MOLECULAR FLUXES IN THE LUNAR ATMOSPHERE

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Abstract—The properties of a neutral lunar atmosphere are investigated theoretically. A non-uniformity is shown to result from the temperature variations and non-uniform gas source distribution on the surface of the Moon. An integral equation governing the distribution of molecular fluxes, in the steady state, is formulated. This equation is solved by computer and analytical methods. Solutions are obtained and discussed for mass numbers ranging from hydrogen to the heavy gases. Characteristic relaxation times for approach to the steady state are estimated and found generally to be a small fraction of the synodic period. It is concluded that in all cases a marked anisotropy of molecular fluxes can be expected. By measuring these fluxes conclusions can be drawn about the distribution of gas sources, the physical properties of the surface and the composition of the lunar atmosphere.

1. INTRODUCTION

In considerations of the lunar atmosphere in the literature[1,2] the models used generally assume a constant surface temperature or local equilibrium. The known strong variations of the surface temperature[3] however, suggest that the atmosphere cannot be in local equilibrium and that one can at most, expect a lunar atmosphere in the steady state.

In this paper we investigate the properties of the atmosphere that are related to the anisotropy of molecular fluxes. This anisotropy, which is the manifestation of the lack of local equilibrium, is a result of the non-uniform surface temperature and the transport of gases around the Moon. Under certain assumptions these can be treated mathematically. This treatment provides us with a consistent theory and quantitative estimates for measurable effects not obtained by other methods.[1,2,4,5]

The underlying assumptions in the present development are:

(a) The lunar atmosphere is an exosphere with the lunar surface as base.[1,2]

(b) The only field of force acting on neutral particles is the gravitational potential of a perfectly spherical Moon.

(c) The Moon's angular velocity can be neglected.

(d) Gas sources originate underneath the surface.

(e) Radiations from space have second-order effects only.

Assumption (a) is significant since the lack of intermolecular collisions enables the atmospheric molecules to preserve the anisotropy imparted to them by the uneven temperature distribution and by the gravitational field. Assumption (d) includes the contribution of gas particles from outer space by considering them after they strike the surface and are re-emitted into the atmosphere. Assumption (e) refers in a certain sense not to a secondary factor since it is radiation from the Sun which governs the average density of the atmosphere.[1,2]

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However, it will be shown that the characteristic transient and flight times of atmospheric molecules are of the order of hours or days so that they can hardly be affected by solar radiation which allows much larger life times (of the order of months).[1,2] For that reason we shall deal only with the neutral atmosphere from which the ionized fraction can be derived.[1,2]

Molecular fluxes near the lunar surface will be our main concern in this paper since they can be directly related to the transport of gas, to the surface temperature, and to gas emission laws. Other atmospheric parameters (such as numbers densities), both at the surface and at any distance from it, can be derived by straightforward calculation once the fluxes are known. Also, fluxes are important from an experimental point of view since they are directly related to the response of pressure or density sensors.

2. SPREAD OF GAS ORIGINATING IN POINT SOURCES

We consider a gas source of strength, $f_1 \, d\sigma_1$ where f_1 is the velocity distribution function of flux emitted from a unit area of the lunar surface and $d\sigma_1$ is the source area. Molecules emitted from this source describe elliptical trajectories due to assumptions (a) and (b). Figure 1 is a sketch of a trajectory of such a molecule, emitted at point 1 and striking the
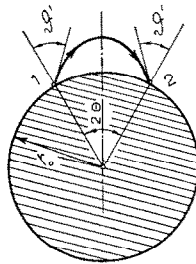

FIG. 1. GEOMETRIC PARAMETERS OF A MOLECULAR TRAJECTORY.

surface at point 2. It can be shown (see Appendix A) that given an initial velocity v the following relation holds

$$\gamma = v^2/v_{\text{esc}}^2 = \sin \Theta/\{\sin \Theta + \sin (2\vartheta' - \Theta)\} \tag{1}$$

where Θ is half the angular range and ϑ' is the angle the initial velocity vector makes with the normal to the surface (or the polar angle of the spherical coordinates ϑ', φ') as shown in Fig. 1; v_{esc} is the escape velocity; γ is the dimensionless kinetic energy and is equal to unity for the escape energy.

An element of flux emitted at point 1 within the range of energies γ, $\gamma + d\gamma$ and the solid angle $d\Omega' = \sin \vartheta' \, d\vartheta' \, d\varphi'$ is given by

$$d\Phi_1 \, d\sigma_1 = f \, d\sigma_1 \, d\gamma \, d\Omega' \tag{2}$$

where Φ_1 is the flux density emitted at point 1. This flux will spread and strike an element of area $d\sigma_2 = 2r_0^2 \sin 2\Theta \, d\Theta \, d\varphi'$ around point 2. Dividing equation (2) by $d\sigma_2$ we get

$$d\Phi_2 = f_1 \frac{d\sigma_1}{2r_0^2} \frac{\sin \vartheta'}{\sin^2 \Theta} \frac{\partial \gamma}{\partial \Theta} \, d\vartheta' \tag{3}$$

where Φ_2 is the flux density striking the surface at point 2 and r_0 is the lunar radius. Combining equations (1) and (3) we get

$$d\Phi = f_1 \frac{d\sigma_1}{2r_0{}^2} \gamma^2 \frac{\sin\vartheta' \sin 2\vartheta'}{\sin 2\Theta \sin^2\Theta} d\vartheta' \tag{3a}$$

To find the total flux density at point 2 due to the point source at 1 we integrate equation (3a) trough ϑ'.

$$d\Phi_2 = \frac{d\sigma_1}{2r_0{}^2 \sin^2\Theta \sin 2\Theta} \left\{ \int_{\Theta/2}^{\pi/2} (\gamma^2 f_1)_\Theta \sin\vartheta' \sin 2\vartheta' \, d\vartheta' \right.$$

$$\left. + \int_{\pi/2-\Theta/2}^{\pi/2} (\gamma^2 f_1)_{\pi-\Theta} \sin\vartheta' \sin 2\vartheta' \, d\vartheta' \right. \tag{4}$$

Here γ is regarded as a function of ϑ' and Θ (equation 1). f_1 is the velocity distribution function and is therefore a function of γ and ϑ'. The limits of the integrals express the fact that molecules with $\gamma > 1$ escape to space and do not return to strike the surface. The two integrals take into account molecules that start *towards* their target (angular distance 2Θ) and those that start *away* from it but strike it from the other side of the Moon (angular distance $2\pi - 2\Theta$). Equation (4) enables us to calculate the flux density at any point on the lunar surface once we know or assume the velocity distribution function f_1.

The spread of flux from a single point on the Moon creates an anisotropic pattern of molecular fluxes. To see this we look for the unidirectional flux density[6] which is a function of orientation in space (or the spherical coordinates ϑ', φ'). From equation (1) and (2), and the symmetry of the molecular trajectories we have

$$\Delta g_2 = g_1 \Delta \gamma = g_1 \gamma^2 \frac{\sin 2\vartheta'}{\sin^2\Theta} \Delta\Theta \tag{5}$$

where $g_1 = f_1/\cos\vartheta'$ is the unidirectional source flux density and is the unidirectional striking flux density due to $\Delta\Theta$ which is half the angular extent of the source along the great circle between the points 1 and 2.

As an illustrative example we assume a Maxwell–Boltzmann distribution for molecules emitted at the source. The source flux density can therefore be expressed by

$$g_1 = c\gamma e^{-E\gamma} \tag{6}$$

where

$$E = \frac{mv_{esc}^2}{2kT} = \frac{340°K}{T} \mu \tag{6a}$$

and C is independent of γ. T and μ are the surface temperature and the molecule's mass number respectively. Combining equations (5) and (6), we get

$$\Delta g_2 = c\gamma^3 e^{-E\gamma} \frac{\sin 2\vartheta'}{\sin^2\Theta} \Delta\Theta \tag{7}$$

Figure 2 is a polar diagram for Δg_2 when hydrogen atoms strike a point at an angular distance of 20° from the source. Figure 3 is a similar diagram for argon. The case of hydrogen is particularly interesting and will be discussed later.
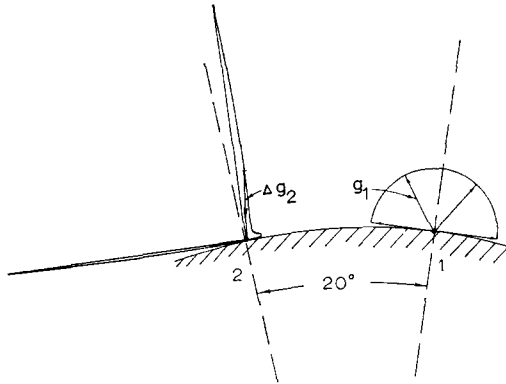
FIG. 2. UNIDIRECTIONAL FLUX DENSITY OF HYDROGEN ATOMS EMITTED FROM A POINT SOURCE AT AN ANGULAR DISTANCE OF $2\Theta = 20°$ (600 km).

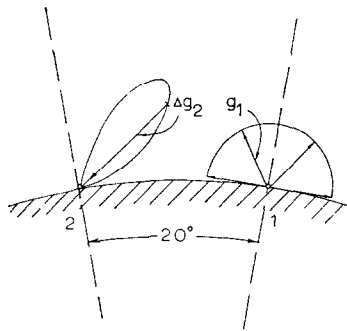FIG. 3. UNIDIRECTIONAL FLUX DENSITY OF ARGON ATOMS EMITTED FROM A POINT SOURCE AT AN ANGULAR DISTANCE OF $2\Theta = 20°$.
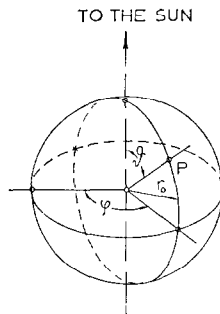

FIG. 4. COORDINATES OF A POINT P ON THE SURFACE OF THE MOON.

3. EXTENDED SOURCES OF GAS

To calculate the striking flux for any extended sources of gas, we integrate equation (4) over the surface of the Moon. We introduce the coordinates ϑ and φ as shown in Fig. 4. The element of surface area is then $d\sigma_1 = r_0^2 \sin\vartheta \, d\vartheta \, d\varphi$ so that from equation (4) we get

$$\Phi_2(\vartheta_0, \varphi_0) = \int\int \frac{\sin\vartheta \, d\vartheta \, d\varphi}{2\sin^2\Theta \sin 2\Theta} \left\{ \int_{\Theta/2}^{\pi/2} \gamma^2 f_1 \sin\vartheta' \sin 2\vartheta' \, d\vartheta' \right.$$
$$\left. + \int_{\pi/2-\Theta/2}^{\pi/2} \gamma^2 f_1 \sin\vartheta' \sin 2\vartheta' \, d\vartheta' \right\} \quad (8)$$

where ϑ_0, φ_0 are the coordinates of any fixed point on the surface. $\cos 2\Theta = \cos\vartheta_0 \cos\vartheta + \sin\vartheta_0 \sin\vartheta \cos(\varphi - \varphi_0)$ and $\gamma = \gamma(\Theta, \vartheta')$ as in equation 1.

For the relation between the fluxes f_1 and Φ_2 we now make the following assumptions:

(f) For every molecule that strikes the surface there is a probability $(1 - \varepsilon)$ to be re-emitted into the atmosphere. (ε is the fraction of molecules absorbed after each such strike.)

(g) Molecules that strike the surface accommodate to it so that the law of emission depends only upon the surface temperature and the molecular mass.

Following assumption (g) we can express $d\Phi_1$ by

$$d\Phi_1 = \Phi_1(\vartheta, \varphi) f(\mu, \gamma, T, \vartheta' \varphi') \, d\gamma \, d\Omega_i \quad (9)$$

where Φ_1 is the total flux density of molecules emitted from the surface. Due to assumption (f), $\Phi/(1 - \varepsilon)$ is the total flux density of molecules which struck the surface. The flux distribution function, f, is normalized with respect to γ and the spherical coordinates ϑ', φ'. It is a function of ϑ and φ since the temperature T is also a function of ϑ and φ.

Starting from any initial flux distribution $\Phi_i = \Phi_i(\vartheta, \varphi)$, we get from equations (8) and (9)

$$\Phi_{i+1}(\vartheta_0, \varphi_0) = (1 - \varepsilon)\int\int \Phi_i(\vartheta, \varphi) K(\vartheta_0, \varphi_0, \vartheta, \varphi) \, d\vartheta \, d\varphi \quad (10)$$

where

$$K = \frac{\sin\vartheta}{2\sin^2\Theta \sin 2\Theta} \left\{ \int_{\Theta/2}^{\pi/2} \gamma^2 f \sin\vartheta' \sin 2\vartheta' \, d\vartheta' + \int_{\pi/2-\Theta/2}^{\pi/2} \gamma^2 f \sin\vartheta' \sin 2\vartheta' \, d\vartheta' \right\} \quad (10a)$$

Φ_{i+1} is the flux distribution after all the molecules, emitted with Φ_i have completed one jump through the atmosphere. The integer i is the number of jumps accomplished and is a discrete parameter of Φ. We ignore temporarily the time which is a continuous parameter of Φ. The relationship between i and "time" will be investigated in section 6.

If the recurrence relation (10) is applied successively the distributions $\Phi_{i\to\infty}$ will approach a limit which satisfies the integral equation

$$\Phi(\vartheta_0, \varphi_0) = (1 - \varepsilon)\int\int \Phi(\vartheta, \varphi) K(\vartheta_0, \varphi_0, \vartheta, \varphi) \, d\vartheta \, d\varphi \quad (11)$$

It can be shown, by the method of contraction mapping, that the solution of equation (11) is $\Phi = 0$. (The "contraction" is ensured since a fraction of the molecules is absorbed in the

surface and molecules with energies greater than the escape energy are lost to space so that any finite flux distribution will eventually decay to zero.) Therefore, to maintain a steady-state flux distribution there must be a steady source of gas. This we express by writing

$$\Phi(\vartheta_0, \varphi_0) = (1 - \varepsilon) \int \int \Phi(\vartheta, \varphi) K(\vartheta_0, \varphi_0, \vartheta, \varphi) \, d\vartheta \, d\varphi + \Phi_s(\vartheta_0, \varphi_0) \tag{11a}$$

where Φ_s is the source flux distribution. Equation (10) (with the addition of Φ_s) is then the solution, by successive approximations of equation (11a).

4. STEADY STATE FLUX DISTRIBUTIONS

Setting out to solve equation (11a), we assume for the function f of equation (9)

$$f = f_\gamma f_{\Omega'} = \frac{E^{n+1}}{\Gamma(n+1)} \gamma^n e^{-E\gamma} \frac{\cos \vartheta'}{\pi} \tag{12}$$

where $E = 340/T$ is defined in equation (6a). When $n = 1$, f is identical with the Maxwell–Boltzmann distribution, assumed by Öpik and Singer.[4]

We can now calculate the average angular range of a molecule,

$$\overline{\Theta} = \int \int \int \Theta f \sin \vartheta' \, d\gamma \, d\vartheta' \, d\varphi' \tag{13}$$

For molecules with $\mu \gg 1$ we know from 6a that $E \gg 1$ so that the effective γ satisfies $\gamma \ll 1$ and equation (1) can be approximated by

$$\gamma = \Theta/\sin 2\vartheta' \tag{14}$$

Combining equations (12–14) we get

$$\overline{\Theta} = \pi(n+1)/4E$$

The average number of jumps to spread over the entire surface of the Moon is now estimated by

$$I = 2\pi/2\overline{\Theta} = 4E/(n+1) \tag{15}$$

or in the case of $n = 1$,

$$I = 2\overline{E} \tag{15a}$$

where $\overline{E}$ is some average value of $E = 340 \, \mu/T$. I can be considered as a characteristic
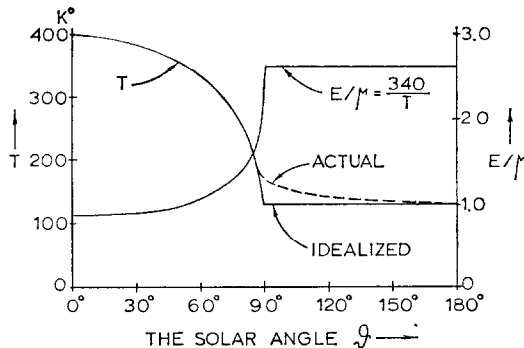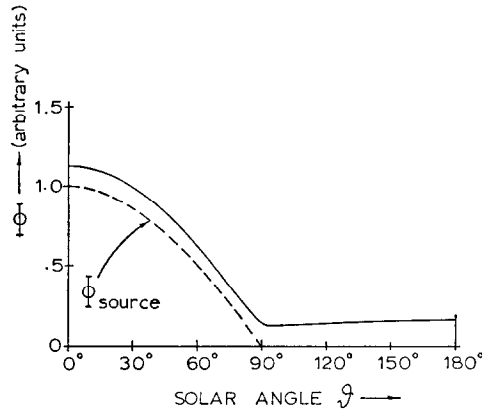

Fig. 5. The temperature T of the lunar surface.

FIG. 6. STEADY-STATE FLUX DENSITY DISTRIBUTION FOR HYDROGEN ATOMS.
A steady source cos ϑ is assumed for the sunlit side of the Moon. The law of emission is given by equation (12) with $n = 1$.
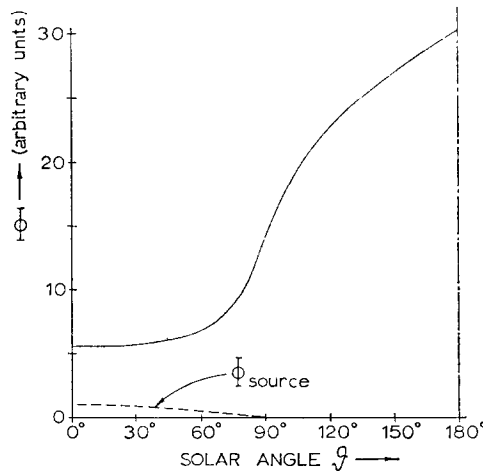

FIG. 7. STEADY-STATE FLUX DENSITY DISTRIBUTION FOR HELIUM.
A steady source cos ϑ is assumed for the sunlit side of the Moon. The law of emission is given by equation (12) with $n = 1$.

number of iterations in the approach to the steady state from any transient flux distribution. For an average temperature of 200°K we get $I_H = 4$ for hydrogen atoms and $I_{He} = 14$ for helium. Such iterations, as specified by equation (10), were performed by a computer and the results are shown in Figs. 6 and 7. In these calculations, the surface temperature was assumed to have axial symmetry with the direction of the Sun as axis. Figure 5 shows the temperature as a function of solar angle based on Wesselink.[3] Since the thermal escape is the predominant escape mechanism $\varepsilon = 0$ was also assumed.

For molecules with $\mu \gg 1$, we have $E \gg 1$ and $\gamma \ll 1$ (equations 12 and 13). This means that only low velocities and small angular ranges are effective in gas transfer from point to point. We therefore assume, for the present, that the lunar surface is an infinite plane around any given point and we calculate the flux density striking it. Figure 8 is a sketch of
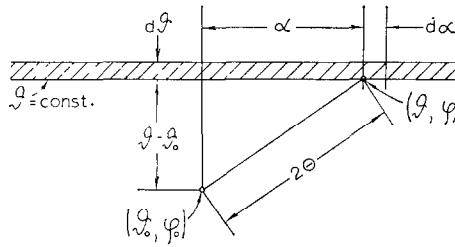
FIG. 8. SKETCH OF FLATTENED NEIGHBORHOOD OF POINT (ϑ_0, φ_0).
Equation (16b) is the element of flux of molecules originating in the sectioned strip.

such a flattened neighborhood of point (ϑ_0, φ_0). The element of flux striking this point is

$$d\Phi_2 = \int \frac{d\vartheta \, d\alpha}{4\Theta^3} \, \Phi_1 \int f\gamma^2 \sin \vartheta' \sin 2\vartheta' \, d\vartheta' \tag{16}$$

From equation (12) we have $f = f_\gamma (\cos \vartheta')/\pi$, so that integrating equation (16) through $-\infty < \alpha < +\infty$ we obtain

$$d\Phi_2 = \int d\vartheta \, \frac{\Phi_1}{4} \int_{(\vartheta - \vartheta_0)/2}^{\infty} \frac{f_\gamma}{\gamma} \, d\gamma \tag{16a}$$

Due to axial symmetry, which we denote as assumption (h) Φ_1 is a function of ϑ only. We limit ourselves to the case $n = 1$ in equation (12), and get

$$d\Phi_2 = \Phi_1 \frac{E}{4} \exp\left(-\frac{E}{2} |\vartheta - \vartheta_0|\right) d\vartheta \tag{16b}$$

This is the contribution to Φ_2 of a strip of surface between the lines $\vartheta = $ const. and $\vartheta + d\vartheta = $ const. At this point we have to consider the curvature of the surface. In Appendix B it is shown that the curvature contributes a factor $\sqrt{(\sin \vartheta/\sin \vartheta_0)}$ to equation (16b) so that we can write, to first order

$$\Phi_{i+1}(\vartheta_0) = (1 - \varepsilon) \int_{-\infty}^{+\infty} \Phi_i(\vartheta) K(\vartheta_0, \vartheta) \, d\vartheta \tag{17}$$

where

$$K(\vartheta_0, \vartheta) = \left\{1 + \frac{1}{2} (\vartheta - \vartheta_0) \cot \vartheta_0\right\} \frac{E}{4} \exp\left(-\frac{E}{2} |\vartheta - \vartheta_0|\right) \tag{17a}$$

Equation (17) is a special case of equation (10) and the same considerations that led to equation (11) now lead to the integral equation

$$\Phi(\vartheta_0) = (1 - \varepsilon) \int_{-\infty}^{+\infty} \Phi(\vartheta) K(\vartheta_0, \vartheta) \, d\vartheta + \Phi_s(\vartheta_0) \tag{18}$$

where Φ is the steady-state flux density.

As before, Φ can be obtained by iterations (equation 17), but here we may simplify the process by developing it into a power series. The integral equation 18 is converted to the differential equation

$$\left\{I_0 - 1 + I_1 \frac{\cot \vartheta_0}{2}\right\} \Phi(\vartheta_0) + \left\{I_1 + I_2 \frac{\cot \vartheta_0}{2}\right\} \Phi'(\vartheta_0)$$

$$+ \frac{I_2}{2} \Phi''(\vartheta_0) - \varepsilon\Phi(\vartheta_0) + \Phi_s(\vartheta_0) = 0 \tag{19}$$

where

$$I_n = \int_{-\infty}^{+\infty} (\vartheta - \vartheta_0)^n \frac{E}{4} \exp\left(-\frac{E}{2}|\vartheta - \vartheta_0|\right) d\vartheta, \qquad n = 0, 1, 2,$$

and we assume that $\varepsilon \ll 1$. On the sunlit side of the Moon we may approximate the integrals I_n to first order and get $I_0 = 1$, $I_1 = -(16/E^3) E'$ and $I_2 = 8/E^2$. Equation (19) can therefore be approximated by

$$\Phi'' + \left\{2 \cot\vartheta_0 - \frac{4E'}{E}\right\} \Phi' - \left\{\cot\vartheta_0 \frac{4E'}{E}\right\} \Phi = 0 \qquad (20)$$

where Φ, E and Φ_s are functions of ϑ_0. Here we have neglected $-\varepsilon\Phi + \Phi_s$ as small compared to

$$\left\{\cot\vartheta_0 \frac{4E'}{E}\right\} \Phi.$$

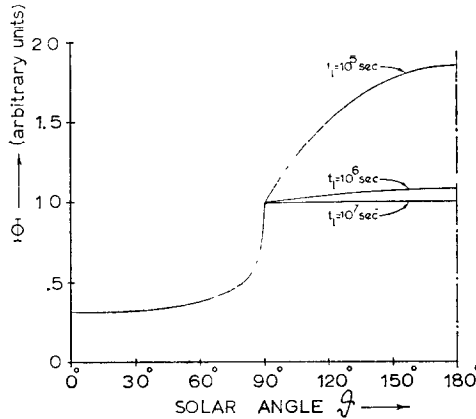Equation (20) was solved numerically and the results are plotted in Figs. 9 and 10.

FIG. 9. STEADY-STATE FLUX DENSITY DISTRIBUTION FOR MASS $\mu \simeq 40$ ($E = 100$).
A uniform source is assumed on the dark side of the Moon. The values t_1 represent assumed lifetimes of molecules on the sunlit side. The curves are normalized to $\Phi = 1$ at the terminator.

FIG. 10. STEADY-STATE FLUX DENSITY DISTRIBUTION FOR MASS $\mu \simeq 40$.
Values of ε are assumed surface absorption coefficients for the dark side of the Moon. A constant flux crosses the terminator from the sunlit side. The gas is presumed to arise from the solar wind or radiation. The curves are normalized to $\Phi = 1$ at the terminator and rounded off in the transition region $\pi/2 - 2/E < \vartheta < \pi/2 + 2/E$.

On the dark side of the Moon the coefficient of Φ in equation (20) is zero and we can no longer neglect the terms $-\varepsilon\Phi + \Phi_s$ of equation (19). These terms are closely related to the net transport of gas from sources to sinks. We can see this by writing the condition for the balance of fluxes in the steady state

$$\int_0^\pi \Phi_s \sin\vartheta \, d\vartheta = \int_0^\pi \varepsilon\Phi \sin\vartheta \, d\vartheta \tag{21}$$

equation (21) can be used to determine the boundary conditions for the solution of equation (19). We shall consider two cases:

(1) $\varepsilon = 0$ on the dark side.

(2) $\Phi_s = 0$ on the dark side.

Case 1 applies when gas steadily permeates from underneath the surface and the buildup of the atmosphere is checked by the steady depletion of gas, on the sunlit side, by solar radiations. Case 2 applies when the surface is absorbing a fraction ε of the flux striking it and, in order to establish a steady state, a constant flux crosses the terminator from the sunlit side. The gas source, in this case, is presumed to originate in the solar wind or radiation.

In case 1 we have $\varepsilon = 0$, $I_0 = 1$, $I_1 = 0$ and $I_2 = 8/E^2$ so that equation (19) becomes

$$\Phi'' + \cot\vartheta\Phi' + \frac{E^3}{4}\Phi_s = 0 \tag{22}$$

In Appendix B it is shown that equation (21) is equivalent to the values of

$$\Phi'_{\pi/2} = \frac{E^2}{4}\overline{\Phi}_s \qquad \text{and} \qquad \Phi_{\pi/2} = \frac{\sqrt{(E)}t_i}{500}\overline{\Phi}_s$$

at the terminator. $\overline{\Phi}$ is the average source flux density and t_i is the lifetime, in seconds, of a molecule exposed to the Sun (such molecules are assumed to have a certain probability—due to solar radiations—of being ejected out of the atmosphere). Solutions of equation (22) are plotted in Fig. 9 for $E = 100$ (about mass 40) and $t_i = 10^5$, 10^6 and 10^7 sec.

In case 2 we have for equation (19)

$$\Phi'' + \cot\vartheta\Phi' - \varepsilon\frac{E^2}{4}\Phi = 0 \tag{23}$$

In Appendix B it is shown that equation (21) for the boundary conditions is now equivalent to $\Phi_\pi' = 0$ at $\vartheta = \pi$ and

$$\Phi'_{\pi/2} = -\frac{E^2}{4}\int_0^{\pi/2}\Phi_s \sin\vartheta \, d\vartheta \qquad \text{at} \qquad \vartheta = \pi/2$$

Numerical solutions of equation (23) are plotted in Fig. 10 for $E = 100$ and $\varepsilon = 10^{-4}$, 10^{-3} and 10^{-2}.

These cases demonstrate the strong dependence of the distribution of heavy gases on the disposition of their sources and sinks. This dependence can be expressed in terms of the parameters $(E^2/4)\Phi_s$ and $\varepsilon(E^2/4)$ (equations 22 and 23, respectively) showing the relation between the mass number (μ or E), the absorption by the surface (ε), the source intensity (Φ_s) and the non-uniformity of the flux density (Φ) on the dark side of the Moon.

5. UNIDIRECTIONAL FLUX DENSITIES

The non-uniform distribution of flux densities over the surface of the Moon, discussed in the previous section, results in an anisotropy of the molecular fluxes at any given point in the atmosphere. To obtain a measure of this anisotropy, we calculated the unidirectional flux density at the lunar surface.

We apply equation (5) to extended sources by integration

$$g_2 = \sin 2\vartheta' \int_{\vartheta' = \text{const.}} g_1 \gamma^2 \frac{d\Theta}{\sin^2 \Theta} \tag{24}$$

With equations (9) and (12) this becomes

$$g_2 = \sin 2\vartheta' \int_{\vartheta' = \text{const.}} \Phi \frac{E^{n+1} \gamma^{n+2} e^{-E\gamma}}{\pi \Gamma(n + 1) \sin^2 \Theta} d\Theta \tag{24a}$$

with the integration along a great circle. For molecules with $\mu \gg 1$ equation (14) applies and we get

$$g_2(\vartheta') = \int_0^\infty \frac{\Phi(\vartheta + 2\Theta) E^{n+1}(\vartheta + 2\Theta) \Theta^n \exp\left[-E\Theta/(\sin 2\vartheta')\right]}{\pi \Gamma(n + 1) \sin^{n+1} 2\vartheta'} d\Theta \tag{24b}$$

Due to axial symmetry, E and Φ are functions of ϑ only. For greatest anisotropy, the integration path here chosen is a great circle through the axis of symmetry. Expanding Φ and E to first order we get from equation (24b)

$$g_2(\vartheta) = \frac{1}{\pi} \Phi(\vartheta) + \Phi'(\vartheta) \frac{2(n + 1)}{\pi E} \sin 2\vartheta' \tag{25}$$

It is seen that, to first order, the anisotropy is directly related to the flux density gradient $\Phi'(\vartheta)$. In addition there is a marked dependence on the surface temperature and on the molecular mass: the anisotropy varies directly with the temperature and inversely with the mass (equation 6a).

6. TRANSIENT FLUX DISTRIBUTIONS

In previous sections we dealt with the gas distribution resulting from molecules jumping from point to point. To take into account the time of flight associated with each such jump, we write equation (10) with a time dependent Φ.

$$\Phi(\vartheta_0, \varphi_0, t) = (1 - \varepsilon) \int \int \int \Phi(\vartheta, \varphi, t - \Delta t) \frac{\sin \vartheta \gamma^2 f \sin \vartheta' \sin 2\vartheta'}{2 \sin^2 \Theta \sin 2\Theta} \cdot d\vartheta' \, d\vartheta \, d\varphi \tag{26}$$

where t is time and Δt is the time of flight (a function of Θ, ϑ'). We do not solve equation (26) here, since many conclusions regarding transients can be drawn from the results already obtained in this paper; we direct our attention to the iterations of previous sections, each of which has a certain spread in time associated with it (due to the distribution of velocities).

To calculate this spread in time we assume a gas heavy enough so that equation (14) is valid. The time of flight is then

$$t = 2r_0 \Theta / v \sin \vartheta' \tag{27}$$

Introducing a dimensionless time of flight we have from (27)

$$\tau = [\pi/\sqrt{(2)}] t / t_{\text{orb}} = \gamma^{1/2} \cos \vartheta' \tag{27a}$$

where $t_{orb} = 108$ min is the period of a lunar satellite. We now calculate the probability that a molecule, emitted at time $\tau = 0$, strikes the surface in the interval $d\tau$ around τ. From equations (9) and (12) we know that the probability of a molecule being in the element $d\vartheta'\,d\varphi'\,d\gamma$ is:

$$df = \{E^{n+1}/2\pi\Gamma(n+1)\}\gamma^n e^{-E\gamma} \sin 2\vartheta'\,d\vartheta'\,d\varphi'\,d\gamma \qquad (28)$$

Transforming ϑ' to τ and integrating through φ' we get

$$df = \frac{2E^{n+1}}{\Gamma(n+1)}\gamma^{n+1}e^{-E\gamma}\,\tau\,d\tau\,d\gamma \qquad (28a)$$

Putting $n = 1$ and integrating through γ we get

$$df = \tau\,d\tau \int_{\tau^2}^{\infty} 2E^2 e^{-E\gamma}\,d\gamma = 2E\tau e^{-E\tau^2}\,d\tau \qquad (28b)$$

The lower limit of the integral $\gamma_{min} = \tau^2$ follows from equation (27a). The probability density $p_1(\tau)$ now follows from equation (28b).

$$p_1(\tau) = 2E\tau e^{-E\tau^2} \qquad (29)$$

We introduce the variable

$$S = E^{1/2}\tau \qquad (30)$$

and get

$$p_1(S) = 2Se^{-S^2} \qquad (29a)$$

For successive jumps a recurrence relation follows

$$p_{i+1}(S_0) = \int_0^{S_0} p_i(S)2(S_0 - S)e^{-(S_0-S)^2}\,dS \qquad (31)$$

where i, as before, is the number of jumps. When i is large we may use equation (30) with some average value for E so that S is also a dimensionless time but is independent of the molecular mass. It can be shown from equations (29) and (31) that

$$p_i(S) < 4S^{2i-1}e^{-S^2/i}/2^i\Gamma(i)\Gamma\left(i + \frac{1}{2}\right) \qquad (32)$$

It therefore follows that

$$\int_0^{S_a} p_i(S)\,dS \underset{i\to\infty}{\to} 0 \qquad \text{for} \qquad S_a < \frac{\sqrt{(2)}}{e}\cdot i \qquad (32a)$$

$$\int_{S_b}^{\infty} p_i(S)\,dS \underset{i\to\infty}{\to} 0 \qquad \text{for} \qquad S_b > \sqrt{(2)}\cdot i \qquad (32b)$$

This means that for $i \gg 1$, the probability of finding a molecule in its ith jump is confined to the interval $\sqrt{(2)}/e \cdot i < S < \sqrt{(2)}\cdot i$ or expressed in dimensionless time

$$\frac{\sqrt{(2)}}{e}iE^{1/2} < \tau < \sqrt{(2)}\,iE^{1/2} \qquad (33)$$

We conclude therefore that the ith jump or iteration, though spread in time, is confined to the interval (33).

Assuming as an average time, $\tau_i = iE^{-1/2}$ for iteration i; we get, with the aid of equation

(15a), an estimate for the characteristic transient time $\tilde{\tau}$

$$\tilde{\tau} = IE^{-1/2} = 2\bar{E}^{+1/2} \tag{34}$$

For an average surface temperature of 200°K, we have $\bar{E} = 1\cdot7\ \mu$ and the result

$$\tilde{\tau} = 2\cdot6\ \mu^{1/2} \tag{34a}$$

For hydrogen and xenon, we get $\tilde{\tau}_h = 2\cdot6$ and $\tilde{\tau}_{xe} = 30$ respectively, or from 27a, $\tilde{t}_h \simeq 2$ hr and $\tilde{t}_{xe} \simeq 24$ hr. These characteristics times are quite short compared to the synodic period (29·5 days). We can therefore expect the lunar atmosphere to be almost in a steady state; the flux distributions discussed in previous sections will follow the slowly rotating axis of symmetry which is pointing toward the sun.

In calculating the spread in time we have ignored, in this section, the dwell times of molecules trapped in the solid lattices of the lunar surface. Taking these into account may increase the characteristic times to the order of the synodic period and more, so that the steady state may never be established for certain types of molecules. Consequently, it is seen that the physical and chemical composition of the surface are the factors which most affect the transient behavior of the lunar atmosphere. The inclusion of dwell times, however, would not contradict assumption (f), section 3, or any of the results up to section 6. In general, therefore, the steady state (when established) depends essentially on the temperature variations and the distribution of gas sources on the lunar surface.

7. DISCUSSION

We can now indicate some outstanding properties of the lunar atmosphere which arise primarily from the lack of collisions, surface temperature variations, and gas source distributions.

1. In the case of point sources there is a considerable anisotropy of molecular fluxes. This is particularly striking for hydrogen atoms (Fig. 2) where the average velocity allows a sizable fraction of the emitted atoms to orbit around the Moon and to strike their targets from behind (Fig. 2). Letting $\Theta \to \pi$ in equation (5) we get $\Delta g_2 = \frac{1}{2}g_1$ which means that all the molecules with velocities $\frac{1}{2} < \gamma < 1$ are collected at the target. We see then that the Moon, assumed to be a perfect sphere, acts as a double focussing lens with object and image points which coincide. This suggests that lunar-based molecular beam experiments can be used to measure perturbations in the gravitational field and other forces.

2. In the case of extended sources of the light gasses, hydrogen and helium, there is a non-uniform distribution of flux densities over the surface. This is governed mainly by the thermal escape flux in the case of hydrogen. Figure 6 shows that the flux density of hydrogen follows closely the source flux. In this case the escape flux is so high that the atmosphere has to be maintained by a substantial source and only a small accumulation is possible on the dark side of the Moon. (The source is assumed to be the solar wind.)

In the case of helium the escape flux is small enough to allow a sizable accumulation, particularly on the dark side of the Moon. This accumulation is largely a result of the temperature difference across the terminator. Atoms emitted on the sunlit side strike deeper into the dark side than vice versa. Therefore, a higher flux density is required on the dark side to maintain, in the steady state, the balance of fluxes across the terminator. Once inside the dark region the gradually diminishing area of rings of increasing latitudes concentrate the flux density more and more in order to maintain the balance of fluxes. In Appendix B this is shown to be directly related to the asymmetry of the kernel.

3. For the heavy gases the thermal escape flux is extremely small and very large accumulations of gas are possible. The magnitudes of the steady state fluxes may now be determined by factors other than thermal escape.[1,2] The considerations of section 4 show that, again, as in the case of the light gases, non-uniform distributions of gas over the surface of the Moon can be expected. On the sunlit side the flux distribution is largely governed by the surface temperature variations. On the dark side, where the temperature is practically constant, the flux distributions depend in a very sensitive way on the gas source (and sink) distributions. Due to the short range of heavy molecules, any net transport of gas along the surface results on an appreciable density gradient. This is clearly demonstrated in Figs. 9 and 10 where the sources are assumed to originate on the sunlit and the dark sides respectively.

4. The non-uniform distribution of gas over the surface of the Moon is coupled with a marked anisotropy of molecular fluxes. This is shown in section 5 (equation 25) and is essentially a consequence of the lack of collisions in the lunar atmosphere. The transport of gas from sources to sinks (absorptions by the surface or loss to space) is governed by density and temperature gradients on the surface. No "average translation velocity" can therefore be imparted to the molecules and the transport of gas can only be maintained by such gradients of considerable magnitude. Gas "pressure" is conspicuously absent as a relevant physical parameter.

5. The establishment of the steady state is a result of a continuous migration of molecules jumping from point to point on the lunar surface. As shown in sections 2–5, the steady state is achieved without reference to time and depends only upon the way molecules spread around their points of emission. This spread, expressed by equation (10–12), depends upon the physical nature of the lunar surface through the velocity distribution function f. The fact that gas molecules striking the surface can be expected to accommodate to it thermally makes f strongly dependent on the surface temperature. Information from other physical parameters is obliterated by the random nature of thermal agitation. Molecules trapped in the solid lattices of the surface, for instance, may be slowed considerably in their reemission to the atmosphere. As far as the steady state is concerned this retardation can be compensated for by a greater (steady) density of molecules Φ. Transients in Φ, on the other hand, may preserve information on the physics of the lunar surface, as indicated in the conclusion of section 6. The steady state, then, depends on the temperature, but not in a unique way since the accommodation of molecules to the surface may not be complete and may be dependent upon local surface conditions. One should therefore consider distributions f other than the Maxwell–Boltzmann distribution. Whatever the distribution function may be, it strongly reflects the properties of the lunar surface and its environment. Hence such properties could be determined experimentally by measuring fluxes in the lunar atmosphere. The methods used in this paper, by relating such fluxes to the distribution function should therefore prove valuable in the design and interpretation of such experiments.

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APPENDIX A

The relation between the parameters γ, Θ and ϑ' of a molecular trajectory

The equation of an ellipse describing the trajectory can be written in polar coordinates r, ψ

$$r = R/(1 - \varepsilon_0 \cos \psi) \tag{A-1}$$

where ε_0 is the eccentricity and R is the radius of curvature at $\psi = 0$. The axis of the polar coordinates is the axis of symmetry in Fig. 1. At the peak of the trajectory ($\psi = 0$) the acceleration is normal to the trajectory and is given by

$$v_m^2/R = gr_0^2/r_m \tag{A-2}$$

where g is the acceleration of gravity, v_m and r_m are the velocity and the radius vector respectively at that point. But from equation (A-1) we have $r_m = R/(1 - \varepsilon_0)$ and $r_0 = R/(1 - \varepsilon_0 \cos \Theta)$ so that equation (A-2) becomes

$$v_m^2 = gr_0(1 - \varepsilon_0^2)/(1 - \varepsilon_0 \cos \Theta) \tag{A-2a}$$

From the conservation of the energy we have

$$v^2/2 - gr_0^2/r_0 = v_m^2/2 - gr_0^2/r_m \tag{A-3}$$

Introducing the new variable $\gamma = v^2/2gr_0$, equations (A-2a) and (A-3) yield

$$\gamma = 1 - r_0/r_m + \gamma_m = 1 - (1 - \varepsilon_0^2)/2(1 - \varepsilon_0 \cos \Theta) \tag{A-3a}$$

For $\varepsilon_0 = 1$ the trajectory extends to infinity (equation A-1). From equation (A-3a) this corresponds to $\gamma = 1$ or $v = (2gr_0)^{1/2}$ which is indeed the escape velocity. To determine ε_0 we refer to equation (A-1) and Fig. 1. We have $\tan \vartheta' = r_0/(dr/d\psi)_0 = (1 - \varepsilon_0 \cos \Theta)/\varepsilon_0 \sin \Theta$ or

$$\varepsilon = \cos \vartheta'/\cos(\vartheta' - \Theta) \tag{A-4}$$

Combining equations (A-3a) and (A-4) we get

$$\gamma = 1 - \frac{\cos^2(\vartheta' - \Theta) - \cos^2 \vartheta'}{2 \sec(\vartheta' - \Theta) \sin \vartheta' \sin \Theta}$$

$$= 1 - \frac{\sin(2\vartheta' - \Theta)\sin \Theta}{\sin \Theta + \sin(2\vartheta' - \Theta) \sin \Theta} = \frac{\sin \Theta}{\sin \Theta + \sin(2\vartheta' - \Theta)}. \tag{A-5}$$

APPENDIX B

Heavy gas distribution on the dark side of the Moon

We assume a constant, large value for $E = 340/(T)\mu$ and develop the kernel of equation (10) in terms of inverse powers of E. We also assume the velocity distribution function f

of equation (12) with $n = 1$ and get

$$K = \frac{\sin\vartheta}{2\sin^2\Theta\sin 2\Theta}\int_{\Theta/2}^{\pi/2}\gamma^2 f\,\text{din}\,2\vartheta'\sin\vartheta'\,d\vartheta'$$

$$= \frac{\sin\vartheta}{2\sin^2\Theta\sin 2\Theta}\int_{\Theta/2}^{\pi/2}\frac{E^2}{2\pi}\gamma^3 e^{-E\gamma}\sin^2 2\vartheta'\,d\vartheta' \quad (B-1)$$

[The second term of the kernel in equation (10) can be neglected because it accounts only for values of $\gamma \geqslant \frac{1}{2}$ which would make it of order $e^{-E/2}$ compared to the term retained in equation (B-1).]

We transform ϑ' to γ with Θ held constant, using the relation

$$\gamma = \sin\Theta/\{\sin\Theta + \sin(2\vartheta' - \Theta)\}$$

(equation 1), and get

$$d\vartheta' = \pm\tan\Theta\,d\gamma/2\gamma\sqrt{[(\gamma + \tan^2\Theta)^2 - \tan^2\Theta/\cos^2\Theta]}$$

and

$$\sin^2 2\vartheta' = \frac{\sin^2\Theta\cos^2\Theta}{\gamma^2}\{1 - \gamma \pm \sqrt{[(b + \tan^2\vartheta)^2 - \tan^2\Theta/\cos^2\Theta]}\}^2$$

With new variables x and a given by $x = \gamma + \tan^2\Theta$ and $a = \tan\Theta/\cos\Theta$ equation (B-1) becomes

$$K = \frac{E^2}{8\pi}\sin\vartheta e^{E\tan^2\Theta}\int_a^\infty\frac{e^{-EX}\,dx}{\sqrt{(x^2 - a^2)}}\left[\frac{1}{\cos^4\Theta} + a^2 + 2(x^2 - a^2) - \frac{2x}{\cos^2\Theta}\right]$$

$$= \frac{E^2}{8\pi}\sin\vartheta e^{E\tan^2\Theta}\left[\left(\frac{1}{\cos^4\Theta} + a^2\right)\cdot K_0(Ea) + \frac{2(Ea)}{E^2}\cdot K_1(Ea) - \frac{2(Ea)}{E\cos^2\Theta}\cdot K_1(Ea)\right]$$

$$(B-2)$$

where K_0 and K_1 are the Modified Bessel Functions of order 0 and 1 respectively. The upper limit of the integral in (B-2) replaces

$$\tfrac{1}{2} + \tan^2\Theta \quad \text{and} \quad 1 + \tan^2\Theta \quad \text{by} \quad \infty$$

because only the neglect of factors of order $e^{-E/2}$ is involved. Note that factors in the square brackets with $\pm\sqrt{(x^2 - a)}$ cancel upon integration and are therefore omitted in equation (B-2). We expand $e^{E\tan^2\theta}$ in a power series and get

$$K = \frac{E^2\sin\vartheta}{8\pi}\left[1 + \frac{E^2\tan^2\Theta}{E} + \frac{E^4\tan^4\Theta}{2E^2} + \cdots\right]$$

$$\cdot\left[\left(\frac{1}{\cos^4\Theta} + a^2\right)\cdot K_0(Ea) + \frac{2(Ea)}{E}\cdot K_1(Ea) - \frac{2(Ea)}{E^2\cos^2\Theta}\cdot K_1(Ea)\right] \quad (B-3)$$

With a new variable t given by

$$t = E\sin\Theta \quad (B-4)$$

we have $\quad Ea = t(1 + t^2/E^2),\ E\tan\Theta = t(1 + t^2/2E^2)\ 1/\cos^4\Theta = 1 + 2t^2/E^2$
and

$$K_0(Ea) = K_0(t) - \frac{t^3}{E^2}\cdot K_1(t).$$

In these expansions we neglect terms or order $1/E^2$ or higher. Opening the brackets of equation (B-3) and rearranging terms we get

$$K = \frac{E^2 \sin \vartheta}{8\pi} \left\{ K_0(t) + \frac{1}{E} [t^2 \cdot K_0(t) - 2t \cdot K_1(t)] \right.$$
$$\left. + \frac{1}{E^2} \left[\left(\frac{t^4}{2} + 3t^2 \right) \cdot K_0(t) - (3t^3 - 2t) \cdot K_1(t) \right] \right\} \quad \text{(B-5)}$$

Assuming axial symmetry for our problem we note that the flux density Φ of the integral equation (11) is a function of ϑ only and get

$$\Phi(\vartheta_0) = (1 - \varepsilon) \int_0^\pi K(\vartheta_0, \vartheta) \Phi(\vartheta) \, d\vartheta + \Phi_s(\vartheta_0) \quad \text{(B-6)}$$

where

$$K(\vartheta_0, \vartheta) = \int_{\varphi_0 - \pi}^{\varphi_0 + \pi} K \, d\varphi$$

From spherical trigonometry we know that φ and Θ are related by:

$$\cos 2\Theta = \cos \vartheta_0 \cos \vartheta + \sin \vartheta_0 \sin \vartheta \cos (\varphi - \varphi_0) \quad \text{(B-7)}$$

Therefore, from equations (B-4) and (B-7) we obtain:

$$d\varphi = 4t \, dt/E \bigg/ \sqrt{\left[\sin \vartheta_0 \sin \vartheta \, (t^2 - b^2) \left(1 - \frac{t^2 - b^2}{E^2 \sin \vartheta_0 \sin \vartheta} \right) \right]} \quad \text{(B-8)}$$

where

$$b = \left| E \sin \frac{\vartheta - \vartheta_0}{2} \right| \quad \text{(B-9)}$$

The kernel $K(\vartheta_0, \vartheta)$ of equation (B-6) now becomes

$$K(\vartheta_0, \vartheta) = \frac{E}{2\pi} \sqrt{\left(\frac{\sin \vartheta}{\sin \vartheta_0} \right)} \int_b^\infty \frac{t \, dt}{\sqrt{(t^2 - b^2)}} \left[1 + \frac{t^2 - b^2}{2E^2 \sin \vartheta_0 \sin \vartheta} \right] \{ K_0(t) + \dots \}$$

where the term $\sqrt{[1 - (t^2 - b^2)/E^2 \sin \vartheta_0 \sin \vartheta]}$ has been expanded in powers of $1/E^2$. As before, changing the upper limit introduces only errors of order $e^{-E/2}$. We write the above in the form

$$K(\vartheta_0, \vartheta) = \frac{E}{4} \sqrt{\left(\frac{\sin \vartheta}{\sin \vartheta_0} \right)} \left\{ A + \frac{B}{E} + \frac{C}{E^2} \right\}$$

where

$$A = \frac{2}{\pi} \int_b^\infty \frac{t \, dt}{\sqrt{(t^2 - b^2)}} \cdot K_0(t) = e^{-b} \quad \text{(B-10)}$$

$$B = \frac{2}{\pi} \int_b^\infty \frac{t \, dt}{\sqrt{(r^2 - b^2)}} \{ t^2 \cdot K_0(t) - 2t \cdot K_1(t) \} = e^{-b} \{ b^2 - b - 1 \}$$

$$C = \frac{2}{\pi} \int_b^\infty \frac{t \, dt}{\sqrt{(t^2 - b^2)}} \left\{ \left(\frac{t^4}{4} + 3t^2 \right) \cdot K_0(t) - (3t^3 - 2t) \cdot K_1(t) + \frac{t^2 - b^2}{2 \sin \vartheta_0 \sin \vartheta} \cdot K_0(t) \right\}$$
$$= \frac{1}{2} e^{-b} \left\{ b^4 - 4b^3 - b^2 + b + 1 + \frac{b + 1}{\sin \vartheta_0 \sin \vartheta} \right\}$$

2

We note from equation (B-9) that A, B and C are symmetric functions with respect to ϑ and ϑ_0. However, the kernel $K(\vartheta_0, \vartheta)$ is asymmetric due to the factor $\sqrt{(\sin \vartheta/\sin \vartheta_0)}$. This expresses the fact that flux originating from an elementary ring with constant ϑ and striking a ring with ϑ_0 will spread or contract depending whether $\vartheta_0 < \vartheta$ or $\vartheta > \vartheta_0$ respectively. To find the steady state flux distribution we apply the integral equation (11), expanding Φ in inverse powers of E. First we have

$$\Phi(\vartheta_0) = (1 - \varepsilon) \int_0^\pi K(\vartheta_0, \vartheta)\Phi(\vartheta) \, \mathrm{d}\vartheta + \Phi_s(\vartheta_0) = (1 - \varepsilon) \int_0^\pi K(\vartheta_0, \vartheta) \, \mathrm{d}\vartheta \Big\{ \Phi(\vartheta_0)$$

$$+ (\vartheta - \vartheta_0)\Phi'(\vartheta_0) + \frac{(\vartheta - \vartheta_0)^2}{2} \Phi''(\vartheta_0) + \ldots \Big\} + \Phi_s(\vartheta_0)$$

We now introduce a new variable

$$\beta = \frac{E}{2}(\vartheta - \vartheta_0) \tag{B-11}$$

apply equation (B-10) for $K(\vartheta_0, \vartheta)$, and get

$$\Phi(\vartheta_0) = (1 - \varepsilon) \int_{-\infty}^{+\infty} \frac{E}{4} \sqrt{\left(\frac{\sin \vartheta}{\sin \vartheta_0}\right)} \left\{ A + \frac{B}{E} + \frac{C}{E^2} \right\} \Big\{ \Phi(\vartheta_0)$$

$$+ \frac{2\beta}{E} \Phi'(\vartheta_0) + \frac{2\beta^2}{E^2} \Phi''(\vartheta_0) \Big\} \frac{2}{E} \, \mathrm{d}\beta + \Phi_s(\vartheta_0) \tag{B-12}$$

The limits of the integral replace $(E/2)(\pi - \vartheta_0)$ and $-(E/2)\vartheta_0$. This approximation is valid provided $\vartheta_0 \gg 2/E$ and $\pi - \vartheta_0 \gg 2/E$. If we now assume $\varepsilon \ll 1$, and expand $\sqrt{(\sin \vartheta/\sin \vartheta_0)}$ in a power series in β, we obtain the following from equation (B-12)

$$\varepsilon\Phi(\vartheta) = \Phi(\vartheta) \Big\{ -1 + \frac{1}{2} \int_{-\infty}^{+\infty} A \, \mathrm{d}\beta + \frac{1}{2E} \int_{-\infty}^{+\infty} [\beta A \cot \vartheta + B] \, \mathrm{d}\beta$$

$$+ \frac{1}{2E^2} \int_{-\infty}^{+\infty} \Big[C - \beta^2 A \Big(1 + \frac{\cot^2 \vartheta}{2} \Big) + \beta B \cot \vartheta \Big] \, \mathrm{d}\beta \Big\}$$

$$+ \Phi'(\vartheta) \Big\{ \frac{1}{E} \int_{-\infty}^{+\infty} \beta A \, \mathrm{d}\beta + \frac{1}{E^2} \int_{-\infty}^{+\infty} [\beta A + \beta^2 A \cot \vartheta] \, \mathrm{d}\beta \Big\}$$

$$+ \Phi''(\vartheta) \Big\{ \frac{1}{E^2} \int_{-\infty}^{+\infty} \beta^2 A \, \mathrm{d}\beta \Big\} + \Phi_s(\vartheta) \tag{B-13}$$

This is a second order differential equation that the flux density Φ has to satisfy. Derivatives of order higher than 2 will contribute terms of order $1/E^3$ which for large E can be neglected, except near a possible singularity.

In evaluating the coefficients of this differential equation integrals containing integrand(which are odd functions of β vanish by inspection, so that we have

$$\varepsilon\Phi = \Phi \Big\{ -1 + \int_0^\infty A \, \mathrm{d}\beta + \frac{1}{E} \int_0^\infty B \, \mathrm{d}\beta + \frac{1}{E^2} \int_0^\infty \Big[C - \beta^2 A \Big(1 + \frac{\cot^2 \vartheta}{2} \Big) \Big] \, \mathrm{d}\beta \Big\}$$

$$+ \Phi' \Big\{ \frac{2}{E^2} \int_0^\infty \beta^2 A \cot \vartheta \, \mathrm{d}\beta \Big\} + \Phi'' \Big\{ \frac{2}{E^2} \int_0^\infty \beta^2 A \, \mathrm{d}\beta \Big\} + \Phi_s \tag{B-14}$$

Using now the expressions (B-10) for A, B and C and (B-9) and (B-11) for the relation between b and β we obtain zero for the right hand side coefficient of Φ so that equation (B-14) reduces to

$$\Phi'' + \cot \vartheta \Phi' - \varepsilon \frac{E^2}{4} \Phi + \frac{E^2}{4} \Phi_s = 0 \qquad \text{(B-15)}$$

In the steady state the boundary conditions for this equation are obtained from the following relation which expresses the balance between the total source flux and that absorbed in the surface

$$\int_0^\pi \Phi_s \sin \vartheta \, d\vartheta = \int_0^\pi \varepsilon \Phi \sin \vartheta \, d\vartheta \qquad \text{(B-16)}$$

Equations (B-15) and (B-16) determine the steady state distribution on the dark side of the Moon.

The net flux, F, can be calculated directly, utilizing the definition of K (equation 10). We have

$$F = \int_0^{\vartheta_0} 2\pi r_0^2 \sin \vartheta_1 \, d\vartheta_1 \int_{\vartheta_0}^\pi K(\vartheta_1, \vartheta) \Phi(\vartheta) \, d\vartheta$$

$$- \int_{\vartheta_0}^\pi 2\pi r_0^2 \sin \vartheta \, d\vartheta \int_0^{\vartheta_0} K(\vartheta, \vartheta_1) \Phi(\vartheta_1) \, d\vartheta_1 \quad \text{(B-17)}$$

where $\vartheta_0 \leqslant \vartheta \leqslant \pi$ and $0 \leqslant \vartheta_1 \leqslant \vartheta_0$. This is the net flux crossing latitude ϑ_0 from all over that part of the Moon for which $\vartheta > \vartheta_0$. From equations (B-10) and (B-9) we have, for large E, near the terminator

$$F \cong 2\pi r_0^2 \left\{ \int_0^{\vartheta_0} d\vartheta_1 \int_{\vartheta_0}^\pi \frac{E}{4} \exp\left(-\frac{E}{2}|\vartheta_1 - \vartheta|\right) [\Phi(\vartheta_0) + (\vartheta - \vartheta_0)\Phi'(\vartheta_0)] \, d\vartheta \right.$$

$$\left. - \int_{\vartheta_0}^\pi d\vartheta \int_0^{\vartheta_0} \frac{E}{4} \exp\left(-\frac{E}{2}|\vartheta - \vartheta_1|\right) [\Phi(\vartheta_0) + (\vartheta_1 - \vartheta_0)\Phi'(\vartheta_0)] \, d\vartheta_1 \right\}$$

where $\Phi(\vartheta)$ was expanded, to first order, in powers of $\vartheta - \vartheta_0$. Performing the integrations we get

$$F = 2\pi r_0^2 \frac{4}{E^2} \Phi_{\pi/2} \qquad \text{(B-18)}$$

We now show that in the case when $\varepsilon = 0$ on the dark side (case 1 of section 4) equation (B-16) gives, as boundary conditions, the values of Φ and Φ' at $\vartheta = \pi/2$. To see this we calculate the net flux crossing the terminator from the dark to the sunlit sides. This flux is

$$F = \int_{\pi/2}^\pi 2\pi r_0^2 \sin \vartheta \, d\vartheta \Phi_s(\vartheta) \equiv 2\pi r_0^2 \overline{\Phi}_s \qquad \text{(B-19)}$$

where $\overline{\Phi}_s$ is the average source flux density and r_0 is the radius of the Moon. Combining equations (B-18) and (B-19) we have

$$\Phi'_{\pi/2} = \frac{E^2}{4} \overline{\Phi}_s \qquad \text{(B-20)}$$

To get $\Phi_{\pi/2}$ we estimate the loss of flux on the sunlit side. Assume a life time t_l for molecules exposed to the Sun. From equations 27a and 33 of section 6 we have for the average

number of jumps during a lifetime: $\bar{l}_l \cong E^{1/2}\tau_l = E^{1/2}(\pi)/\sqrt{(2)}t_l/t_{orb}$. We have therefore an estimate for ε on the sunlit side by putting

$$\varepsilon = 1/\bar{l}_l = \frac{\sqrt{(2)}}{\pi}\, t_{orb}/\sqrt{E}t_l \sim 3 \times 10^3\, \mathrm{sec}/\sqrt{(E)}t_l \qquad \text{(B-21)}$$

Equation (B-16) now becomes

$$2\overline{\Phi}_s = \int_0^{\pi/2} \frac{\Phi \sin\vartheta}{\sqrt{[E(\vartheta)]}}\, d\vartheta.$$

The integral was evaluated numerically from the results of section 4 and found to be $0 \cdot 31 \Phi_{\pi/2}/\sqrt{[E(\pi/2)]}$ so that we have

$$\Phi_{\pi/2} \cong \sqrt{(E)}\frac{t_l}{500_{\mathrm{sec}}}\,\overline{\Phi}_s \qquad \text{(B-22)}$$

Equations (B-20) and (B-22) are the boundary conditions that uniquely determine the solution of equation (B-15) (or equation 21) with $\varepsilon = 0$ on the dark side of the Moon. Solutions for $E = 100$ and $t_l = 10^5$, 10^6 and 10^7 sec are plotted in Fig. 9.

We now consider the case when there is absorption on the dark side (case 2 of section 4) and evaluate the right hand side of equation (B-16). From equation (B-15) with $\Phi_s = 0$ we have

$$\frac{E^2}{4}\int_{\pi/2}^\pi \varepsilon \Phi \sin\vartheta\, d\vartheta = \int_{\pi/2}^\pi d(\Phi' \sin\vartheta) = \lim_{\vartheta\to\pi}(\Phi' \sin\vartheta) - \Phi'_{\pi/2} \qquad \text{(B-23)}$$

The integral on the left hand side of equation (B-16) is the normalized net flux crossing the terminator from the sunlit to the dark side. From equation (B-18) this is just $-(4/E^2)\Phi'_{\pi/2}$. Equation (B-16) therefore reduces to $-\Phi'_{\pi/2} = \lim(\Phi' \sin\vartheta) - \Phi'_{\pi/2}$. This relation is satisified only by the non-singular solution of equation (B-15), i.e. the one for which $\Phi_\pi' = 0$.

Резюме—Теоретически исследуются свойства нейтральной лунной атмосферы. Обнаруживается, что неравномерность является результатом температурных колебаний и неравномерного распределения источников газа на поверхности Луны. Формулируется интегральное уравнение, регулирующее распределение молекулярных потоков, в установившемся состоянии. Это уравнение решается методами вычисления и анализа. Получаются и обсуждаются решения для массовых чисел в диапазоне от водорода до тяжелых газов. Уточняются характерные периоды реляксации, предшествующие установившемуся состоянию, которые являются, в общем, небольшой долей синодического периода. Заключают, что во всех случаях можно ожидать заметную анизотропию молекулярных потоков. Путем измерения этих ротоков могут быть сделаны выводы относительно распределения источников газа, физических свойств поверхности и состава лунной атмосферы.