

## On Two Discrete-Time System Stability Concepts and Supermartingales\*

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A random discrete-time system  $\{x_n\}$ ,  $n = 0, 1, 2, \dots$  is called stochastically stable if for every  $\epsilon > 0$  there exists a  $\lambda > 0$  such that the probability  $P[(\sup_n \|x_n\|) > \epsilon] < \epsilon$  whenever  $P[\|x_0\| > \lambda] < \lambda$ . A system is shown stochastically stable if some local Lyapunov function  $V(\cdot)$  satisfies the supermartingale definition on  $\{V(x_n)\}$  in a neighborhood of the origin; earlier proofs of stochastic stability require additional restrictions. A criterion for  $x_n \rightarrow 0$  almost surely is developed. It consists of a global inequality on  $\{U(x_n)\}$  stronger than the supermartingale defining inequality, but applied to a  $U(\cdot)$  that need not be a Lyapunov function. The existence of such a  $U(\cdot)$  is exhibited for a stochastically unstable nontrivial stochastic system. This indicates that our criterion for  $x_n \rightarrow 0$  is "tight," and that the two stability concepts studied are substantially distinct.

The application of deterministic Lyapunov functions to stability analysis for nonrandom systems suggests that stochastic forms of such functions may be used in corresponding investigations on random systems. Although the possibility of stochastic Lyapunov functions was recognized over ten years ago, it remained for Bucy [2] and Kushner [6] to separately initiate a systematic theory. They realized that the definition of positive supermartingales [10, p. 131] embodies properties usually ascribed to deterministic Lyapunov functions. In subsequent work, Kushner [7] and Bucy and Joseph [3] refined their analyses, arriving at conclusions that require somewhat different hypotheses.

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One of the aims of the present note is to strengthen and unify some of the theorems appearing in [7] and [3], in particular showing that a stochastic version of Lyapunov stability holds under less demanding hypotheses than the intersection of the respective hypotheses demanded in [3] and [7]. We also find a sufficiency condition under which the system state approaches zero almost surely; this form of stability is comparable to "weak" asymptotic stability for deterministic systems, see [4, pp. 4-5 and 96]. It is of interest to relate the latter type of (stochastic) stability to the stochastic asymptotic stability [3, p. 84] motivated by the analogous deterministic concept. We illustrate this relationship, which is the subject of a new theorem, by an example.

We consider here, as in [2] and [6], the stability of the discrete-time system  $\{x_n\}$ ,  $n = 0, 1, \dots$ , where the  $x_n$  are random variables in the  $m$ -dimensional space  $R^m$ . In systems theory,  $x_0$  is interpreted as a random initial state, from which the system evolves successively to states  $x_1, x_2, \dots$ . Each new state depends in general not only on  $x_0$  and the preceding  $x_k$  but also on current random inputs and/or stochastic system parameters.<sup>1</sup> A discrete-time dynamical system is often described by a stochastic difference equation, e.g.  $x_{n+1} = f_n(x_0, \dots, x_n, y_n)$ , where the random process  $\{y_n\}$  is regarded as the forcing function of the system. A specific application is to dynamical feedback systems with randomly timed instantaneous sampling, which is modeled by  $x_{n+1} = [g_n(x_n, \tau_n)] x_n$ , the  $\tau_n$  being random sampling intervals. The stability of the latter system has been investigated both by direct probabilistic [1] and stochastic Lyapunov function [8] methods.

An intuitively appealing probabilistic version of Lyapunov stability is given by the following definition.

DEFINITION 1a ([3], p. 82). A system is stochastically stable at the origin if for each  $\epsilon > 0$  there exists a  $\lambda > 0$  such that the probability

$$P[\sup_n \|x_n\| > \epsilon] < \epsilon, \quad (1)$$

whenever

$$P[\|x_0\| > \lambda] < \lambda. \quad (2)$$

The properties of the system which assure stochastic stability are expressed in [3] by a stochastic Lyapunov function  $V(\cdot)$ . It is assumed that  $V(\cdot)$  is a scalar function which is zero at the origin, continuous in  $R^m$ , and satisfying an inequality

$$h(\|x\|) \leq V(x), \quad (3)$$

<sup>1</sup> We do not in general assume  $\{x_n\}$  to be Markov, but require some discretionary choice over the distribution of the initial state  $x_0$ .

where  $h(\cdot)$  is continuous and nondecreasing, with  $h(u) = 0$  iff  $u = 0$ . Furthermore,  $\{V(x_n)\}$  is supposed a supermartingale, that is

$$E[V_{n+1} | V_0, V_1, \dots, V_n] - V_n \leq 0 \quad \text{a.s.}, \tag{4}$$

in which "a.s." is the abbreviation of "almost surely," and  $V_n$  replaces  $V(x_n)$  purely for notational convenience.

The value of the result quoted from [3] is compromised by the demand that (4) hold globally, since certain prospective Lyapunov functions satisfy (4) only in some vicinity of the origin, i.e., only for  $\|x_n\| \leq q$ ,  $0 < q < \infty$  (see [6, p. 10], for example). Kushner has recognized the enhanced generality attained by requiring (4) to apply only on an open set of  $R^m$  containing the origin [6, Theorem 1]. On the other hand, Kushner sets more restrictive conditions on  $x_0$  than (2), requiring instead of (2) that expectation

$$EV(x_0) < \rho \tag{5}$$

and

$$\|x_0\| < \eta \quad \text{a.s.}, \tag{6}$$

where  $\rho$  depends on  $\epsilon$ , and  $\eta$  on both  $\rho$  and  $\epsilon$ .

In this note, we prove that (1) actually follows from assumptions most succinctly described as the intersection of the hypotheses of Kushner [6] and Bucy and Joseph [3]. That is, we require (2) as in [3] instead of the stronger (5) and (6) from [6], while demanding that (4) hold only in a neighborhood of the origin (as in [6]) in place of the global requirement of [2, 3]. We could adopt (3), following Kushner [7], but shall only need  $V(x) \geq 0$  and

$$V(x) \rightarrow 0 \quad \text{implies} \quad x \rightarrow 0, \tag{7}$$

which is more convenient than but completely equivalent to (3).<sup>2</sup>

It is assumed hereafter that  $V(\cdot)$  is the function on  $R^m$  previously defined, with (7) replacing Kushner's inequality (3). The supermartingale defining property is presumed to hold locally; for each  $n$ ,

$$E[V_{n+1} | \mathcal{F}_n] - V_n \leq 0 \quad \text{a.s.} \tag{8}$$

on the set  $A_n = \{\omega: \|x_n(\omega)\| \leq q\}$  in probability space, for some (fixed)  $q > 0$ . Here  $\mathcal{F}_n$  is the completion of the  $\sigma$ -field in probability space generated by  $x_0, x_1, \dots, x_n$ .

Our first result extends (8) to a true supermartingale defining relation.

<sup>2</sup> The hypotheses of Bucy and Joseph [3] on  $V(\cdot)$  do not preclude  $V(x) = 0$  in a finite region which may not include the origin. Then  $\alpha(\epsilon) = 0$  for all sufficiently small  $\epsilon$ , and so their proof fails.

LEMMA 1. Define  $\{W_n\}$  to be the random process

$$W_n = V_n \wedge r, \tag{9}$$

where  $r = \inf_{\|x\| \geq q} V(x)$ , and  $u \wedge v = \min(u, v)$ . If  $\{V_n\}$  satisfies (8),  $\{W_n\}$  is a supermartingale adapted to  $\{\mathcal{F}_n\}$ .

*Proof.* Take  $B_n$  to be given by  $B_n = \{\omega: V_n(\omega) < r\}$ .  $B_n$  is clearly measurable on  $\mathcal{F}_n$ , and  $B_n \subset A_n$ . Thus, (8) holds a.s. on  $B_n$ , and since  $V_n = W_n$ ,  $V_{n+1} \geq W_{n+1}$ , the same inequality is valid for  $W_n$  and  $W_{n+1}$ . The supermartingale inequality holds for  $\{W_n\}$  on the complement of  $B_n$  also. In fact, if  $V_n \geq r$ ,  $W_n = r$  and so  $W_{n+1} \leq W_n$ ; hence,  $E[W_{n+1} | \mathcal{F}_n] \leq W_n$ .

A positive supermartingale, such as  $\{W_n\}$  is subject to the inequality [9, p. 81]

$$P[(\sup_n W_n) \geq \epsilon] \leq \epsilon^{-1}EW_0; \tag{10}$$

it is this inequality which is utilized to show stochastic stability. However, the right side of (10) cannot be made small by direct application of a probability statement such as (2). Bucy and Joseph [3] circumvent this problem by an ad hoc calculation on their equivalent of our  $EW_0$ , but we prefer to proceed more systematically. We start by noting that stochastic stability at the origin (Definition 1a) can be expressed in a different way. For this purpose, we shall write  $Z_n \xrightarrow{p} Z$  to indicate that  $\{Z_n\}$  converges to  $Z$  in probability. Then we have as an alternative definition of stochastic stability the following:

DEFINITION 1b. Let  $\{x_n^k\}$ ,  $n = 1, 2, \dots$  be a family of stochastic processes constituting successive states of a system, whose corresponding initial states are respectively  $x_0^k$ ,  $k = 1, 2, \dots$ . The system is stochastically stable at the origin if

$$\|x_0^k\| \xrightarrow{p} 0 \tag{11}$$

implies

$$(\sup_n \|x_n^k\|) \xrightarrow{p} 0. \tag{12}$$

The latter definition facilitates proof of the stochastic stability theorem, for we have the following lemma.

LEMMA 2.  $\|x_0^k\|$ ,  $V_0^k$  and  $W_0^k$  all converge to zero in probability together, where

$$V_n^k = V(x_n^k) \quad \text{and} \quad W_n^k = V_n^k \wedge r. \tag{13}$$

Similarly,  $(\sup_n \|x_n^k\|)$ ,  $(\sup_n V_n^k)$  and  $(\sup_n W_n^k)$  converge in probability to zero together as  $k \rightarrow \infty$ .

*Proof.* The result follows immediately from (7) and the consideration of sufficiently small neighborhoods of  $x = 0$ .

The facts already derived make it possible to use statements on convergence in probability to deduce convergence in probability mean, and in particular to estimate  $EW_0^k$ .

LEMMA 3. *If  $\|x_0^k\| \xrightarrow{p} 0$ , then  $EW_0^k \rightarrow 0$ .*

*Proof.* By Lemma 2,  $W_0^k \xrightarrow{p} 0$ . But each  $W_0^k$  is bounded by  $r$ , and so  $\{W_0^k\}$  is uniformly integrable. A standard result (see [10, Proposition II.5.4]) then yields the desired conclusion. The theorem on sufficiency conditions for stochastic stability at the origin is now within easy grasp.

THEOREM 1. *Let  $V(\cdot)$  be a nonnegative continuous function on  $R^m$  satisfying (7). If (8) holds for every solution sequence  $\{x_n\}$  on the sets  $A_n = \{\omega: \|x_n(\omega)\| \leq q\}$ , the system is stochastically stable at the origin.*

*Proof.* Since we prefer to apply Definition 1b of stochastic stability, we consider  $\{x_0^k\}$  such that  $\|x_0^k\| \rightarrow 0$  as  $k \rightarrow \infty$ . We apply the inequality (10) to the positive supermartingale  $\{W_n^k\}$ , noting that  $EW_0^k \rightarrow 0$  by virtue of Lemma 3. It follows that  $(\sup_n W_n^k) \xrightarrow{p} 0$ . Application of Lemma 2 then yields  $(\sup_n \|x_n^k\|) \xrightarrow{p} 0$ , and the proof is complete.

For deterministic systems, conventional asymptotic stability requires not only  $x_n \rightarrow 0$ , but also Lyapunov stability [4, pp. 4–5]. It is, therefore, natural that a sufficiency condition for asymptotic stability be phrased in terms of a more stringent requirement on the Lyapunov function than is needed to merely insure Lyapunov stability. This deterministic situation again serves as an analog to stochastic systems, for which Bucy and Joseph [3, Theorem 6.3] again offer a stochastic extension.<sup>3</sup> However, we may wish to inquire whether  $x_n \rightarrow 0$  without regard to Lyapunov stability, these two stability notions being quite distinct for deterministic systems [4, p. 96]. In what follows, we shall state a condition assuring  $x_n \rightarrow 0$  in a stochastic system; while the requirement superficially appears more demanding than that on  $\{V_n\}$  in Theorem 1, it does not after all imply the assumptions of that Theorem. Indeed, we shall exhibit a system such that  $x_n \rightarrow 0$  for every initial distribution of  $x_0$ , but which is stochastically unstable in the sense of Definition 1. The property  $x_n \rightarrow 0$  is formalized by the following.

DEFINITION 2. The system  $\{x_n\}$  is weakly almost surely asymptotically stable (WASAS for short) relative to a family  $K$  of random variables, if for any  $x_0 \in K$  we have  $x_n \rightarrow 0$  a.s.

<sup>3</sup> The hypotheses of the referenced theorem require a minor modification, since the stated assumptions are inadequate to assure that  $x \rightarrow 0$  follows from  $\gamma(\|x\|) \rightarrow 0$ . Our  $k(\cdot)$  corresponds to the  $\gamma(\cdot)$  of [3].

This behavior, in combination with stochastic stability at the origin (see Definition 1), leads to this next definition.

**DEFINITION 3.** The system  $\{x_n\}$  is almost surely asymptotically stable relative to  $K$  if it is stochastically stable at the origin as well as WASAS relative to  $K$ .

We now present the proposition giving sufficient conditions that a system is WASAS. The upcoming inequality on  $U(\cdot)$  differs in one vital aspect from (8);  $U(\cdot)$  need not be a Lyapunov function (as in Theorem 1), and indeed may lack most of the properties of the latter.

**THEOREM 2.** Let  $U(\cdot)$  be a measurable scalar function on  $R^m$  which is bounded from below.<sup>4</sup> Assume  $U_n = U(x_n)$  satisfies

$$E[U_{n+1} | \mathcal{F}_n] - U_n \leq -k(\|x_n\|) \quad \text{a.s.}, \tag{14}$$

where  $k(\cdot)$  is a nonnegative function for which  $k(u) \rightarrow 0$  implies  $u \rightarrow 0$ . Then  $\{x_n\}$  is WASAS relative to  $K = \{x_0: EU_0 < \infty\}$ .

**COROLLARY.** If  $U(\cdot)$  is continuous, nonnegative, zero at the origin, and satisfies (7) and (14), the system  $\{x_n\}$  is almost surely asymptotically stable with respect to  $K$ .

*Proof.* The corollary is immediate (from Theorem 1 and Definition 3) once the main Theorem is proved, so we proceed to the latter. Since  $\{U_n\}$  is a supermartingale [by (14)] that is bounded from below, it has a Riesz decomposition (see [9, p. 89] for definitions and proof)

$$U_n = S_n + T_n, \tag{15}$$

where  $\{S_n\}$  is a martingale and  $\{T_n\}$  is a potential. From this decomposition, together with the martingale property, we obtain

$$E[U_{n+1} | \mathcal{F}_n] - U_n = E[T_{n+1} | \mathcal{F}_n] - T_n. \tag{16}$$

We assert that the expression (16) approaches zero a.s. as  $n \rightarrow \infty$ . Indeed,  $T_n \rightarrow 0$  a.s. because  $\{T_n\}$  is a potential; but since a potential is also a nonnegative supermartingale

$$0 \leq E[T_{n+1} | \mathcal{F}_n] \leq T_n \rightarrow 0 \quad \text{a.s.} \tag{17}$$

Hence, (16) approaches zero as claimed, and the same is true of the left side of (14). But  $k(\cdot)$  is nonnegative, so we must have  $k(\|x_n\|) \rightarrow 0$  a.s. and, consequently,  $x_n \rightarrow 0$  a.s. also.

<sup>4</sup> While it suffices for  $U_n \geq Z_n$ , where  $\{Z_n\}$  is a submartingale, there seems little application for the additional generality implied thereby.

An example will serve to illustrate application of Theorem 2 as well as the distinction between WASAS and almost sure asymptotic stability.

EXAMPLE. Consider the scalar system relations

$$x_{n+1} = \begin{cases} \frac{x_n + y_{n+1}}{1 - x_n y_{n+1}}, & x_n \neq -1, \\ 0, & x_n = -1, \end{cases} \quad (18)$$

where  $y_{n+1}$  depends on  $x_0, \dots, x_n$  only through  $x_n$  via a specified conditional probability distribution. For instance, we may assume that  $y_{n+1} = 0$  or  $y_{n+1} = -|x_n|$ , each with probability one-half. Under this supposition, a positive  $x_0$  yields a sequence of positive  $x_n$  which decreases monotonely to zero. On the other hand, if  $-1 < x_0 < 0$  the  $x_n$  become successively more negative until  $x_n < -1$  for some  $n$ . The next state variable,  $x_{n+1}$ , will then be positive, and succeeding ones will decrease monotonically to zero as before.

It is seen from the above that the system described is WASAS relative to the class of all random variables. Nonetheless, there exists for every  $\lambda > 0$  and every positive  $M$  an  $x_0$  such that  $-\lambda < x_0 < 0$ , and for some  $n$ ,  $x_n < -M$ , as a direct calculation will readily verify. Thus, this system is WASAS without being stochastically stable.

Although verification of WASAS is straightforward in this instance. Theorem 2 could have been used for the same purpose. We may take  $U(x) = \tan^{-1} x$ , noting that this definition yields for  $x_n \neq -1$

$$U_{n+1} = U_n + (\tan^{-1} y_n). \quad (19)$$

This  $U(\cdot)$  demonstrates WASAS, since

$$E[U_{n+1} | \mathcal{F}_n] - U_n \leq -\frac{1}{2}(\tan^{-1} |x_n|); \quad (20)$$

moreover,  $K$  is the collection of all possible  $x_0$ , by the boundedness of  $U(\cdot)$ . It is evident that  $U(\cdot)$  is not a stochastic Lyapunov function nor—since  $\{x_n\}$  is not stochastically stable—does a function satisfying the conditions of the corollary exist.

Less transparent elaborations of the preceding example are possible through the specification of other conditional statistics for the  $y_n$ . If (for instance) the conditional density of  $y_{n+1}$ , given  $x_n$ , is uniform from  $-|x_n|$  to  $+\frac{1}{2}|x_n|$ , WASAS seems plausible, but appears difficult to verify in direct fashion. However, an application of the same  $U(\cdot)$  as before, i.e.,  $U(x) = \tan^{-1} x$ , yields

$$E[U_{n+1} | \mathcal{F}_n] - U_n \leq -\frac{1}{3}(\tan^{-1} \frac{1}{2} |x_n|). \quad (21)$$

Equation (21) clearly satisfies condition (14) of Theorem 2. Moreover,  $-\pi \leq U_n \leq \pi/2$ , so that this  $U(\cdot)$  not only meets the other hypothesis of the theorem, but also guarantees WASAS for all initial distributions on  $x_0$ .

We assert that the WASAS system just discussed is again stochastically unstable. To show this, we consider the Markov chain  $\{U_n\}$  over the state space  $[-\pi, 0)$ . For any  $\epsilon > 0$ , let  $A = [-\pi, -\epsilon]$  in this state space. Then for any  $U_0 < 0$  (i.e., any  $x_0 < 0$ ), there exists an  $n$  such that the  $n$  step transition probability from  $U_0$  into  $A$  satisfies

$$p^{(n)}(U_0, A) > 0. \quad (22)$$

It follows (see [5, Lemma 5.1, p. 194]) that the probability of an eventual return to  $A$  can be made arbitrarily close to one by taking a sufficiently large number of steps. In other words,  $x_n$  eventually leaves the  $\epsilon$ -neighborhood of the origin a.s., regardless of the choice of negative initial position  $x_0$ . In fact, one can verify that actually  $U_n \rightarrow -\pi$  a.s. for any  $x_0 < 0$ , which in turn requires  $(\inf_n x_n) < -1$ ; for a proof, one combines the law of large numbers for Markov chains [5, p. 220] with the convergence of the supermartingale  $\{U_n\}$  to a random variable taking on only the values zero and  $-\pi$ .

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