JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 36, 377–431 (1971)

Critical Point Theory in Hilbert Space under Regular Boundary Conditions

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1. INTRODUCTION

In recent years, particularly by the work of Palais and Smale, considerable progress was made in extending the Morse Theory of critical points of real-valued functions defined in a finite dimensional space to functions defined in a Hilbert space. Surveys of this development may be found in [19, Chap. IV] and in [2].

The present paper deals with the following situation: let E be a real Hilbert space with elements $x, y, \ldots$ with scalar product $\langle x, y \rangle$, and norm $\| x \| = \sqrt{\langle x, x \rangle}$. Let V be a bounded open set in E satisfying the following

ASSUMPTION A. (i) V is arcwise connected, and there exists a positive constant C of the following property: To every couple x_0, x_1 of points in V there corresponds a continuously differentiable curve $x = x(t) \subset V$ with $x(0) = x_0$, $x(1) = x_1$ such that

$$\int_0^1 \left| \frac{dx}{dt} \right| dt \leqslant C \| x_1 - x_0 \|; \tag{1.1}$$

(ii) the boundary $\dot{V}$ of V is a smooth hypermanifold in the sense of [17, Definition 3.2]. (We will need the property of such manifolds to have a unique exterior unit normal at every point [17, Theorem 4.1].)

Let f be a real valued function defined on the closure $\bar{V}$ of V.

ASSUMPTION B. f is not constant in any ball. Moreover, f has a continuous and uniformly bounded differential $df(x; h)$ for $x \in V$ ("differential" means Fréchet differential). Moreover we assume that for some constant M, $g = \operatorname{grad} f$ (defined by $df(x; h) = \langle g(x), h \rangle$) satisfies the inequality

$$\| g(x) \| < M \qquad \text{for } x \in \bar{V}. \tag{1.2}$$

DEFINITION 1.1. The point $x \in \bar{V}$ is called a critical point of f if

$$g(x) = \Theta, \tag{1.3}$$

where Θ denotes the zero element of E. A real number c is called a critical value (or critical level) of f if there exists a critical point x such that

$$f(x) = c. \tag{1.4}$$

ASSUMPTION C (Palais-Smale condition). If on the set $S \subset \bar{V}$, f is bounded while $\| g \|$ is not bounded away from zero on S then the closure $\bar{S}$ of S contains a critical point [11, Condition C].

ASSUMPTION D. The critical levels of f are isolated.

ASSUMPTION E (Regular boundary condition). g is exteriorly directed on $\dot{V}$, i.e.,

$$\langle g(x), n(x) \rangle > 0 \qquad \text{for all } x \in \dot{V}, \tag{1.5}$$

where $n(x)$ denotes the exterior unit normal to $\dot{V}$ at x.

For any real number r, we set

$$\{f > r\} = \{x \in V \mid f(x) > r\}, \qquad f_r = \{x \in V \mid f(x) < r\},$$
$$\bar{f}_r = \{x \in V \mid f(x) \leqslant r\}, \qquad \{f = r\} = \{x \in V \mid f(x) = r\},$$

and for any couple of real numbers $b > a$, $H_q(\bar{f}_b, \bar{f}_a)$ denotes the q-th singular homology group of the couple $(\bar{f}_b, \bar{f}_a)$ with a fixed coefficient group over a principal ideal domain [3, Chap. VII].

In the classical theory of Morse there is attached to every critical level c and every nonnegative integer q a group by the following

DEFINITION 1.2. If $a < c < b$, and if c is the only critical level in the closed interval $[a, b]$ then the q-th critical group $C_q(c)$ at level c is defined by

$$C_q(c) = H_q(\bar{f}_b, \bar{f}_a). \tag{1.6}$$

It is, therefore, the first task to establish the legitimacy of this definition i.e., its independence of a and b under the assumption made above. This is done in Theorem 2.3 in whose proof (via Theorem 2.1) Assumption E plays a decisive role. (In the special situation where V is a ball, where $g(x) - x$ is completely continuous, and where the number of critical values is finite, these results are contained in [16]).

We conclude Section 2 with two theorems on the existence of minima and of critical levels and with a few simple examples.

By Assumption D the critical values form a countable set. If we denote them by c_1, c_2,..., we set

$$C_q(c_i) = C_q^{\,i}, \qquad M_q^{\,i} = \operatorname{rank} C_q^{\,i}, \qquad M_q = \sum_i M_q^{\,i}. \tag{1.7}$$

Section 3 deals with sufficient conditions for the validity of the inequalities

$$R_q \leqslant M_q, \tag{1.8}$$

and of the equality

$$R_q = M_q, \tag{1.9}$$

where R_q denotes the q-th Betti number of V i.e., the rank of $H_q(V)$). We recall the following

DEFINITION 1.3. Let (B, A) and (D, C) be two pairs of sets in E with $(B, A) \supset (D, C)$. Let I denote the unit interval, and let δ:

$$(B \times I, A \times I) \to (B, A)$$

be a continuous map. Then δ is said to deform (B, A) into (D, C) if it has the following additional properties: $\delta(x, 0)$ is the identity map, and under δ

$$(B \times 1, A \times 1) \to (D, C), \qquad (D \times I, C \times I) \to (D, C).$$

Theorem 3.1 states that (1.8) is true if in addition to assumptions A-E the following one is satisfied.

ASSUMPTION F. Let $b > c$, and suppose that c is the only critical level in $[c, b]$. Then there exists a deformation deforming $\mathring{f}_b$ into $\mathring{f}_c$. (We follow the convention by which [ ] indicates a closed, and ( ) an open interval.)

The proof of Theorem 3.1 is based on a result of Seifert and Threlfall [20, Section 5, Satz II] according to which (1.8) is true if their axiom I [20, p. 24] is satisfied. It will be shown that the validity of this axiom follows from our assumption (Lemma 3.3).

Similarly the discussion of (1.9) is based on a result of Seifert and Threlfall [20; Section 6].

Assumption F is automatically satisfied if the set Γ of critical points is finite (Theorem 3.3).

The addition of Assumption F to our previous assumptions allows us to establish the following facts by the same method by which they are established in the finite-dimensional case [12, Sections 7 and 8]: If $\sigma(c)$ denotes the set of critical points at level c, then

$$C_q(c) \approx H_q(\mathring{f}_c, \mathring{f}_c - \sigma(c)) \tag{1.10}$$

(Theorem 3.4), and from this, by excision,

$$C_q(c) = H_q(\bar{f}_c \cap W, f_c \cap W - \sigma(c)) \tag{1.11}$$

for any neighborhood W of $\sigma(c)$ whose closure contains no other critical points than those of $\sigma(c)$. (Theorem 3.5; the symbol $\approx$ denotes isomorphism.)

So far no nondegeneracy assumptions have been made. In Section 4 the definition of nondegeneracy of order $p \geqslant 2$ is recalled (Definition 4.2; for $p = 2$, this definition coincides with the usual definition of nondegeneracy). A critical point which is nondegenerate of order p is isolated (Lemma 4.3); therefore, it follows easily from Assumption C that there are at most a finite number of such critical points. Consequently, if we assume that each critical point is nondegenerate of some finite order, there will be only a finite number, say N, of critical levels. (There exists at least one critical level by Theorem 2.5). Our notation will be such that

$$c_1 < c_2 < \cdots c_N \tag{1.12}$$

(cf. the line above (1.7)). Moreover, we set

$$C_q(c_i) = C_q{}^i \qquad \text{for } i = 1, 2, \ldots N, \tag{1.13}$$

and denote by $\sigma_j{}^i$, $j = 1, 2, \ldots n_i$, the critical points at level c_i. If $W_j{}^i$ is a neighborhood of $\sigma_j{}^i$ whose closure contains $\sigma_j{}^i$ as the only critical point, then $C_q^{i,j}$ defined by

$$C_q^{i,j} = H_q(\bar{f}_{c_i} \cap W_j{}^i, f_{c_i} \cap W_j{}^i - \sigma_j{}^i) \tag{1.14}$$

is independent of $W_j{}^i$, and is called the critical group at $\sigma_j{}^i$. We have

$$C_q{}^i \approx \text{direct} \operatorname*{sum}_j C_q^{i,j} \tag{1.15}$$

(cf. the corollary to Theorem 3.5). Therefore,

$$M_q{}^i = \sum_{j=1}^{n_i} M_q(\sigma_j{}^i), \tag{1.16}$$

where $M_q{}^i$ and $M_q(\sigma_j{}^i)$ denote the rank of $C_q{}^i$ and $C_q^{i,j}$, respectively (cf. (1.7)). $M_q(\sigma_j{}^i)$ is called the q-th type number of the critical point $\sigma_j{}^i$.

We are thus led to consider a nondegenerate critical point of order $p \geqslant 2$ which we assume to be Θ and a neighborhood W of Θ whose closure contains Θ as the only critical point. If, in addition. we assume that

$$f(\Theta) = 0, \tag{1.17}$$

then the q-th critical group $C_q(\Theta)$ at Θ is given by

$$C_q(\Theta) = H_q(\bar{f}_0 \cap W, \bar{f}_0 \cap W - \Theta), \tag{1.18}$$

and the q-th type number $M_q(\Theta)$ at Θ is given by

$$M_q(\Theta) = \text{rank of } C_q(\Theta). \tag{1.19}$$

When it is necessary to emphasize that these quantities refer to the function f, we write $C_q(\Theta; f)$, $M_q(\Theta, f)$, etc. Theorem 4.1 then states that

$$C_q(\Theta, f) \approx C_q(\Theta, \psi), \tag{1.20}$$

if $\psi(x)$ is the p-form given by the p-th differential of f at Θ. This is a familiar result if E is finite-dimensional and if $p = 2$ in which case it is proved by bringing the symmetric quadratic form ψ to its diagonal form. In the general case the proof of Theorem 4.1 is based on the approximation Theorem 4.2 which gives sufficient conditions for an approximation ψ to f to satisfy (1.20) (this theorem seems to be new even in the finite dimensional case); it is then shown that these conditions are satisfied if ψ is the p-th differential of f.

The proof of the approximation theorem is given in Section 6 and is based on the econcept of a "cylindrical neighborhood" of an isolated critical point which was introduced by Seifert and Threlfall [20; Section 9] in the finite-dimensional case. Section 5 is devoted to its generalization to Hilbert space.

In Section 4 the approximation Theorem 4.2 is applied to a Taylor approximation ψ of f. In Section 7 it is applied to obtain a reduction to a finite dimensional space. Here, in addition to the assumptions of Section 4, we assume f to be of the special form

$$f(x) = (p)^{-1} \| x \|^p + F(x), \qquad p \geqslant 2 \text{ and even}, \tag{1.21}$$

where

$$G(x) = \operatorname{grad} F(x) \tag{1.22}$$

is completely continuous. (Scalars of this form were treated in [15], and Section 7 is closely related to some of the results of that paper but care has been taken that the present section may be read independently.) It will be shown (Theorems 7.1 and 7.2): There exists an n_0-dimensional linear subspace E^{n_0} of E such that to every linear subspace E^n of E containing E^{n_0}, there corresponds an approximation f_n to f of the following properties: If $\tilde{f}_n$ denotes the restriction of f_n to E^n then, grad $\tilde{f}_n \subset E^n$, and

$$C_q(\Theta, f) = C_q(\Theta, f_n) = C_q(\Theta, \tilde{f}_n). \tag{1.23}$$

From (1.23) it can be deduced that $C_q(\Theta, \tilde{f}_n)$ is finitely generated (Theorem 7.3).

In Section 8, we return to the global situation under the assumption that each critical point σ_j^i (defined in the line following (1.13)) satisfies the assumptions made in Section 7 concerning the critical point Θ, i.e. σ_j^i is nondegenerate of even order $p_{ij} \geqslant 2$, and in some neighborhood W_j^i of σ_j^i, f is of the form

$$f(x) = (p_{ij})^{-1} \| x - \sigma_j^i \|^{p_{ij}} + F_{ij}(x - \sigma_j^i), \tag{1.23}$$

where

$$G_{ij}(x - \sigma_j^i) = \operatorname{grad} F_{ij}(x - \sigma_j^i) \tag{1.24}$$

is completely continuous. The goal is to prove the Morse relations [10, p. 143]

$$\begin{aligned} M_0 &\geqslant R_0 \\ M_1 - M_0 &\geqslant R_1 - R_0 \\ M_k - M_{k-1} + \cdots + (-1)\, M_0 &\geqslant R_k - R_{k-1} + \cdots + (-1)^k R_0 \\ M_n - M_{n-1} + \cdots + (1)^n M_0 &= R_n - R_{n-1} + \cdots + (-1)^n R_0, \\ &\qquad \text{for } n = n_0 . \end{aligned} \tag{1.25}$$

Now the proof for these relations given in [12, Section 11] shows that it is sufficient to verify that certain groups are finitely generated. Under the additional assumption that the homology groups $H_q(V)$ are finitely generated, this verification is carried out in Section 8 by the use of Theorem 7.2.

The "principal parts" (1.23) cannot be given arbitrarily since they determine the M_k which must satisfy the relations (1.25). This author has not treated the problem whether these relations represent the only restriction on the principal parts. (The related question whether the inequalities (1.25) are the only relations between the M_q and R_q has in the finite-dimensional case an affirmative answer if all $p_{ij} = 2$ as was proved by F. John [5a] (see also [10, p. 145]).

The Appendix (Section 9) contains continuity proofs for the deformations used in the earlier sections.

2. The Critical Groups

As pointed out in the introduction, the main object of this section is to legitimize Definition 1.2.

Lemma 2.1. *$f(x)$ is bounded in V.*

Indeed, if x is an arbitrary point in V and if M is as in (1.2), one easily derives from (1.1) the inequality $|f(x)| \leqslant |f(x_0)| + MC \| x - x_0 \|$ for any x in the bounded set V.

LEMMA 2.2. *There are no critical points on $\dot{V} = \bar{V} - V$.*

This is obvious from Assumption E.

LEMMA 2.3. (i) *The set Γ of critical points of f in V is compact;* (ii) *The set Λ of critical levels is finite.*

Proof of (i). Let $x_1, x_2, \ldots,$ be a sequence of points in Γ. We have to prove the existence of a convergent subsequence. Since by Assumption B, f is not constant in any ball and since g is continuous, it is easily seen that Γ contains no ball. Using again the continuity of g, we see that there exists a sequence $y_1, y_2, \ldots,$ of points in V such that

$$g(y_n) \neq \Theta, \qquad \| g(y_n)\| = \| g(y_n) - g(x_n)\| < n^{-1}, \qquad \| x_n - y_n \| < n^{-1}. \tag{2.1}$$

Then if S denotes the set consisting of the elements of the sequence $\{y_n\}$, the closure $\bar{S}$ of S contains a critical point y_0 (Assumption C and Lemma 2.1), and there exists a sequence of integers n_i such that y_{n_i} converges to y_0. But then by (2.1), x_{n_i} converges also to y_0.

Proof of (ii). Suppose the assertion is not true. Then there exists a sequence $\{c_n\}$ of different critical levels, and a sequence $\{x_n\}$ of different critical points such that $c_n = f(x_n)$. By (i) there exists a sequence n_i of integers such that x_{n_i} converges. The limit point x_0 is critical, and, therefore, a point of V (Lemma 2.2). But $c_{n_i} = f(x_{n_i})$ converges to $f(x_0)$. Thus the critical level $f(x_0)$ is not isolated in contradiction to assumption D.

LEMMA 2.4. *For $\mu > 0$, let*

$$K_\mu = \{x \in V \mid \delta(x) \geqslant \mu\}, \tag{2.2}$$

where $\delta(x)$ denotes the distance of the point x from the critical set Γ. Then there exists a positive $m = m(\mu)$ such that

$$\| g(x)\| \geqslant m(\mu) \qquad \textit{for } x \in K_\mu . \tag{2.3}$$

Proof. If the assertion were wrong there would exist a sequence of points x_n in K_μ such that $\lim g(x_n) = \Theta$, and by Assumption C and Lemma 2.1, some point of Γ would be in the closure of the sequence $\{x_n\}$. This contradicts the fact that, by definition of K_μ, the distance of x to the set Γ is not smaller than the positive constant μ.

LEMMA 2.5. *Let P be a closed set of real numbers which has no point in common with the set Λ of critical levels. Let μ_0 be a positive number not greater* *than the distance between the sets P and Λ. (Such μ_0 exists by Lemma 2.3). Finally, let*

$$S_P = \{x \in V \mid f(x) \in P\}. \tag{2.4}$$

Then

$$S_P \subset K_\mu \tag{2.5}$$

for

$$0 < \mu \leqslant \mu_0(MC)^{-1}, \tag{2.6}$$

where M and C are as in (1.2) and (1.1) respectively.

Proof. Let $x_0 \in V$, but

$$x_0 \notin K_\mu \tag{2.7}$$

for some μ satisfying (2.6). We have to prove then that

$$x_0 \notin S_P\,. \tag{2.8}$$

By definition of K_μ, (2.7) implies that

$$\delta(x_0) < \mu \leqslant \mu_0(MC)^{-1}. \tag{2.10}$$

Since Γ is compact, there exists a $\gamma_0 \in \Gamma$ such that

$$\delta(x_0) = \| x_0 - \gamma_0 \|\,. \tag{2.11}$$

Now from (1.1) and (1.2) it is easily seen that

$$|f(x_0) - f(\gamma_0)| \leqslant MC\,\|(x_0) - \gamma_0\|\,.$$

From this, (2.10) and (2.11), the inequality

$$|f(x_0) - f(\gamma_0)| < \mu_0 \tag{2.12}$$

follows. But $\gamma_0 \in \Gamma$, and $f(\gamma_0) \in \Lambda$. Thus (2.12) implies that $f(x_0)$ has a distance less than μ_0 from Λ. Thus, by definition of μ_0, $f(x_0) \notin P$ which by (2.4) proves (2.8).

THEOREM 2.1. *In addition to the assumptions made previously, we assume g to be Lipschitz. Let $x(t) = x(t, x_0)$ be the solution of the problem*

$$\frac{dx}{dt} = -g(x), \qquad x(0, x_0) = x_0 \in V. \tag{2.13}$$

Then, $x(t, x_0) \in V$ for all positive t.

Proof. The assertion is obvious if x_0 is a critical point, for then $x(t) = x_0$ for all t.

Let $x_0 \in V$ be not critical and suppose the theorem to be wrong.

Then, there exists a positive t_1, such that $x_1 = x(t_1) \in \dot{V}$ while $x(t) \in V$ for $0 \leqslant t < t_1$, and from (2.13) and (1.5) we see that

$$\left\langle n(x_1), \left(\frac{dx}{dt}\right)_{t=t_1} \right\rangle = - \langle n(x_1)\, g(x_1) \rangle < 0.$$

To arrive at a contradiction and thereby prove our theorem, we will show that

$$\left\langle n_1, \left(\frac{dx}{dt}\right)_{t=t_1} \right\rangle \geqslant 0 \qquad \text{where} \qquad n_1 = n(x_1). \tag{2.14}$$

Now the tangent space T to $\dot{V}$ at x_1 is a linear subspace of E of codimension 1 [17, Theorem 3.1]. Consequently, every $z \in E$ has the unique representation

$$z = \sigma n_1 + \eta, \sigma \text{ real}, \qquad \eta \in T. \tag{2.15}$$

In particular,

$$\frac{x(t) - x(t_1)}{t - t_1} = \sigma(t)\, n_1 + \eta(t), \qquad 0 \leqslant t < t_1 . \tag{2.16}$$

Let us assume first that there exists a t_2 in the open interval $(0, t_1)$ such that $\sigma(t) \geqslant 0$ for $t_2 \leqslant t < t_1$. Since η and n_1 are orthogonal, we see from (2.16) that

$$\left\langle n_1, \frac{x(t) - x(t_1)}{t - t_1} \right\rangle = \sigma(t) \geqslant 0 \qquad \text{for } t_2 \leqslant t < t,$$

which obviously proves (2.14).

If the assumption just made is not true, then there exists a monotone increasing sequence of positive real numbers t_2, t_3,..., converging to t_1 and such that $\sigma(t_m) < 0$, or by (2.16),

$$\left\langle \frac{x(t_m) - x(t_1)}{t - t_1}, n_1 \right\rangle < 0. \tag{2.17}$$

We will show that

$$\lim_{m\to\infty} \left\langle \frac{x(t_m) - x(t_1)}{t_m - t_1}, n_1 \right\rangle = 0, \tag{2.18}$$

which implies (2.14).

Now let $\xi \in \dot{V}$ and, as before, $x_1 = x(t_1)$. Then from (2.15), with $z = \xi - x_1$,

$$\xi - x_1 = \sigma n_1 + \eta. \tag{2.19}$$

We recall the following facts [17, Theorem 3.2]: η is, for a small enough neighborhood of x_1, an admissible parameter for $\dot{V}$:

$$\xi = \xi(\eta), \tag{2.20}$$

with the additional properties

$$x_1 = \xi(\Theta), \qquad d\xi(\Theta, \eta) = \eta. \tag{2.21}$$

Moreover [17, pp. 370, 371], there exists a positive ϵ such that the "cylinder"

$$Z = \{z = \xi(\eta) + \lambda n_1 \mid \| \eta \| < \epsilon, \mid \lambda \mid < \epsilon\} \tag{2.22}$$

has the following properties:

(a) (2.20) and (2.21) hold in $\dot{V} \cap Z$ such that, in particular, by (2.21)

$$\xi(\eta) - x_1 = \eta + r(\eta) \qquad \text{with} \qquad \lim_{\eta \to \Theta} \frac{r(\eta)}{\| \eta \|} = \Theta \tag{2.23}$$

(b) the points of Z are interior or exterior with respect to V according to whether λ is negative or positive

(c) Z contains a spherical neighborhood N of x_1.

With the notation $x(t_m) = x_m$, we see from (2.15), with $z = x_m - x_1$, that

$$x_m - x_1 = \sigma_m n_1 + \eta_m, \sigma_m \text{ real}, \qquad \eta_m \in T.$$

Since $t_m \to t_1$, there exists an integer $m_0 \geqslant 2$ such that $x_m = x(t_m) \subset N \in Z$ for $m \geqslant m_0$. Therefore (see (2.22))

$$x_m = \xi(\eta_m) + \lambda_m n_1, \lambda_m \text{ real}, \qquad m \geqslant m_0.$$

Subtracting x_1 and taking the scalar product with n_1, we obtain

$$\langle x_m - x_1, n_1 \rangle = \langle \xi(\eta_m) - x_1, n_1 \rangle + \lambda_m.$$

From this equality, from (2.17) and (2.23), and by noting that $t_m - t_1 < 0$ for $m \geqslant m_0$, and that η_m and n_1, are orthogonal we see that

$$\begin{aligned} 0 < \langle x_m - x_1, n_1 \rangle &= \langle \eta_m + r(\eta_m), n_1 \rangle + \lambda_m \\ &= \langle r(\eta_m), n_1 \rangle + \lambda_m. \end{aligned} \tag{2.24}$$

But by assumption, $x_m = x(t_m) \in V$ for $m \geqslant 2$. Therefore, $\lambda_m < 0$, by property (b) above, and we see from (2.24) that

$$0 < \langle x_m - x_1 , n_1 \rangle < \langle r(\eta_m), n_1 \rangle .$$

Thus for the proof of (2.18) it will be sufficient to show that $\| r(\eta_m)/(t_m - t_1)\|$ converges to zero. For this again, it will, on account of the second part of (2.23), be sufficient to prove that

$$\frac{\| \eta_m \|}{| t_m - t_1 |} = \frac{\| \eta_m \|}{\| x_m - x_1 \|} \cdot \frac{\| x_m - x_1 \|}{| t_m - t_1 |}$$

is bounded in m. But the first factor at the right is $\leqslant 1$ since η_m is the projection of $x_m - x_1$ on the tangent space T while the second factor approaches

$$\left(\frac{dx}{dt}\right)_{t=t_1} = - g(x_1)$$

as $m \to \infty$.

THEOREM 2.2. *Let $a < b$, and suppose that the closed interval $[a, b]$ contains no critical level. Then*

$$H_q(\bar{f}_b , \bar{f}_a) = 0 \tag{2.25}$$

(*for the notation used, see the lines following the statement of Assumption* E *in the Introduction*).

Proof. If $\bar{f}_b = \phi$, the empty set, then $\bar{f}_b = \bar{f}_a$ since $\bar{f}_a \subset \bar{f}_b$, and (2.25) is true (see [3; I, Lemma 8.1]).

Let now $\bar{f}_b \neq \phi$, and $x_0 \in \bar{f}_b$. Then by Theorem 2.1, $x(t, x_0)$ stays in the domain V of f and g for all positive t. Moreover, from Lemma 2.5 (with $P = [a, b]$) and Lemma 2.4 follows the existence of a constant m such that

$$\| g(x)\| \geqslant m > 0, \tag{2.26}$$

if

$$a \leqslant f(x) \leqslant b. \tag{2.27}$$

These facts imply that the deformation Δ defined by

$$\Delta(x_0 , \tau, a, b) = \Delta(x_0 , \tau) = x\left(\frac{\tau(b - a)}{m^2}, x_0\right), \qquad 0 \leqslant \tau \leqslant 1, \tag{2.28}$$

deforms in the sense of Definition 1.3 the couple $(\bar{f}_b , \bar{f}_a)$ into the couple $(\bar{f}_a , \bar{f}_a)$. For the proof of this assertion we refer to [16; pp. 242, 243; in particular, Lemma 4.12]. This proves the theorem by [16, Corollary to Lemmas 2.1 and 2.3].

THEOREM 2.3. *Let $b \geqslant \beta > c > \alpha \geqslant a$, and let c be the only critical value in $[a, b]$. Then*

$$H_q(\bar{f}_b, \bar{f}_a) \approx H_q(\bar{f}_\beta, \bar{f}_\alpha) \tag{2.29}$$

$$(\bar{f}_b, \bar{f}_a) \sim (\bar{f}_\beta, \bar{f}_\alpha), \tag{2.30}$$

where the symbols $\approx$ and $\sim$ denote isomorphism and homotopy equivalence [3, I, 11.1 or 16, Definition 2.3, resp.].

Proof. (2.30) implies (2.29) [3, I, 11.2]. To prove (2.30), we will show that

$$(\bar{f}_\beta, \bar{f}_\alpha) \sim (\bar{f}_\beta, \bar{f}_a) \sim (\bar{f}_b, \bar{f}_a). \tag{2.31}$$

Now from the properties referred to in the proof of Theorem 2.2 it is easily seen that $\Delta(x_0, \tau, a, \alpha))$ (defined by (2.28)) deforms the first couple in (2.31) into the second. This proves the equivalence of these couples by [16, Lemma 2.1]. The equivalence between the second and third couples in (2.31) is shown correspondingly.

Obviously, Theorem 2.2 makes Definition 1.2 legitimate.

We conclude this section with applications of the previous results and some simple examples.

THEOREM 2.4. *There exists at least one critical point in V.*

Proof. Let us assume that the assertion is wrong. We claim first that under this assumption there exists a constant m such that (2.26) is true for all $x \in V$. Indeed, otherwise there would, by assumption C, exist a critical point $\bar{x}$ in the closure $\bar{V}$ of V. But then by Lemma 2.2, $\bar{x} \in V$ in contradiction to the assumption made in the first line of this proof.

Thus (2.26) is true in V for some m. If $x(t)$ is as in Theorem 2.1, we see from (2.13) that $df(x(t))/dt = -\|g(x)\|^2 \leqslant -m^2$. Therefore, $f(x(t)) \to -\infty$ as $t \to +\infty$. This contradicts Lemma 2.1, since $x(t) \in V$ for all positive t.

THEOREM 2.5. (i) *There exists an $\bar{x} \in \bar{V}$ such that*

$$f(\bar{x}) \leqslant f(x) \qquad \text{for all } x \in \bar{V}. \tag{2.32}$$

(ii) *if $\bar{x}$ satisfies* (i), *then $\bar{x} \in V$.*

(iii) *if $\bar{x}$ satisfies* (i) *then $\bar{c} = f(\bar{x})$ is a critical level, and every x satisfying $\bar{c} = f(x)$ is a critical point.*

Proof of (i). By Lemma 2.1, f has in V a finite greatest lower bound a. Under the assumption that our assertion is wrong, a is not in the range of f and, therefore, not a critical value. On the other hand, there exists a smallest critical value c (Theorem 2.4 and Lemma 2.3). Then $a < c$. If b is such that $a < b < c$, there exists an $x_0 \in V$ such that

$$a < f(x_0) < b < c. \tag{2.33}$$

Now the interval $[a, b]$ has from the set Λ of critical levels the positive distance $c - b$. Therefore, there exists a positive m such that (2.27) implies (2.26). Now, again, if $x(t, x_0)$ denotes the solution of the problem (2.13), then by Theorem 2.1 and by the definition of a

$$a < f(x(t, x_0)) \qquad \text{for } t \geqslant 0. \tag{2.34}$$

The inequalities (2.33), (2.34) together with the fact that $f(x(t, x_0))$ is decreasing in t imply that (2.27) holds with $x = x(t, x_0)$ for all positive t. But then (2.26) holds with $x = x(t, x_0)$ for all positive t, and from this we conclude as in the last paragraph of the proof of Theorem 2.4 that $f(x(t, x_0) \to -\infty$ as $t \to +\infty$. But this contradicts (2.34).

Proof of (ii). Let us assume there exists an $\bar{x} \in \dot{V}$ satisfying (2.32). Then for σ positive and small enough, $\bar{x} - \sigma n(\bar{x})$ is an interior point of V since $-n(\bar{x})$ is the interior unit normal to $\dot{V}$ at $\bar{x}$ [16, Definition 4.1]. Therefore, we see from (2.31) that $f(\bar{x} - \sigma n(\bar{x})) - f(\bar{x}) \geqslant 0$, which implies

$$\left[\frac{d}{d\sigma} f(\bar{x} - \sigma n(\bar{x}))\right]_{\sigma=0} \geqslant 0. \tag{2.35}$$

On the other hand, the left member of (2.35) equals $- \langle(g(\bar{x}), n(\bar{x})\rangle$ and is, therefore, negative by (1.5) in contradiction with (2.35).

This finishes the proof of (ii). Assertion (iii) is an obvious corollary to assertions (i) and (ii).

EXAMPLES. Let E be the Hilbert space of points $x = (x_1, x_2, \ldots)$ with $\sum x_i^2 < \infty$ and with $\langle x, y\rangle = \sum x_i y_i$, and let $f(x) = (2)^{-1} \sum x_i^2(1 + \lambda_i)$, where $\lambda_i \downarrow 0$.

If $V = V_1$ is the open unit ball it is easily verified that all assumptions of Theorem 2.5 are satisfied, and that $\bar{x} = \Theta \in V$ is the only minimum and the only critical point of f in $\bar{V}$. We note that the point $(1, 0, 0, \ldots) \in \dot{V}$ is the only maximum point in $\bar{V}$. This shows that Theorem 2.5 becomes wrong if the minimum condition (2.32) is replaced by the corresponding maximum condition.

If $V = V_2$ is the ring: $1 < \| x \| < 2$, then it is easily verified that Assumption E is not satisfied at the points of the unit sphere $S_1 \subset \dot{V}_2$, and that f takes no minimum in $\bar{V}_2$.

V_1 is convex, V_2 is not. But the validity of the assumptions of Theorem 2.5 is not restricted to convex domains as is easily verified in the following example:

$$f(x) = (2^{-1}(\| x \| - 1)^2, \qquad V = V_3 = \{x \mid 2^{-1} < \| x \| < 2\}.$$

Here $g(x) = x(\| x \| - 1)/\| x \|$. The critical and minimum points are the points of the unit sphere $S_1 \subset V_3$.

3. On the Relations (1.8) and (1.9)

Definition 3.1. Let s_q be a singular q-simplex in E, i.e. a map $s_q : \Delta_q \to E$, where Δ_q denotes the closed Euclidean q-simplex whose vertices are the unit points on the axes of the $(q + 1)$-dimensional Euclidean space E^{q+1} [3, VII, 2.2]. Then the support $| s_q |$ of s_q is the point set $s_q(\Delta_q) \subset E$. If, for $j = 1, 2,..., i$, $s_q{}^j$ are singular q-simplices in E, and if g^j are nonzero elements of the coefficient group, then the support $| c_q |$ of the singular q-chain $c_q = \sum_{j=1}^{i} s_q{}^j g^j$ is the set $\bigcup_{j=1}^{i} | s_q{}^j | \subset E$. The support of the zero q-chain is the empty set ϕ. For any set S, "$c_q \subset S$" means that $| c_q | \subset S$.

Lemma 3.1. (a) $| c_q |$ *is compact*, (b) *a real-valued continuous function defined on* $| c_q |$ *takes there a maximum and a minimum.*

Indeed, (a) is obvious since Δ_q is compact, each $S_q{}^j$ is continuous, and the union defining $| c_q |$ is finite. (b) follows from (a).

Definition 3.2. Let V and f be as in the previous sections. Then, for any singular q-chain c_q, whose support is in V, we set

$$\mu(c_q) = \max_{x \in |c_q|} f(x). \tag{3.1}$$

Moreover, if $\hat{z}_q$ is a nonzero element of $H_q(V)$, i.e., a homology class of nonbounding singular cycles $z_q \subset V$, then

$$\mu(\hat{z}_q) = \inf_{z_q \in \hat{z}_q} \mu(z_q) \tag{3.2}$$

(Cf. [20, p. 24]).

It follows from Lemma 2.1 that $\mu(\hat{z}_q)$ is finite. Obviously,

$$\mu(z_q) \geqslant \mu(\hat{z}_q) \qquad \text{for all } z_q \in \hat{z}_q, \tag{3.3}$$

and for $\epsilon > 0$

$$\mu(z_q) < \mu(\hat{z}_q) + \epsilon \qquad \text{for some } z_q \in \hat{z}_q . \tag{3.4}$$

LEMMA 3.2. *$\mu(\hat{z}_q)$ is a critical value.*

Proof. Let $c_1 < c_2 < \cdots c_N$ be the critical values of f (cf. Lemma 2.3). We will first show that

$$\mu(\hat{z}_q) \geqslant c_1 . \tag{3.5}$$

Suppose this inequality is not true. Then, by (3.3) and (3.4), there exists a $z_q \in \hat{z}_q$ such that

$$\mu(\hat{z}_q) \leqslant \mu(z_q) < c_1 . \tag{3.6}$$

Now $\mu(z_q) = f(x_0)$ for some $x_0 \in | z_q | \subset V$ by Lemma 3.1. But

$$c_1 = \min_{x \in V} f(x)$$

by Theorem 2.5. Thus $\mu(z_q) = f(x_0) \geqslant c_1$ which contradicts (3.6).

From (3.5) thus proved we see: If our lemma is not true, then $c_\zeta < \mu(\hat{z}_q) < c_{\zeta+1}$, for some positive integer ζ not greater than N provided we choose for c_{N+1} a number greater than $\max(c_N , \mu(\hat{z}_q))$. Let now a, b be such that $c_\zeta < a < \mu(\hat{z}_q) < b < c_{\zeta+1}$. Then by (3.3) and (3.4), there exists a $z_q \in \hat{z}_q$ such that

$$c_\zeta < a < \mu(\hat{z}_q) \leqslant \mu(z_q) < b < c_{\zeta+1} . \tag{3.7}$$

Since $[a, b]$ contains no critical value, the deformation (2.28) deforms $\bar{f}_b$ into $\bar{f}_a$. In particular, the cycle z_q is deformed into a cycle $z_q' \subset \bar{f}_a$. Then $\mu(z_q') \leqslant a$, which, by (3.7), implies $\mu(z_q') < \mu(\hat{z}_q)$. This contradicts (3.3), since $z_q' \in \hat{z}_q$.

DEFINITION 3.3. Let $\hat{z}_q$ be a nonzero element of $H_q(V)$. A cycle $z_q \in \hat{z}_q$, for which $\mu(z_q) = \mu(\hat{z}_q)$, is called a minimal cycle for $\hat{z}_q$ [20, p. 24].

LEMMA 3.3. *Provided that the Assumptions* A-F *stated in the Introduction are satisfied, the homology class $\hat{z}_q$ of the preceding definition contains a minimal cycle.*

Proof. By Lemma 3.2, $\mu(\hat{z}_q)$ is a critical value. Therefore, by Lemma 2.3, there exists a $b' > \mu(\hat{z}_q)$ such that $\mu(\hat{z}_q)$ is the only critical value in $[\mu(\hat{z}_q), b']$. Then by (3.3) and (3.4), there exists a cycle $z_q \in \hat{z}_q$ such that $\mu(\hat{z}_q) \leqslant \mu(z_q) < b'$. If the equality holds here, the lemma is true. If the inequality holds, let δ be a deformation with the property postulated in assumption F with $c = \mu(\hat{z}_q)$ and $b = \mu(z_q)$. Then the support of the cycle z_q' obtained from z_q by the deformation δ lies in $\bar{f}_c$, i.e., $\mu(z_q') \leqslant c = \mu(\hat{z}_q)$. Here the equality sign must hold, since $z_q' \in \hat{z}_q$. Thus z_q' is a minimal cycle.

THEOREM 3.1. *Under the assumptions of Lemma 3.3, the relation* (1.8) *is true.*

Proof. The theorem is a consequence of Lemma 3.3 since as proved by Seifert and Threlfall in [20, p. 25, Satz 2], the existence of a minimal cycle implies (1.8).

Remark. Seifert and Threlfall work with singular chains as defined by them [20, Section 2] and with integers modulo 2 as coefficients. However, their proof, referred to above, remains valid if, as in the present paper, singular chains as defined in [3, VII] with coefficient groups over a principal ideal domain are used. Let us mention that in other respects their proof covers a more general situation than the present one: E is a neighborhood space and f is continuous.

Our next goal is to find sufficient conditions for the validity of (1.9).

DEFINITION 3.4. We say that Assumption F is strictly satisfied if, among the deformations whose existence is required by Assumption F, there is one leaving every point of $\bar{f}_c$ fixed during the deformation. (In other words, $\bar{f}_c$ is a deformation retract of $\bar{f}_b$).

LEMMA 3.4. *Let Assumptions* A-E *be satisfied, and suppose that the closed interval* $[a, b]$ *contains no critical value. Then* $\bar{f}_a$ *is a deformation retract of* $\bar{f}_b$.

Proof. For $x_0 \in \bar{f}_b - \bar{f}_a$, $0 \leqslant t \leqslant 1$, let $x(t, x_0)$ be the solution of the problem

$$\frac{dx}{dt} = -(f(x_0) - a)\frac{g(x)}{\|g(x)\|^2}, \qquad x(0, x_0) = x_0. \tag{3.8}$$

We then set, for $0 \leqslant t \leqslant 1$,

$$\delta(x_0, t) = \begin{cases} x(t, x_0) & \text{for } x \in \bar{f}_b - \bar{f}_a \\ x_0 & \text{for } x_0 \in \bar{f}_a. \end{cases} \tag{3.9}$$

Noting that (2.26) holds if (2.27) is true, and that

$$\frac{df(x(t))}{dt} = \left\langle g(x(t)), \frac{dx}{dt} \right\rangle = -(f(x_0) - a), \tag{3.10}$$

and that, therefore,

$$f(x(t)) = f(x_0) - (f(x_0) - a)\, t, \tag{3.11}$$

one easily verifies that $\delta(x_0 t)$ is the required deformation retraction (cf. [16, Section 5]).

LEMMA 3.5. *Let Assumptions* A-F *be satisfied, the last one strictly. Let θ_q be the zero element of $H_q(V)$, and let z_q be a bounding cycle in V, i.e., $z_q \in \theta_q$. Then there exists a number $\gamma = \gamma(z_q)$ such that*

(i) *z_q bounds on $\bar{f}_\gamma$, i.e., there exists a chain $c_{q+1} \subset \bar{f}_\gamma$ with $\dot{c}_{q+1} = z_q$.*

(ii) *z_q does not bound on f_γ.*

Proof. We set

$$\gamma = \inf_{\dot{c}_{q+1} = z_q} \max_{x \in |c_{q+1}|} f(x) \tag{3.12}$$

and claim that γ has the required properties. Since

$$\mu(z_q) = \max_{x \in |z_q|} f(x) \leqslant \max_{x \in |c_{q+1}|} f(x)$$

for every c_{q+1} whose boundary is z_q we see that

$$\mu(z_q) \leqslant \gamma. \tag{3.13}$$

Now since there are only a finite number of critical values (Lemma 2.3), there exists a $b > \gamma$ such that the half open interval $(\gamma, b]$ contains no critical value (whether γ is a critical value or not), and by definition of γ, there exists an α in this interval and a c_{q+1} such that

$$\max_{x \in |c_{q+1}|} f(x) \leqslant \alpha \qquad \text{and} \qquad \dot{c}_{q+1} = z_q .$$

Now there exists a deformation which retracts $\bar{f}_b$ onto $\bar{f}_\gamma$, by Lemma 3.4 if γ is not a critical value, and, by assumption (cf. Definition 3.4), if γ is a critical value. Then, if c'_{q+1} denotes the deformed c_{q+1}, we have $\dot{c}'_{q+1} = z_q$ since, due to (3.13), z_q remains pointwise fixed under the deformation. This proves assertion (i). Assertion (ii) follows directly from the Definition (3.12) of γ.

DEFINITION 3.5. Let $\hat{z}_q \in H_q(\bar{f}_b, f_b)$, and $z_q \in \hat{z}_q$. Then z_q is called a relative q-cycle on $\bar{f}_b$ modulo f_b; such z_q is said to be extendable below b (to an absolute cycle) if there exists a q-chain $c_q \subset f_b$ such that $z_q + c_q$ is a cycle [20, p. 26].

If b is not a critical value and if Assumptions A-E hold, then it is easily seen from Lemma 3.4 that every relative cycle on $\bar{f}_b$ modulo f_b is extendable below b.

THEOREM 3.2. *Let Assumptions* A-F *be satisfied, the last one strictly. Suppose that for every critical value c any relative cycle on $\bar{f}_c$ modulo f_c is extendable below c. Then* (1.9) *holds.*

Proof. By Lemma 3.3 every nonzero homology class contains a minimal cycle; moreover, to every bounding cycle there corresponds a number γ of the two properties stated in Lemma 3.5. But these two properties together with the condition stated in the second sentence of the theorem imply (1.9) as was proved by Seifert and Threlfall in [20, pp. 26, 27]. (Cf. the remark following the proof of Theorem 3.1.)

We now give a sufficient condition for Assumption F to be satisfied strictly.

THEOREM 3.3. *Suppose that Assumptions* A-E *are satisfied and that the set Γ of critical points is finite. Then Assumption* F *is strictly satisfied.*

Proof. Let $b > c$, and let c be the only critical level in $[c, b]$. We have to define a deformation $\delta(x_0, t)$ retracting $\bar{f}_b$ onto $\bar{f}_c$. This is done by using the $x(t, x_0)$ defined in the proof of Lemma 3.4 (with a replaced by c in (3.8)) with the difference that in the present case $x(t, x_0)$ may not be defined for $t = 1$ since $\| g(x)\| = 0$ for some points x on the level c. However, $f(x(t, x_0))$ is decreasing for $0 \leqslant t < 1$, and $\lim_{t\to 1^-} f(t, x_0)) = c$ by [16, Lemma 5.3]. Moreover, on account of the assumed finiteness of Γ, $\lim_{t\to 1^-} x(t, x_0)$ exists [16, Theorem 5.1]. Due to these properties, we may define

$$\delta(x_0, t) = \begin{cases} x(t, x_0) & \text{if} \quad x_0 \in \bar{f}_b - \bar{f}_c \quad 0 \leqslant t < 1 \\ \lim\limits_{t\to 1^-} x(t, x_0) & \text{if} \quad x_0 \in \bar{f}_b - \bar{f}_c, \quad t = 1 \\ x_0 & \text{if} \quad x_0 \in \bar{f}_c \quad 0 \leqslant t \leqslant 1, \end{cases} \tag{3.14}$$

and δ is seen to be a deformation retracting $\bar{f}_b$ onto $\bar{f}_c$ (cf. the proof of Lemma 5.4 in [16]). For the continuity of $\delta(x_0, t)$ see Lemma 9.2.

In establishing Definition 1.2 of the critical group $C_q(c)$ at level c we used Assumptions A-E. By the use of the additional Assumption F it is possible to write $C_q(c)$ in a new form useful for later purposes and also to define a critical group at an isolated critical point. The rest of this section is devoted to this task. The technique used is essentially the same as the one employed in [12] for the finite dimensional case (cf. also [18]).

THEOREM 3.4. *Suppose Assumptions* A-F *to be satisfied, the last one strictly.*

Let c be a critical value and let $\sigma = \sigma(c)$ denote the set of critical points at level c. Then,

$$C_q(c) \approx H_q(\bar{f}_c, f_c - \sigma).$$

Proof. Let a, b be such that $a < c < b$ and such that c is the only critical level in $[a, b]$. We have to prove

$$H_q(\bar{f}_c, f_c - \sigma) \approx H_q(\bar{f}_b, \bar{f}_a). \tag{3.15}$$

Now by assumption, there exists a deformation which retracts $\bar{f}_b$ onto $\bar{f}_c$. Since $\bar{f}_a \subset \bar{f}_c$ such deformation retracts the couple $(\bar{f}_b, \bar{f}_a)$ onto the couple $(\bar{f}_c, \bar{f}_a)$. Consequently, (see, e.g., [16, corollary to Lemmas 2.1 and 2.2]) $H_q(\bar{f}_b, \bar{f}_a) \approx H_q(\bar{f}_c, \bar{f}_a)$, and it remains to prove

$$H_q(\bar{f}_c, f_c - \sigma) \approx H_q(\bar{f}_c, \bar{f}_a). \tag{3.16}$$

Now if c is a minimum value then $\bar{f}_a$ is empty and the right member of (3.16) equals $H_q(\bar{f}_c, \phi) = H_q(\bar{f}_c)$. But, by Theorem 2.5, all points at the minimum level c are critical. Therefore, $\bar{f}_c - \sigma$ is empty, and the left member of (3.16) also equals $H_q(\bar{f}_c)$.

Let then the critical value c be not a minimum value for f. Then, by Theorem 2.5 there exists a minimal critical value $c' < c$. By definition of a, we see that $c' < a < c$. Thus $\bar{f}_a \supset \bar{f}_{c'}$, which shows that $\bar{f}_a$ is not empty. We now consider the triple $\bar{f}_c \supset f_c - \sigma \supset \bar{f}_a$. Since $\bar{f}_a$ is not empty we conclude from [3, I Theorem 10.4] that (3.16) is a consequence of

$$H_q(\bar{f}_c - \sigma, \bar{f}_a) \approx H_q(\bar{f}_a, \bar{f}_a). \tag{3.17}$$

To prove (3.17) we note that $g(x_0) \neq \Theta$ for $x_0 \in (\bar{f}_c - \sigma) - \bar{f}_a$. Thus one sees easily that the deformation obtained by replacing in (3.9), $\bar{f}_b$ by $\bar{f}_c - \sigma$ retracts the couple $(\bar{f}_c - \sigma, \bar{f}_a)$ into the couple $(\bar{f}_a, \bar{f}_a)$. This proves (3.17).

Remark to Theorem 3.4. In a similar way, it may be proved that

$$C_q(c) \approx H_q(f_c \cup \sigma, f_c).$$

THEOREM 3.5. *With the notations and assumptions of Theorem 3.4, let W be a neighborhood of σ whose closure contains no other critical points than those of σ. (The existence of such a W follows easily from Lemma 2.3). Then*

$$C_q(c) \approx H_q(\bar{f}_c \cap W, \bar{f}_c \cap W - \sigma). \tag{3.18}$$

Proof. This follows from Theorem 3.4 and the excision-theorem [3, VII, 9.1] since the couple at the right member of (3.18) is obtained from the couple $(\bar{f}_c, \bar{f}_c - \sigma)$ by excising $\bar{f}_c - W$.

Corollary to Theorem 3.5. *With the assumptions and notations of Theorem 3.5 we assume that* $\sigma = \bigcup_1^r \sigma_j$, *where the* σ_j *are closed and disjoint. Let* W_j *be a neighborhood of* σ_j *such that the closures* $\bar{W}_j$ *are disjoint and such that the closure of the neighborhood* $W = \bigcup_1^r W_j$ *of* σ *contains no critical points not belonging to* σ. *Then* $C_q(c)$ *is isomorphic to the direct sum of the groups* $H_q(\bar{f}_c \cap W_j, \bar{f}_c \cap W_j - \sigma))$. *Indeed, this direct sum is isomorphic to the group at the right member of* (3.18) *by the "direct sum theorem"* [3, I, 13.2]. *It is easy to see that the groups* $H_q(\bar{f}_c \cap W_j, \bar{f}_c \cap W_j - \sigma_j)$ *are independent of the particular choice of* W_j.

With the notations used in (1.7) and (1.12) let σ^i, σ_j^i, W_j^i, W^i ($i = 1, 2, \dots N$; $j = 1, 2, \dots r_i$) be defined with respect to c_i as σ, σ_j, W_j, W were just defined with respect to c. Moreover, let

$$M_q(\sigma_j^i) = \text{rank of } H_q(\bar{f}_c \cap W_j^i, \bar{f}_c \cap W_j^i - \sigma_j^i). \tag{3.19}$$

Then, by the above corollary

$$M_q^i = \sum_{j=1}^{r_i} M_q(\sigma_j^i). \tag{3.20}$$

Definition 3.6. If the set Γ of critical points is finite then the component σ_j^i of Γ introduced above will always denote a set consisting of a single point. In this case, we call the group whose rank is the right member of (3.19) the q-th critical group of f at the critical point σ_j^i and denote it by $C_q(\sigma_j^i)$; its rank (3.19) is called the q-th type number at σ_j^i. If there is need to emphasize that we deal with the function f, we write $C_q(\sigma_j^i; f)$, $M_q(\sigma_j^i; f)$ etc.

4. Non Degenerate Critical Points of Order p

For $p \geqslant 2$ let E_p denote the product of E p-times by itself. Let $Q(h_1, h_2, \dots, h_p)$ be a map of E_p into the reals which is linear and continuous in each h_j. Then, as is well known,

$$| Q(h_1, h_2, \dots, h_p) \leqslant C \, | h_1 | \, | h_2 | \cdots | h_p | \tag{4.1}$$

for some constant C. If, in addition, Q is symmetric, it is called a p form. In this case, we set

$$Q(h) = Q(h_1, h_2, \dots, h_p) \quad \text{if} \quad h_1 = h_2, \dots, h_p = h \tag{4.2}$$

and

$$Q(h; k) = Q(h_1, h_2, \dots, h_{p-1}, k) \quad \text{if} \quad h_1 = h_2 = \cdots h_{p-1} = h. \tag{4.3}$$

The differential $dQ(h; k)$ of $Q(h)$ with "increment k" exists and is given by

$$dQ(h; k) = pQ(h; k). \tag{4.4}$$

(This is easily seen from the binomial formula for $Q(x + h)$; see, e.g., [9, Section 40 (3)]). Since, by definition of the gradient,

$$dQ(h; k) = \langle \operatorname{grad} Q(h), k \rangle,$$

we see from (4.4) that grad Q is homogenous of order $p - 1$.

DEFINITION 4.1. The p-form Q is called nondegenerate if there exists a positive constant m such that

$$\| \operatorname{grad} Q(h) \| \geqslant m \| h \|^{p-1}. \tag{4.5}$$

We recall some facts about differentials. Let $N = N(x_0)$ be a convex neighborhood of the point $x_0 \in E$, and let ψ be a map of N into a Hilbert space F. For the definition of the j-th differential $d^j\psi(x_0; h_1, ..., h_j)$ of ψ at x_0, we refer to the literature (see, e.g., [1; VIII.12] or [4]). We note that continuity of the differential at x_0 means that to each positive ϵ there corresponds a δ such that

$$\begin{aligned} &\| d^j\psi(x; h_1, ... h_j) - d^j\psi(x_0; h_1, ... h_j) \| \\ &\quad < \epsilon \| h_1 \| \cdots \| h_j \| \qquad \text{for } \| x - x_0 \| < \delta. \end{aligned} \tag{4.6}$$

We set $d^0\psi(x) = \psi(x)$, and for $j = 0, 1, ...$, we write $\psi \in C^j(x_0)$ to indicate that $d^j\psi(x; h_1, ..., h_j)$ exists and is continuous for all x in some neighborhood of x_0. For every set $S \subset E$, we write $\psi \in C^j(S)$ to indicate that $\psi \in C^j(x_0)$ for every $x_0 \in S$. Since the j-th differential is linear and continuous in each h_i, it follows from (4.1) and (4.6) that for a $\psi \in C^j(x_0)$ there exists a neighborhood N of x_0 and a positive constant K such that

$$\| d^j\psi(x; h_1, ..., h_j) \| < K \| h_1 \| \cdots \| h_j \| \qquad \text{for } x \in N(x_0). \tag{4.7}$$

We recall further that if $\psi \in C^j(N)$ for some open convex neighborhood N of x_0, then $d^j\psi(x; h_1, ..., h_j)$ is symmetric in $h_1, ..., h_j$ for each $x \in N$. Thus for such x, $d^j\psi(x; h)$ (see (4.2)) is a p-form in h. From (4.7) we see that for some $N(x_0)$ and some positive K,

$$\| d^p\psi(x; h) \| < K \| h \|^p \qquad \text{for } x \in N(x_0). \tag{4.8}$$

LEMMA 4.1. *Let $\psi = f$ be a real-valued function in $C^{p+1}(x_0)$ $(p \geqslant 0)$. Then $g = \operatorname{grad} f \in C^p(x_0)$ and*

$$d^jf(x; h_1, ..., h_j) = \langle d^{j-1}g(x; h_1, ..., h_{j-1}), h_j \rangle \qquad \textit{for } j = 1, 2, ..., p + 1. \tag{4.9}$$

We omit the proof since there is no difficulty in extending (by induction) the proof given in [17, Lemma 2.2] for the case $p = 1$ to arbitrary p.

LEMMA 4.2. *Let f be as in the preceding lemma with $x_0 = \theta$, and let $g = \operatorname{grad} f$. We suppose that*

$$d^j f(\Theta; h) = 0 \qquad \text{for } j = 1, \dots, p-1; p \geqslant 2. \tag{4.10}$$

Then there exists a constant K and a neighborhood N of x_0 such that for $x \in N$

$$|f(x) - f(\Theta)| < K \| x \|^p \tag{4.11}$$

and

$$\| g(x) \| \leqslant K \| x \|^{p-1}. \tag{4.12}$$

Proof. By Taylor's theorem [1, VIII, 14; 5] we see from (4.10) that,

$$f(x) - f(\Theta) = \int_0^1 d^p f(tx; x) \frac{(1-t)^{p-1}}{(p-1)!} dt.$$

This together with (4.8) proves (4.11).

To prove (4.12) we note that by Lemma 4.1, $g \in C^p(\Theta)$, and that by (4.9) and (4.10)

$$d^j g(\Theta; h) = \Theta \qquad \text{for } j = 0, 1, \dots, p-2. \tag{4.13}$$

Therefore, (4.12) follows again from Taylor's theorem (applied to g) and (4.8) (with $\psi = g$ and with p replaced by $p - 1$).

DEFINITION 4.2 (cf. [14, Definition 2.6]). The real-valued function f is said to have Θ as nondegenerate critical point of order $p \geqslant 2$ if $f \in C^{p+1}(\Theta)$, if (4.10) is satisfied, and if the p-form $d^p f(\theta; h)$ is nondegenerate in the sense of Definition 4.1.

LEMMA 4.3. *Θ is a nondegenerate critical point of $f \in C^{p+1}(\Theta)$ if and only if there exists a neighborhood $N(\theta)$ of θ and two positive constants k and K such that*

$$k \| x \|^{p-1} < \| g(x) \| < K \| x \|^{p-1} \qquad \text{for } x \in N(\Theta). \tag{4.14}$$

Proof. Let the symbols d_2 and grad_2 written in front of $d^p f(x; h)$ denote differential and gradient operations, resp., operating on the second variable h, while, as before, d and grad refer to the first variable x. With this notation,

$$d_2 d^p f(\Theta; h_1, \dots, h_p; k) = p d^p f(\Theta; h_1, \dots, h_{p-1}; k)$$

if $h_1 = h_2, \ldots, h_p = h$ (cf. (4.4) and (4.3)). From this and (4.9), we obtain

$$d_2 d^p f(\Theta; h_1, \ldots, h_p; k) = \langle p d^{p-1} g(\Theta; h_1, \ldots, h_{p-1}), k \rangle \quad \text{for } h_1 = \cdots = h_p = h;$$

and from this we conclude that

$$\operatorname{grad}_2 d^p f(\Theta; h) = p d^{p-1} g(\Theta; h). \tag{4.15}$$

This equality together with Definitions 4.1 and 4.2 shows that a necessary and sufficient condition for Θ to be a nondegenerate critical point of order p is that, in addition to (4.10),

$$d^{p-1} g(\Theta; h) \geqslant m_1 \| h \|^{p-1} \tag{4.16}$$

for some positive constant m_1.

Suppose now that Θ is nondegenerate of order p, and let us prove (4.14). Then, since (4.10) holds by assumption and since, as shown in the proof of the preceding lemma, (4.10) implies (4.13), we see from Taylor's theorem that

$$g(x) = \frac{d^{p-1} g(\Theta; x)}{(p-1)!} + \int_0^1 d^p g(tx; x) \frac{(1-t)^{p-1}}{(p-1)!} \, dt. \tag{4.17}$$

From this, from (4.8) (with $\psi = g$), and from (5.16) (with h replaced by x), we see that in some neighborhood of Θ,

$$\| g(x) \| \geqslant \| x \|^{p-1} \frac{(m_1 - K \| x \|)}{(p-1)!},$$

which obviously proves the assertion pertaining to the left part of (4.14). The right part of (4.14) follows from (4.12).

Conversely, suppose (4.14) to be true. We have to prove (4.10) and (4.16). For the proof of (4.10), it will, on account of (4.9), be sufficient to prove (4.13). Suppose, (4.13) not to be true, and let j_0 be the smallest nonnegative integer $\leqslant p - 2$ such that the j_0-th differential of g at Θ is not zero. Then the equation which is obtained from (4.17) by replacing $p - 1$ by j_0 holds. From that equation, from the left part of (4.14), and from the fact that $j_0 \leqslant p - 2$, one easily obtains an inequality of the form

$$\| d^{j_0} g(\Theta; x) \| \leqslant \text{const} \, \| x \|^{j_0+1} \qquad \text{for } \| x \|$$

small enough which is in contradiction with $d^{j_0} g(\Theta; x)$ 'being homogeneous of degree j_0.

From (4.13) thus proved we see that (4.17) holds, and from this equation, from (4.14), and from (4.8), an inequality of the form (4.16) (with h replaced by x) is easily derived for small enough $\| x \|$.

THEOREM 4.1. *Let Θ be a nondegenerate critical point of order p for f. For convenience, we assume*

$$f(\Theta) = 0. \tag{4.18}$$

Let

$$f_p(x) = \frac{d^p f(\Theta; x)}{p!}. \tag{4.19}$$

Then, for every $q = 0, 1, 2, \ldots$,

$$C_q(\Theta; f) = C_q(\Theta; f_p), \tag{4.20}$$

where $C_q(\Theta; f)$ and $C_q(\Theta; f_p)$ denote the q-th critical groups at Θ of f and f_p resp. (cf. Definition 3.6).

The proof is based on the following.

THEOREM 4.2 (Approximation Theorem). *Let Θ be a nondegenerate critical point of order p for f. Let* (4.18) *be satisfied. Let $\bar{R}$ and m be two positive numbers such that*

$$\| g(x) \| \geqslant 2m \| x \|^{p-1} \qquad \text{for } \| x \| < \bar{R}. \tag{4.21}$$

(Such m and $\bar{R}$ exist by Lemma 4.4). Let $\psi \in C^{p+1}(\Theta)$ with $\psi(\Theta) = 0$. We suppose that $\gamma(x) = \operatorname{grad} \psi(x)$ has the following property: To each $\eta > 0$, there corresponds a positive $R_\eta < \bar{R}$ such that

$$\| g(x) - \gamma(x) \| < \eta \| x \|^{p-1} \qquad \text{for } \| x \| < 2R_\eta . \tag{4.22}$$

Under these assumptions, (i) *Θ is a nondegenerate critical point of order p for ψ, and* (ii)

$$C_q(\Theta; f) = C_q(\Theta; \psi) \qquad \text{for } q = 0, 1, 2, \ldots . \tag{4.23}$$

The proof of this theorem will be given in Section 6. We now prove that Theorem 4.2 implies Theorem 4.1 by showing that $\psi = f_p$ satisfies the assumptions of Theorem 4.2. Obviously, $\psi(\Theta) = 0$. Moreover, by (4.15) and (4.19),

$$\gamma(x) = \operatorname{grad}_2 \frac{d^p f(\Theta; x)}{p!} = \frac{d^{p-1} g(\Theta; x)}{(p-1)!}, \tag{4.24}$$

which shows that $\gamma(\Theta) = \Theta$. Finally, it follows from (4.24) and (4.17) that in some neighborhood of $\Theta, g(x) - \gamma(x)$ equals the integral in (4.17). Its norm is, therefore, (cf. (4.8)) majorized by $(K \| x \| / p!) \| x \|^{p-1}$, which shows that, for η given, (4.22) holds if $K 2R_\eta / p! < \eta$.

5. The Cylindrical Neighborhood of an Isolated Critical Point

This useful concept was introduced by Seifert and Threlfall in [20, Section 9] in the finite-dimensional case. In the present section this concept is generalized to the Hilbert space case. In the next section it will be applied to the proof of the Approximation Theorem 4.2.

Let Θ be an isolated critical point of f. We assume (4.18). For $R > 0$, we denote by $B(R) = B(\Theta, R)$ the open ball with center Θ and radius R. We assume R to be such that Θ is the only critical point in $\overline{B(2R)}$. We consider the solution $\xi(t) = \xi(t, x_0)$ of the problem

$$\frac{d\xi}{dt} = -\frac{g(\xi)}{\| g(\xi)\|^2}, \qquad \xi(0, x_0) = x_0 \in B(R) - \{\Theta\}. \tag{5.1}$$

Since

$$\frac{df(\xi(t))}{dt} = \left\langle g(\xi), \frac{d\xi}{dt} \right\rangle = -1, \qquad f(\xi(t)) - f(x_0) = -t, \tag{5.2}$$

we may introduce a new parameter

$$\tau = f(\xi(t)) \qquad \text{and set} \qquad x(\tau) = x(\tau, x_0) = \xi(t), \qquad \tau_0 = f(x_0). \tag{5.3}$$

With these notations, the problem (5.1) takes the form

$$\frac{dx}{d\tau} = \frac{g(x)}{\| g(x)\|^2}, \qquad x(\tau_0) = x_0 \in B(R) - \{\Theta\}. \tag{5.4}$$

We refer to the solution $x(\tau)$ of (5.4) as the gradient line through x_0.

Lemma 5.1. *Let $x(\tau)$ and $B(R)$ be as above.*

I. We consider $x(\tau)$ for $\tau \leqslant \tau_0 = f(x_0)$. Then either there exists a point $\bar{x} \in \dot{B}(R)$ such that

$$\lim_{\tau \downarrow f(\bar{x})} x(\tau) = \bar{x}$$

while $x(\tau) \in B(R)$ for $f(\bar{x}) < \tau \leqslant f(x_0)$, or

$$\lim_{\tau \downarrow 0} x(\tau) = \Theta. \tag{5.5}$$

In this case we say that the gradient line $x(\tau)$ ends at Θ.

II. We consider $x(\tau)$ for $f(x_0) = \tau_0 \leqslant \tau$. Then either there exists a point $\bar{x} \in \dot{B}(R)$ such that

$$\lim_{\tau \uparrow f(\bar{x})} x(\tau) = \bar{x},$$

while $x(\tau) \in B(R)$ for $f(x_0) \leqslant \tau < f(\bar{x})$, or

$$\lim_{\tau \uparrow 0} x(\tau) = \Theta. \tag{5.6}$$

In this case, we say that $x(\tau)$ starts from Θ.

Proof. The proof given for I in [15, Lemma 4.8] remains valid under the present assumptions if in that proof the reference to the corollary to Theorem 2.2 of [15] is replaced by reference to Lemma 2.1 of the present paper while the reference to Lemma 4.5 of [15] is replaced by reference to Lemma 2.4 of the present paper. The proof of II is essentially the same as that of I.

DEFINITION 5.1. Let R be as above, let $0 < R_1 < R$, and let ϵ be a positive number. Then the cylindrical (R_1, ϵ)-neighborhood $C(R_1, \epsilon)$ of Θ is defined as follows:

$$C(R_1, \epsilon) = C^+(\epsilon) \cup C^-(\epsilon) \cup Z(R_1, \epsilon) \cup \{\Theta\}, \tag{5.7}$$

where

$C^+(\epsilon)$ is the set of those points x_0 on gradient lines ending at Θ for which $0 < f(x_0) < \epsilon$,

$C^-(\epsilon)$ is the set of those points x_0 on gradient lines starting from Θ for which $-\epsilon < f(x_0) < 0$,

and where $Z(R_1, \epsilon)$ is the set of points x_0 satisfying two conditions: (i) $-\epsilon < f(x_0) < \epsilon$, (ii) x_0 lies on a gradient line which intersects the set $Z(R_1) = \{x \mid f(x) = 0 \text{ and } 0 < \| x \| < R_1\}$. If we want to emphasize the role of f, we write $C(R_1, \epsilon, f)$ for $C(R_1, \epsilon)$, and use the corresponding notation for the other sets in (5.7).

Remark 1. The designation of $C(R_1, \epsilon)$ as "neighborhood" will be justified later (Lemma 5.4).

Remark 2. If the critical point θ is a maximum or a minimum, then $C(R_1, \epsilon)$ reduces to $C^-(\epsilon) \cup \{\Theta\}$ or $C^+(\epsilon) \cup \{\Theta\}$ resp.

Remark 3. The sets at the right member of (5.7) are mutually disjoint.

LEMMA 5.2. *To a positive $R_1 < R$, there corresponds an $\epsilon > 0$ such that*

$$C(R_1, \epsilon) \subset B(R). \tag{5.8}$$

Proof. The proof is based on

LEMMA 5.3. *Let R be as in Lemma 5.2. Let ζ_1, ζ_2 be real numbers satisfying the inequality*

$$0 < \zeta_1 < \zeta_2 \leqslant R. \tag{5.9}$$

Finally, let $x(\tau)$ be a gradient line satisfying

$$\zeta_1 \leqslant \| x(\tau)\| \leqslant \zeta_2 \qquad \text{for } \tau_1 \leqslant \tau \leqslant \tau_2 . \tag{5.10}$$

Then there exists a positive constant $m = m(\zeta_1, \zeta_2)$ such that

$$\| x(\tau_2) - x(\tau_1)\| \leqslant \frac{\tau_2 - \tau_1}{m(\zeta_1, \zeta_2)} . \tag{5.11}$$

Proof. The ring $P(\zeta_1, \zeta_2) : \zeta_1 \leqslant \| x \| \leqslant \zeta_2$ has a positive distance from the set of critical points of f. Therefore, by Lemma 2.4 there exists a positive $m = m(\zeta_1, \zeta_2)$ such that

$$\| g(x)\| \geqslant m \qquad \text{for } x \in P(\zeta_1, \zeta_2). \tag{5.12}$$

Since, by (5.4), $\| dx/d\tau \| = \| g(x)\|^{-1}$, we see from (5.4) and (5.12) that

$$\| x(\tau_2) - x(\tau_1)\| = \left| \int_{\tau_1}^{\tau_2} \frac{dx(\tau)}{d\tau} \, d\tau \right| \leqslant \int_{\tau_1}^{\tau_2} m^{-1} \, d\tau,$$

which proves (5.11).

We now return to the proof of Lemma 5.2. We claim that (5.8) is true if

$$0 < \epsilon < (R - R_1) \, m(R_1, R), \tag{5.13}$$

where $m(R_1, R)$ is as in Lemma 5.3. We have to show that each of the summands at the right member of (5.7) is contained in $B(R)$.

We start with $Z(R_1, \epsilon)$. Let $x_0 \in Z(R_1)$, i.e.,

$$0 < \| x_0 \| < R_1 < R, \qquad \tau_0 = f(x_0) = 0, \tag{5.14}$$

and let $x(\tau)$ be the gradient line through x_0, i.e., satisfying $x(0) = x_0$. We have to show that

$$\| x(\tau)\| < R \qquad \text{for } -\epsilon < \tau < \epsilon. \tag{5.15}$$

Suppose (5.15) is not true. Then because of (5.14),

$$\| x(\tau)\| = R, \tag{5.16}$$

for some τ in one of the open intervals $(0, \epsilon)$, $(-\epsilon, 0)$. It will be sufficient to carry out the proof for the case that the first of these intervals contains a τ satisfying (5.16). Clearly, then there exists a τ_2 in that interval such that

$$\| x(\tau_2)\| = R, \qquad \| x(\tau)\| < R \qquad \text{for } 0 \leqslant \tau < \tau_2 < \epsilon \tag{5.17}$$

since

$$\| x(0)\| = \| x_0 \| < R_1 < R. \tag{5.18}$$

We see from (5.18) and (5.17) that $\| x(\tau)\| = R_1$ for some τ in the interval $(0, \tau_2)$, and from this the existence of a τ_1 follows easily for which

$$\| x(\tau_1)\| = R_1 , \qquad R_1 < \| x(\tau)\| < R \qquad \text{for } \tau_1 < \tau < \tau_2 . \tag{5.19}$$

From this, (5.17), and Lemma 5.3 we obtain

$$R - R_1 = \| x(\tau_2)\| - \| x(\tau_1)\| \leqslant \| x(\tau_2) - x(\tau_1)\| \leqslant \frac{\tau_2 - \tau_1}{m(R_1 , R)} < \frac{\epsilon}{m(R_1 , R)}, \tag{5,20}$$

an inequality which contradicts (5.13).

This finishes the proof of the inclusion $Z(R_1 , \epsilon) \subset B(R)$, and we turn to the proof of

$$C^+(\epsilon) \subset B(R). \tag{5.21}$$

Let $x(\tau)$ be a gradient line for which (5.5) holds. We have to prove that

$$\| x(\tau)\| < R \tag{5.22}$$

for

$$0 < \tau < \epsilon. \tag{5.23}$$

Suppose this assertion not to be true. Since by (5.5), $\| x(\tau)\| < R$ for small enough τ, we infer the existence of a τ_2 in the interval (5.23) such that

$$\| x(\tau_2)\| = R, \qquad \| x(\tau)\| < R \qquad \text{for } 0 < \tau < \tau_2 . \tag{5.24}$$

But, again by (5.5), $\| x(\tau)\| < R_1$ for small enough τ. It follows the existence of a τ_1 for which (5.19) holds, and from this and (5.24) we infer by Lemma 5.3 the inequality (5.20) which contradicts (5.13). Thus (5.21) is proved. The inclusion $C^-(\epsilon) \subset B(R)$ is proved the same way.

This finishes the proof of (5.8) since, obviously, $\{\Theta\} \subset B(R)$.

LEMMA 5.4. *Let* (5.8) *be satisfied. Then* $C(R_1 , \epsilon)$ *is open.*

We start the proof with the following remark which for later reference we formulate as

LEMMA 5.5. *Let $\psi : B(x_0, \sigma) \to E$. Suppose that ψ is Lipschitz and that there exists a constant $m > 0$ such that*

$$\| \psi(x) \| < m^{-1} \qquad \text{for } x \in B(x_0, \sigma). \tag{5.25}$$

Then, the unique solution $x = x(t)$ of

$$\frac{dx}{dt} = \psi(x), \qquad x(t_0) = x_0 \tag{5.26}$$

is defined at least for $| t - t_0 | < \sigma m$ and satisfies there the inequality

$$\| x(t) - x_0 \| < m^{-1} | t - t_0 | < \sigma. \tag{5.27}$$

This lemma is simply a statement of the classical local existence theorem for (autonomous) differential equations (for the validity of this theorem in Hilbert space, see, e.g., [15, Lemma 4.3]).

We return to the proof of Lemma 5.4. We have to show that every $x_0 \in C(R_1, \epsilon)$ is an interior point of $C(R_1, \epsilon)$:

A. Let $x_0 = \Theta$. We choose an R_2 such that $0 < R_2 < R_1 < R$. Since (4.18) is assumed, there exists a positive $R_3 < R_2$ such that

$$| f(x) | < \min\{\epsilon, (R_1 - R_2)\, m(R_2, R_1)\} \qquad \text{for } \| x \| < R_3. \tag{5.28}$$

We will show that

$$x_1 \in C(R_1, \epsilon) \tag{5.29}$$

if

$$0 < \| x_1 \| < R_3, \tag{5.30}$$

which will obviously prove that $x_0 = \Theta$ is an interior point.

For x_1 satisfying (5.30), let $x(\tau)$ be the gradient line satisfying

$$x(\tau_1) = x_1, \qquad \text{where} \qquad \tau_1 = f(x_1). \tag{5.31}$$

We distinguish three cases:

I. $x(\tau)$ ends at Θ. Then, $x_1 \in C^+(\epsilon) \subset C(R_1, \epsilon)$ since $| f(x_1) | < \epsilon$ by (5.30) and (5.28).

II. $x(\tau)$ starts from Θ. Then, $x_1 \in C^-(\epsilon) \subset C(R_1, \epsilon)$ by the same argument.

III. $x(\tau)$ neither ends at nor starts from Θ. Then we distinguish two subcases:

IIIa. $\tau_1 = f(x_1) = 0$. Now, by (5.30),

$$0 < \| x_1 \| < R_3 < R_2 < R_1 , \tag{5.32}$$

and we see that

$$x_1 \in Z(R_1) \subset Z(R_1 , \epsilon) \subset C(R_1 , \epsilon).$$

IIIb. $\tau_1 = f(x_1) \neq 0$. We consider the case $\tau_1 > 0$, the proof in the case $\tau_1 < 0$ being essentially the same. We will show that

$$x_1 \subset Z(R_1 , \epsilon) \subset C(R_1 , \epsilon). \tag{5.33}$$

Since $\| x_1 \| < R_1$ (cf. (5.32)) and since we are not in Case I or II, it follows from Lemma 5.1 that there exists a τ_2 such that

$$\lim_{\tau \downarrow \tau_2} x(\tau) \in \mathring{B}(R_1) \quad \text{while} \quad \| x(\tau)\| < R_1 \quad \text{for} \quad \tau_2 < \tau < \tau_1 . \tag{5.34}$$

Denoting this limit by x_2, we have

$$x_2 = x(\tau_2), \qquad \| x(\tau_2)\| = R_1 , \tag{5.35}$$

and it follows from (5.32) that $\| x(\tau)\| = R_2$ for some τ in (τ_2 , τ_1). Therefore, there exists a τ_3 such that

$$x(\tau_3) = R_2 , \qquad \text{and} \qquad R_2 < \| x(\tau)\| < R_1 \quad \text{for } \tau_2 < \tau < \tau_3 < \tau_1 . \tag{5.36}$$

From (5.35), (5.36) and Lemma 3.3, we conclude

$$\begin{aligned} R_1 - R_2 = \| x(\tau_2)\| - \| x(\tau_3)\| &\leqslant \| x(\tau_2) - x(\tau_3)\| \\ &\leqslant \frac{\tau_3 - \tau_2}{m(R_2 , R_1)} < \frac{\tau_1 - \tau_2}{m(R_2 , R_1)} . \end{aligned} \tag{5.37}$$

But $0 < \tau_1 = f(x_1) < (R_1 - R_2)\, m(R_2 , R_1)$ by (5.32) and (5.28). From this inequality together with (5.37) we see that $f(x_2) = \tau_2 < 0$. Since $f(x_1) = \tau_1 > 0$ the existence of a $\bar{\tau}$ in (τ_2 , τ_1) follows such that $f(\bar{x}) = 0$ for $\bar{x} = x(\bar{\tau})$. Moreover, $\| \bar{x} \| < R_1$, by (5.34). Thus, $\bar{x} \subset Z(R_1)$. Now x_1 lies on the gradient line through $\bar{x}$ and $|f(x_1)| < \epsilon$ by (5.30) and (5.28). These two properties prove (5.33), and the proof that $x_0 = \Theta$ is an interior point is finished.

B. $x_0 \in Z(R_1)$, i.e.,

$$0 < \| x_0 \| < R_1 < R \tag{5.38}$$

and

$$f(x_0) = 0. \tag{5.39}$$

Let

$$\zeta = \min \{\| x_0 \|/4, \qquad (R_1 - \| x_0 \|/4\}, \tag{5.40}$$

and let $P(x_0, 2\zeta)$ be the ring

$$P(x_0, 2\zeta) = \{x \mid \| x_0 \| - 2\zeta < \| x \| < \| x_0 \| + 2\zeta\}. \tag{5.41}$$

It is easily verified that the closure of this ring is contained in $B(R_1)-\{\Theta\}$. Therefore, there exists, by Lemma 2.4, a constant $m = m(\zeta)$ such that

$$\| g(x)\| \geqslant m(\zeta) > 0 \qquad \text{for } x \in P(x_0, 2\zeta). \tag{5.42}$$

But on account of (5.39) there exists a positive $\zeta_0 \leqslant \zeta$ such that

$$| f(x)| < \min\{\zeta m(\zeta), \epsilon\} \qquad \text{for } x \in B(x_0, \zeta_0). \tag{5.43}$$

We will show that

$$B(x_0, \zeta_0) \subset Z(R_1, \epsilon) \subset C(R_1, \epsilon). \tag{5.44}$$

We note first that because of $\zeta_0 \leqslant \zeta$, the inclusion

$$B(x_1, \zeta) \subset P(x_0, 2\zeta) \qquad \text{for } x_1 \in B(x_0, \zeta_0) \tag{5.45}$$

is easily verified. Consequently, the inequality (5.42) holds for $x \in B(x_1, \zeta)$, and it follows from Lemma 5.5 that the solution $x(\tau)$ of the differential Eq. (5.4), with the initial condition $x(\tau_1) = x_1$, $\tau_1 = f(x_1)$, satisfies

$$x(\tau) \in B(x_1, \zeta) \qquad \text{for } | \tau - \tau_1 | < \zeta m(\zeta). \tag{5.46}$$

But, for $x_1 \in B(x_0, \zeta_0)$, it follows from (5.43) that

$$| 0 - \tau_1 | = | \tau_1 | = | f(x_1)| < \zeta m(\zeta).$$

Therefore, (5.46) shows that

$$x(0) \in B(x_1, \zeta) \subset P(x_0, 2\zeta) \subset B(R_1).$$

Thus $0 < \| x(0)\| < R_1$, and we see that $x(0) \in Z(R_1)$, since $f(x(0)) = 0$. But x_1 lies on the gradient line through $x(0)$. Moreover, for $x_1 \in B(x_0, \zeta_0)$, we conclude from (5.43) that $| \tau_1 | = | f(x_1)| < \epsilon$. This completes the proof of (5.44).

C. $x_0\ Z(R_1, \epsilon)$, $\epsilon \neq 0$. Let $x = x(\tau, x_0)$ be the solution of the differential Eq. (5.4) satisfying $x(\tau_0, x_0) = x_0$, $\tau_0 = f(x_0)$. Then, by definition of $Z(R_1, \epsilon)$,

$$\bar{x} = x(0, x_0) \in Z(R_1). \tag{5.47}$$

Thus,

$$0 < \| \bar{x} \| = \| x(0, x_0) \| < R_1, \qquad \bar{\tau} = 0 = f(\bar{x}). \tag{5.48}$$

From (5.47) and the proof given for case B we conclude the existence of a positive ζ_0 such that (5.44) holds with x_0 replaced by $\bar{x}$:

$$B(\bar{x}, \zeta_0) \subset Z(R_1, \epsilon). \tag{5.49}$$

We claim the existence of a positive ζ_1 such that

$$B(x_0, \zeta_1) \subset Z(R_1, \epsilon) \subset C(R_1, \epsilon). \tag{5.50}$$

We define ζ_1 as follows: Since $|f(x_0)| < \epsilon$, by definition of $Z(R_1, \epsilon)$, we may choose ζ_1 such that

$$|f(x_1)| < \epsilon, \tag{5.51}$$

if

$$x_1 \subset B(x_0, \zeta_1). \tag{5.52}$$

But since $x(\tau, x_1)$ depends continuously on the initial value $x_1 = x(\tau_0, x_1)$, we may subject ζ_1 to the additional restriction that (5.52) implies the inequality $\| x(\bar{\tau}, x_1) - x(\bar{\tau}, x_0) \| < \zeta_0$, or by (5.48), (5.47) that

$$x(0, x_1) \in B(\bar{x}, \zeta_0). \tag{5.53}$$

By (5.49), this implies that $\bar{x}_1 = x(0, x_1) \subset Z(R_1, \epsilon)$. Consequently, $\bar{x}_1$ lies on the gradient line through some point y_0 of $Z(R_1)$, i.e., there exists a solution $x = y(\theta, y_0)$ of the differential Eq. (5.4) with the initial condition $y(0, y_0), f(y_0) = 0$, such that, for some θ, say, $\theta = \theta_1$, $x(0, x_1) = \bar{x}_1 = y(\theta, y_0)$. By the uniqueness theorem for differential equations this implies that $x(\tau, x_1) = y(\tau + \theta_1, y_0)$. Therefore, $x(-\theta_1, x_1) = y(0, y_0) \in Z(R_1)$ for x_1 satisfying (5.52). This proves (5.50).

D. $x_0 \in C^+(\epsilon) \cup C^-(\epsilon)$. It will be sufficient to treat the case that $x_0 \in C^+(\epsilon)$. If $x = x(\tau, x_0)$ is the solution of (5.4) satisfying

$$x(\tau_0, x_0) = x_0 \in C^+(\epsilon), \qquad 0 < \tau_0 = f(x_0) < \epsilon, \tag{5.54}$$

then

$$\lim_{\tau \downarrow 0} x(\tau, x_0) = \Theta. \tag{5.55}$$

Now, by A, there exists a positive ζ such that

$$B(\theta, \zeta) \subset C(R_1, \epsilon), \tag{5.56}$$

and by (5.55) there exists a $\bar{\tau}$ such that

$$\bar{x} = x(\bar{\tau}, x_0) \in B(\Theta, \zeta), \qquad 0 < \bar{\tau} < \tau_0 . \tag{5.57}$$

We claim that there exists a positive ζ_1 such that

$$B(x_0, \zeta_1) \subset C(R_1, \epsilon). \tag{5.58}$$

We determine ζ_1 as follows: Because of (5.54) we may choose ζ_1 such that

$$0 < f(x_1) < \epsilon, \tag{5.59}$$

if

$$x_1 \in B(x_0, \zeta_1). \tag{5.60}$$

Moreover, with the usual notation $x(\tau, x_1)$ for the solution of the differential Eq. (5.4) with the initial condition $x(\tau_0, x_1) = x_1$, we may, because of (5.57), require

$$\| x(\bar{\tau}, x_1) - \bar{x} \| = \| x(\bar{\tau}, x_1) - x(\bar{\tau}, x_0) \|$$

to be so small that

$$\bar{x}_1 = x(\bar{\tau}, x_1) \in B(\Theta, \zeta) \subset C(R_1, \epsilon), \tag{5.61}$$

provided x_1 satisfies (5.60).

Now let $y(\theta, \bar{x}_1)$ be the solution of the differential Eq. (5.4) satisfying

$$y(f(\bar{x}_1), \bar{x}_1) = \bar{x}_1 = x(\bar{\tau}, x_1). \tag{5.62}$$

Because of (5.61), $y(\theta, \bar{x}_1)$ is one of the gradient lines used in the construction of $C(R_1, \epsilon)$. The same is true for $x(\tau, x_1)$, since, as seen from (5.62) and the uniqueness theorem for differential equations, $x(\tau, x_1) = y(\tau + f(\bar{x}_1) - \bar{\tau}, \bar{x}_1)$. This together with (5.59) proves that $x_1 \in C(R_1, \epsilon)$ if x_1 satisfies (5.60). Thus (5.58) holds.

6. Proof of the Approximation Theorem 4.2

Assertion (i) of this theorem follows in an obvious way from the assumption made on g, from (4.22), and from Lemma 4.3.

We turn to the proof of (4.24). By Definition 3.6 we have to exhibit neighborhoods $N(f)$ and $N(\psi)$ of Θ, both contained in $B(\bar{R})$ for which

$$H_q(\bar{f}_0 \cap N(f), \bar{f}_0 \cap N(f) - \{\Theta\} \approx H_q(\bar{\psi}_0 \cap N(\psi), \bar{\psi}_0 \cap N(\psi) - \{\Theta\}). \tag{6.1}$$

Now by Lemma 4.3 we may suppose $\bar{R}$ and a constant K be chosen in such a way that in addition to (4.21) with $m = k$

$$\frac{K \| x \|^{p-1}}{2} \geqslant \| g(x) \| \qquad \text{for } x \in B(\bar{R}), \tag{6.2}$$

If we set

$$\zeta = \min \left\{ \eta, \frac{K}{2}, k3^{-p-1} \right\}, \tag{6.3}$$

it is easily verified from (4.21), (6.2), and (4.22), with η replaced by ζ, that

$$K \| x \|^{p-1} \geqslant \| \gamma(x) \| \geqslant k \| x \|^{p-1} \qquad \text{for } x \in B(R_\zeta) \in B(\bar{R}). \tag{6.4}$$

We note that because of (6.3) also

$$\| g(x) - \gamma(x) \| \leqslant \eta \| x \|^{p-1} \qquad \text{for } x \in B(R_\zeta). \tag{6.5}$$

We now keep η, and therefore ζ, fixed and set

$$P(R) = \{x \mid R \leqslant \| x \| \leqslant 2R\}, \qquad R = \frac{R_\zeta}{4}. \tag{6.6}$$

We then see from (4.21) (with $m = k$) and (6.4) that

$$\left. \begin{matrix} \| g(x) \| \\ \| \gamma(x) \| \end{matrix} \right\} \geqslant kR^{p-1} \qquad \text{for } x \in P(R). \tag{6.7}$$

It now follows from Lemma 5.2 and its proof (see, in particular, (5.13) with R replaced by $2R$, and R_1 by R) that

$$\left. \begin{matrix} C(R, \epsilon, f) \\ C(R, \epsilon, \psi) \end{matrix} \right\} \subset B(2R), \tag{6.8}$$

if we choose ϵ in such a way that

$$\frac{kR^p}{2} < \epsilon < kR^p. \tag{6.9}$$

We will now prove (6.1) with

$$N(f) = \overline{C(R, \epsilon, f)}, \qquad N(\psi) = \overline{C(R, \epsilon, \psi)}, \tag{6.10}$$

where, as usual, the bar denotes closure. We set

$$\begin{gathered} N^-(f) = f_0 \cap N(f), \qquad N^-(\psi) = \psi_0 \cap N(\psi), \\ \Sigma = N^-(f) \cup N^-(\psi). \end{gathered} \tag{6.11}$$

Then (6.1) will be proved, once it is shown that

$$H_q(\Sigma, \Sigma - \{\Theta\}) \approx H_q(N^-(f), N^-(f) - \{\Theta\}), \tag{6.12}$$

and

$$H_q(\Sigma, \Sigma - \{\Theta\}) \approx H_q(N^-(\psi), N^-(\psi) - \{\Theta\}). \tag{6.13}$$

It is sufficient to prove (6.12), as is clear from the fact that (6.4) holds if γ is replaced by g (as follows from (4.21) and (6.2)), that (6.7), (6.8) are the same for g and γ, and that (6.5) is symmetric in g and γ.

As a first step towards proving (6.12), we show that

$$H_q(\Sigma, \Sigma - \{\Theta\}) \approx H_q(\Sigma \cap \bar{f}_0, \Sigma \cap \bar{f}_0 - \{\Theta\}). \tag{6.14}$$

To this end, we construct a deformation

$$\delta(x_0, t) : \Sigma \times [0, 1] \to \Sigma \tag{6.15}$$

such that

$$\delta(x_0, t) = x_0 \qquad \text{for } x_0 \in \Sigma \cap \bar{f}_0, \quad 0 \leqslant t \leqslant 1, \tag{6.16}$$

while for all $x_0 \in \Sigma$

$$\delta(x_0, 0) = x_0, \qquad \delta(x_0, 1) \in \Sigma \cap \bar{f}_0. \tag{6.17}$$

Because of (6.16), we have to define $\delta(x_0, t)$ only for $x_0 \in \Sigma - (\Sigma \cap \bar{f}_0)$, i.e., for

$$x_0 \in N^-(\psi) \cap \{f > 0\}. \tag{6.18}$$

To do this, we note first that for such x_0 by (6.11), (6.10) and (6.8)

$$x_0 \in N^-(\psi) \subset \overline{C(R, \epsilon, \psi)} \subset B(2R), \qquad f(x_0) > 0, \tag{6.19}$$

and thus

$$-\epsilon \leqslant \psi(x_0) \leqslant 0 < f(x_0). \tag{6.20}$$

We now consider the "ψ-gradient line" $x = \xi(\tau, x_0)$ defined by

$$\frac{d\xi}{d\tau} = \frac{\gamma(\xi)}{\| \gamma(\xi)\|^2}, \qquad \xi(\tau_0, x_0) = x_0, \qquad \tau_0 = \psi(x_0) \tag{6.21}$$

and recall that

$$\psi(\xi(\tau, x_0) = \tau. \tag{6.22}$$

We will prove the existence of one and only one $\tau_1 = \tau_1(x_0)$ in the interval

$$-\epsilon \leqslant \tau \leqslant \tau_0 \tag{6.23}$$

for which

$$f(\xi(\tau_1, x_0)) = 0, \tag{6.24}$$

and then define, for x_0 as in (6.18),

$$\delta(x_0, t) = \xi((\tau_1 - \tau_0)\, t + \tau_0, x_0), \qquad 0 \leqslant t \leqslant 1. \tag{6.25}$$

To prove our assertion concerning τ_1, we consider the ring

$$P(x_0) = \left\{ x \;\middle|\; \frac{\| x_0 \|}{2} < \| x \| < 3 \frac{\| x_0 \|}{2} \right\}. \tag{6.26}$$

Now $\| x_0 \| < 2R$ by (6.19). Therefore, $P(x_0) \subset B(3R) \subset B(R_\zeta)$, the last inclusion following from (6.6). Consequently, we see from (6.4) that

$$\| \gamma(x) \| \geqslant k \| x \|^{p-1} \geqslant k \left\| \frac{x_0}{2} \right\|^{p-1} \qquad \text{for } x \in P(x_0), \tag{6.27}$$

and this inequality holds a fortiori in the ball

$$\| x - x_0 \| < \left\| \frac{x_0}{2} \right\| \tag{6.28}$$

which is contained in $P(x_0)$. It, therefore, follows from Lemma 5.5 that

$$\| \xi(\tau, x_0) - x_0 \| < \left\| \frac{x_0}{2} \right\|, \tag{6.29}$$

if

$$| \tau - \tau_0 | < k \left\| \frac{x_0}{2} \right\|^p. \tag{6.30}$$

We will now show that

$$f(\xi(\tau, x_0)) < 0, \tag{6.31}$$

if τ is in the interval

$$\tau_0 - \left\| \frac{x_0}{2} \right\|^p k < \tau < \tau_0 - \left\| \frac{x_0}{2} \right\|^p \frac{k}{2} \tag{6.32}$$

Indeed, if we add the inequality $f(x_0) - 2\psi(x_0) > 0$ (which follows from (6.20)) to the identity $f(\xi(\tau, x_0) = f(\xi(\tau, x_0)) - \psi(\xi(\tau, x_0)) + \psi(\xi(\tau, x_0))$ and observe (6.22), we see that

$$f(\xi(\tau, x_0) < f(\xi(\tau, x_0)) - \psi(\xi(\tau, x_0)) + f(\xi(\tau_0, x_0)) - \psi(\xi(\tau_0, x_0)) + \tau - \tau_0 . \tag{6.33}$$

Now if τ is in the interval (6.32), and, therefore, in the interval (6.30), we know that $x = \xi(\tau, x_0) \in P(x_0) \subset B(R_\zeta)$. But for any $x \in B(R_\zeta)$

$$|f(x) - \psi(x)| < \frac{\zeta \| x \|^p}{2} \tag{6.34}$$

for

$$\begin{aligned} |f(x) - \psi(x)| &= \left| \int_0^1 \langle g(x\theta) - \gamma(x\theta), x \rangle \, d\theta \right| \\ &\leqslant \| x \| \int_0^1 \| g(x\theta) - \gamma(x\theta) \| \, d\theta \end{aligned}$$

from which (6.34) follows on account of (4.22) with η replaced by ζ. Now from (6.33), (6.34), (6.29), and (6.32) we see that

$$\begin{aligned} f(\xi(\tau, x_0) &< \frac{\zeta}{2} [\| \xi(\tau, x_0) \|^p + \| x_0 \|^p] + \tau - \tau_0 \\ &< \frac{1}{2} \left\| \frac{x_0}{2} \right\|^p [\zeta(3^p + 2^p) - k]. \end{aligned}$$

Here the bracket is negative by (6.3). This proves (6.31) and also the existence of a τ_1 satisfying (6.24) since $f(\xi(\tau_0, x_0) = f(x_0) > 0$ by (6.20). It remains to prove the asserted uniqueness of such τ_1. We do this by showing that

$$\frac{df}{d\tau}(\xi(\tau, x_0)) > 0. \tag{6.35}$$

Now, by (6.22), we have (in obvious notation)

$$\frac{df}{d\tau} = 1 + \frac{d(f - \psi)}{d\tau} . \tag{6.36}$$

But by (6.21), (6.4), (4.23) (with η replaced by ζ), and (6.3),

$$\left| \frac{d(f - \psi)}{d\tau} \right| = \left| \left\langle g - \gamma, \frac{d\xi}{d\tau} \right\rangle \right| \leqslant \frac{\| g - \gamma \|}{\| \gamma \|} < \frac{\zeta}{k} < \frac{1}{2} ,$$

which together with (6.36) proves (6.35).

Thus the deformation defined by (6.25) satisfies the requirements (6.15)-(6.17). This proves (6.14).

Now, by Lemma 5.4, there exists a positive r such that

$$B_r = B(\Theta, r) \subset C(R, \epsilon, f). \tag{6.37}$$

By excising from $\Sigma \cap \bar{f}_0$ the intersection of this set with the complement of B_r, we see that

$$H_q(\Sigma \cap \bar{f}_0, \Sigma \cap \bar{f}_0 - \{\Theta\}) \approx H_q(\Sigma \cap \bar{f}_0 \cap B_r, \Sigma \cap \bar{f}_0 \cap B_r - \{\Theta\}). \tag{6.38}$$

But it is easily seen from (6.37), (6.10), and (6.11) that

$$\Sigma \cap \bar{f}_0 \cap B_r = N^-(f) \cap B_r .$$

Therefore, the right member of (6.38) is $H_q(N^-(f) \cap B_r, N^-(f) \cap B_r - \{\Theta\})$, and this group is isomorphic to $H_q(N^-(f), N^-(f) - \Theta)$ as is again seen by excision. This shows that (6.38) together with (6.14) implies (6.12).

7. Sufficient Conditions for the Critical Group $C_q(\Theta, f)$ to be Finitely Generated

We assume now that Assumptions (1.21) and (1.22) are satisfied in addition to those made in Section 6. Then

$$g(x) = \operatorname{grad} f(x) = x \| x \|^{p-2} + G(x), \tag{7.1}$$

where G is completely continuous.

Example. Let E be the Hilbert space of real functions $x(t)$ whose square is Lebesgue integrable over [0, 1]. Let p be an integer $\geqslant 2$, and let

$$F(x) = \sum_{n=p}^{\infty} n^{-p} \int_0^1 \cdots \int_0^1 K_n(t_1, ..., t_n)\, x(t_1) \cdots x(t_n)\, dt_1 \cdots dt_n ,$$

where each K_n is symmetric in its arguments and square summable over its domain of integration and where

$$\sum_{n=p}^{\infty} \left(\int_0^1 \cdots \int_0^1 [K_n(t_1, ..., t_n)]^2\, dt_1 \cdots dt_n \right)^{1/2} < \infty. \tag{7.2}$$

Then

$$G(x) = \sum_{n=p}^{\infty} n^{-p+1} \int_0^1 \cdots \int_0^1 K_n(s, t_1, ..., t_{n-1})\, x(t_1) \cdots x(t_{n-1})\, dt_1 \cdots dt_{n-1} \tag{7.3}$$

(see, e.g., [21, Section 21.2]). Assume

$$\Big\| x \| x \|^{p-2} + p^{-p+1} \int_0^1 \cdots \int_0^1 K_p(s, t_1, \ldots, t_{p-1})\, x(t_1) \cdots x(t_{p-1}) \, dt_1 \cdots dt_{p-1} \Big\| \geqslant m \| x \|^{p-1}$$

for some positive m. Using (7.1)-(7.3) and Lemma (4.3), it is not hard to see that f is nondegenerate of order p at Θ.

We return to the general case. To obtain conditions as indicated in the title of this section we use finite-dimensional approximations.

If E^n is a given finite dimensional (linear) subspace of E of dimension n, we denote by y^n or $(y)^n$ the projection of the element $y \in E$ into E^n. Moreover, we set

$$f_n(x) = p^{-1} \| x \|^p + F_n(x), \tag{7.4}$$

where $F_n(x) = F(x^n)$ and

$$g_n(x) = x \| x \|^{p-2} + G_n(x), \tag{7.5}$$

where $G_n(x) = (G(x^n))^n$. Then, by [15, Lemma 2.3],

$$g_n = \operatorname{grad} f_n . \tag{7.6}$$

We state the following fact whose routine verification we omit as

LEMMA 7.1. *For $r = 1, 2, \ldots, p$, the r-th differential of G_n exists and is given by*

$$d^r G_n(x; h_1, \ldots, h_r) = (d^r G(x_n; h_1{}^n, \ldots, h_r{}^n))^n. \tag{7.7}$$

THEOREM 7.1. *There exists an n_0-dimensional subspace E^{n_0} of E of the following property: If $E^n \supset E^{n_0}$, then Θ is a nondegenerate critical point of f_n of order p, and*

$$C_q(\Theta; f) = C_q(\Theta; f_n). \tag{7.8}$$

Proof. We will show that $\psi = f_n$ satisfies the assumptions of the approximation Theorem 4.2. By (7.6), $\gamma = \operatorname{grad} \psi = g_n$, and, by (7.4), (4.18), and (7.5), $\psi(\Theta) = 0$, $\gamma(\Theta) = \Theta$ for any $E^n \subset E$. Moreover, $\psi \in C^{p+1}$ by Definition 4.2 and Lemma 7.1.

It remains to prove that to every $\eta > 0$ there corresponds an E^{n_0} and an R_η such that (4.22) is satisfied, i.e., for $E^n \supset E^{n_0}$ and $x \in B(2R_\eta)$,

$$\| g(x) - g_n(x) \| = \| G(x) - G(x^n))^n < \eta \| x \|^{p-1}. \tag{7.9}$$

We will prove, for suitable E^n and R_η, the inequalities

$$\| G(x) - G(x^n)\| < \frac{\eta}{2} \| x \|^{p-1} \qquad \text{for } x \in B(2R_\eta) \tag{7.10}$$

and

$$\| G(x^n) - (G(x^n))^n \| < \frac{\eta}{2} \| x \|^{p-1} \qquad \text{for } x \in B(2R_\eta), \tag{7.11}$$

which together imply (7.9).

Starting with the proof of (7.11), we recall the following well-known fact (which follows, e.g., from [8, 2nd lemma, p. 51]): If $\bar{N}$ is a bounded closed neighborhood of Θ and if $\bar{G}$ is a completely continuous map $\bar{N} \to E$, then to each positive η_1 there corresponds an E^{n_0} such that for $E^n \supset E^{n_0}$

$$\| \bar{G}(x) - (\bar{G}(x))^n \| < \eta_1 (x \in \bar{N}). \tag{7.12}$$

We want to apply this for some N with

$$\bar{G}(x) = d^{p-1} G(\Theta, x). \tag{7.13}$$

That this $\bar{G}$ is (for small enough N) completely continuous follows from Lemma 7.2 which for later use is formulated in more general terms than would be necessary for the present purpose. (It is a modification of a well-known lemma by Krasnolseskii [7; II, Lemma 4.1].)

LEMMA 7.2. *Let N_1 be a neighborhood of Θ. Let G be a completely continuous map $N_1 \to E$ which is an element of $C^p(N_1)$. Then there exists a neighborhood $N \subset N_1$ of Θ such that for $r = 1, 2,..., p-1$, $d^r G(x; h_1, ..., h_r)$ is completely continuous as map of $N \times E_1 \times \cdots \times E_r$ into E.*

Proof. With $d^0 G = G$, the statement of the lemma is true for $r = 0$ by assumption. We assume it to be true for $r - 1 < p - 1$ for some neighborhood $N_{r-1} \subset N_1$. We may assume N_{r-1} to be spherical and its radius ζ to be so small that for some constant K

$$\| d^{r+1} G(x; h_1, ..., h_{r+1})\| \leqslant \tfrac{1}{2} K \| h_1 \| \cdots \| h_{r+1} \| \qquad \text{for } x \in N_{r-1} \tag{7.14}$$

(cf. (4.7)). Let $N_r \subset N_{r-1}$ be a spherical neighborhood of Θ with radius

$$\sigma \leqslant \min \left\{ \frac{\zeta}{2}, 1, \frac{2\delta}{3K} \right\}. \tag{7.15}$$

Suppose now the lemma not to be true for r. Then taking into account the linearity of $d^r G$ in each h_ρ, we see that there exist a $\delta > 0$ and sequences x^i, $h_\rho{}^i$ ($\rho = 1,..., r$, $i = 1, 2,...$) such that

$$d^r G(x^i; h_1{}^i, ..., h_r{}^i) - d^r (G(x^j; h_1{}^j, ..., h_r{}^j) \geqslant \delta, \quad x^i \in N_r, \quad \| h_\rho{}^i \| = 1. \tag{7.16}$$

By Taylor's Theorem for arbitrary $h_1, \ldots, h_{r-1}$, and for x and $h_r \in N_r$,

$$\begin{aligned} d^{r-1}G(x + h_r; h_1, \ldots, h_{r-1}) - d^{r-1}G(x; h_1, \ldots, h_{r-1}) \\ = d^rG(x; h_1, \ldots, h_{r-1}, h_r) + R(x; h_1, .., h_{r-1}, h_r), \end{aligned} \tag{7.17}$$

where

$$\begin{aligned} R(x; h_1, \ldots, h_{r-1}, h_r) \\ = \int_0^1 d^{r+1}G(x + th_r; h_1, \ldots, h_{r-1}, h_r, h_r)(1 - t)\, dt. \end{aligned} \tag{7.18}$$

From (7.18) and (7.14) we see that

$$\| R(x; h_1, \ldots, h_{r-1}, h_r)\| \leqslant \frac{K}{2} \| h_1 \| \cdots \| h_{r-1} \| \| h_r \|^2 \tag{7.19}$$

for arbitrary $h_1, \ldots, h_{r-1}$, and h_r, x having norms not greater than σ. We now set in (7.17), $x = x^i$, $h_\rho = h_\rho{}^i$ for $\rho = 1, \ldots, r - 1$, and $h_r = \sigma h_r{}^i$, and from the equality thus obtained we subtract the one obtained from it by replacing the index i by the index j. We then see that

$$\begin{aligned} &\| d^r \ \ G(x^i + \sigma h_r{}^i; h_1{}^i, \ldots, h_{r-1}^i) - d^{r-1}G(x_1{}^j + \sigma h_r{}^j; h_1{}^j, \ldots, h_{r-1}^j)\| \\ &\quad + \| d^{r-1}G(x^i; h_1{}^i, \ldots, h_{r-1}^i) - d^{r-1}G(x^j; h_1{}^j, \ldots, h_{r-1}^j)\| \\ &\geqslant \| d^rG(x^i; h_1{}^i, \ldots, h_{r-1}^i, \sigma h_r{}^i) - d^rG(x^j; h_1{}^j, \ldots, h_{r-1}^j, \sigma h_r{}^j)\| \\ &\quad - \| R(x^i; h_1{}^i, \ldots, h_{r-1}^i, \sigma h_r{}^i)\| - \| R(x^j; h_1{}^j, \ldots, h_{r-1}^j, \sigma h_r{}^j)\| . \end{aligned}$$

Now by (7.16), (7.19), and (7.15), the right member of this inequality is not less than $\delta\sigma/3$. Thus the inequality is in contradiction with our induction assumption for $r - 1$.

We return to the proof of (7.11). By Lemma 7.2, the $\bar{G}$ defined by (7.13) satisfies (7.12). We may assume the neighborhood N of (7.12) to be spherical of radius ζ and choose $\eta_1 < \zeta^{p-1}\eta/4$. Then, taking into account the fact that $\bar{G}(x)$ is homogeneous in x of degree $p - 1$, we conclude the existence of an E^{n_0} such that for $E^n \supset E^{n_0}$

$$\| \bar{G}(x) - (\bar{G}(x))^n \| \leqslant \frac{\eta}{4} \| x \|^{p-1} \qquad \text{for all } x \in E,$$

and, therefore,

$$\| \bar{G}(x^n) - (\bar{G}(x^n))^n \| < \frac{\eta}{4} \| x^n \|^{p-1} \leqslant \frac{\eta}{4} \| x \|^{p-1}. \tag{7.20}$$

We now apply (4.17) with g replaced by G. We then see (cf. (7.13)) that

$$G(x^n) - (G(x^n))^n = \frac{1}{(p-1)!}[\bar{G}(x^n) - (\bar{G}(x^n))^n] + R(x^n) - (R(x^n)^n, \quad (7.21)$$

where $R(x)$ denotes the integral in (4.17). From (7.21), in conjunction with (7.20) and (4.8), we conclude that for some constant K and for x in some neighborhood of Θ

$$\| G(x^n) - (G(x^n))^n \| \leqslant \frac{1}{(p-1)!}\left[\frac{\eta}{4} \| x \|^{p-1} + 2K \| x \|^p\right].$$

This inequality implies (7.11), if $\| x \| < \eta/8K$.

Turning to the proof ot (7.10), we note that again by (4.17) with g replaced by G:

$$G(x) - G(x^n) = \frac{1}{(p-1)!}[\bar{G}(x) - \bar{G}(x^n)] + R(x) - R(x^n), \qquad (7.22)$$

where $\bar{G}(x)$ and $R(x)$ have the same meaning as in (7.21). As in the preceding paragraph, we see that for $\| x \|$ small enough

$$\| R(x) - R(x^n)\| \leqslant 2K \| x \|^p. \qquad (7.23)$$

To estimate the first term at the right member of (7.22), we use the following

LEMMA 7.3. *Let N be a spherical neighborhood of θ, and let $\tilde{G}$ be a map $N \to E$ of the following properties:*

(α) *$\tilde{G}$ is a completely continuous gradient map,*

(β) *$d\tilde{G}(x; k)$ exists and is completely continuous considered as map $N \times E \to E$. Then to given positive ϵ there corresponds an E^{n_0} such that, for $E^n \supset E^{n_0}$,*

$$\| \tilde{G}(x) - \tilde{G}(x^n)\| < \frac{\eta}{4}. \qquad (7.24)$$

The lemma is a restatement of [15, Lemma 2.5]. We now verify that $\tilde{G} = \bar{G}$ (cf. (7.13)) has properties (α) and (β). Property (α) follows from (7.13), Lemma 7.2, and (4.15). As to property (β) we note that

$$d\bar{G}(x; k) = d_2 d^{p-1} G(\Theta; x_1, \ldots, x_{p-1}; k)_{x_1 = \cdots x_{p-1} = x}$$

(cf. the definition of d_2 given at the beginning of the proof for Lemma 4.3). Therefore, taking into account that $\bar{G}(x)$ is homogeneous of degree $p-1$, we see from (4.4) that

$$d\bar{G}(x; k) = (p-1)\, d^{p-1}G(\theta; x_1, ..., x_{p-2}; k)_{x_1=\cdots=x_{p-2}=x}\,.$$

On account of Lemma 7.2 and the complete continuity of G, this equality proves (β).

Thus the conclusion (7.24) of Lemma 7.3 holds with $\tilde{G} = \bar{G}$. But since $\bar{G}(x)$ is homogeneous of degree $p-1$ and since for any positive α, $(\alpha x)^n = \alpha x^n$, we conclude that

$$\| \bar{G}(x) - \bar{G}(x^n)\| < \frac{\eta}{4} \| x \|^{p-1}.$$

Combining this inequality with (7.22) and (7.23), we see that

$$\| G(x) - G(x^n)\| < \| x \|^{p-1} \left[\frac{\eta}{4} + 2K \| x \|\right].$$

This obviously proves (7.10) for $\| x \| < \eta/8K$, and Theorem 7.1 is proved.

For any $E^n \subset E$, we denote by $\tilde{f}_n$ and $\tilde{g}_n$ the restriction to E^n of f_n and g_n, resp. ($\tilde{f}_n$ equals the restriction of f to E^n, but $\tilde{g}_n$ is, in general, not the restriction of g to E^n). With these notations, we have

THEOREM 7.2. *There exists an $E^{n_0} \subset E$ such that, for $E^n \supset E^{n_0}$, Θ (as zero point of E^n) is a nondegenerate critical point of order p for $\tilde{f}_n$, and*

$$C_q(\Theta, \tilde{f}_n) = C_q(\Theta, f). \tag{7.25}$$

Proof. We choose E^{n_0}, according to Theorem 7.1. Then Θ is a nondegenerate point of order p for f_n and, therefore, by Lemma 4.3, also for $\tilde{f}_n$. To prove (7.25), we note that, by [15, Lemma 3.1], for any small enough spherical neighborhood N of Θ, the couple $(N \cap (\tilde{f}_n)_0, N \cap (\tilde{f}_n)_0 - \{\Theta\})$ is a deformation retract of the couple $(N \cap (f_n)_0, N \cap (f_n)_0 - \{\Theta\})$. Therefore, $C_q(\theta, \tilde{f}_n) \approx C_q(\Theta, f_n)$, which together with (7.8) implies (7.25).

THEOREM 7.3. *$C_q(\Theta, f)$ is finitely generated for every nonnegative integer q. For $q > n_0$, the q*-th *type number of f at θ* (see Definition 3.6) *is* 0.

This is a corollary to Theorem 7.2 (see, e.g. [20, Section 10].

8. Return to the Global Situation

We suppose conditions (A)–(E) of the Introduction to be satisfied. In addition, we assume each critical point of f to be nondegenerate of some order $\geqslant 2$. Then, by Lemma 4.3 all critical points are isolated. It follows easily from Lemmas 2.3 and 2.2 that the set Γ of critical points is finite. In addition to the notations explained in Definition 3.6 and in the paragraph preceding it, we will use the following ones (cf. [12, p. 17]): a_0, a_1,..., a_N are real numbers satisfying

$$a_0 < c_1 < a_1 < \cdots < a_{i-1} < c_i < a_i \cdots < a_{N-1} < c_N < a_N ,$$

where a_N is supposed to be larger than $\sup f$ in V (cf. Lemma 2.1). Moreover, let $A_i = V \cap \bar{f}_{a_i}$ for $i = 1, 2,..., N$, and $A_0 = \Phi$, the empty set.

Theorem 8.1. *In addition to the assumptions made above, we suppose that $H_q(V)$ is finitely generated and that for every critical point σ_j^i of f the representation* (1.23) *holds in some neighborhood of σ_j^i, where p_{ij} is an integer $\geqslant 2$ and where $G_{ij} = \operatorname{grad} F_{ij}$ is completely continuous. Then* (α) *the groups $H_q(A_i)$ ($i = 0, 1,..., N$) are finitely generated, and* (β) *the Morse inequalities* (1.25) *hold.*

Proof. $H_q(A_i, A_{i-1})$ is the critical group at level c_i, and, therefore, (see corollary to Theorem 3.5) isomorphic to the finite direct sum over j of the groups $C_q(\sigma_j^i)$, each of which is finitely generated by Theorem 7.3. Thus the groups $H_q(A_i, A_{i-1})$ are finitely generated. To prove (α), we consider the part

$$H_{q+1}(A_i, A_{i-1}) \to H_q(A_{i-1}) \to H_q(A_i) \tag{4.2}$$

of the homology sequence. If we first set $i = N$, then the two extreme groups of (4.2) are finitely generated, the one at the left as just proved and the one at the right by assumption, since $A_N = V$. From the exactness of the sequence (4.2), it follows easily that $H_q(A_{N-1})$ is also finitely generated. Now, setting $i = N - 1$ in (4.2), we see, by the same argument, that $H_q(A_{N-2})$ is finitely generated. Continuation of this procedure proves (α). But (α) implies (β) as Pitcher's proof of the Morse relations [12, Section 11] shows.

9. Appendix

In the preceding sections, various facts concerning existence and continuation of solutions of ordinary differential equations in Hilbert space were used. Concerning these, we refer to the remarks made in [15, Lemmas 4.3, 4.4] and [6, Section 3].

Those remarks, however, are not sufficient to guarantee the joint continuity in x_0 and t of the deformations used in the present paper, viz., the deformations (2.28), (3.8), (3.14), and the deformation defined by (6.16) and (6.25).

This Appendix is devoted to the continuity proof for the two last-named of these deformations, the proof for the first two being considerably simpler because of the absence of interfering critical points.

We start with the following lemma and its corollary both of which represent obvious generalizations of theorems classical in the finite-dimensional case.

LEMMA 9.1. *Let $u = u(t)$ be a map of the real interval $[\alpha, \beta]$ into the Hilbert space E. We assume the existence and continuity of du/dt in this interval. Moreover, we assume the existence of a positive constant λ and a nonnegative constant μ such that*

$$\left\| \frac{du}{dt} \right\| \leqslant \lambda \| u(t) \| + \mu \qquad \text{for } t \in [\alpha, \beta]. \tag{9.1}$$

Then, for any couple t_0, t in $[\alpha, \beta]$:

$$\| u(t) \|^2 \leqslant \| u(t_0) \|^2 e^{2\lambda |t-t_0|} + \frac{M_0 \mu (e^{2\lambda |t-t_0|} - 1)}{\lambda}, \tag{9.2}$$

where M_0 denotes the maximum of $\| u(t) \|$ for $t \in [\alpha, \beta]$.

Proof. From (9.1) we obtain

$$\left| \frac{d \| u(t) \|^2}{dt} \right| = 2 \left| \left\langle u(t), \frac{du(t)}{dt} \right\rangle \right| \leqslant 2\lambda \| u(t) \|^2 + 2M_0 \mu.$$

By a well-known lemma, this inequality implies (9.2) (see, e.g., [6, p. 93, Hilfssatz 2]).

COROLLARY TO LEMMA 9.1. *Let $\xi_1(t)$ and $\xi_2(t)$ be solutions of the differential equation in Hilbert space $d\xi/dt = \chi(\xi)$. Suppose that the $\xi_i(t)$ exist for $t \in [\alpha, \beta]$, and that χ satisfies, in this interval, a Lipschitz condition with Lipschitz constant λ. Then,*

$$\| \xi_1(t) - \xi_2(t) \| \leqslant \| \xi_1(t_0) - \xi_2(t_0) \| e^{\lambda |t-t_0|} \qquad \text{for } t, t_0 \in [\alpha, \beta]. \tag{9.3}$$

Indeed, $u = \xi_1 - \xi_2$ satisfies (9.1) with $\mu = 0$, and (9.3) follows from (9.2) with $\mu = 0$.

LEMMA 9.2. *Under the assumptions of Theorem 3.3 the deformation $\delta(x_0, t)$ defined by (3.14) is jointly continuous in x_0 and t.*

Proof. Let (x_0, t_0) be the point at which we want to establish the continuity. We distinguish four cases:

(a) $x_0 \in f_c, 0 \leqslant t_0 \leqslant 1$

(b) $x_0 \in \bar{f}_b - \bar{f}_c, 0 \leqslant t_0 < 1$

(c) $x_0 \in \{f = c\}, 0 \leqslant t_0 \leqslant 1$

(d) $x_0 \in \bar{f}_b - \bar{f}_c, t_0 = 1.$

Proof for Case (a). Since f_c is open, there exists a neighborhood $N(x_0)$ of x_0 such that $N(x_0) \in f_c$. Then, $\delta(y_0, t) = y_0$ for $y_0 \in N(x_0)$ and $t \in [0, 1]$, by (3.14). Thus, $\delta(y_0, t) - \delta(x_0, t_0) = y_0 - x_0$.

Proof for Case (b). We recall that $x(t, x_0)$ denotes the solution of the initial-value problem (3.8) (with a replaced by the critical value c). We write shortly $x(t)$ for $x(t, x_0)$ and set $y(t) = x(t, y_0)$, where y_0 denotes another initial value.

We will first show that $x(t)$ and, with proper choice of y_0, also $y(t)$ are bounded away from the set Γ of critical points, if

$$0 \leqslant t \leqslant t_0 < 1. \tag{9.9}$$

Indeed, from (3.11) (with a replaced by c) and from (9.9), we see that

$$b - c \geqslant f(x(t)) - c = (1 - t)(f(x_0) - c) \geqslant (1 - t_0)(f(x_0) - c).$$

This inequality shows that $f(x(t))$ for t satisfying (9.9) is bounded away from the set Λ of critical levels. Therefore, $x(t)$ is bounded away from Γ by Lemma 2.5.

We will now define a positive ϵ such that $y(t)$ is bounded away from Γ, if

$$\| x_0 - y_0 \| < \epsilon, \quad 0 \leqslant t \leqslant t_1 < 1, \qquad \text{where} \qquad | t_1 - t_0 | < \epsilon. \tag{9.10}$$

Again, by Lemma 2.5 it will be sufficient to choose ϵ such that $f(y(t))$ is bounded away from Λ. Firstly, we require ϵ to be so small that $t_1 < 1$.

Our next requirement on ϵ is that, for y_0 satisfying (9.10),

$$|f(x_0) - f(y_0)| < \zeta, \tag{9.11}$$

where the positive number ζ is chosen in such a way that the interval $[b, b + \zeta]$ contains no critical level. Noting that $f(y(t))$ is decreasing in t, we see that $b + \zeta \geqslant f(y(t)) \geqslant f(y(t_1))$, for t satisfying (9.10). This inequality shows that it will be sufficient to subject ϵ to the third requirement that, for y_0 and t_1 satisfying (9.10),

$$f(y(t_1)) - c > 0. \tag{9.12}$$

To show that it is possible to satisfy this requirement, we note first that, by assumption,

$$f(x(t_0)) - c > 0. \tag{9.13}$$

But

$$f(y(t_1)) - f(x(t_0)) = (1 - t_0)(f(y_0) - f(x_0) + (f(y_0) - c)(t_1 - t_0), \tag{9.14}$$

as is seen by subtracting from (3.11), with $a = c$ and $t = t_0$, the equation obtained by replacing $x(t)$ by $y(t)$, and t_0 by t_1. Now the absolute value of the right member of (9.14) is less that $|f(y_0) - f(x_0)| + (b + \zeta - c)\,|t_1 - t_0|$. Clearly, ϵ may be chosen in such a way that this quantity is less than one half of the left member of (9.13) for y_0 and t_1 satisfying (9.10). But with this choice of ϵ, we see from (9.13) and (9.14) that (9.12) is satisfied.

Our goal is to find, for every $\rho > 0$, an ϵ, such that

$$\| y(t_1) - x(t_0) \| < \rho, \tag{9.15}$$

for y_0 and t_1 satisfying (9.10). We assume that ϵ satisfies the three requirements made above. Since then, as just proved, $x(t)$ and $y(t)$ are bounded away from Γ, it follows from Lemma 2.4 that

$$\left.\begin{matrix} \| g(x(t)) \| \\ \| g(y(t)) \| \end{matrix}\right\} \geqslant m \qquad \text{for } \begin{matrix} 0 \leqslant t \leqslant t_0 \\ 0 \leqslant t \leqslant t_1 \end{matrix} \tag{9.16}$$

for some positive constant m. Since $y(t)$ satisfies (3.8), we see from (9.16) that

$$\| y(t_1) - y(t_0) \| = (f(y_0) - c) \left\| \int_{t_0}^{t_1} \frac{dy}{dt}\, dt \right\| \leqslant (b + \zeta) \frac{|t_1 - t_0|}{m}.$$

This shows that ϵ can be chosen in such a way that $\| y(t_0) - y(t_0) \| < \rho/2$ for t satisfying (9.10).

Obviously, it will, for proving (9.15), be sufficient to choose ϵ such that

$$\| y(t_0) - x(t_0) \| < \frac{\rho}{2}, \tag{9.17}$$

for y_0 satisfying (9.10). To this end, we set $u(t) = x(t) - y(t)$ and $\psi = g/\| g \|^2$. We then easily obtain, from (3.8),

$$\frac{du}{dt} = \psi(y)\,[f(y_0) - f(x_0)] + (f(x_0) - c)(\psi(y) - \psi(x)). \tag{9.18}$$

Now, it follows easily from (9.16) that g satisfies (in the intervals indicated) a uniform Lipschitz condition (see, e.g. [16, p. 245]). Calling the Lipschitz constant L, we see, from (9.18) and (9.16), that

$$\left\| \frac{du}{dt} \right\| \leqslant m^{-1} |f(y_0) - f(x_0)| + (b - c) L \| u \| .$$

Applying Lemma 9.1, we obtain the inequality

$$\| u(t)\|^2 \leqslant \| u(0)\|^2 e^{2L(b-c)t} + \frac{2R\,|f(y_0) - f(x_0)|}{m^{-1}(b-c)L} (e^{2L(b-c)t} - 1), \tag{9.19}$$

where R denotes the diameter of our domain such that

$$\| u(t)\| \leqslant \| x(t)\| + \| y(t)\| \leqslant 2R.$$

It is evident from (9.19) that ϵ can be chosen as required.

Proof of Case (c). In this case, $\delta(x_0, t) = x_0$ for all t in [0, 1]. We will prove: To every $\rho > 0$, there corresponds a positive ϵ such that, for all t in [0, 1],

$$\| \delta(y_0, t) - x_0 \| \leqslant \rho, \tag{9.20}$$

if

$$\| x_0 - y_0 \| < \epsilon. \tag{9.21}$$

Our assertion is trivial if $y_0 \in \bar{f}_c$; for, then, $\delta(y_0, t) = y_0$. Let $y_0 \in \bar{f}_b - \bar{f}_c$. We will first prove our assertion for t in the half open interval [0, 1). The interval $[c, b]$ has a positive distance from Λ–$\{c\}$, the set of critical levels other than c. From this it follows easily (by a slight generalization of Lemma 2.5) that there exists an ϵ_1 such that, for all $x \in \bar{f}_b - f_c$,

$$\| x - y \| \geqslant \epsilon_1 > 0 \qquad \text{for all } y \in \Gamma - \{f = c\}. \tag{9.22}$$

We may and will assume that ϵ_1 satisfies the additional inequality

$$\epsilon_1 < \rho. \tag{9.23}$$

We note that (9.22) holds, in particular, for $x = x_0$ and $x = y_0$. Now $f(x_0) = c$ is a critical level. Thus x_0 may or may not be critical point. But, in any case, since by assumption Γ is finite, there exist constants ϵ_2, ϵ_3 such that

$$\rho > \epsilon_1 > \epsilon_2 > \epsilon_3 > 0, \tag{9.24}$$

and such that the ring

$$P(x_0) = \{x \mid \epsilon_3 \leqslant \| x - x_0 \| \leqslant \epsilon_2\} \tag{9.25}$$

has a positive distance from Γ. Consequently, by Lemma 2.4 there exists a constant m such that

$$\| g(x) \| \geqslant m > 0 \qquad \text{for } x \in P(x_0). \tag{9.26}$$

Let now ϵ be a number satisfying the requirements

$$\epsilon_3 > \epsilon > 0, \qquad | f(y_0) - c | < (\epsilon_2 - \epsilon_3)\, m \tag{9.27}$$

for y_0 satisfying (9.21). Clearly, such ϵ exists since $f(x_0) - c = 0$ and since f is continuous.

With such choice of ϵ we claim that (9.21) implies (9.20) for $0 \leqslant t < 1$. Because of (9.24) it will be sufficient to show that (9.21) implies

$$\| \delta(y_0, t) - x_0 \| \leqslant \epsilon_2 \qquad \text{for } 0 \leqslant t < 1. \tag{9.28}$$

Suppose this not to be true. Then, there exists a y_0 satisfying (9.21) and a $t' \in [0, 1)$ such that

$$\| \delta(y_0, t') - x_0 \| > \epsilon_2 . \tag{9.29}$$

But

$$\| \delta(y_0, 0) - x_0 \| = \| y_0 - x_0 \| < \epsilon < \epsilon_3 < \epsilon_2 . \tag{9.30}$$

The last two inequalities together imply the existence of a t'' such that

$$\| \delta(y_0, t'') - x_0 \| = \epsilon_2 , \qquad 0 < t'' < t' < 1. \tag{9.31}$$

This equality together with (9.30) implies the existence of a t''' such that

$$\| \delta(y_0, t''') - x_0 \| = \epsilon_3 , \qquad 0 < t''' < t''. \tag{9.32}$$

Clearly, then, there exist also t″, t''' which, in addition to satisfying (9.31) and (9.32), have the property

$$\epsilon_3 \leqslant \| \delta(y_0, t) - x_0 \| \leqslant \epsilon_2 \qquad \text{for } t''' \leqslant t \leqslant t'',$$

i.e. that $\delta(y_0, t) \in P(x_0)$ for the t values indicated. Therefore, the inequality (9.26) holds with $x = \delta(y_0, t) = x(t, y_0) = y(t)$, and we see, from (9.31), (9.32), (9.26), (3.8), and (3.11) (with $a = c$), that

$$\begin{aligned} 0 < \epsilon_2 - \epsilon_3 &= \| \delta(y_0, t'') - x_0 \| - \| \delta(y_0, t''') - x_0 \| \\ &\leqslant \| \delta(y_0, t'') - \delta(y_0, t''') \| = \left\| \int_{t'''}^{t'''} \frac{dy(t)}{dt}\, dt \right\| \\ &\leqslant (f(y_0) - c)\,(t'' - t''')\, m^{-1} = (f(y(t'')) - f(y(t''')))\, m^{-1}. \end{aligned}$$

Observing that $f(y(t))$ is decreasing in t, that $y(0) = y_0$, and that by (3.11) (with $a = c$), $\lim_{t\to 1^-} f(y(t) = c$ we see, from the preceding inequality, that $0 < \epsilon_2 - \epsilon_3 \leqslant (f(y_0) - c)\, m^{-1}$. But this inequality contradicts (9.27).

This finishes the proof for $t \in [0, 1)$. But

$$\| \delta(y_0, 1) - x_0 \| \leqslant \| \delta(y_0, 1) - \delta(y_0, t)\| + \| \delta(y_0, t) - x_0 \| . \tag{9.28}$$

Note now that the ϵ, which we constructed to some given ρ, did not depend on t for $t \in [0, 1)$. Thus the second term at the right member of (9.28) will not be greater than ρ for $t \in [0, 1)$ if y_0 satisfies (9.21) with this ϵ. But by [16, Theorem 5.1] and the Definition (3.14) of $\delta(y_0, 1)$, the first term at the right member of (9.28) tends to zero as $t \to 1^-$. This completes the proof of (9.20) for $t = 1$.

Proof for Case (d). We set

$$\bar{x} = \delta(x_0, 1) = \lim_{t\to 1^-} x(x_0, t). \tag{9.29}$$

We have to prove: To every $\rho > 0$ corresponds an $\epsilon > 0$ such that

$$\| \delta(y_0, t) - \bar{x} \| < \rho \tag{9.30}$$

if

$$\| y_0 - x_0 \| < \epsilon, \qquad 0 < 1 - t < \epsilon. \tag{9.31}$$

Now since $f(\bar{x}) = c$, the arguments and definitions leading up to (9.26) hold literally in the present case with x_0 replaced by $\bar{x}$. Moreover, we see from (9.29) that there exists a $\bar{t}$ such that

$$\| x(x_0, t) - \bar{x} \| < \frac{\epsilon_3}{2} \qquad \text{for } \bar{t} \leqslant t < 1. \tag{9.32}$$

In addition, we require of $\bar{t}$ that

$$(b + \zeta - c)(1 - \bar{t}) < (\epsilon_2 - \epsilon_3)\, m \tag{9.33}$$

where ζ is as in (9.11). Let now ϵ be a positive number satisfying the following two requirements: Firstly,

$$f(y_0) > c \qquad \text{for } y_0 \text{ satisfying (9.31)}. \tag{9.34}$$

(This is possible since $f(x_0) > c$). Secondly,

$$\| x(y_0, t) - x(x_0, t)\| < \frac{\epsilon_3}{2} \qquad \text{for } 0 \leqslant t \leqslant \bar{t} \quad \text{and for } y_0 \text{ satisfying (9.31)}. \tag{9.35}$$

That this is possible is simply the assertion of case (b) with t_0 replaced by $\bar{t}$.

We claim that with this choice of ϵ, (9.11) implies (9.30). In the first place, we have for y_0 satisfying (9.31)

$$\| x(y_0, \bar{t}) - \bar{x} \| < \epsilon_3 . \tag{9.36}$$

Indeed, the left member of (9.36) is not greater than

$$\| x(y_0, \bar{t}) - x(x_0, \bar{t})\| + \| x(x_0, \bar{t}) - \bar{x} \| .$$

Thus (9.36) follows from (9.35) and (9.32).

We will now show that (9.31) implies the inequality obtained from (9.30) by replacing ρ by the smaller number ϵ_1 (see (9.23)). If this inequality were not true we could, by the arguments used in the corresponding part of the proof for case (c), conclude the existence of two constants t'' and t''' such that

$$\begin{gathered} \bar{t} < t''' < t'' < 1, \qquad x(y_0, t''') = \epsilon_3, \qquad x(y_0, t'') = \epsilon_2 \\ x(y_0, t) \in P(\bar{x}) \qquad \text{for } t''' \leqslant t \leqslant t'', \end{gathered} \tag{9.37}$$

and derive the inequality $0 < \epsilon_2 - \epsilon_3 \leqslant (f(y_0) - c)(t'' - t''') m^{-1}$. But here the right member is, by (9.37) adn (9.33), smaller than

$$(f(y_0) - c)(1 - \bar{t}) m^{-1} \leqslant (b + \zeta - c)(1 - \bar{t}) m^{-1} < \epsilon_2 - \epsilon_3 .$$

We thus arrive at a contradiction.

Lemma 9.3. *The deformation δ defined by* (6.16) *and* (6.25) *is jointly continuous in x_0 and t if the assumptions of Theorem 4.2 are satisfied.*

Proof. We denote the solution of the initial value problem (6.21) by $\xi(\tau, \tau_0, x_0)$ such that $\xi(\tau_0, \tau_0, x_0) = x_0$ and $\tau_0 = \psi(x_0)$. Then, (6.25) reads

$$\delta(x_0, t) = \xi((\tau_1(x_0) - \tau_0) t + \tau_0, \tau_0, x_0), \tag{9.38}$$

where $\tau_1(x_0)$ denotes the solution of (6.24).

We should prove the following three statements:

(a) $\tau_1(x_0)$ depends continuously on x_0.

(b) The right member of (9.38) depends continuously on its third argument.

(c) The δ given by (6.25) "fits continuously" at $(f = 0\}$ with the identity map (6.17).

Now (b) follows easily from the corollary to Lemma 9.1, and the proof of (b) is routine once (a) is proved. Therefore we restrict ourselves to giving the more complicated proof for statement (a).

Let then $\xi(\tau, \bar{\tau}, \bar{x})$ be the solution of the differential Eq. (6.21) satisfying

$$\xi(\bar{\tau}, \bar{\tau}, \bar{x}) = \bar{x}, \qquad \bar{\tau} = \psi(\bar{x}), \tag{9.39}$$

and let $\tau = \tau_1(\bar{x})$ be the root of $f(\xi(\tau, \bar{\tau}, \bar{x}) = 0$.

We have to prove: To every positive ρ, there corresponds a positive ϵ such that

$$| \tau_1(\bar{x}) - \tau_1(x_0)| < \rho, \tag{9.40}$$

if

$$| \bar{x} - x_0 | < \epsilon. \tag{9.41}$$

We will subject ϵ to a successive number of compatible restriction. The first of these is

$$0 < \epsilon < \frac{\| x_0 \|}{11}. \tag{9.42}$$

It is then a matter of direct verification that, for $\bar{x}$ satisfying (9.41),

$$B\left(\bar{x}, 3\,\frac{\| \bar{x} \|}{8}\right) \subset B\left(x_0, \frac{\| x_0 \|}{2}\right) \subset P(x_0). \tag{9.43}$$

(We recall that $B(x, a)$ denotes the ball with center x and radius a; see also (6.26)). Therefore, by (6.27),

$$\left\| \frac{\gamma(x)}{\| \gamma(x) \|^2} \right\| = \frac{1}{\| \gamma(x) \|} \leqslant \frac{1}{k \| x_0/2 \|^{p-1}} \qquad \text{for } x_0 \in B\left(\bar{x}, 3\,\frac{\| \bar{x} \|}{8}\right). \tag{9.44}$$

It follows from Lemma 5.5 that

$$\| \xi(\tau, \bar{\tau}, \bar{x}) - \bar{x} \| < \frac{| \tau - \bar{\tau} |}{k \| x_0/2 \|^{p-1}}, \tag{9.45}$$

for

$$| \tau - \bar{\tau} | < \frac{3}{8} \| \bar{x} \| \, k \left\| \frac{x_0}{2} \right\|^{p-1}. \tag{9.46}$$

We now remark that

$$\tau_0 - \frac{1}{2} k \left\| \frac{x_0}{2} \right\|^{p-1} < \tau_1(x_0) < \tau_0, \tag{9.47}$$

as follows from (6.31) and (6.32) together with the fact that $f(x_0) > 0$ (see (6.20)).

Our second requirement on ϵ is that, for $\bar{x}$ satisfying (9.41),

$$|\bar{\tau} - \tau_0| = |\psi(\bar{x}) - \psi(x_0)| < \frac{1}{32} k \left\| \frac{x_0}{2} \right\|^p. \tag{9.48}$$

With this choice, the interval

$$|\tau - \tau_1(x_0)| < \frac{1}{16} k \left\| \frac{x_0}{2} \right\|^p \tag{9.49}$$

is contained in the interval (9.46) as is seen by a direct verification, using (9.48), (9.47), (9.49), and the fact that $\| x_0 \| < 11 \| \bar{x} \|/10$ by (9.42).

We now describe our last two requirements on the ϵ corresponding to the given ρ (cf. (9.40), (9.41)). First, let ρ_1 be such that

$$0 < \rho_1 < \min \left\{ \rho, \frac{1}{32} k \left\| \frac{x_0}{2} \right\|^p, \tau_0 - \tau_1(x_0) \right\}. \tag{9.50}$$

Since $f(\xi(\tau, \tau_0, x_0)$ is increasing in τ and vanishes for $\tau = \tau_1(x_0)$, the inequality

$$f(\xi(\tau_1(x_0) - \rho_1, \tau_0, x_0) < 0 < f(\xi(\tau_1(x_0) + \rho_1, \tau_0, x_0)$$

holds. Consequently, there exists a positive ζ such that

$$f(x') < 0 < f(x''), \tag{9.51}$$

if

$$\| x' - \xi(\tau_1(x_0) - \rho_1, \tau_0, x_0) \| < \zeta \tag{9.52}$$

and

$$\| x'' - \xi(\tau_1(x_0) + \rho_1, \tau_0, x_0) \| < \zeta. \tag{9.53}$$

Our additional requirements on ϵ then are, for x_0 satisfying (9.41),

$$|\bar{\tau} - \tau_0| = |\psi(\bar{x}) - \psi(x_0)| \leqslant \zeta \frac{k}{2} \left\| \frac{x_0}{2} \right\|^{p-1} \tag{9.54}$$

for x_0 satisfying (9.41) and

$$0 < \epsilon < \frac{\zeta}{2} \exp\left(- \lambda k \left\| \frac{x_0}{2} \right\|^p \right), \tag{9.55}$$

where λ denotes a Lipschitz constant for $\gamma(x)/\| \gamma(x) \|^2$ for $x \subset B(x_0, \| x_0/2 \|)$.

We will show that for such ϵ with $\bar{x}$ satisfying (9.41),

$$x' = \xi(\tau_1(x_0) - \rho_1, \bar{\tau}, \bar{x}) \quad \text{and} \quad x'' = \xi(\tau_1(x_0) + \rho_1, \bar{\tau}, \bar{x})$$

satisfy (9.52) and (9.53), resp., and, consequently, (9.51). Thus there will be a zero of $f(\xi(\tau, \bar{\tau}, \bar{x})$ in the interval $(\tau_1(x_0) - \rho_1), \tau_1(x_0) + \rho_1)$ which will prove our statement (a).

It will be sufficient to prove the first of the two assertions just made the proof for the second one being essentially the same.

Now,

$$\begin{aligned} &\| \xi(\tau_1(x_0) - \rho_1, \bar{\tau}, \bar{x}) - \xi(\tau_1(x_0) - \rho_1, \tau_0, x_0)\| \\ &\quad \leqslant \| \xi(\tau_1(x_0) - \rho_1, \bar{\tau}, \bar{x}) - \xi(\tau_1(x_0) - \rho_1, \tau_0, \bar{x})\| \\ &\qquad + \| \xi(\tau_1(x_0) - \rho_1, \tau_0, \bar{x}) - \xi(\tau_1(x_0) - \rho_1, \tau_0, x_0)\| . \end{aligned} \tag{9.56}$$

We will prove that both terms at the right member are majorized by $\zeta/2$. We note, first, that $\xi(\tau, \tau_0, \bar{x}) = \xi(\tau + \bar{\tau} - \tau_0, \bar{\tau}, \bar{x})$, both members of this equality satisfying the differential Eq. (6.21) and both reducing to $\bar{x} =$ for $\tau = \tau_0$. Thus, the first term at the right member of (9.56) equals

$$\begin{aligned} &\| \xi(\tau_1(x_0) - \rho_1, \bar{\tau}, \bar{x}) - \xi(\tau_1(x_0) - \rho_1 + \bar{\tau} - \tau_0, \bar{\tau}, \bar{x})\| \\ &\quad = \left\| \int_{\tau_1(x_0) - \rho_1 + \bar{\tau} - \tau_0}^{\tau_1(x_0) - \rho_1} \left(\frac{\gamma}{\|\gamma\|^2}\right) d\tau \right\| . \end{aligned} \tag{9.57}$$

Now, by (9.48) and (9.50), the interval of integration is contained in the interval $[\tau_1(x_0) - \frac{1}{16} k \| x_0/2 \|^p, \tau_1(x_0)]$, and, therefore, in the interval (9.49) which in turn (as proved above) is contained in the interval (9.46). Consequently, it follows from (9.45) and (9.43) that

$$x = \xi(\tau, \bar{\tau}, \bar{x}) \subset B\left(x_0, \left\| \frac{x_0}{2} \right\|\right)$$

for τ in the interval of integration. Thus the estimate (9.44) holds for the integrand and the norm of the integral in (9.57) is less than $| \bar{\tau} - \tau_0 | (k \| x_0/2 \|^{p-1})^{-1}$ which is less than $\zeta/2$ by (9.54).

This proves our assertion concerning the first term at the right member of (9.56). But the second term is, by the corollary to Lemma 9.1, not greater than $\| \bar{x} - x_0 \| \exp(\lambda \mid \tau_1(x_0) - \rho_1 - \tau_0 \mid)$, and, therefore, by (9.47), (9.50), and (9.55), not greater than $\zeta/2$.

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