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Rank 3 Extensions of Frobenius Groups

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INTRODUCTION

Recently the study of finite permutation groups of rank 3 has received much attention. No doubt this is due to the fact that the new simple groups of Higman and Sims, McLaughlin, and Suzuki were constructed as rank 3 extensions of known permutation groups. Rank 3 extensions of multiply transitive groups have also been carefully studied. Tsuzuki [11] studied rank 3 extensions of S_n, the symmetric group on n letters. Montague [8] classified the rank 3 extensions of $PSL_2(q)$, $PSU_3(q)$, $Sz(q)$, $R(q)$, and A_n. This paper is concerned with the rank 3 extensions of Frobenius groups. The analogous problem for doubly transitive groups is the classification of all transitive extensions of Frobenius groups, or equivalently the determination of all Zassenhaus groups which are not sharply doubly transitive. The problem for rank 3 groups is much easier than the classification of Zassenhaus groups. This is due to the fact that only a small number of groups occur. The main result of this paper is the following theorem.

THEOREM 1. *Let G be a primitive permutation group of rank 3 on a finite set $\mathscr{G}$. For $A \in \mathscr{G}$ let $\{A\}$, $I(A)$, and $J(A)$ denote the three orbits of G_A. Assume that G has even order and G_A acts faithfully on $I(A)$ as a Frobenius group. Then G satisfies one of the following.*

(1) *G is isomorphic to A_5 acting on the unordered pairs of* $\{1, 2, 3, 4, 5\}$.

(2) *G is solvable and has a regular normal subgroup N of order* 16. *G_A acts on N like $Sz(2)$, the degenerate Suzuki group, or like the Frobenius group of order* 36.

The method of proof of this result is to find the values of certain parameters. The resulting values must be considered one at a time in order to determine which groups actually occur. The parameter conditions come from counting involutions in G and from the standard conditions on the graph of a rank 3 permutation group. These conditions are listed in the next section.

We remark that in view of the Feit-Thompson Theorem [2], the assumption that $|G|$ is even is not very restrictive. Also, the assumption that G_A acts faithfully on $I(A)$ rules out only one group, S_5 acting on the unordered pairs of $\{1, 2, 3, 4, 5\}$. (See [10] for a proof.) All sets and groups considered in this paper are finite.

1. Notation, Parameters, and Involutions

Let $\mathscr{G}$ be a nonempty set. Denote the points of $\mathscr{G}$ by capital letters A, B, C, D, and E, subscripted if necessary. Let G be a group which acts on $\mathscr{G}$ on the right as a transitive permutation group. (All group actions will be right actions.) G has a natural action on $\mathscr{G} \times \mathscr{G}$ given by $(A, B)g = (Ag, Bg)$, for $A, B \in \mathscr{G}$ and $g \in G$. The number of G-orbits of $\mathscr{G} \times \mathscr{G}$ is called the rank of G as a group acting on $\mathscr{G}$. We are concerned with the case in which G has rank 3. Then in addition to the diagonal orbit $\{(A, A) \mid A \in \mathscr{G}\}$, there are two other G-orbits of $\mathscr{G} \times \mathscr{G}$, denoted by I and J. For $A \in \mathscr{G}$ let $I(A) = \{B \in \mathscr{G} \mid (A, B) \in I\}$ and $J(A) = \{C \in \mathscr{G} \mid (A, C) \in J\}$. Then $\{A\}$, $I(A)$, and $J(A)$ are the three orbits of G_A acting on $\mathscr{G}$.

It is possible to associate a graph with the action of G on $\mathscr{G}$. The point set of this graph is just $\mathscr{G}$. The edge set is the orbit I. $\mathscr{G} = \mathscr{G}_I$ will also denote this graph. Since G has even order, the orbit I is self-paired, and so $\mathscr{G}$ is undirected. (See [5].) G acts on $\mathscr{G}$ as a group of collineations of $\mathscr{G}$ which is transitive on the set of edges and the set of nonedges of $\mathscr{G}$. Consequently, $\mathscr{G}$ has a great deal of symmetry. In particular, $\mathscr{G}$ has the following properties.

1.1 (1) $\mathscr{G}$ is regular, i.e., the number of points adjacent to a fixed point of $\mathscr{G}$ is constant.

(2) $\mathscr{G}$ is strong. This means that the number of points adjacent to each of two distinct points A and B in $\mathscr{G}$ depends only on whether or not A and B are adjacent.

These two properties show that the following parameters are well-defined and have a geometric interpretation in the graph.

$$n = |\mathscr{G}|$$
$$i = |I(A)|$$
$$j = |J(A)|$$
$$\lambda = |I(A) \cap I(B)|, \qquad \text{where} \quad A \in I(B).$$
$$\mu = |I(A) \cap I(B)|, \qquad \text{where} \quad A \in J(B).$$

These parameters are not completely independent. The following relations are satisfied.

1.2 (1) $n = 1 + i + j$

(2) $i(i - \lambda - 1) = \mu j$

The second relation comes from enumerating the set $\{(B, C) \mid B \in I(A)$ and $C \in J(A) \cap I(B)\}$ in two ways. Valuable information about the parameters of $\mathscr{G}$ comes from a computation of the eigenvalues of the adjacency matrix of $\mathscr{G}$ and their multiplicities.

1.3 $d = (\lambda - \mu)^2 + 4(i - \mu)$ is a square, or else $\lambda = \mu - 1$ and $i = j = 2\mu$. The degrees of the two nontrivial constituents of the permutation character of G are $h = [(\lambda - \mu + d^{1/2})(n - 1) + 2i]/2d^{1/2}$ and $k = n - h - 1$. In particular h and k are integers.

A proof of 1.3 is contained in D. G. Higman's work on rank 3 permutation groups [5]. Also found in [5] is a proof of the following simple result.

1.4 The following are equivalent.

(1) G is primitive.

(2) $\mathscr{G}$ and its complementary graph are connected.

(3) $\lambda \neq i - 1$, and $\mu \neq i$

If x is an element of G, let $K(x)$ denote the conjugacy class of G which contains x. Computation of $|K(x)|$, especially when x is an involution, leads to additional information about the parameters of G. The best known way of computing $|K(x)|$ is the formula $|K(x)| = G : C_G(x)$. Another way of computing $|K(x)|$ occurs in a paper of Montague [8]. Suppose G is a transitive permutation group on a set $\mathscr{G}$. Let y be an element in G. Consider the set $S = \{(B, z) \mid z \in K(y)$ and $Bz = B\}$. Since G is transitive, the number c of conjugates of y which fix B is independent of the point B. Similarly, the number f of fixed points of z is independent of $z \in K(y)$. Consequently, $|\mathscr{G}|\, c = |S| = |K(y)| f$. If $y = x$ is an involution which fixes at least one point, then $|K(x)|$ is given by:

1.5 $$|K(x)| = |\mathscr{G}|\, c/f.$$

Finally we show how $|K(x)|$ can be computed according to the double cosets of a subgroup $H < G$. Again suppose that G is a transitive permutation group on a set $\mathscr{G}$. Choose $A \in \mathscr{G}$, and let $H = G_A$. Let $I_1, I_2, \ldots, I_m$ be the nontrivial orbits of G_A. For $k = 1, 2, \ldots, m$ choose a point $B_k \in I_k$. Let R_k

be the subgroup of G which fixes A and B_k. Let T_k be a right transversal for R_k in H. (i.e., T_k contains exactly one element from each right coset of R_k in H.) Finally let y_k be an element of G which permutes A to B_k. Then $G = H \cup Hy_1H \cup \cdots \cup Hy_mH$ is the decomposition of G into $H - H$ double cosets. Moreover, $Hy_kH = Hy_kT_k$, and every element $g \in Hy_kH$ has a unique expression $g = hy_kt$ with $h \in H$ and $t \in T_k$.

Let x be an involution in G. Consider a double coset Hy_kH containing a conjugate of x. Set $y = y_k$, $R = R_k$, $B = B_k$, and $T = T_k$. We can assume that y is conjugate to x. Then y interchanges A and B, and so y normalizes $R = G_{A,B}$. Suppose hyt is an involution in HyT, where $h \in H$ and $t \in T$. Then $Ahythyt = A$. But h and $t \in G_A$ implies $Ayth = Ay$, or $Bth = B$. Thus, th fixes both A and B and so is contained in R. Also, $yth = h^{-1}(hyt)h$ is an involution. Since y itself is an involution, y must invert th. Together these two remarks imply that the set of involutions in HyH equals $\{hyt \mid h \in H,\ t \in T, \text{ and } th \in R \text{ is inverted by } y\}$. For an element $z \in R$, there are exactly $|T| = H : R$ elements hyt with $h \in H$ and $t \in T$ such that $th = z$ (i.e., for $t \in T$ let $h = t^{-1}z$). Thus, there are $a\,|T|$ involutions in HyH, where a equals $|\{z \in R \mid z^y = z^{-1}\}|$. In particular, there are at most $|R|\,|T| = |H|$ involutions in HyH.

Actually the above argument proves even more. An involution hyt in HyT is conjugate to the involution $yth \in yR$. Let $b = |K(x) \cap yR|$. Then $|K(x) \cap HyH| = b\,|T|$. If $K(x) \cap Hy_kH = \varnothing$, let $b_k = 0$. Otherwise let $b_k = |K(x) \cap y_kR_k|$, where y_k is a conjugate of x. Then the above argument yields:

$$1.6 \qquad |K(x)| = c + (H : R_1)\, b_1 + \cdots + (H : R_m)\, b_m$$

Here as in 1.5 c equals $|K_H(x)|$, the number of conjugates of x which lie in H. 1.5 and 1.6 can be combined to give the following equation:

$$1.7 \qquad |\mathscr{G}|\, c/f = c + (H : R_1)\, b_1 + \cdots + (H : R_m)\, b_m$$

This equation plays a fundamental role in the proof of Theorem 1. We shall refer to it as the involution equation. Suppose G has rank 3. Let $Q = R_1$, $R = R_2$, $b_Q = b_1$, and $b_R = b_2$. In this case the involution equation becomes:

$$nc/f = c + ib_Q + jb_R,$$

where i, j, and n are the usual parameters of a rank 3 group. This form of the involution equation is used later in the proof.

For the remainder of this paper assume that G is a primitive rank 3 permutation of even order on a set $\mathcal{G}$ such that for $A \in \mathcal{G}$ G_A acts faithfully on $I(A)$ as a Frobenius group. In addition we use the following notation.

$$H = G_A$$
$$Q = G_{A,B}\text{, where } B \text{ is a point in } I(A)$$
$$R = G_{A,C}\text{, where } C \text{ is a point in } J(A)$$
$$M \text{ is the Frobenius kernel of } H$$
$$m = |M|$$
$$q = |Q|$$

The group R is a (possibly degenerate) Frobenius group. By proper choice of the point C we can write $R = M_1Q_1$, where $M_1 = R \cap M$ and $Q_1 = R \cap Q$. M_1 is the Frobenius kernel of R, and Q_1 is a complement. Let $m_1 = |M_1|$ and $q_1 = |Q_1|$. $F(X)$ denotes the set of fixed points of a group $X \leqslant G$. The proof of Theorem 1 breaks down into cases depending on whether $|M|$ is even or odd.

2. The Case $|M|$ Even

In this section we show that there are no groups G with $|M|$ even. Suppose that M_1 has odd order. Then an involution in H fixes exactly one point. Since M is nilpotent of even order, M has exactly one Sylow 2-subgroup S, which is normal in H. Hence, the Sylow 2-subgroups of G are independent. A result of Suzuki implies that G acts doubly transitively on the cosets of $N_G(S)$. (See Bender [1], Proposition 2.3, for a short proof of this result.) But $N_G(S) = H$ since H is a maximal subgroup of G. Since G does not act doubly transitively on the cosets of H, M_1 has even order. In particular $M_1 \neq 1$.

2.1 *M_1 contains no nontrivial normal subgroup of H.*

Proof. Suppose $P \leqslant M_1$ is a normal subgroup of H. Since M_1 fixes C and P is normal in H, P fixes every element of $J(A)$. Since G is primitive, there are points $D \neq E$ in $J(A)$ such that $D \in I(E)$. Then $P \leqslant G_{D,E}$, which is conjugate to $G_{A,B} = Q$. Since $|Q|$ and $|M_1|$ are relatively prime, P must be the identity.

This result implies that M is a Hall subgroup of G. For suppose p is a prime divisor of m. $O_p(M)$, the Sylow p-subgroup of H, is not contained in M_1 by 2.1. Hence, p divides m/m_1, and $n = 1 + m + (mq/m_1q_1)$ is congruent to 1 modulo p.

2.2 *$N_G(M_1)$ acts doubly transitively on $F(M_1)$.*

Proof. $F(M_1)$ consists of A and the fixed points of M_1 in $J(A)$. Choose $C \in F(M_1) \cap J(A)$, and let M^* denote the Frobenius kernel of G_C. M_1 is a proper subgroup of M^* by 2.1. Since M^* is nilpotent, $N_G(M_1) \cap M^*$ properly contains M_1. Consequently, $N_G(M_1)$ does not fix A. But $N_H(M_1)$ acts transitively on $F(M_1) \cap J(A)$ since M_1 is a normal Hall subgroup of $R = H_C$. Hence, $N_G(M_1)$ acts doubly transitively on $F(M_1)$.

2.3 $Q_1 = 1$.

Proof. Suppose $Q_1 \neq 1$. The Frattini argument implies that Q_1 fixes $N_H(Q_1)R : R = N_Q(Q_1) : Q_1$ points in $J(A)$. Then Q_1 fixes $2 + N_Q(Q_1) : Q_1$ points altogether. Since $|Q|$ is odd, Q_1 fixes an odd number of points. Let C be a point in $J(A)$ which is fixed by $R = M_1Q_1$. By 2.2 there is an element $g \in G$ which interchanges A and C and so normalizes R. Thus, $N_G(R) : R$ is even, and the Frattini argument implies that $N_G(Q_1)$ has even order. Let x be an involution in $N_G(Q_1)$. Since Q_1 fixes an odd number of points, there is a point D which is fixed by both Q_1 and x. Then $x \in N_G(Q_1) \cap G_D$. But $N_G(Q_1) \cap G_D$ has odd order since G_D is a Frobenius group with kernel of order $|M|$ and complement of order $|Q|$. This contradiction implies that $Q_1 = 1$.

Let $N = N_G(M_1)$. If $X \leqslant N$, let $\bar{X} \simeq XM_1/M_1$ denote the image of X acting on $F(M_1)$. According to 2.2 and 2.3 $\bar{N}$ is a doubly transitive Frobenius group. Let $Z = \Omega_1(Z(O_2(M)))$, the group generated by the involutions in $Z(O_2(M))$. Z is normal in H and so is not contained in M_1. Thus, $\bar{Z}$ is a nontrivial elementary abelian subgroup of $\bar{M} \cap \bar{N}_A$. Since $\bar{N}$ is a Frobenius group, $\bar{Z}$ is a cyclic group of order 2.

2.4 *Every involution in N centralizes $Z \cap M_1$.*

Proof. Let $U = C_N(Z(M_1))$. U is a normal subgroup of N containing ZM_1. Then $\bar{U} \geqslant \bar{Z} \neq 1$ is a nontrivial normal subgroup of $\bar{N}$ and so is transitive on $F(M_1)$. Since $\bar{Z}$ contains the unique involution in $\bar{N}_A$, $\bar{U}$ contains all involutions in $\bar{N}$. Hence, U contains all involutions in N. The result now follows since $Z(M_1)$ contains $Z \cap M_1$.

2.5 *G contains one class of involutions.*

Proof. The idea of the proof is to show that G contains a normal, simple subgroup G_2 which has a self-centralization system of type 2. Then exceptional character theory can be applied to give the desired result.

Let B denote the fixed point of Q in $I(A)$. Since every G-conjugate of Q in H is already conjugate to Q in H, $N_G(Q)$ acts transitively on $\{A, B\}$. Hence, $N_G(Q) = Q\langle x \rangle$ where x is an involution which interchanges A and B. Q is a Hall subgroup of H. Since q is odd and divides $m - 1$, q is relatively prime to

$n = 1 + m + (mq/m_1)$. Thus, Q is a Hall subgroup of G. Also, since Q is a Frobenius complement of odd order, Q' is a cylic Hall subgroup of Q with a cyclic complement S. Choose S so that $x \in N_G(S)$.

Let $1 \neq X$ be a subgroup of Q. According to 2.3 X fixes only A and B. Consequently, $N_G(X) \leqslant G_{\{A,B\}} = N_G(Q)$. Let $V = C_Q(x)$ and $K = \{y^{-1}y^x \mid y \in Q\}$. Then $KV = Q$, $K \cap V = 1$, and K consists of the elements of Q which are inverted by x. (See [3], page 341.) $K \cap Q'$, $V \cap Q'$, $K \cap S$, and $V \cap S$ are each Hall subgroups of Q. Moreover, $K = (K \cap Q')(K \cap S)$ is a subgroup of Q. Now $N_G(V \cap S) = N_Q(V \cap S)\langle x\rangle = N_{Q'}(V \cap S)S\langle x\rangle \leqslant C_G(V \cap S)$. Since $V \cap S$ is a Hall subgroup of G, Burnside's Transfer Theorem implies that $V \cap S$ has a normal complement G_1 in G. $G_1 \cap Q = (V \cap Q')K$. $V \cap Q'$ centralizes K since $V \cap Q'$ and K are each normal in Q. Thus, the normalizer of $V \cap Q'$ in G_1 equals $(G_1 \cap Q)\langle x\rangle \leqslant C_G(V \cap Q')$. Another application of Burnside's Transfer Theorem yields a normal complement G_2 for V in G.

Suppose $V \neq 1$. Let θ denote the set of primes dividing $|G_2|$. Then V acts on G_2 as a θ'-group of automorphisms. V normalizes $O_2(G_A)$ and $O_2(G_B)$, each of which is a Sylow 2-subgroup of G. The Schur–Zassenhaus Theorem implies that any V-invariant 2-subgroup of G_2 is contained in a V-invariant Sylow 2-subgroup of G_2 and $C_{G_2}(V)$ acts transitively on the V-invariant Sylow 2-subgroups of G_2. (See [3], page 224). Since $C_{G_2}(V) \leqslant K\langle x\rangle \leqslant G_{\{A,B\}}$, $O_2(G_A)$ and $O_2(G_B)$ are the only V-invariant Sylow 2-subgroups of G_2. But then $x \in C_{G_2}(V)$ is contained in $O_2(G_A)$ or $O_2(G_B)$, which is not possible since x does not fix A or B. Hence, $V = 1$ and $Q = K$ is inverted by x. In fact Q is a self-centralization system of type 2 in G. This means that $N_G(Q) : Q = 2$ and $Q = C_G(y)$ for $1 \neq y \in Q$.

Finally we show that G is simple. G' contains $M = [M, Q]$ and $Q = [Q, x]$. Thus G' contains H, whence $G' = G$ by the primitivity of G. Suppose $1 \neq N$ is a normal subgroup of G. Since G is primitive, $NH = G$. Then $G/N = NH/N \simeq H/H \cap N$ is solvable. $G = G'$ implies that $G = N$.

Thus, Q is a self-centralization system of type 2 in the simple group G. A result from exceptional character theory (see [7], page 16) now implies that G has one class of involutions.

Since $O_2(M)$ is a Sylow 2-subgroup of G, Burnside's Fusion Theorem (see [3], page 240) implies that the involutions in Z are conjugate in $N_G(O_2(M)) = H$. Thus, Z contains q involutions, $|Z| = q + 1$, and $|Z \cap M_1| = \frac{1}{2}(q + 1)$. Choose $1 \neq z \in Z \cap M_1$. Then z fixes

$$\tfrac{1}{2}(q - 1) \mid C_H(z)|/| M_1 \mid = (q - 1)m/2m_1$$

points of $J(A)$ and $1 + (q - 1)m/2m_1$ points altogether. The number of G-conjugates of z in H equals aq, a multiple of q.

We are now ready to apply the involution equation of 1.7 to z.

$$\begin{aligned} nc/f &= c + ib_Q + jb_R \\ naq &= f(aq + mb_Q + (mq/m_1)b_R) \\ 1 + m + mq/m_1 &= 1 + m(q-1)/2m_1 + f/a[(m/q)b_Q + (m/m_1)b_R] \\ m_1 + (q+1)/2 &= (f/a)[(m_1/q)b_Q + b_R] \end{aligned}$$

Also, $naq/f = |K(z)| = G : C_G(z) = nmq/|C_G(z)|$. Then $a \mid |C_G(z)| = mf$. Since $C_G(z)$ contains M, a must divide f.

Since q is odd, $(q+1)/2$ is integral, and the above equation implies that fm_1b_Q/aq is integral. But $(q, m_1) = 1$ and a divides f, so that fb_Q/aq must be integral. In particular fm_1b_Q/aq is a multiple of m_1. Since G has one class of involutions, b_R equals the number of elements of $R = M_1$ inverted by an involution $y \in N_G(R) - R$. But 2.4 implies that any involution in $N_G(R)$ centralizes and hence inverts $Z \cap M_1$. It follows that $b_R \geqslant |Z \cap M_1| = (q+1)/2$. Since a divides f, the above equation implies $f = a$, $b_Q = q$ and $b_R = (q+1)/2$. In particular $|C_G(z)| = mf/a = m$, a contradiction to 2.4.

3. The Case m Odd and $M_1 \neq 1$

In this section we show that there is only one group with m odd and $M_1 \neq 1$. The group is solvable and has order 16×36. The method of proof is to study the action of $N_G(M_1)$ on the set F of fixed points of M_1 and apply the resulting information to estimate the parameters of G.

By hypothesis $|G| = (1 + i + j)|H|$ is even. Thus, either $|H|$ is even or else $1 + i + j$ is even. In the latter case i or j is even. Since i and j divide $|H|$, $|H| = |M||Q|$ is even in either case. Hence, Q has even order and so contains a unique involution x. M is abelian since x inverts M.

Let $N = N_G(M_1)$, and for $X \leqslant N$ let $\bar{X}$ denote the image of X acting on the fixed points of M_1. The proof of 2.2 shows that N acts doubly transitively on F.

3.1 *$\bar{N}$ has a regular normal subgroup.*

Proof. Let $U = C_G(M_1)$. Since H is a Frobenius group with an abelian kernel, $U \cap H = M$. $M > M_1$ by 2.1 and so $\bar{U} \geqslant \bar{M}$ is a nontrivial normal subgroup of $\bar{N}$. The double transitivity of $\bar{N}$ implies that $\bar{U}$ is a (3/2)-transitive permutation group on F. Since $\bar{U}_{A,C} = \bar{M}_C = \bar{M}_1 = 1$, $\bar{U}$ is a Frobenius group. Let V denote the regular normal subgroup of $\bar{U}$. V is characteristic in $\bar{U}$ and so is normal in $\bar{N}$.

V is an elementary abelian r-group for some prime r. Suppose $|V| = r^e$.

$\bar{N}_A = \overline{N_H(M_1)} = \overline{MN_Q(M_1)}$ acts on V as a group of linear transformations. Moreover, $\bar{M}$ is a normal abelian subgroup of $\bar{N}_A$. Since $\bar{N}_A$ acts transitively on the nonzero vectors of V and hence primitively on V, $\bar{M}$ has exactly one Wedderburn component. i.e., V is a sum of isomorphic irreducible $\bar{M}$ modules. Identify V with the additive group of $GF(r^e)$. Let $K = GF(r)$, and let L denote the subring of $\mathrm{End}_K(V)$ generated by $\bar{M}$. Then V is a sum of isomorphic irreducible L modules. Since M is abelian, L is commutative, and so L is contained in $\mathrm{End}_L(V)$. It follows that V is irreducible as an $\mathrm{End}_L(V)$ module. Schur's Lemma implies that the endomorphism ring of V as an $\mathrm{End}_L(V)$ module is a division ring. Since L is contained in this ring, L is a finite field. Let k be the dimension of V as a vector space over L. Suppose $L = GF(r^c)$. Since $C_Q(\bar{M}) = 1$, $\overline{N_Q(M_1)}$ acts faithfully on L as a group of field automorphisms. $\mathrm{Aut}(L)$ is a cyclic group of order c. $\overline{N_Q(M_1)}$ contains an involution $\bar{x}$, where x is the unique involution in Q. Hence, $c = 2b$ is even. $\bar{x}$ inverts $r^b + 1$ elements of $L^* = L - \{0\}$. Since $\bar{x}$ inverts $\bar{M}$, $\bar{M}$ has order at most $r^b + 1$. $\bar{N}_A$ acts transitively on $V - \{0\}$, and so $|V| - 1$ divides $|\bar{N}_A|$. Then $2b(r^b + 1) \geqslant |\bar{M}|\,|\overline{N_Q(M_1)}| = |\bar{N}_A| \geqslant (r^{2bk} - 1)$. The only possibilities are $b = k = 1$ and $r = 2$ or 3. Since $|\bar{M}|$ is odd, $r = 2$. Thus, $L = GF(2^2)$, and $|F| = |V| = 4$. Since $|\bar{N}_A| = 6$, $\bar{N}$ has order 24 and so acts on F like S_4. The proof of the following result is now complete.

3.2 (1) $\bar{N}$ acts on $F(M_1)$ like S_4.

(2) $|\bar{M}| = M : M_1 = 3$, and M is a 3-group by 2.1.

Suppose $z \in N_Q(M_1)$ and $\bar{z} = 1$. Then $1 = [\bar{M}, \bar{z}] = \overline{[M, z]}$. It follows that z is contained in $C_Q(M/M_1) = 1$. Since $|\overline{N_Q(M_1)}| = 2$, Q_1 has order 1 or 2. But the involution $x \in Q$ normalizes M_1, and so permutes the 4 fixed points of M_1, which consist of A and 3 points in $J(A)$. Since x fixes A, x must fix one of the three points in $F(M_1) \cap J(A)$, whence $Q_1 = \langle x \rangle$ has order 2. Some conjugate of x normalizes either $Q = G_{A,B}$ or $R = G_{A,C}$. Since both Q and R have even order, $|G|$ is divisible by 4. But $|G| = n\,|H|$, and $n = 1 + m + (3/2)q$ is congruent to $(1/2)q$ modulo 2. Hence, q is divisible by 4.

The remainder of this analysis consists of a consideration of the parameters of G. As above let x denote the involution in Q. Since $x \in Q_1 < R$, x fixes $(1/2)q$ points in $J(A)$ and $2 + (1/2)q$ points altogether. Since x has m conjugates in H, the involution equation implies:

$$nm/[2 + (q/2)] = m + mb_Q + (3/2)\,qb_R$$
$$m - 1 + q = q[2 + (q/2)][(b_Q/|Q|) + (b_R/|R|)]$$

3.3 $b_R = m_1$ or $m_1 + 1$.

Proof. Recall that M_1 is a 3-group, and $N_G(M_1)$ acts on $F = \{A, C, D, E\}$

like S_4. x acts on F like the transposition (DE). There is a conjugate y of x which acts on F like the transposition (AC). Then y normalizes $M_1\langle x\rangle = R = G_{A,C}$. Moreover, y inverts M_1 since y is conjugate to x in $N_G(M_1)$ and x inverts M_1. Hence, y inverts $m_1 + 1$ elements of R, the elements of M_1 and exactly one of the m_1 involutions in $R = M_1\langle x\rangle$. Since M_1 has odd order, all of the m_1 involutions in yM_1 are conjugate. It follows that $b_R = m_1$ or $m_1 + 1$.

The above result shows that m is approximately equal to $q^2/4$. The next result shows that $m \leqslant 6q - 6$. Together these two results enable us to compute m and q.

3.4 $m \leqslant 6q - 6$.

Proof. The parameters of $\mathscr{G}$ have the following values: $i = m$, $j = mq/m_1q_1 = 3q/2$, and $\mu = i(i - \lambda - 1)/j = 2m_1(m - \lambda - 1)/q$. Since q divides $m - \lambda - 1$ and $\mu < m = 3m_1$, q must equal $m - \lambda - 1$. Then $\mu = 2m_1$ and $\lambda = m - q - 1$. We get:

$$\begin{aligned} d &= (\lambda - \mu)^2 + 4(i - \mu) \\ &= (m_1 - q - 1)^2 + 4m_1 \\ &= (m_1 - q)^2 + 2m_1 + 2q + 1 \end{aligned}$$

Since d is a square, there is a positive integer b such that $(m_1 - q + b)^2 = d$. Then $2b(m_1 - q) + b^2 = 2m_1 + 2q + 1$, and we get the following equations:

$$\begin{aligned} 2m_1(b - 1) + b^2 - 1 &= 2q(b + 1) \\ 2m_1 + b + 1 &= 2q[(b + 1)/(b - 1)] \\ m_1 + (b + 1)/2 &= q[1 + 2/(b - 1)] \end{aligned}$$

Clearly, $b \geqslant 3$, whence $2/(b - 1) \leqslant 1$. Hence, $2m_1 + 4 \leqslant 4q$, or equivalently $m = 3m_1 \leqslant 6q - 6$.

Suppose $b_R = m_1$. Then the involution equation implies $m - 1 + q = (2 + q/2)(b_Q + q/2)$. Since $m \leqslant 6q - 6$, it follows that $(2 + q/2)(b_Q + q/2) \leqslant 7q - 7$. In particular $(2 + q/2)\, q/2 \leqslant 7q - 7$, and so $q < 24 - 28/q$. This inequality implies $q \leqslant 22$. Since 4 divides q but 3 does not, only $q = 4$, 8, 16, or 20 are possible. If $q = 20$, then $m + 19 = 12(b_Q + 10)$, contrary to the fact that 3 divides m. If $q = 16$, then $m + 15 = 10(b_Q + 8)$, a contradiction since 5 does not divide m. If $q = 8$, then $m + 7 = 6(b_Q + 4)$, again contradicting the fact that 3 divides m. Hence, only $q = 4$ is possible. Since $m \geqslant 9$ is a power of 3, $b_Q = 1$ and $m = 9$.

G has parameters $i = m = 9$, $j = 6$, $n = 16$, $\lambda = 4$, and $\mu = 6$. The order of G equals $nmq = 16 \times 36$. G is solvable and so has a regular normal elementary abelian subgroup N of order 16. The Frobenius group H acts on N as a group of linear transformations having 3 orbits of vectors of lengths 1, 6, and 9. Since there is exactly one conjugacy class of such Frobenius groups of order 36 in $GL_4(2)$, G is determined up to permutation equivalence.

A similar analysis can be applied to the case $b_R = m_1 + 1$. There are no solutions to all of the parameter equations, and so no groups occur in this case.

4. The Case m Odd and $R \leqslant Q$

In this section we show that A_5 and a solvable group of order 320 are the only groups in which m is odd and $R \leqslant Q$. The proof amounts to examining the solutions to the involution equation. Unfortunately, a great deal of computation is necessary.

Let $t = Q : R$, so that $j = mt$ and $n = 1 + (t + 1)m$. H acts on $J(A)$ as a transitive permutation group. The cosets of Q in H induce a system of imprimitivity for H acting on $J(A)$. Each imprimitive block contains t points, and the action of H on these blocks is equivalent to the action of H on $I(A)$. Q stabilizes one block and is transitive on the points of this block. Let x denote the unique involution in Q. There are m conjugates of x in H, one in each of the m conjugates of Q in H.

4.1 If $x \notin R$, then $t = 2$, $q = m - 1$, and $\lambda = 0$.

Proof. Suppose $x \notin R$. Then x fixes only A and B. The involution equation yields:

$$nm/2 = m + mb_Q + mtb_R$$
$$(t + 1)(m - 1) + t = 2(b_Q + tb_R) \leqslant 4q$$

Since q divides $m - 1$ and $t > 1$, only $t = 2$ and $m = q - 1$ is possible. $| I(A) \cap I(B)| = \lambda$ is divisible by q since $Q = G_{A,B}$ acts semiregularly on $I(A) \cap I(B)$. The primitivity of G implies $\lambda < m - 1$ and so $\lambda = 0$.

For the remainder of the analysis we assume $x \in R$. Then $x \in Z(Q)$ implies that x fixes each point of the block which x stabilizes. Thus, $| F(x)| = t + 2$. Since H has only one class of involutions, $C_G(x)$ acts transitively on $F(x)$. $C_G(x)_A = C_H(x) = Q$ fixes A and B and acts transitively on $F(x) - \{A, B\} = F(x) \cap J(A)$. Hence, $\{A, B\}$ is an imprimitive block for $C_G(x)$ acting on $F(x)$, and we have the following result.

4.2 (1) 2 divides t.

(2) $C_G(x)$ acts doubly transitively on the set $\bar{F}$ of $C_G(x)$ images of $\{A, B\}$.

Since x fixes $t + 2$ points and has m conjugates in H, the involution equation applied to x yields:

$$nm/(t + 2) = m + mb_Q + mtb_R$$
$$m - 1 = [(t + 2)/(t + 1)](b_Q + tb_R) \leqslant [(t + 2)/(t + 1)]\, 2q$$

Note that the second equation implies that $t + 2$ divides $m - 1$. Since q divides $m - 1$ and $t > 1$, the only possibilities are $m - 1 = q$ or $2q$.

4.3 *Suppose z is an involution acting on a group G. If z inverts more that $(3/4) \mid G \mid$ elements of G, then G is abelian and z inverts G.*

Proof. Let $K = \{y \in G \mid y^z = y^{-1}\}$. Suppose g, h, and gh are contained in K. Then $h^{-1}g^{-1} = (gh)^{-1} = (gh)^z = g^z h^z = g^{-1}h^{-1}$, whence g and h commute. It follows that the group generated by $K \cap Kg$ centralizes g. Since $\mid K \mid > (3/4) \mid G \mid$, $\mid K \cap Kg \mid > (1/2) \mid G \mid$, and so $K \cap Kg$ generates G. Thus, $K \leqslant Z(G)$. The only possibility is that $K = G$ is an abelian group which is inverted by z.

4.4 *If $m - 1 = 2q$, then $\lambda = 0$ and $t \in \{2, 4, 6\}$.*

Proof. Suppose $m - 1 = 2q$ and $t > 6$. Then $(t + 1)(m - 1) = (t + 2)(b_Q + tb_R) \leqslant (t + 2)(b_Q + q)$ implies $b_Q \geqslant [t/(t + 2)]q > (3/4)q$. Let z denote a conjugate of x which interchanges A and B. Then z inverts at least b_Q elements of Q. According to 4.3 Q is abelian and z inverts Q. Hence, Q is cyclic and $b_Q/\mid Q \mid = 1$ or $1/2$. Similarly, $b_R/\mid R \mid = 1$ or $1/2$. Thus, only $t = 2$ is possible. This contradiction implies $t \leqslant 6$. Since G is primitive and $m - 1 = 2q$, $\lambda = 0$ or q. It follows easily from the fact that d is a square that $\lambda = q$ is impossible. Hence, $\lambda = 0$ and $t \in \{2, 4, 6\}$.

Now we assume $m - 1 = q$. The primitivity of G implies $\lambda = 0$. Also, H is a doubly transitive Frobenius group. The involution equation reduces to:

$$q = [(t + 2)/(t + 1)](b_Q + tb_R)$$
$$1 = 1/(t + 2) + (b_Q/\mid Q \mid) + (b_R/\mid R \mid)$$

The idea now is to solve the above equation for t. We will show:

4.5 $t \in T = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 22, 28, 34, 38, 46, 82\}$

Suppose Q is isomorphic to the multiplicative group of one of the 7 exceptional near-fields (See [4], page 391). From a knowledge of the subgroups of $SL_2(5)$ and the fact that both t and $t + 2$ divide $m - 1 = q$, it follows that $t \leqslant 12$. Hence, $t \in T$. For the remainder of this analysis we assume that Q is isomorphic to the multiplicative group of a nonexceptional near-field. A Sylow 2-subgroup S of Q is cyclic or generalized quaternion. In either case Q has a normal 2-complement. In addition, if S is generalized quaternion, then Q has a normal subgroup of index 2 which is the direct product of $O(Q)$ and a cyclic 2-group. (See [4], page 390.) R also has these properties, and the following analysis applies to R as well as Q.

Recall that b_Q is just the number of involutions in $zQ \cap K(x)$, where z is a conjugate of x which interchanges A and B. In particular b_Q is the number of involutions in some union of Q-classes of involutions in zQ. We show that all numbers of this form can be written as $(1/2)[(1/a) + (1/b)] \mid Q \mid$, where a and b are positive integers. Since Q has an odd number of Sylow 2-subgroups, we can assume that z normalizes S. If $y \in O(Q)$ and zy is an involution, then zy is $O(Q)$—conjugate to z by Sylow's Theorem. Hence, we are really interested in the S-classes of involutions in zS.

Suppose $S = \langle y, w \mid y^2 = w^{2^{k-1}}, w^y = w^{-1}\rangle, k > 1$, is generalized quaternion. Table 1 below lists the possible actions of z on S (with respect to appropriate choices of y and w) and the resulting Q-classes of involutions in zQ. Let $c = 2^{k-1}$, and for $g \in Q\langle z\rangle$ let $K_g = \{h \in O(Q) \mid h^g = h^{-1}\}$. x denotes the unique involution in Q.

Suppose $S = \langle u\rangle$ is a cyclic group of order 2^k. Table 2 lists the possible actions of z on S and the resulting Q-classes of involutions in zQ. Again let $c = 2^{k-1}$.

TABLE 1

Action of z	Q-Classes of Involutions in zQ
1. $y^z = y$ $w^z = w$	$\{zv \mid v \in K_z\}, \{zxv \mid v \in K_z\}$
2. $y^z = y^{-1}$ $w^z = w$	$\{zv, zxv \mid v \in K_z\}, \{zyw^av \mid v \in K_{zy}, a \text{ odd}\}$ $\{zyw^av \mid v \in K_{zy}, a \text{ even}\}$
3. $y^z = y^{-1}$ $w^z = w^{-1}$	$\{zw^av \mid v \in K_z, a \text{ even}\}, \{zw^av \mid v \in K_z, a \text{ odd}\}$ $\{zyv, zyxv \mid v \in K_{zy}\}$
4. $y^z = yw$ $w^z = w^{-1}$	$\{zw^av \mid v \in K_z\}$
5. $y^z = y^{-1}$ $w^z = w^{c+1}$	$\{zv, zxv \mid v \in K_z\}, \{zyw^av \mid v \in K_{zy}, a \text{ even}\}$
6. $y^z = y^{-1}$ $w^z = w^{c-1}$	$\{zyv, zyxv \mid v \in K_{zy}\}, \{zw^av \mid v \in K_z, a \text{ even}\}$

TABLE 2

Action of z	Q-Classes of Involutions in zQ
1. $u^z = u$	$\{zv \mid v \in K_z\}, \{zxv \mid v \in K_z\}$
2. $u^z = u^{-1}$	$\{zu^av \mid a \text{ even}, v \in K_{zu^a}\}, \{zu^av \mid a \text{ odd}, v \in K_{zu^a}\}$
3. $u^z = u^{c-1}$	$\{zu^av \mid a \text{ even}, v \in K_{zu^a}\}$
4. $u^z = u^{c+1}$	$\{zv, zxv \mid v \in K_z\}$

If g is an involution acting on $O(Q)$, then $|K_g|$ equals the number of involutions in $O(Q)\langle g\rangle$, and so $|O(Q)| = |K_g|\,|C_{O(Q)}(g)|$ by Sylow's Theorem. In particular $|K_g|$ divides $|O(Q)|$. Also, $|K_g| = |K_h|$ for any element h which is Q-conjugate to g. Thus, tables 1 and 2 show that the number of involutions in any Q-class of involutions in zQ can be written as $(1/2a)\,|Q|$. Moreover, $b_Q/|Q|$ can be written as $(1/2)[(1/a) + (1/b)]$, where a and b are (not necessarily distinct) positive integers. Similarly, $b_R/|R| = (1/2)[(1/c) + (1/d)]$. The involution equation reduces to:

$$1 = [1/(t + 2)] + (1/2)[(1/a) + (1/b) + (1/c) + (1/d)]$$

The above equation has only finitely many solutions. These solutions yield values of t which lie in T.

4.5 *The possible values for the parameters of* $\mathcal{G}$ *are*:

1. $t = 2, \quad \mu = 1, \quad m = 3$
2. $t = 2, \quad \mu = 2, \quad m = 5$
3. $t = 2, \quad \mu = 4, \quad m = 9$
4. $t = 6, \quad \mu = 4, \quad m = 25$
5. $t = 18, \quad \mu = 20, \quad m = 361$

Proof. The possible values of t must be examined one at a time. We use the facts: $\mu = (m - 1)/t$, $d = (\lambda - \mu)^2 + 4(i - \mu)$ is a square, $t + 2$ divides $m - 1$ unless $t = 2$, and $h = [(\lambda - \mu + d^{1/2})(n - 1) + 2i]/2d^{1/2}$ is an integer. Also, since M is abelian and has at most 2 Q-classes of nonidentity elements, M is elementary abelian. In particular m is a prime power.

Suppose $t = 2$. A simple computation yields $h = 4(2\mu + 1)/(\mu + 2)$. Hence, $\mu + 2$ divides 12, and $\mu = 1, 2, 4$, or 10. $\mu = 10$ is impossible since then $m = 2\mu + 1 = 21$ is not a prime power.

Suppose $t > 2$. Since $d = \mu^2 + 4(t - 1)\mu + 4$ is a square, there is an integer b such that $\mu^2 + 4(t - 1)\mu + 4 = (\mu + b + 2)^2$. Then $4(t - 2)\mu = 2b\mu + b^2 + 4b$, and so $b = 2c$ is even. Since $t - 2 \neq 0$, $\mu = (c^2 + 2c)/(t - 2 - c)$. For a fixed value of t there are only finitely many values of c which yield integral values of μ. The only values of t and c which satisfy all other parameter conditions are listed in 4.5. We illustrate this process in the case $t = 18$.

Suppose $t = 18$. Then $m = 18\mu + 1$, and $\mu = (c^2 + 2c)/(16 - c)$. Clearly, $0 < c < 16$ since μ is positive. The possible integral values of μ are 2, 7, 10, 20, 42, 65, 112, and 255. The corresponding values of m are 37, 127, 181, 361, 757, 1171, 2017, and 4591. However, since $t + 2 = 20$ divides $m - 1$, only $m = 181$ or 361 are possible. If $m = 181$, then $h =$

$181 \cdot 344/56$ is not integral. Hence, the only possibility is $\mu = 20$ and $m = 361 = 19^2$.

4.6 $G \simeq A_5$, or G is solvable of order 320.

Proof. If $t = 2$ and $\mu = 1$, then $n = 10$ and $|G| = 60$. Since n is not a prime power, G is not solvable. The only possibility is that G is isomorphic to A_5 acting on the unordered pairs of $\{1, 2, 3, 4, 5\}$. If $t = 2$ and $\mu = 2$, then $n = 16$ and G has order 160 or 320. In either case G is solvable and so has a regular normal elementary abelian subgroup N of order 16. $H = G_A$ acts on N as a subgroup of $GL_4(2)$ of order 10 or 20. The only possibility is $|H| = 20$ and H acts on N as a degenerate Suzuki group $Sz(2)$.

Suppose $t = 2$ and $\mu = 4$. Then $n = 28$ and $|G| = 1008$ or 2016. G is not solvable since $n = 28$ is not a prime power. Consequently, G must have a normal subgroup N isomorphic to $PSL_2(7)$ or $PSL_2(8)$, and G is isomorphic to a subgroup of $\mathrm{Aut}(N)$. Since $\mathrm{Aut}(PSL_2(7)) \simeq P\Gamma L_2(7)$ has order $2 \cdot 168$ and $\mathrm{Aut}(PSL_2(8)) \simeq P\Gamma L_2(8)$ has order $3 \cdot 504$, this is not possible.

Suppose $t = 6$ and $\mu = 4$. Then $m = 25$ and $q = 12$ or 24. Also, $\theta_1(1) = 55$ and $\theta_2(1) = 120$, where θ_1 and θ_2 are the two nontrivial constituents of the permutation character of G acting on $\mathscr{G}$. θ_1 and θ_2 are integral characters, and $\theta_1 + \theta_2$ vanishes on $M^\# = M - \{1\}$. Assume $q = 24$. Then all elements of $M^\#$ are conjugate. Hence, there is an integer b such that $\theta_1(g) = b = -\theta_2(g)$ for all $g \in M^\#$. $(\theta_1|_M, 1_M)_M = 2 + b + [(5 - b)/25]$ and $(\theta_2|_M, 1_M)_M = 5 - b - [(5 - b)/25]$ are nonnegative integers. The only possibility is $b = 5$. But then $(\theta_2|_M, 1_M)_M = 0$, whereas by Frobenius reciprocity $(\theta_2|_M, 1_M)_M \geqslant (\theta_2|_H, 1_H)_H = 1$. This contradiction shows that $q = 24$ does not occur. A similar argument rules out $q = 12$.

Finally suppose $t = 18$ and $\mu = 20$. In this case $q = m - 1 = 360$ and $|R| = q/t = 20$. Consider the action of $U = C_G(x)$ on the right cosets of $N = N_G(Q)$. According to 4.2(2), U acts doubly transitively on the 10 right cosets of N in U. Let $\bar{U}$ denote the image of U acting on the right cosets of N. Since a Sylow 5-subgroup of Q is normal in Q and contained in R, $\bar{U}$ has order 360 or 720. The only possibility is that $\bar{U}$ has a normal subgroup of index at most 2 which acts like $PSL_2(9)$. But $PSL_2(9)$ has elementary abelian Sylow 3-subgroups, whereas $\bar{U}$ has cyclic Sylow 3-subgroups. This contradiction shows that no groups occur in this case.

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