A method will be discussed for evaluating the errors introduced in functional subroutines by the numerical approximation employed and by the hybrid data link. The method provides a spectral representation of the error (gain and phase-shift). Analog error estimates are made frequently on the same basis. Thus, the functional subroutine error can be treated as an equivalent analog subprogram error. In some instances an exact analog model can be found for the digital subroutine.

THE ANALYSIS OF ERRORS DUE TO SAMPLING RATE VARIATION

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Sensitivity analysis methods have recently been applied to the study of the effects of perturbations in sampling frequency on the performance of sampled-data control systems [1, 2]. The resulting sensitivity functions have been used to implement sampling-rate adjustment techniques which minimize the number of samples per unit time consistent with appropriate response and accuracy criteria. The purpose of this discussion is to present some tentative ideas on the application of these techniques to error control in hybrid systems by adjustment of sampling frequency. Some results with digital integration formulas will be presented.

REFERENCES

[1] TOMOVIC, R. and BEKEY, G.A. : « Adaptive Sampling Based on Amplitude Sensitivity », *IEEE Trans. Auto. Control*, AC-11 : 282-284, April 1966.

[2] BEKEY, G.A. and TOMOVIC, R. : « Sensitivity of Discrete Systems to Variation of Sampling Interval », *IEEE Trans. Auto. Control*, AC-11 : 284-287, April 1966.