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MODELLING REGIONAL WASTE WATER TREATMENT SYSTEMS

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Abstract—The economies of scale of waste water treatment plants favor regional systems. This paper presents a mathematical formulation of this problem, and suggests an algorithm for solution. Several case examples are shown.

INTRODUCTION

WITH the demand of the public for a cleaner environment the cost of water pollution control will increase. In the U.S. alone, the former Federal Water Pollution Control Administration (1969) has estimated that in the next 5 years capital outlays for municipal treatment plants should be close to $8 billion, and about $6 billion for sanitary sewer construction. These large expenditures will have to compete with expenditures for other public needs and it is only reasonable to expect that these funds should be used in an economic and efficient way.

In general, no great breakthrough in sewage treatment technology can be expected, at least not cost-wise. One of the hopes of lowering the costs is the joining of industries and municipalities in common treatment facilities to take advantage of the economies of scale in waste water treatment.

It is the purpose of this paper to show how the methods of operations research and systems analysis may be used to plan for more efficient new water pollution control facilities or how to upgrade existing systems. The major question investigated is this: Given a number of communities and/or industries in a geographical area, where should treatment plants be built, how many, at what time, and which intercepting sewers are necessary to connect the municipalities and industries to these plants, such that the total cost of waste water collection and treatment is a minimum.

There are many other advantages to regional systems which are not easily expressed in economic terms. Among them are that there is a central authority with complete responsibility for systems expansion and operation, which eliminates the many problems of adequately staffing and training the operators of small treatment plants. This should lead to better qualified personnel, better management of the plants, and top performance of the treatment plants. Studies by WESTON (1971) indicate, for example, that there is a high correlation between the size of the treatment plant and the percentage of time during which the plant fails to perform according to design standards. The larger the treatment plant, the more reliable it is. An important economic aspect is that larger authorities will generally receive higher bond ratings and thus may borrow at lesser cost. But none of these considerations are explicitly considered in this study.

In the first part of the paper general cost data of the various parts of pollution control works are presented. This is followed by typical example situations. The mathematical formulation of the problems is then presented and an approach for a solution is indicated.

Cost of waste water treatment and conveyance

The cost of waste water treatment is composed of the amortization of the construction costs and the annual operation and maintenance costs.

For the U.S. a recent study by SMITH (1968) provides good data. They show that the construction costs of activated sludge plants can be expressed approximately as

$$C_c = 0.56 \times Q^{0.78}, \qquad (1)$$

where:

C_c = construction cost in million dollars
Q = design flow in MGD.

The operation and maintenance costs follow similar economies of scale. Based on the assumption that the plant is financed by bonds bearing 4.5 per cent interest over a period of 25 yr, the total annual costs may be expressed by the following formula:

$$C_t = 67 \times Q^{0.78} \qquad (2)$$

C_t = annual costs in thousands of dollars
Q = design flow in MGD.

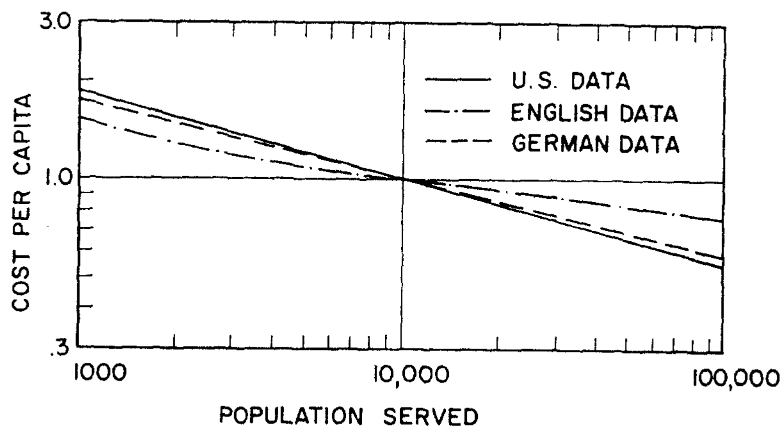

FIG. 1. Relative construction costs of activated sludge plants.

For other types of treatment plants similar economies of scale exist. What should be noted, is that the total cost of waste water treatment in a community of 10,000 people (corresponding to a plant size of 1 MGD) is roughly 18.4 cents per 1000 gal treated. In a city of 1 million people the cost falls to 6.6 cents, or roughly one-third of the cost of treatment in the smaller community.

That these large economies of scale are not only typical for the U.S., but also for other countries, has been shown in a study by DEININGER (1969). FIGURES 1 and 2 show the relative construction, and operation and maintenance costs, for activated sludge plants in the U.S., England and Germany. Since it is quite difficult to compare absolute costs in this world of changing monetary systems, the construction cost and operation cost *per capita* in a city of 10,000 was arbitrarily set equal to one. It should be noted that these graphs also show that the *per capita* costs of waste treatment are

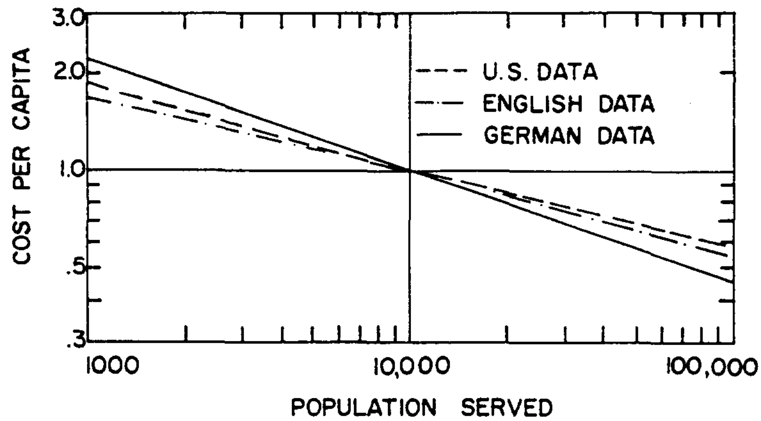

FIG. 2. Relative operation costs of activated sludge plants.

three times higher in a small town of 1000 compared to the *per capita* costs in a city of 100,000.

The costs of main trunk sewers and interceptors depend on a number of factors. Among those are the costs of right-of-way, cost of the pipes, costs of excavation, prevailing soil conditions and the slopes of the terrain. Hydraulic considerations, such as maximum and minimum velocities in the pipes determine the allowable flow. Little data is available on the general costs of trunk sewers as a function of their capacity. BAUER (1962) in a study of regional sewerage systems in the Chicago area reported the following equation as being typical:

$$C_s = 40{,}000 \times Q^{0.50} \tag{3}$$

C_s = costs per mile of trunk sewer
Q = amount of waste water conveyed (MGD).

This cost is based on an ENR index of about 1000. Again, its use and applicability for a specific case will have to be investigated, but in general it should be adequate for evaluating alternate solutions.

In another study by SPENCER (1958) cost data were reported for the Buffalo area in New York. This data shows, that the construction cost of trunk sewers based on an ENR index of about 1000 could be described by

$$C_s = 46{,}000 \times Q^{0.55} \tag{4}$$

If waste water has to be transported uphill, from one community to another one, or if the sewers are becoming too deep, pumping stations will be necessary to lift the waste water. General data on the costs of such stations are very scarce, but they also show large economies of scale as indicated in a study by BENJES (1960).

In certain cases pressure mains may be required. A study by LINAWEAVER and CLARK (1968) indicates that the cost of water transmission in mains can be expressed as

$$C_m = 1865 \times Q^{0.6}, \tag{5}$$

where

C_m = annual cost per mile
Q = daily flow in MGD.

Again, this formula is based on various assumptions regarding interest rate, power costs, efficiencies, roughness coefficient, and an ENR index of 877. Its applicability to a local problem has to be evaluated carefully, but again the economies of scale should not change significantly and the data are reasonably representative for initial planning purposes.

EXAMPLES AND PROBLEM FORMULATION

The general location problem of treatment plants can not be viewed as one with an infinite solution space; that is plants may not be located at will on the plane. The number of possible sites is usually rather restricted due to zoning regulations, the location of a body of water into which discharge is permissible, and other considerations. And thus the problem is one of selecting from a finite number of candidate sites the ones which will be most economical.

Example 1

Consider, for example, the case of two communities each with 10,000 people located along a major river as shown in FIG. 3. Assume furthermore, that there are only two sites available for treatment plants close to the cities. The question is therefore should each community build its individual treatment plant, or is a joint treatment plant more economical?

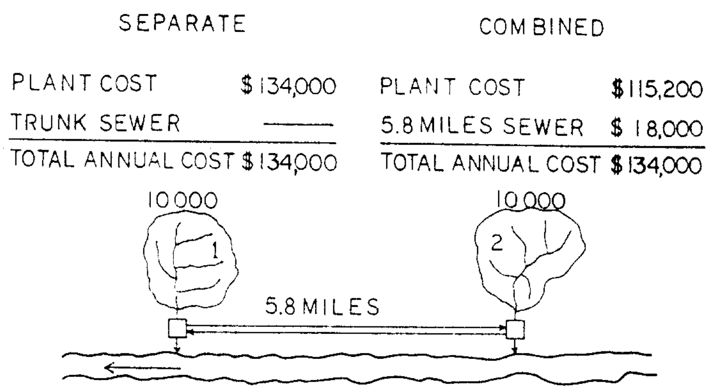

FIG. 3. Cost comparison of two communities.

Based on equation (2) the annual cost of waste treatment for a city of 10,000 is about \$67,000. This assumes a *per capita* flow of about 100 gal day^{-1}. If the wastes of both communities would be treated at a central plant, the total annual cost would be \$67,000 $\times$ $2^{0.78}$ = \$115,200.

The annual savings due to joint treatment would be about \$(2 $\times$ 67,000 − 155,200), or roughly \$18,800.

Based on equation (3) the construction cost of a trunk sewer to carry 1 MGD is \$40,000. Using a 4.5 per cent interest rate and a 25-yr amortization period, and adding 20 per cent of the amortization costs as an estimate of the maintenance cost, the annual cost of a trunk sewer is about \$3240 mile^{-1}. And thus if the distance between the two communities is less than 5.8 miles, a joint treatment plant is more economical.

Example 2

Consider now the case with more than two communities along a river as shown in Fig. 4. First consider only downstream transfer of wastes.

The problem here is locating m treatment plants ($1 \leqslant m \leqslant n$) at the n locations. This can be done in

$$\binom{n-1}{m-1}^*$$

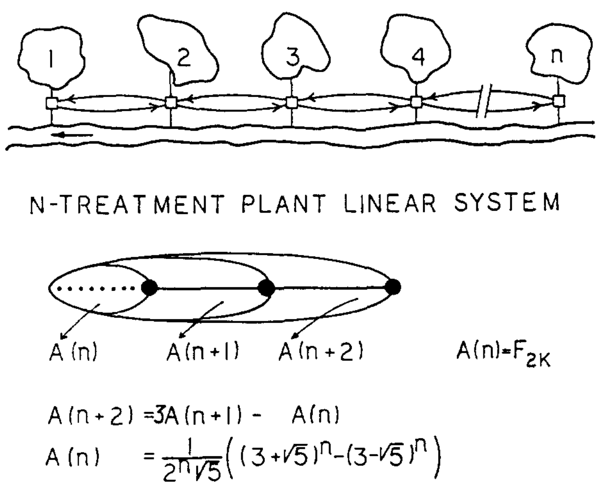

Fig. 4. A linear system of N treatment plants.

ways. This follows from the fact that one treatment plant must be located at the furthest downstream community and that there are

$$\binom{n-1}{m-1}$$

ways in which to locate m-1 treatment plants at n-1 locations. Since we do not know *a priori* how many treatment plants should be built, we have to investigate all combinations for $m = 1, 2, 3, \ldots, n$-1 plants. Thus the total number of combinations to be investigated is

$$A(n) = \sum_{m=1}^{n-1} \binom{n-1}{m-1}.$$

* $\binom{a}{b} = \dfrac{a!}{b!\,(\kappa - b)!}$.

It can be shown that this summation is equal to 2^{n-1}. To show the order of magnitude of the number of combinations to be investigated, a table of $A(n)$ vs. n follows.

Number of communities (n)	Number of combinations ($A(n)$)
2	2
4	8
6	32
8	128
10	512
15	16,854
20	524,288

For a small number of plants a straight forward combinatorial approach is feasible with present day computing systems, however, for large n an evaluation of all combinations is not feasible. In the latter case a dynamic programming formulation is possible which will lead to the optimal solution without evaluating all combinations. This approach has been reported previously by DEININGER (1966).

Consider now that also upstream transfer of the wastes is considered. Due to economies of scale it is known that it would not be economical to "split" the waste flow of one city, that is, one would not simultaneously transfer the wastes upstream and downstream. An interesting question is how many economical solutions exist? The only variables in the problem formulation are the interconnections between cities. Assume a zero for no transport within cities, a 1 for upstream transport for wastes, and a 2 for downstream transport of wastes. For n cities there are $n - 1$ branches between the cities, each of which may take on three different values.

So the total number of economical solutions would be $3^{(n-1)}$ were it not the requirement that a city may not simultaneously transport wastes upstream *and* downstream. For three cities the total number of solutions may be represented as:

00
01
02
10
12
20
21*
22.

The eighth solution here is ruled out, since we do not allow transport of water from city 2 simultaneous to 1 and 3. Thus the total number of economical solutions will be A (3) = 8. How many economical solutions are there for n cities?

Let $A(n + 2)$ stand for the number of solutions for $n + 2$ cities, $A(n + 1)$ for the solution to $n + 1$ cities, and $A(n)$ for the solutions for n cities.

Then, the following recursive relation can be established:

$$A(n + 2) = 3\ A(n + 1) - A(n).$$

This relation may be deduced by the following reasoning. Given the value of $A(n+1)$, the adding of one city increases the number of solutions to $3\,A(n+1)$ since the new branch may assume the values of 0, 1, or 2. However, of these total numbers there are some which are not economical, namely, all those which end in a 2 1 sequence. But the number of those is exactly $A(n)$.

Upon generating the number of solutions for varying n the similarity of the series to the Fibonacci number series was noted:

n	1	2	3	4	5	6
$A(n)$	1	3	8	21	55	144

And thus it was concluded that the total number of economical solutions for n cities is

$$A(n) = F_{2n},$$

where F_k stands for the kth Fibonacci number. This still does not indicate which of the F_{2n} solutions is the most economical one, but places an upper bound on the total number of solutions, and allows to calculate the number of solutions based on Moivre's formula as

$$A(n) = \frac{1}{2^n\sqrt{5}}\left[(3+\sqrt{5})^n - (3-\sqrt{5})^n\right]$$

Example 3

Consider now the sample problem shown on FIG. 5. Seven Communities are located along a main river and its tributary. Seven sites are available for building a treatment plant. The objective is to find the number and location of treatment plants such that the total costs are minimized.

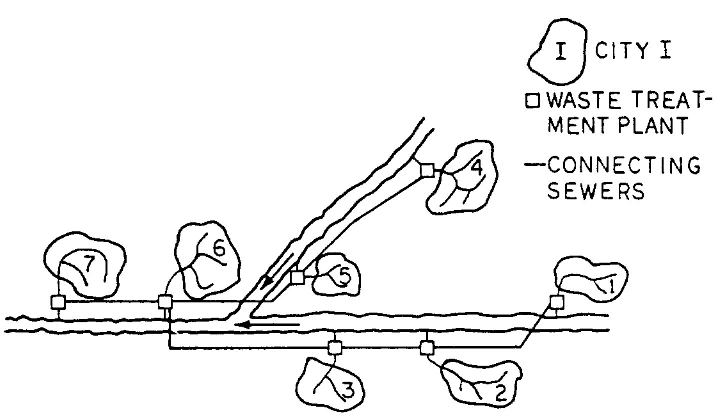

FIG. 5. A branched system.

FIGURE 6 shows the abstraction of this problem into a general network problem. The branches connecting the individual cities represent the interconnecting sewers,

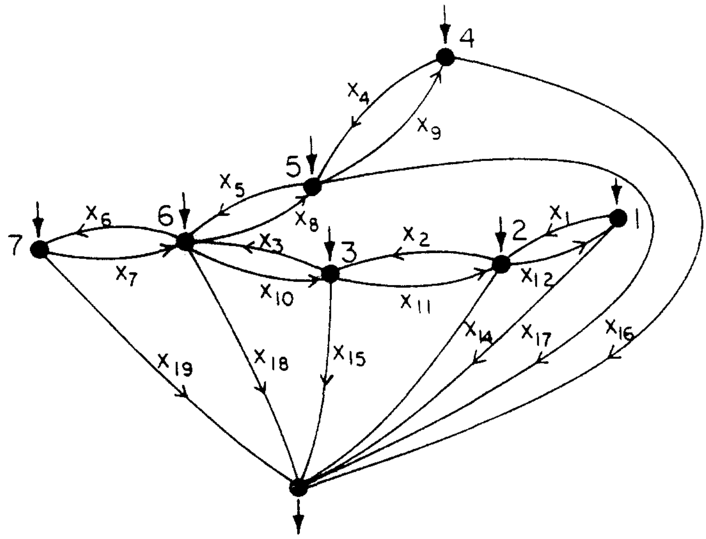

FIG. 6. The abstract network.

while the branches connected to the unnumbered node represent the possible treatment plants. Flows are entering the network at nodes 1–7, and the objective is to find the least cost paths for these flows to the effluent node.

MATHEMATICAL FORMULATION

All of the examples presented previously are network problems. Mathematically, they can be stated in the following general form:

$$\text{minimize:} \quad \sum_i \sum_j f_{ij}(x_{ij})$$

$$\text{subject to:} \quad \sum_j x_{ij} - \sum_j x_{ji} = \sum_j (x_{ij} - x_{ji}) = a_i$$

$$x_{ij} \geq 0; \quad \sum_i a_i = 0$$

where:

x_{ij} = amount of wastes transported from node i to node j
$f_{ij}(x_{ij})$ = costs of transporting x_{ij} units of waste from node i to node i
a_i > 0 implies an input node
a_i = 0 implies a transhipment node
a_i < 0 implies an output node.

SOLUTION APPROACHES

A general solution of the problems formulated poses considerable difficulty, since all the cost functions are concave and the objective is to minimize. Previous work by Graves (1969) and Young *et al.* (1970) lead only to local optima.

In some earlier work by Deininger (1966) the nonlinear functions were approximated by a linear one and the resultant linear optimization problem was solved. Based on the solution of the problem a new linear approximation was obtained and the network problem was solved again. This process was repeated until the new solution would not differ from the previous one. This approach again leads to a local optimal solution. Choosing different starting points leads to different local optima, of which the best was selected. Procedure was not very satisfactory, and alternate approaches were explored.

Among those a method proposed by Cabot and Francis (1970) appeared to be promising. Since the original method dealt only with a quadratic cost function, it was extended and generalized. Briefly, the method proceeds as follows:

Define as problem P1:

$$\text{minimize } f(x) = \sum_{j=1}^{n} f_j(x_j)$$

$$Ax = b$$

$$0 \leqslant x \leqslant B,$$

where A is a given matrix of order $m \times n$, b is a given vector of order $n + 1$, B is a given vector of order $n \times 1$, and x is a vector of variables of order $n \times 1$. Note that all $f_j(x_j)$ in P1 are of the form

$$f_j(x_j) = c_j x_j^{d_j}, j = 1,2, \ldots, n$$

where $0 \leqslant d_j \leqslant 1$, and c_j, d_j are given constants.

Now let S be the set of feasible solutions of P1, and let f^* be the optimal value of P1. It is well known that (i) the summation of the concave functions $f_j(x_j)$ is still a concave function, (ii) if S is nonempty, and $f(x)$ is concave, then the optimal solution of P1 occurs at the extreme point of the convex set S. Because there possibly exist several local minima for P1, any ordinary convex programming algorithms usually lead to a stationary point which is not necessarily a global minimum. The procedure is therefore to formulate a related linear program for P1 and then to apply Murty's (1968) ranking extreme point approach to it to obtain an optimal solution to the original problem. This procedure may be described as follows:

Each of the functions $f_j(x_j)$ may be rewritten as

$$f_j(x_j) = (c_j/x_j^{1-d_j})x_j.$$

Now let $u_j = \min(c_j/x_j^{1-d_j})$, subject to $x \in S$, then a related linear programming problem can be formulated:

P2:

$$\text{minimize } g(x) = \sum_{j=1}^{n} u_j x_j$$

$$\text{subject to } x \in S.$$

Since all the variable x_j are bounded, the problem of minimizing $(c_j/x_j^{1-d_j})$ is trivial. Every x_j is set equal to B_j; that is

$$u_j = c_j/B_j^{1-d_j}.$$

Now it can be shown that

(i) For any $x \in S$, $g(x) \leqslant f(x)$;
(ii) if x^0 is an optimal solution to P2, then $f_l = g(x^0)$ is a lower bound on f^*, and $f_u = f(x^0)$ is an upper bound on f^*;
(iii) given any upper bound f_u on f^*, let x^k be the set of all extreme points of P2, such that $g(x^k) \leqslant f_u$, then P1 has an optimal solution x^*, such that $x^* \in \{x^k\}$.

The algorithm starts with an initial basic feasible solution of P1, namely the optimal solution of P2, and then generates continuously upper and lower bound solutions until both bounds converge or the number of iterations exceeds prespecified limits.

The advantage of the algorithm is that once the procedure terminates the true optimum is obtained. The disadvantage of this algorithm is that the optimal solution may not be obtained until all the extreme points of S have been enumerated by the ranking algorithm.

In general, there is no way to estimate the number of extreme points which have to be ranked. However, in the author's experience the worst case amounted to about 40 per cent of the possible extreme points.

A NUMERICAL EXAMPLE

To show a numerical example, the network shown in FIG. 6 will be used.

It is assumed that the construction costs of waste treatment plants are adequately described by equation (1), and that the costs of conveying the waste water may be described by equation (3). In the branches where upstream transfer is possible, an arbitrary 20 per cent will be added to the costs. Since the algorithm requires that each variable be bounded, an upper bound B_j will be set equal to the maximum possible flow in each arc. The waste inputs from every city, the distances between cities, the cost functions, and the upper bounds are all summarized in TABLE 1.

FIGURE 7 shows the optimal solution. For this particular example, the optimal solution is to build only one treatment plant at city 6 and to convey all the wastes to that plant.

Based on a computer code written in FORTRAN IV for an IBM 360/67 the optimal solution was reached in 233 iterations. Over 900 extreme points were generated and the computation time was about 5 s.

CAPACITY EXPANSION AND OTHER ASPECTS

The problems and formulations described up to now consider the regional waste treatment as independent of time. That is, the inputs to the network are not varying

TABLE 1. COST FUNCTIONS AND DATA FOR THE NUMERICAL EXAMPLE OF FIG. 6

j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
a_j	0.3	0.2	2.1	0.7	0.1	2.5	0.7												
M_j	1	3	2.5	2	3	5	5	3	3	2.5	3	1							
B_j	0.3	0.5	2.6	0.7	0.8	5.9	0.7	3.2	3.3	3.2	4.3	4.5	6.6	6.6	6,6	6.6	6.6	6.6	6.6

a_j = Wastewater from city j (MGD).
M_j = Distance in miles for arc j.
B_j = Maximum possible flow in arc j (upper bound).

Waste conveyance costs:

$(x_j) = M_j \cdot (0.04 \times X_j^{0.50})$ for $j = 1$–6

$= 1.2\, M_j \cdot (0.04 \times X_j^{0.50})$ for $j = 7$–12

Waste treatment costs:

$f(x_j) = 0.56 \cdot X_j^{0.78}$ for $j = 13$–19

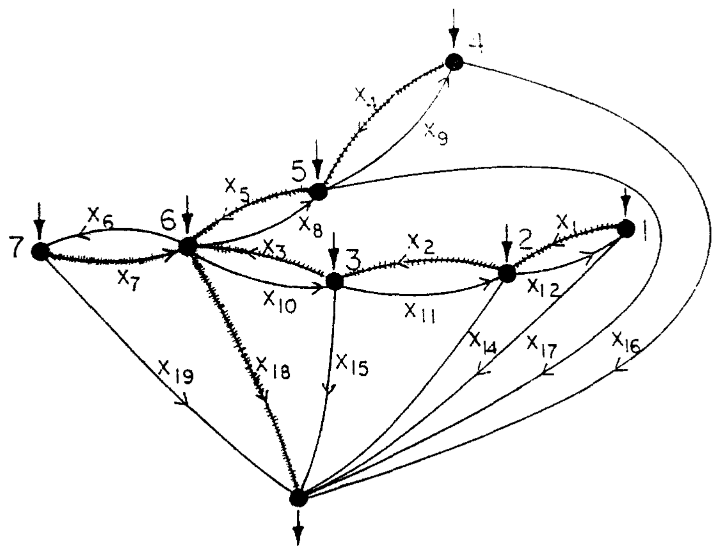

FIG. 7. Abstract network and optimal solution.

with time. If one considers the growth of communities and the resulting increase in waste water flow, then the question of an optimal capacity expansion of the network arises.

These considerations may be dealt with by establishing a finite planning horizon, which is subdivided into m individual time spans. The magnitude of these time spans should be such that the increase in waste water flow within the time span is measurable. The problem may then be stated formally as:

$$\text{minimize} \quad \sum_{i}^{n} \sum_{j}^{n} f_{ij}(K_{ij}) + \sum_{i}^{n} \sum_{j}^{n} \sum_{k}^{m} f_{ijk}(\Delta K_{ijk})$$

subject to:

$$\left.\begin{array}{l} \sum_{j} (x_{ijk} - x_{jik}) = a_{ik} \\ \sum_{i} a_{ik} = 0 \\ x_{ijk} \leqq K_{ij} + \sum_{p=1}^{k} \Delta K_{ijp} \end{array}\right\} \begin{array}{l} i = 1, 2, \ldots n \\ j = 1, 2, \ldots n \\ k = 1, 2, \ldots m \end{array}$$

where:

x_{ijk} = amount of waste water moved from node i to node j in time span k
a_{ik} = amount of waste water entering node i at time span k

K_{ij} = original capacity connecting node i to node j
ΔK_{ijk} = increase in capacity between node i and j during time span k
$f_{ij}(K_{ij})$ = cost of providing capacity K_{ij}
$f_{ijk}(\Delta K_{ijk})$ = cost of providing additional capacity during span k.

This problem is in its structure similar to the previous ones and the same algorithms for a solution apply. The number of variables has increased significantly, and the computation times will increase, but should in general pose no difficulty for large computer systems. The same formulations, of course, also applies to the expansion of existing systems.

One other aspect of regional systems deserves consideration. In the models described previously the degree of treatment was considered as fixed and no consideration was given to the impact of the waste discharges on the water course. As long as the possible discharge points to a river are close to each other (say within 10 miles), the degree of treatment is determined by the required water quality in this reach of stream. However, if the possible discharge points are distributed over a considerable length of the river, the natural self-purification of the river may come into play and the quality in the stream depends then on the location of the waste water discharge. The water quality of a stream at any one point can be described as a linear function of all upstream waste discharges. If certain water quality standards are imposed, one may write linear inequalities for a number of control points, and add those to the models described previously. However, the current trend to install everywhere secondary and possibly tertiary treatment makes these considerations less important.

SUMMARY AND CONCLUSION

The large economies of scale in waste water treatment and transport indicate that regional solutions for pollution control may be more economical than individual solutions. This paper describes a general formulation of the problems and an approach for solving the resulting minimization problems with concave objective functions. Based on the experience with the computer code written it appears that small to medium sized problems can be solved in reasonable time. The bottleneck for the implementation of such regional plans will be the political and institutional arrangements.

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