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THE CUTTING CENTER THEOREM FOR TREES

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Abstract. We introduce the *cutting number* of a point of a connected graph as a natural measure of the extent to which the removal of that point disconnects the graph. The *cutting center* of the graph is the set of points of maximum cutting number. All possible configurations for the cutting center of a tree are determined, and examples are constructed which realize them. Using the lemma that the cutting center of a tree always lies on a path, it is shown specifically that (1) for every positive integer n, there exists a tree whose cutting center consists of all the n points on this path, and (2) for every nonempty subset of the points on this path, there exists a tree whose cutting center is precisely that subset.

The *cutting number* $c(\mathrm{v})$ of a point v of a connected graph G has been defined in the preliminary report [2] as the number of pairs of points $\{\mathrm{u,w}\}$ of G such that $\mathrm{u,w} \neq \mathrm{v}$ and every u - w path contains v. Obviously $c(\mathrm{v}) > 0$ if and only if v is a cutpoint of G. The *cutting number* of G is $c(\mathrm{G}) = \max c(\mathrm{v})$ and the *cutting center* of G, denoted by C(G), is the set of all points v such that $c(\mathrm{v}) = c(\mathrm{G})$. We shall determine all possible configurations for the cutting center of a tree. Except for new concepts, we follow the terminology of [1].

The smallest tree with a cutting center is the 3-point tree of fig. 1(a). The trees in figs. 1(b, c) have cutting centers which are the paths P_2 and P_3 of two and three points, respectively. We shall see that these are the smallest examples of the result that for every positive integer n, there is a tree T such that C(T) induces P_n, the path with n points.

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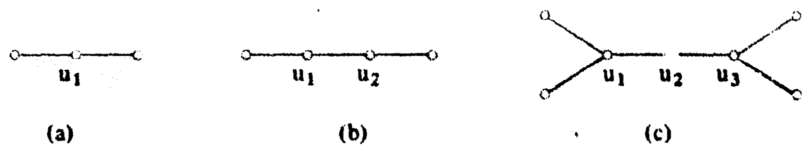

Fig. 1. Trees with small cutting centers.

The following terminology will prove useful. For a tree T and a point v of T, the *(reduced) branches** of T at v are the components of the subgraph T − v. Furthermore, if we have singled out a particular subgraph S of T and v is a point of S, then the branches of T at v which contain points of S will be called *S-branches* and the remainder are the *other branches*, where S will be indicated by context.

Let u, v, w be points of a tree T with p points. It is well known that every pair of points of a tree are joined by a unique path. Obviously u and w belong to the same branch of T at v if and only if the u - w path does not contain v. Consequently, if in T the k branches at v contain $p_1, p_2, ..., p_k$ points, then $c(\mathrm{v}) = \binom{p-1}{2} - \sum_{i=1}^{k} \binom{p_i}{2}$. Note that we have adopted the convention that $\binom{1}{2} = 0$.

Let A be a subset of the point set V of a tree T. There is a unique minimal subtree of T which contains A, namely, the union of all paths which join pairs of points in A.

Theorem 1. *For every tree* T, *the minimal subtree containing the cutting center* C *of* T *is a path.*

For the proof of Theorem 1 we need a lemma.

Lemma 1. *Let* T *be a tree with $p \geq 3$ points,* u *and* w *points of* T, P *the* u - w *path and* v ≠ u,w *a point of* P. *Let r and s be the number of points in the* P*-branches of* T *at* v *which contain* u *and* w, *respectively. If* $c(\mathrm{u}) = c(\mathrm{w}) \geq c(\mathrm{v})$ *then* $3s \geq 2(p-r)$.

Proof of Lemma 1. Let $t = p - r - s$.

If a point of T is not in the P-branch at u, then it is in the P-branch at

* This differs from the conventional definition of the branches of T at v (see [1] p. 35) as these include the point v itself. However, reduced branches are more useful here, and so will be called "branches" in this paper.

v which contains u. Consequently there are at least $p - r = s + t$ points in the P-branch at u. Likewise there are at least $p - s = r + t$ points in the P-branch at w. Thus we have (1) which gives (2) by symmetry:

(1) $$c(\mathrm{u}) \leq \binom{p-1}{2} - \binom{s+t}{2},$$

(2) $$c(\mathrm{w}) \leq \binom{p-1}{2} - \binom{r+t}{2}.$$

There are $t - 1$ points in the other branches at v and $c(\mathrm{v})$ is minimum if they all belong to the same other branch, so

(3) $$c(\mathrm{v}) \geq \binom{p-1}{2} - \binom{r}{2} - \binom{s}{2} - \binom{t-1}{2}.$$

By hypothesis $c(\mathrm{v}) \leq c(\mathrm{u}), c(\mathrm{w})$ and $r^2 - r + 2 < (r + 1)^2$, $s^2 - s + 2 < (s + 1)^2$ because $r, s \geq 1$. Thus (1), (2) and (3) yield

$$2t(s+1) < (r+1)^2,$$

$$2t(r+1) < (s+1)^2,$$

which imply

(4) $$8t^3(s+1) < 4t^2(r+1)^2 < (s+1)^4.$$

It follows from (4) that

(5) $$2t \leq s,$$

The desired conclusion $3s \geq 2(p-r)$ is an immediate consequence of (5) and the definition of t as $p - r - s$.

Proof of Theorem 1. Let T be a tree with p points and S the minimal subtree containing its cutting center C. Suppose S is not a path. Then there is a point v of degree at least 3 in S. Every S-branch at v has at least one point of C. Thus there are points $\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3 \neq \mathrm{v}$ belonging to C which are in distinct S-branches of T at v, containing r, s and t points, respectively. Since these three S-branches are disjoint, we have

(6) $$r + s + t < p.$$

But Lemma 1 implies

$$3s \geq 2(p-r),$$

$$3t \geq 2(p-s),$$

$$3r \geq 2(p-t),$$

which together yield

$$5(r+s+t) \geq 6p$$

which clearly contradicts (6), proving the theorem.

At first we thought that the cutting center of every tree induces a path. There are quite a few counterexamples to this statement, the smallest known case being shown in fig. 2, in which the points in the cutting center are u_1 and u_3.

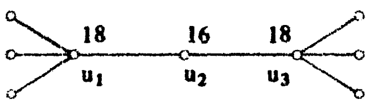

Fig. 2. A tree whose cutting center is not the set of points of a path.

We can now make the strongest possible assertion subject to the restriction imposed by Theorem 1.

Theorem 2. *For every positive integer n and every non-empty subset C of* $\{1, 2, ..., n\}$, *there is a tree* T *containing a path* $u_1 u_2 u_3 ... u_n$ *such that the cutting center of* T *is* $\{u_i | i \in C\}$.

Proof. For $n = 1$, C is necessarily $\{1\}$ and we may take T to be the tree P_3 with 3 points with the point of degree 2 labeled u_1; see fig. 1(a).

The proof for $n \geq 2$ requires some numerical machinery.

For a positive integer n, let $\theta(n)$ be the unique integer $p \geq 2$ such that

$$\binom{p}{2} \leq n < \binom{p+1}{2}. \tag{7}$$

We see immediately that

$$\sqrt{2n} - 1 < \theta(n) < \sqrt{2n} + 1, \tag{8}$$

(9) $$0 \le n - \binom{\theta(n)}{2} \le \theta(n) - 1 .$$

For a non-negative integer n, define $\phi(n)$ inductively:

(10) $$\phi(0) = 0 ,$$
$$\phi(n) = \theta(n) + \phi\left(n - \binom{\theta(n)}{2}\right) , \qquad (n > 0) .$$

The following lemma describes the asymptotic behavior of ϕ, and assists in the proof of Theorem 2.

Lemma 2.
(a). $\sqrt{2n} \le \phi(n) \le 2n$ *for all* $n \ge 0$.

(b) $$\lim_{n\to\infty} \frac{\phi(n)}{\sqrt{2n}} = 1 .$$

Proof. The inequalities (a) are easily verified for $0 \le n \le 5$. We complete the proof of (a) by induction on n, taking $n \ge 6$. Using (9), we know $n \ge \binom{\theta(n)}{2}$. If $n = \binom{\theta(n)}{2}$, then by (10) we have

$$\phi(n) = \theta(n) = \tfrac{1}{2}(1 + \sqrt{1 + 8n}) ,$$

from which the desired conclusion is immediate. If $n > \binom{\theta(n)}{2}$, then (8) and (9) imply

(11) $$1 \le n - \binom{\theta(n)}{2} \le \sqrt{2n} ,$$

from which it follows by the inductive hypothesis that

(12) $$\sqrt{2} \le \phi\left(n - \binom{\theta(n)}{2}\right) \le 2\sqrt{2n} .$$

Combining (8), (10) and (12) yields

(13) $$\sqrt{2n} - 1 + \sqrt{2} \le \phi(n) \le \sqrt{2n} + 1 + 2\sqrt{2n} .$$

Taking the first derivative, we find that the function $2n$ is increasing faster than $3\sqrt{2n} + 1$ for all $n \geq 2$, and for $n = 6$ we have $12 > 1 + 3\sqrt{12}$. Thus $3\sqrt{2n} + 1 < 2n$ for all $n \geq 6$ and we see from (13) that

$$\sqrt{2n} \leq \phi(n) < 2n\ ,$$

proving (a).

Combining (10) with (a), we get

$$(14) \qquad \sqrt{2n} \leq \phi(n) \leq \theta(n) + 2\{n - \tbinom{\theta(n)}{2}\}\ ,$$

and combining (8) and (9) with (14) yields

$$(15) \qquad \sqrt{2n} \leq \phi(n) \leq 3\sqrt{2n} + 1\ ,$$

which improves the bound of (a). Now repeating this process using (15) rather than (a) gives

$$\sqrt{2n} \leq \phi(n) \leq \sqrt{2n} + 3\sqrt{2\sqrt{2n}} + 2\ ,$$

and thereby

$$(16) \qquad 1 \leq \frac{\phi(n)}{\sqrt{2n}} \leq 1 + \frac{3\sqrt{2\sqrt{2n}} + 2}{\sqrt{2n}}\ ,$$

obviously implying (b) and completing the proof of Lemma 2.

The property of the function ϕ which is of interest to us, and which led to its concoction, is that for $n \geq 1$ there is a finite sequence of integers $q_1 \geq q_2 \geq ... \geq q_t \geq 2$ in which $q_1 = \theta(n)$ such that

$$(17) \qquad \sum_{i=1}^{t} q_i = \phi(n)\ ,$$

$$(18) \qquad \sum_{i=1}^{t} \tbinom{q_i}{2} = n\ .$$

This is easily established by induction on n. These same facts can be expressed in a somewhat more usable form. Recall the convention that $\binom{1}{2} = 0$. It then follows immediately from (17) and (18) that for positive integers m and n such that $\phi(n) \leq m$, there is a finite sequence of integers $q_1 \geq q_2 \geq ... \geq q_t \geq 1$ with $q_1 = \theta(n)$ such that

(19) $$\sum_{i=1}^{t} q_i = m ,$$

(20) $$\sum_{i=1}^{t} \binom{q_i}{2} = n .$$

We shall use the following construction in the proof of the theorem: Let $n \geq 2$ be an integer. There is a rational number $\alpha > 2$ such that

(21) $$7^{n-2} \left(1 - \frac{2}{\alpha}\right) < \frac{1}{4} .$$

Lemma 2 insures that there is a positive integer M such that

(22) $$\phi(t) \leq \sqrt{\alpha t} \qquad \text{for all } t \geq M .$$

Condition (21) implies that

$$\lim_{t\to\infty} \left[\binom{t}{2} - 7^{n-2} \left((t+1)^2 - \frac{2}{\alpha}(t-3)^2\right) \left(\frac{2t}{t+1}\right)\right] = \infty .$$

Consequently, there is a positive integer N such that

(23) $$\binom{t}{2} - 7^{n-2} \left((t+1)^2 - \frac{2}{\alpha}(t-3)^2\right) \left(\frac{2t}{t+1}\right) > M \quad \text{for all } t \geq N.$$

Let $\alpha = a/b$ where a, b are relatively prime positive integers. Define

(24) $$p = aN + 3 ,$$

(25) $$b_1 = abN^2 ,$$

(26) $$\beta = \frac{(p+1)^2 - 2b_1}{p+1} ,$$

(27) $$a_1 = p ,$$

(28) $$c = \binom{2p-1}{2} - \binom{p}{2} - b_1 .$$

Obviously (23) implies

(29) $\binom{p}{2} - 2p(7^{n-2}\beta) > M$.

By (24) and (25) we see that

(30) $p > N$,

(31) $(p-3)^2 = \alpha b_1$.

We define a_i and b_i inductively for $i \geq 2$:

(32) $a_{i+1} = \theta(b_i)$,

(33) $b_{i+1} = \binom{2p-1}{2} - \binom{2p-a_{i+1}}{2} - c$,

where we terminate the process at i_0 if $b_{i_0} \leq 0$ or $i_0 = n$.

Lemma 3.

(a). $b_i > \binom{p}{2} - 2p(p-a_i)$ *for* $2 \leq i \leq i_0$,

(b). $p - a_i \leq 7^{i-2}\beta$ *for* $2 \leq i \leq i_0$,

(c). $i_0 = n$,

(d). $\alpha b_i \leq (a_i - 3)^2$ *for* $1 \leq i \leq n$,

(e). $\phi(b_i) \leq a_i - 3$ *for* $1 \leq i \leq n$.

The parts of Lemma 3 of real interest to us are (c) and (e), and the others are used only to establish their validity.

Proof.

(a). (31) implies that $b_1 < \binom{p}{2}$. Consequently, $a_2 = \theta(b_1) < p = a_1$. Combining (28) and (33) we get

(34) $b_i = b_1 + \binom{p}{2} - \binom{2p-a_i}{2}$ for $1 \leq i \leq i_0$,

and in particular we see that $b_2 < b_1$. It follows easily by induction that the sequences $(a_i)_{i=1}^{i_0}$ and $(b_i)_{i=1}^{i_0}$ are non-increasing. Thus it is evident that $b_1 \geq \binom{a_i}{2}$ for $2 \leq i \leq i_0$. Combining this with (34) yields

$$b_i \geq \binom{p}{2} - (2p-1)(p-a_i) \qquad \text{for } 2 \leq i \leq i_0 \,,$$

from which (a) follows directly.

(b). For $1 \leq i < i_0$, (8) and (32) imply

$$(35) \qquad p - a_{i+1} < p + 1 - \sqrt{2b_i} = \frac{(p+1)^2 - 2b_i}{p+1+\sqrt{2b_i}} < \frac{(p+1)^2 - 2b_i}{p+1} \,.$$

In particular,

$$(36) \qquad p - a_2 < \beta \,.$$

In view of (a), (35) implies

$$(37) \qquad p - a_{i+1} < \frac{(p+1)^2 - p(p-1) + 4p(p-a_i)}{p+1} = \frac{3p+1+4p(p-a_i)}{p+1}$$

$$< 3 + 4(p-a_i) \leq 7(p-a_i) \qquad \text{for } 2 \leq i < i_0 \,.$$

(36) and (37) yield (b).

(c). By (a), (b) and (29), $b_i > M$ for $1 \leq i \leq n$ and thus (c) is proved.

(d). By (31) we have $\alpha b_1 \leq (a_1 - 3)^2$. We complete the proof by showing that the sequence $((a_i - 3)^2 - \alpha b_i)_{i=1}^n$ is increasing.

By (34) we have

$$b_i - b_{i+1} = \binom{2p-a_{i+1}}{2} - \binom{2p-a_i}{2} \qquad \text{for } 1 \leq i \leq n-1 \,.$$

Thus

$$\alpha b_i - \alpha b_{i+1} \geq 2(b_i - b_{i+1}) = (a_i - a_{i+1})(4p - a_i - a_{i+1} - 1)$$

$$\geq (a_i - a_{i+1})(a_i + a_{i+1} - 6) = (a_i - 3)^2 - (a_{i+1} - 3)^2 \,,$$

so that

$$(a_{i+1} - 3)^2 - \alpha b_{i+1} \geq (a_i - 3)^2 - \alpha b_i \,.$$

(e). We have seen that $b_i \geq b_n > M$ for $1 \leq i \leq n$. Then by (22) we have $\phi(b_i) \leq \sqrt{\alpha b_i}$, and hence by (d) that $\phi(b_i) \leq a_i - 3$ for $1 \leq i \leq n$.

We are now prepared to complete the proof of Theorem 2.

Proof of Theorem 2 (continued). For $n \geq 2$ let C be a non-empty subset of $\{1, ..., n\}$. Let $p, c, a_1, a_2, ..., a_n, b_1, b_2, ..., b_n$ be the integers determined by n in the preceding construction. We may suppose without loss of generality that $1, n \in C$. Let P be a path with points u_i, $1 \leq i \leq n$, and edges $u_i u_{i+1}$, $1 \leq i \leq n-1$. We shall enlarge P by adding additional branches to each point u_i to obtain a tree T with $2p$ points such that $c(T) = c$ and the cutting center of T is $\{u_i | i \in C\}$.

By Lemma 3 we have $\phi(b_i) \leq a_i - 3$. Thus by (19) and (20) we have for each $1 \leq i \leq n$ a finite sequence of integers $q_{i1} \geq q_{i2} \geq ... \geq q_{i,t_i} \geq 1$ such that $q_{i1} = \theta(b_i)$ and

$$\sum_{k=1}^{t_i} q_{ik} = a_i - 3\,, \tag{38}$$

$$\sum_{k=1}^{t_i} \binom{q_{ik}}{2} = b_i\,. \tag{39}$$

For $i \notin C$, let $s_i = t_i + 1$ and $q_{i,s_i} = 2$. For $i \in C$, let $s_i = t_i + 2$ and $q_{i,s_i - 1} = q_{i,s_i} = 1$. Then by (38) and (39) we see that

$$\sum_{k=1}^{s_i} q_{ik} = a_i - 1 \qquad \text{for } 1 \leq i \leq n\,, \tag{40}$$

$$\sum_{k=1}^{s_i} \binom{q_{ik}}{2} = b_i \qquad \text{for } i \in C\,, \tag{41}$$

$$\sum_{k=1}^{s_i} \binom{q_{ik}}{2} = b_i + 1 \qquad \text{for } i \notin C\,. \tag{42}$$

We shall now add additional branches to each u_i. Each added branch will be a path joined to u_i at an endpoint of itself. To u_1 we add s_1 branches; one with p points and one each with q_{1k} points for $2 \leq k \leq s_1$. For $2 \leq i \leq n-1$, to u_i we add $s_i - 1$ branches; one each with q_{ik} points for $2 \leq k \leq s_i$. To u_n we add s_n branches; one each with q_{nk} points for $1 \leq k \leq s_n$.

Now we shall do some counting. First the total number of points in the other branches at u_i relative to P, i.e. in the branches added at u_i. At u_1 we have

$$p+\sum_{k=2}^{s_1} q_{1k} = p + a_1 - 1 - q_{11} = p + a_1 - 1 - a_2 = 2p - 1 - a_2 .$$

For $2 \le i \le n-1$ we have at u_i

$$\sum_{k=2}^{s_i} q_{ik} = a_i - 1 - q_{i1} = a_i - 1 - a_{i+1} .$$

At u_n we have

$$\sum_{k=1}^{s_n} q_{nk} = a_n - 1 .$$

We note that T has the n points of P and all the points of all the other branches. Thus the number of points in T is

$$n + (2p - 1 - a_2) + \sum_{i=2}^{n-1} (a_i - 1 - a_{i+1}) + a_n - 1 = 2p ,$$

as was claimed earlier.

Next, for $1 \le i \le n-1$, we count the points in the P-branch at u_i which contains u_n. This contains the point u_i and the points of the other branches at u_j for $i+1 \le j \le n$, so we get

$$(n-i) + \sum_{j=i+1}^{n-1} (a_j - 1 - a_{j+1}) + a_n - 1 = a_{i+1} = q_{i1} .$$

Finally, for $2 \le i \le n$, the P-branch at u_i which contains u_1 will have all points of T except those in the P-branch at u_{i-1} which contains u_n. Thus it has $2p - a_i$ points.

Now we are prepared to calculate cutting numbers. The point u_1 has a P-branch with q_{11} points, other branches with q_{1k} points, $2 \le k \le s_1$, and another branch with p points. Thus

$$c(u_1) = \binom{2p-1}{2} - \binom{p}{2} - \sum_{k=1}^{s_1} \binom{q_{1k}}{2} = \binom{2p-1}{2} - \binom{2p-a_1}{2} - \sum_{k=1}^{s_1} \binom{q_{1k}}{2} . \tag{43}$$

For $2 \le i \le n-1$, u_i has P-branches with $2p - a_i$ and q_{i1} points and other branches with q_{ik} points $2 \le k \le s_i$. Thus

(44) $$c(\mathrm{u}_i) = \binom{2p-1}{2} - \binom{2p-a_i}{2} - \sum_{k=1}^{s_i} \binom{q_{ik}}{2} \qquad \text{for } 2 \le i \le n-1\,.$$

Lastly, u_n has a P-branch with $2p - a_n$ points and other branches with q_{nk} points, $1 \le k \le s_n$, so

(45) $$c(\mathrm{u}_n) = \binom{2p-1}{2} - \binom{2p-a_n}{2} - \sum_{k=1}^{s_n} \binom{q_{nk}}{2}\,.$$

Thus for each $1 \le i \le n$, (41), (45) and (33) imply

(46) $$c(u_i) = \binom{2p-1}{2} - \binom{2p-a_i}{2} - b_i = c \qquad \text{if } i \in C\,,$$

(47) $$c(\mathrm{u}_i) = \binom{2p-1}{2} - \binom{2p-a_i}{2} - (b_i+1) = c - 1 \qquad \text{if } i \notin C\,.$$

If v is a point of T not in P then v is of degree either one or two. In the former case, $c(\mathrm{v}) = 0$. In the latter case, v has two branches with a total of $2p-1$ points, so $c(\mathrm{v}) = s(2p-1-s)$ for some $1 \le s \le 2p-2$. Note that

(48) $$\max_{1 \le s \le 2p-2} s(2p-1-s) = p(p-1)\,,$$

and

(49) $$c = \binom{2p-1}{2} - \binom{p}{2} - b_1 = \binom{2p-1}{2} - \binom{p}{2} - \frac{(p-3)^2}{\alpha} > \binom{2p-1}{2} - \binom{p}{2} - \tfrac{1}{2}(p-3)^2$$

By (24) we have $p > 3$, which implies

(50) $$p(p-1) < \binom{2p-1}{2} - \binom{p}{2} - \tfrac{1}{2}(p-3)^2\,.$$

Clearly (48)–(50) imply

(51) $$c(\mathrm{v}) < c \qquad \text{for } \mathrm{v} \notin \mathrm{P}\,.$$

Then we see by (46), (47) and (51) that $c(\mathrm{T}) = c$ and the cutting center of T is $\{\mathrm{u}_i \mid i \in C\}$.

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