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Tensor Products of Closed Operators on Banach Spaces

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Let A and B be closed operators on Banach spaces X and Y. Assume that A and B have nonempty resolvent sets and that the spectra of A and B are unbounded. Let α be a uniform cross norm on $X \otimes Y$. Using the Gelfand theory and resolvent algebra techniques, a spectral mapping theorem is proven for a certain class of rational functions of A and B. The class of admissable rational functions (including polynomials) depends on the spectra of A and B. The theory is applied to the cases $A \otimes I + I \otimes B$ and $A \otimes B$ where A and B are the generators of bounded holomorphic semigroups.

1. INTRODUCTION

Let A and B be bounded operators on Banach spaces X and Y, respectively. Let α be a uniform cross norm [10] on the algebraic tensor product $X \otimes Y$. We will denote the closure of $X \otimes Y$ in α by $X \hat{\otimes}_\alpha Y$. Since α is uniform, $A \otimes B$ is a bounded operator on $X \hat{\otimes}_\alpha Y$, in fact $\| A \otimes B \| = \| A \| \| B \|$, for all bounded operators A and B. Therefore, polynomials $P(A, B)$ in A and B are well-defined bounded operators and it is natural to ask how the spectrum of $P(A, B)$ is related to the spectra of A and B. In the case when X and Y are Hilbert spaces Brown and Pearcy [2] showed that $\sigma(A \otimes B) = \sigma(A)\,\sigma(B)$. Schecter [11] generalized this result to Banach spaces and arbitrary polynomials by proving that

$$\begin{aligned}\sigma(P(A, B)) &= P(\sigma(A), \sigma(B))\\ &= \{P(\mu, \lambda) \mid \mu \in \sigma(A), \lambda \in \sigma(B)\}.\end{aligned}$$

The purpose of this paper is to extend this result to the unbounded case where, however, there are restrictions on $\sigma(A)$ and $\sigma(B)$ and the

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polynomial P. There are also results in the unbounded case by Ichinose (see the discussion below). We remark that the spectral mapping theorem is trivial (even in the unbounded case) if A and B are self-adjoint. So, the importance of this result is in the case of unbounded non-self-adjoint operators like the generators of bounded holomorphic semi-groups.

In Sect. 2 we consider the case where A and B are bounded and prove (Theorem 1) the spectral mapping theorem of Dash–Schecter [3] for a class of rational functions $f(A, B)$ of A and B (called AB-rational functions). The proof of the direction $f(\sigma(A), \sigma(B)) \subset \sigma(f(A, B))$ generalizes the argument of Pearcy [2] and makes use of a corollary of Hartog's theorem. The proof in the direction $\sigma(f(A, B)) \subset f(\sigma(A), \sigma(B))$ uses the Gelfand transform to prove a slightly stronger result than Schecter, namely that the resolvents of $f(A, B)$ are contained in the commutative Banach algebra $\mathscr{R}(A) \hat{\otimes} \mathscr{R}(B)$ generated by the resolvent algebras, $\mathscr{R}(A)$ and $\mathscr{R}(B)$, of A and B (Theorem 3). We also prove (Theorem 2) that all the multiplicative linear functionals on $\mathscr{R}(A) \hat{\otimes} \mathscr{R}(B)$ are of the form $l_1 \otimes l_2$ where l_1 and l_2 are multiplicative linear functionals on $\mathscr{R}(A)$ and $\mathscr{R}(B)$, respectively. These results are necessary for our proof of the unbounded case. Simon [13] has generalized Theorem 2 to prove that every $l \in \sigma(\mathfrak{A}_1 \otimes \mathfrak{A}_2)$ is of the form $l_1 \otimes l_2$ if $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are any commutative algebras of operators.

In Sect. 3 we consider the case where A and B are closed unbounded operators with nonempty resolvent sets. Since A and B are unbounded, it is not even a priori clear how to define $f(A, B)$. We take a very strong definition which essentially requires that "$f(A, B)$" be approximable in norm resolvent sense by elements of $\mathscr{R}(A) \hat{\otimes} \mathscr{R}(B)$. This definition allows us to use the machinery developed in Sect. 2 to prove the main theorem of this paper (Theorem 4), namely, that

$$\begin{aligned}\sigma(f(A, B)) &= \overline{f(\sigma(A), \sigma(B))} \\ &= \overline{\{ f(\lambda, \mu) \mid \lambda \in \sigma(A), \mu \in \sigma(B)\}}.\end{aligned}$$

We remark that it is already clear from considering self-adjoint operators that the closure is necessary in the case of unbounded operators.

In order to use Theorem 4 on a given operator one must prove that it is approximable in our sense by elements of $\mathscr{R}(A) \hat{\otimes} \mathscr{R}(B)$. To show how this may be done, we prove in Sect. 4 two theorems in the case where A and B are the generators of bounded holomorphic semigroups. Theorem 5 states that the closure of $A \otimes I + I \otimes B$

on the "natural" domain D_0 of finite linear combinations of vectors $\varphi \otimes \psi$, $\varphi \in D(A)$, $\psi \in D(B)$, generates a holomorphic semigroup and $\sigma(A \otimes I + I \otimes B) = \sigma(A) + \sigma(B)$. Theorem 6 states that under fairly general hypotheses the closure of $A \otimes B$ on D_0 generates a holomorphic semigroup and $\sigma(A \otimes B) = \sigma(A)\,\sigma(B)$. As far as we know, this is a new result in the theory of semigroups.

It is appropriate to comment on the limitations of Theorem 4 and its relation to the work of Ichinose. It follows from the proof of Theorem 4 that in the case where f is a polynomial P, the range of P on $\sigma(A) \times \sigma(B)$ is closed; i.e., P will not be approximable in our sense unless the range of P on $\sigma(A) \times \sigma(B)$ is closed. The authors originally hoped [9] that Theorem 4 would cover other cases, but stronger techniques are necessary. The requirement that the range of P be closed on $\sigma(A) \times \sigma(B)$ is a hypothesis in the work of Ichinose [6] who also requires that there exists a path in $\mathbb{C}\backslash(\sigma(A) \times \sigma(B))$ which is sufficiently "close" to $\sigma(A) \times \sigma(B)$ and which is not too "long." Under these hypotheses, Ichinose proves the spectral mapping theorem in the case of polynomials. It is not clear how much overlap there is between his result and ours since in any application, our result requires the construction of resolvent approximates and this requires the construction of special paths. We also use integrals of resolvents over paths, but because of the Banach algebra techniques our paths may be chosen to be very simple and need not be "close" to $\sigma(A) \times \sigma(B)$. In any case it is useful to have both methods available.

We will always use the letter σ to denote "spectrum." The kind of spectra involved will be clear from the additional notation involved. For example, if $\mathfrak{A}$ is a commutative Banach algebra, then $\sigma(\mathfrak{A})$ denotes the multiplicative linear functionals on $\mathfrak{A}$; if $A \in \mathfrak{A}$, then $\sigma_{\mathfrak{A}}(A)$ denotes the spectrum of A as an element of $\mathfrak{A}$ and $\sigma(A)$ denotes the spectrum of A as an operator on the underlying space.

2. The Bounded Case

Throughout this section A and B will denote bounded operators on Banach spaces X and Y, respectively. α will always denote an arbitrary but fixed uniform cross norm on $X \otimes Y$. We begin by defining the class of functions which we will deal with:

Definition. By an AB-rational function $f(z, \zeta)$ on $\sigma(A) \times \sigma(B)$ we mean a finite linear combination of functions of two complex variables of the form $g(z)\,h(\zeta)$, where: $g(z)$ is a polynomial in z and a finite number of functions $(\mu_i - z)^{-1}$ with $\mu_i \notin \sigma(A)$; $h(\zeta)$ is a polynomial in ζ and a finite number of functions $(\eta_i - \zeta)^{-1}$ with $\eta_i \notin \sigma(B)$. The bounded operator $f(A, B)$ on $X \hat{\otimes}_\alpha Y$ obtained by substituting A for z and B for ζ will be called an *AB-rational function of A and B.*

Notice that the AB-rational functions are analytic on $\sigma(A) \times \sigma(B)$ and that the class of operators $f(A, B)$ includes the polynomials $P(A, B)$. We now state the spectral mapping theorem in the bounded case:

THEOREM 1. *Let A and B be bounded operators on Banach spaces X and Y, respectively. Let $f(A, B)$ be an AB-rational function of A and B on $X \hat{\otimes}_\alpha Y$. Then,*

$$\sigma(f(A, B)) = f(\sigma(A), \sigma(B)) \equiv \{f(\mu, \eta) \mid \mu \in \sigma(A), \eta \in \sigma(B)\}.$$

The proof of Theorem 1 will be accomplished in two parts. In the first part, we use classical operator theory and an argument from several complex variables to prove that $f(\sigma(A), \sigma(B)) \subset \sigma(f(A, B))$. In the second part we introduce the resolvent algebras $\mathscr{R}(A)$, $\mathscr{R}(B)$, and use Banach algebra techniques to prove that the inclusion is, in fact, an equality. The results on resolvent algebras (Theorems 2 and 3) are the core of the proof of the unbounded case presented in Sect. 3.

Let A be a bounded operator on a Banach space X. Let S be the set of $\lambda \in \mathbb{C}$ such that there is a $c > 0$ with $\|(A - \lambda) x\| \geqslant c \| x \|$ for all $x \in X$. The complement of S, called the approximate point spectrum, is denoted by $\sigma_{\text{a.p.}}(A)$.

$$\sigma_{\text{a.p.}}(A) = \{\lambda \mid \exists x_n \in X, \| x_n \| = 1, (A - \lambda) x_n \to 0\}.$$

For each $\lambda \in S$, $\text{Ran}(A - \lambda)$ is closed since A is closed. We define

$$\sigma_r(A) = \{\lambda \mid \lambda \in S, \text{Ran}(A - \lambda) \neq X\}.$$

$\sigma_r(A)$ is called the residual spectrum. It is clear that $S \backslash \sigma_r(A) = \rho(A)$ the resolvent set of A, so $\mathbb{C}$ is the disjoint union of $\sigma_{\text{a.p.}}(A)$, $\sigma_r(A)$, $\rho(A)$. We need the following standard lemma.

LEMMA 1. *$\sigma_r(A)$ is an open subset of $\mathbb{C}$.*

The second lemma we need is a standard theorem from several complex variables. For a proof see [4] or [8].

LEMMA 2. *Let K be a bounded set in $\mathbb{C}^n$, $n \geqslant 2$, and suppose that f is analytic in a neighborhood of K. If $f(p) = \lambda$ for some point p in K, then there is a point p' on the topological boundary of K so that $f(p') = \lambda$.*

We are now ready to prove the main lemma.

LEMMA 3. *Suppose that A and B are bounded operators on Banach spaces X and Y, respectively. Let $f(A, B)$ be an AB-rational function of A and B on $X \hat{\otimes}_\alpha Y$. Then $f(\sigma(A), \sigma(B)) \subset \sigma(f(A, B))$.*

Proof. Suppose $\lambda = f(\mu, \eta)$ for $\mu \in \sigma(A)$, $\eta \in \sigma(B)$. We want to show that $\lambda \in \sigma[f(A, B)]$. By Lemma 2, we may assume that $\langle \mu, \eta \rangle$ is on the topological boundary of $\sigma(A) \times \sigma(B)$. Thus either μ is on the boundary of $\sigma(A)$ or η is on the boundary of $\sigma(B)$. Without loss we assume the first possibility. Therefore, by Lemma 1, $\mu \in \sigma_{\text{a.p.}}(A)$.

We must consider two cases: suppose first that $\eta \in \sigma_{\text{a.p.}}(B)$. Then there exist sequences $x_n \in X$, $\| x_n \| = 1$, and $y_n \in Y$, $\| y_n \| = 1$, so that $(A - \mu)x_n \to 0$ and $(B - \eta)y_n \to 0$. For $\tau \notin \sigma(A)$, the relation $(A - \tau)(A - \tau)^{-1} x_n = x_n$ implies that

$$(A - \tau)^{-1} x_n - (\mu - \tau)^{-1} x_n \to 0$$

and similarly

$$(B - \tau)^{-1} y_n - (\eta - \tau)^{-1} y_n \to 0$$

if $\tau \notin \sigma(B)$. It follows easily that

$$f(A, B)(x_n \otimes y_n) - f(\mu, \eta)(x_n \otimes y_n) \to 0.$$

Thus, $\lambda = f(\mu, \eta) \in \sigma[f(A, B)]$.

The other case is when $\mu \in \sigma_{\text{a.p.}}(A)$ and $\eta \in \sigma_r(B)$. If we denote the adjoints of A and B by A' and B', then $\sigma(A) = \sigma(A')$ and $\sigma(B) = \sigma(B')$. Since μ is on the boundary of $\sigma(A')$ we have $\mu \in \sigma_{\text{a.p.}}(A')$ by Lemma 1. But $\eta \in \sigma_r(B)$ implies that $\eta \in \sigma_{\text{a.p.}}(B')$. Thus, the same argument as above shows that there are sequences $x_n^* \in X^*$, $\| x_n^* \| = 1$, $y_n^* \in Y^*$, $\| y_n^* \| = 1$, such that

$$f(A', B') x_n^* \otimes y_n^* - f(\mu, \eta) x_n^* \otimes y_n^* \to 0.$$

It is easily checked that the restriction of $f(A, B)'$, the adjoint operator on $(X \hat{\otimes}_\alpha Y)^*$, to vectors of the form $x^* \otimes y^*$ equals $f(A', B')$. Thus $f(\mu, \eta) \in \sigma(f(A, B)')$ which implies $f(\mu, \eta) \in \sigma(f(A, B))$. ∎

We now introduce resolvent algebras.

Definition. Let A be a bounded operator on a Banach space X. Let $\mathscr{R}(A)$ be the smallest norm closed algebra of operators on X containing all the resolvents $(\lambda - A)^{-1}$ for $\lambda \in \rho(A)$. $\mathscr{R}(A)$ is called the resolvent algebra of A.

Using the Neumann series for $(\lambda - A)^{-1}$, the reader can easily check that $I \in \mathscr{R}(A)$ and $A \in \mathscr{R}(A)$, so $\mathscr{R}(A)$ is a commutative Banach algebra containing A and the identity. Also, $\mathscr{R}(A)$ is the closure of the set of polynomials in resolvents of A at different points. Notice that in general $\mathscr{R}(A)$ will be larger than the algebra generated by A which will only contain $(\lambda - A)^{-1}$ if λ can be connected to infinity by a path remaining in the resolvent set. The following lemma states the most important property of $\mathscr{R}(A)$.

Lemma 4. *The Gelfand map $\hat{A}$ is a homeomorphism of $\sigma(\mathscr{R}(A))$ onto $\sigma(A)$, the spectrum of A as an operator on X.*

Proof. By the construction of $\mathscr{R}(A)$, the spectrum of A as an operator on X is equal to its spectrum as an element of $\mathscr{R}(A)$. The Gelfand map A is always onto $\sigma(A)$. Suppose l is a multiplicative linear functional on $\mathscr{R}(A)$. Then if $\lambda \in \rho(A)$, $l((\lambda - A)^{-1})\, l(\lambda - A) = l(I) = 1$ so $l((\lambda - A)^{-1}) = (\lambda - l(A))^{-1}$. Thus if two multiplicative linear functionals agree on A, they also agree on all the resolvents. Since polynomials in the resolvents are dense, they agree on all of $\mathscr{R}(A)$. Thus $\hat{A}$ is one to one. Since $\hat{A}$ is continuous and both sets are compact, the topologies are the same. ∎

In what follows we will suppress the homeomorphism and use $\sigma(\mathscr{R}(A))$ and $\sigma(A)$ interchangeably.

Definition. Let A and B be bounded operators on Banach spaces X and Y. We will denote by $\mathscr{R}(A) \hat{\otimes} \mathscr{R}(B)$ the operator algebra on $X \hat{\otimes}_\alpha Y$ formed by taking the norm closure of the set of AB-rational functions $f(A, B)$ of A and B.

We now state and prove the two crucial properties of $\mathscr{R}(A) \hat{\otimes} \mathscr{R}(B)$.

Theorem 2. $\sigma(\mathscr{R}(A) \hat{\otimes} \mathscr{R}(B)) = \sigma(A) \times \sigma(B)$.

This theorem shows that the multiplicative linear functionals on $\mathscr{R}(A) \hat{\otimes} \mathscr{R}(B)$ are especially simple. The proof in one direction is trivial: If $l \in \sigma(\mathscr{R}(A) \hat{\otimes} \mathscr{R}(B))$, then l restricted to the subalgebras $\mathscr{R}(A) \otimes I$ and $I \otimes \mathscr{R}(B)$ gives multiplicative linear functionals l_1

on $\mathscr{R}(A)$ and l_2 on $\mathscr{R}(B)$. The content of Theorem 2 is the converse statement. Namely, if $l_1 \in \sigma(A)$, $l_2 \in \sigma(B)$ then $l_1 \otimes l_2$ defined in the natural way on finite sums of operators of the form $L \otimes M$, where $L \in \mathscr{R}(A)$, $M \in \mathscr{R}(B)$ can be extended to $\mathscr{R}(A) \hat{\otimes} \mathscr{R}(B)$. That is, we must show that $l_1 \otimes l_2$ is bounded on the rational functions.

Remark. By a completely different method, one can prove that arbitrary $l_1 \otimes l_2 \in \mathscr{R}(A)^* \hat{\otimes} \mathscr{R}(B)^*$ are bounded; see [13].

THEOREM 3. *Let f be an AB-rational function of A and B. Suppose $\lambda \in \rho(f(A, B))$. Then $(\lambda - f(A, B))^{-1} \in \mathscr{R}(A) \hat{\otimes} \mathscr{R}(B)$.*

Proof of Theorems 2 *and* 3. We have the following inclusions:

$$\begin{aligned}
\sigma(f(A, B)) &\subseteq \sigma_{\mathscr{R}(A)\hat{\otimes}\mathscr{R}(B)}(f(A, B)) \\
&= \{l(f(A, B)) \mid l \in \sigma(\mathscr{R}(A) \hat{\otimes} \mathscr{R}(B))\} \\
&\subseteq \{(l_1 \otimes l_2)(f(A, B)) \mid l_1 \in \sigma(A), l_2 \in \sigma(B)\} \\
&= \{f(l_1(A), l_2(B)) \mid l_1 \in \sigma(A), l_2 \in \sigma(B)\} \\
&= f(\sigma(A), \sigma(B)) \\
&\subseteq \sigma(f(A, B)).
\end{aligned}$$

The first $\subseteq$ merely states that the spectrum of the operator $f(A, B)$ on $X \hat{\otimes}_\alpha Y$ is smaller than its spectrum as an element of $\mathscr{R}(A) \hat{\otimes} \mathscr{R}(B)$. The second $\subseteq$ follows from the trivial part of Theorem 2 explained in the remarks above. The last $\subseteq$ is just the statement of Lemma 3. We conclude that all the above sets are equal. This immediately proves Theorem 3 since the statement

$$\sigma(f(A, B)) = \sigma_{\mathscr{R}(A)\hat{\otimes}\mathscr{R}(B)}(f(A, B))$$

implies that if $\lambda - f(A, B)$ has an inverse it is in $\mathscr{R}(A) \hat{\otimes} \mathscr{R}(B)$.

Now, suppose $l_1 \in \sigma(A)$, $l_2 \in \sigma(B)$, and $f(A, B)$ is given. Then there is an $l \in \sigma(\mathscr{R}(A) \hat{\otimes} \mathscr{R}(B))$ so that

$$(l_1 \otimes l_2)(f(A, B)) = l(f(A, B)).$$

Therefore

$$|(l_1 \otimes l_2) f(A, B)| = |l(f(A, B))| \leqslant \| f(A, B)\|$$

so $l_1 \otimes l_2$ is bounded on the rational functions and thus extends to a multiplicative linear functional on $\mathscr{R}(A) \hat{\otimes} \mathscr{R}(B)$. This proves Theorem 2. ∎

Proof of Theorem 1. From Theorem 3 it follows that $\sigma(f(A, B)) = \sigma_{\mathscr{R}(A)\hat{\otimes}\mathscr{R}(B)}(f(A, B))$. And, Theorem 2 implies that

$$\begin{aligned}\sigma_{\mathscr{R}(A)\hat{\otimes}\mathscr{R}(B)}[f(A, B)] &= \{(l_1 \otimes l_2)(f(A, B)) \mid l_1 \in \sigma(A), l_2 \in \sigma(B)\} \\ &= \{f(l_1(A), l_2(B)) \mid l_1 \in \sigma(A), l_2 \in \sigma(B)\} \\ &= f(\sigma(A), \sigma(B)).\end{aligned}$$ ∎

We remark that Theorem 3 implies the following:

COROLLARY. *Let* $C \in \mathscr{R}(A)$, $D \in \mathscr{R}(B)$ *and let* f *be a CD-rational function. Then for each* $\lambda \in \rho(f(C, D))$, *we have* $(\lambda - f(C, D))^{-1} \in \mathscr{R}(A) \hat{\otimes} \mathscr{R}(B)$.

Proof. Because $C \in \mathscr{R}(A)$, there is a sequence A_n of rational functions of A so that $A_n \to C$ in norm. Thus $A_n \to C$ in norm resolvent sense also. Since the resolvents of A_n are all in $\mathscr{R}(A)$ the same is true of the resolvents C. Similarly, the resolvents of D are all in $\mathscr{R}(B)$. Therefore any CD-rational function is in $\mathscr{R}(A) \hat{\otimes} \mathscr{R}(B)$. Since $f(C, D) \in \mathscr{R}(A) \hat{\otimes} \mathscr{R}(B)$, there exists a sequence of AB-rational functions f_n so that $f_n(A, B)$ converges to $f(C, D)$ in norm. But, this implies that $f_n(A, B)$ converges to $f(C, D)$ in norm resolvent sense, so the corollary follows from Theorem 3. ∎

3. THE UNBOUNDED CASE

Throughout this section, A and B will denote unbounded operators with nonempty resolvent sets on Banach spaces X and Y. We will prove a spectral mapping theorem for certain rational functions of A and B on $X \hat{\otimes}_\alpha Y$. In practice, the class of rational functions (in particular, the class of polynomials) permitted will depend on $\sigma(A)$, $\sigma(B)$, and the asymptotic properties of the resolvents of A and B. Since A and B are unbounded, it is not even a priori clear how to define $f(A, B)$. We take a very restrictive definition. The use of resolvent algebras for spectral mapping theorems has been emphasized by Hille and Phillips [5].

DEFINITION. Let T be a closed operator with nonempty resolvent set on a Banach space X. We define $\mathscr{R}(T)$ to be the smallest uniformly closed Banach algebra containing I and all the resolvents of T. A sequence of bounded operators $T_n \in \mathscr{R}(T)$ will be called an $\mathscr{R}(T)$-*approximation* if T_n converges to T in the norm resolvent sense (written $T_n \xrightarrow{\text{n.r.}} T$).

DEFINITION. Let A and B be closed operators with nonempty resolvent sets on Banach spaces X and Y. Let α be a uniform cross-norm on $X \otimes Y$ and let $f(\cdot, \cdot)$ be an AB-rational function. A closed operator C on $X \hat{\otimes}_\alpha Y$ is said to equal $f(A, B)$ if there exists an $\mathscr{R}(A)$-approximation $\{A_n\}$ and an $\mathscr{R}(B)$-approximation $\{B_n\}$ so that $f(A_n, B_n)$ converges to C in the norm resolvent sense.

Several remarks are in order. First, we make this definition of $f(A, B)$ so that the spectral mapping theorem (below) is a relatively straightforward consequence of the machinery we have developed in the bounded case. To apply the theorem in practice some additional work is necessary, namely the construction of the resolvent approximates (see Sect. 4). Secondly, it is not clear in general whether $f(A, B)$ is uniquely defined, that is, whether two different sequences of resolvent approximates, $\{A_n\}$, $\{B_n\}$, and $\{A_n'\}$, $\{B_n'\}$, necessarily give the same definition of $f(A, B)$. However in the applications of Theorem 4 discussed in Sect. 4 where A and B generate holomorphic semigroups, $f(A, B)$ is unique.

THEOREM 4. *Suppose that A and B are closed operators with nonempty resolvent sets on Banach spaces X and Y. We assume that their spectra extend to ∞. Let α be a uniform cross-norm on $X \otimes Y$. Let f be an AB-rational function and suppose that $C = f(A, B)$ in the sense of the above definition. Then*

$$\sigma(C) = \overline{f(\sigma(A), \sigma(B))} = \overline{\{ f(\mu, \eta) \mid \mu \in \sigma(A), \eta \in \sigma(B)\}}.$$

In order to prove Theorem 4 we introduce a definition and two lemmas which will enable us to use the information about the bounded case derived in Sect. 3.

LEMMA 5. *Let T be a closed operator with nonempty resolvent set on a Banach space X. Let $\mu \in \rho(A)$. Then:*

(a) *The map $z \to 1/(\mu - z)$ is a homeomorphism of $\sigma(T) \cup \{\infty\}$ onto $\sigma((\mu - T)^{-1})$ when both sets are regarded as subsets of the Riemann sphere.*

(b) $\mathscr{R}(T) = \mathscr{R}((\mu - T)^{-1})$.

Proof. The proof of (a) is a standard exercise using the first resolvent identity; for a reference see [5; Theorem 5.12.1]. Part (b) follows from part (a) and the formula

$$\left[\frac{1}{z_0 - \mu} - (\mu - T)^{-1}\right]^{-1} = (\mu - z_0) I + (\mu - z_0)^2 (z_0 - T)^{-1}, \qquad z_0 \in \rho(A)$$

which expresses the resolvents of T and $(\mu - T)^{-1}$ in terms of one another. ∎

DEFINITION. Let T be a closed operator with nonempty resolvent set on a Banach space X. Suppose that $\mathfrak{A}$ is a commutative Banach algebra of operators on X containing $\mathscr{R}(T)$. Then we define the Gelfand transform of T to be the map $\hat{T}$ of $\sigma(\mathfrak{A})$ into the Riemann sphere given by $\hat{T}(l) = l(T)$, where

$$l(T) = \begin{cases} \mu - \{l(\mu - T)^{-1})\}^{-1}, & \text{if} \quad l((\mu - T)^{-1}) \neq 0 \\ \infty, & \text{if} \quad l((\mu - T)^{-1}) = 0 \end{cases}$$

where $\mu \in \rho(T)$.

LEMMA 6. *Let T be a closed unbounded operator with nonempty resolvent set. Let $\mathfrak{A}$ be a commutative Banach algebra of operators containing $\mathscr{R}(T)$. Then*

(a) *$\hat{T}$ is well-defined (independent of μ), continuous, and has range $\sigma(T) \cup \{\infty\}$.*

(b) *If $T_n \in \mathfrak{A}$ and $T_n \to T$ in norm resolvent sense, then $l(T_n) \to l(T)$ for all $l \in \sigma(\mathfrak{A})$.*

(c) *If $\mathfrak{A} = \mathscr{R}(T)$, then T is a homeomorphism.*

Proof. By the first resolvent identity, if $\mu \in \rho(T)$ and $l((\mu - T)^{-1}) \neq 0$, then $l((\lambda - T)^{-1}) \neq 0$ for any other $\lambda \in \rho(T)$ and $\lambda + \{l((\lambda - T))^{-1}\}^{-1} = \mu + \{l((\mu - T)^{-1})\}^{-1}$. Conversely, if $l((\mu - T)^{-1}) = 0$, then $l((\lambda - T)^{-1}) = 0$, so $\hat{T}$ is independent of μ. Let $z \in \sigma(A)$, then by part (a) of Lemma 5, $1/(\mu - z) \in \sigma[(\mu - T)^{-1}]$. Thus, by the usual Gelfand theory there is an $l \in \sigma(\mathfrak{A})$ so that $l((\mu - T)^{-1}) = 1/(\mu - z)$, so $l(T) = z$. Since $(\mu - T)^{-1}$ does not have a bounded inverse, there is an $l \in \sigma(\mathfrak{A})$ so that $l((\mu - T)^{-1}) = 0$, i.e., $l(T) = \infty$. Therefore, Ran $\hat{T} = \sigma(T) \cup \{\infty\}$. The proof of continuity is straightforward. This proves (a).

The proof of (b) is straightforward also. To prove (c) we need only note that two functionals l_1, $l_2 \in \mathscr{R}(T)$ which agree on $(\mu - T)^{-1}$ must agree on all polynomials in the resolvents of T. Since these polynomials and the identity are dense in $\mathscr{R}(T)$, l_1 and l_2 must agree on all of $\mathscr{R}(T)$. ∎

Proof of Theorem 4. Let $\mathfrak{A} = \mathscr{R}(A) \hat{\otimes} \mathscr{R}(B)$ and $C = f(A, B)$. Let $\{A_n\}$ and $\{B_n\}$ be the approximating sequences for A and B. Suppose $\lambda \in \rho(C)$. Then by hypothesis, $\lambda \in \rho(f(A_n, B_n))$ for n large enough and $(\lambda - f(A_n, B_n))^{-1} \to (\lambda - C)^{-1}$. By the corollary to Theorem 3, $\lambda - (f(A_n, B_n))^{-1} \in \mathscr{A}$ for each n, so $(\lambda - C)^{-1} \in \mathscr{A}$. This shows that $\mathscr{R}(C) \subset \mathscr{A}$.

By Lemma 5, $\mathscr{R}(A) = \mathscr{R}((\mu - A)^{-1})$ and $\mathscr{R}(B) = \mathscr{R}((\eta - B)^{-1})$ where $\mu \in \rho(A)$ and $\eta \in \rho(B)$. Thus, by Theorem 2, $\sigma(\mathscr{A}) = \sigma(\mathscr{R}(A)) \times \sigma(\mathscr{R}(B))$. But, by Lemma 6 (part c), $\sigma(\mathscr{R}(A))$ is homeomorphic to $\sigma(A) \cup \{\infty\}$ and $\sigma(\mathscr{R}(B))$ to $\sigma(B) \cup \{\infty\}$. Since we are assuming that $\sigma(A)$ and $\sigma(B)$ extend to infinity, the set $M = \hat{A}^{-1}[\sigma(A)] \times \hat{B}^{-1}[\sigma(B)]$ is dense in $\sigma(\mathscr{A})$. Therefore, by Lemma 6 (part a),

$$\sigma(C) = \operatorname{Ran} \hat{C} \backslash \{\infty\} = \overline{\{(l_1 \otimes l_2)(C) \mid l_1 \times l_2 \in M\}}.$$

But for $l_1 \otimes l_2 \in M$, Lemma 6 (part b) implies

$$(l_1 \otimes l_2)\{(\lambda - f(A_n, B_n))^{-1}\} \to (l_1 \otimes l_2)((\lambda - C)^{-1})$$

and

$$\begin{aligned}(l_1 \otimes l_2)&(\lambda - f(A_n, B_n))^{-1} \\ &= \{\lambda - f(l_1(A_n), l_2(B_n))\}^{-1} \to \{\lambda - f(l_1(A), l_2(B))\}^{-1}.\end{aligned}$$

Therefore, $\sigma(C) = \overline{f(\sigma(A), \sigma(B))}$. ∎

A few remarks about the closure operation in the statement $\sigma(C) = \overline{f(\sigma(A), \sigma(B))}$ are appropriate. First, it is already clear in the self-adjoint case that the closure operation is necessary. Let $X = \mathscr{H}_1$ and $Y = \mathscr{H}_2$ be Hilbert spaces. Let A on $\mathscr{H}_1$ and B on $\mathscr{H}_2$ be self-adjoint operators, both with spectra $\{n \mid n = 1, 2, \ldots\}$. The construction of resolvent approximates such that $C = A \otimes (2B/(2B + I))$ is straightforward (it is also a special case of Theorem 6 in Sect. 4). Of course, the spectrum of C can be computed directly from the spectral theorem; we will check how it arises out of Theorem 4. Let $l_n \in \sigma(\mathscr{R}(A))$ and $\tilde{l}_n \in \sigma(\mathscr{R}(B))$ be the functionals so that $l_n(A) = n = \tilde{l}_n(B)$, and let $l_\infty \in \sigma(\mathscr{R}(A))$, $\tilde{l}_\infty \in \sigma(\mathscr{R}(B))$ be the functionals so that $l_\infty(A) = \infty = \tilde{l}_\infty(B)$. $\{l_n \times \tilde{l}_m \mid n = 1, 2, \ldots, m = 1, 2, \ldots\}$ is just the set M in the proof of Theorem 4. Then

$$\begin{aligned}
&(l_n \otimes \tilde{l}_m)(C) = n(2m/(2m+1)) && n = 1, 2, 3, \ldots \\
& && m = 1, 2, 3, \ldots \\
&(l_n \otimes \tilde{l}_\infty)(C) = n && n = 1, 2, \ldots \\
&(l_\infty \otimes \tilde{l}_m)(C) = \infty && m = 1, 2, \ldots \\
&(l_\infty \otimes \tilde{l}_\infty)(C) = \infty &&
\end{aligned}$$

Thus, the range of f on $\sigma(A) \times \sigma(B)$ is just $\{n(2m/(2m+1)) \mid n = 1, 2, \ldots; m = 1, 2, \ldots\}$. The closure operation picks up the odd positive integers which are also spectral points of C.

Secondly, in the case where $f(A, B)$ is a polynomial, the closure is not necessary in the cases where the theorem applies. This can be seen as follows. Suppose that Ran P is not closed on $\sigma(A) \times \sigma(B)$ and let $\lambda \in \overline{\text{Ran } P} \backslash \text{Ran } P$. Then there are sequences $\mu_n \in \sigma(A)$, $\mu_n \to \infty$, and $\eta_n \in \sigma(B)$, $\eta_n \to \infty$, so that $P(\mu_n, \eta_n) \to \lambda$. But it is possible to find subsequences, $\{\tilde{\mu}_n\}$ of $\{\mu_n\}$ and $\{\tilde{\eta}_n\}$ of $\{\eta_n\}$, so that $P(\tilde{\mu}_n, \tilde{\eta}_n) \not\to \lambda$. This means that there will be no operator C which equals $P(A, B)$ in our sense for if $C = P(A, B)$ then by Lemma 6, $\hat{C}$ would be continuous which it is not. Thus for non-normal operators and polynomials where Ran P on $\sigma(A) \times \sigma(B)$ is not closed, the existence of a spectral mapping theorem is an open question.

4. Generators of Bounded Holomorphic Semigroups

In this section we present two theorems which show how Theorem 4 may be applied in the case where A and B generate bounded holomorphic semigroups. The two cases we consider are $A \otimes I + I \otimes B$ and $A \otimes B$; in the second case slightly stronger assumptions on A and B are necessary. The reader will easily see how to generalize the techniques to handle more complicated polynomials. The Hille–Yosida Theorem says that a closed operator T on a Banach space generates a bounded holomorphic semigroup e^{-tT} if, and only if, [7]

(i) There exists $0 \leqslant \theta < \pi/2$ such that

$$\sigma(T) \subseteq W^\theta = \{z \mid |\arg z| \leqslant \theta\};$$

and

(ii) For any sector $W^{\theta+r} = \{z \mid |\arg z| \leqslant \theta + r\}$, $r > 0$, there is a constant M so that for all $z \in \mathbb{C} \backslash W^{\theta+r}$, $(z - T)^{-1}$ satisfies:

$$\|(z - T)^{-1}\| \leqslant M/|z|.$$

We begin by constructing a resolvent approximate for T.

Lemma 7. *Let T be a closed operator on a Banach space satisfying* (i) *and* (ii). *Define $T_n = T(I + T/n)^{-1}$. Then:*

(a) *$T_n \in \mathscr{R}(T)$, $\sigma(T_n) \subseteq W^\theta$, and $\|(z - T_n)^{-1}\| \leqslant M_1/|z|$ for $z \in W^{\theta+r}$, where M_1 is independent of n.*

(b) *T_n converges to T in norm resolvent sense. In fact, given any straight line $L \subset (\mathbb{C}\backslash W^{\theta+r})$ and $0 < \epsilon < 1$, there is a sequence of positive numbers $\{c_n\}$ with $c_n \to 0$ so that*

(iii) $\|(z - T_n)^{-1} - (z - T)^{-1}\| \leqslant c_n/|z|^{1-\epsilon}$ *for all* $z \in L$.

If $0 \notin \sigma(T)$, then L can be chosen in $\overline{\mathbb{C}\backslash W_r^\theta}$.

Proof. Since $T[I + (T/n)]^{-1} = nI - (n + T/n)^{-1}$, $T_n \in R(T)$. By Lemma 5, the spectrum of T_n is just the set of numbers of the form $n - (n + \mu/n)^{-1}$ where $\mu \in \sigma(T)$. Since $\sigma(T) \subseteq W^\theta$, this implies that $\sigma(T_n) \subseteq W^\theta$.

For each $z \in \mathbb{C}\backslash W^{\theta+r}$ and positive integer n, $nz/(n - z) \in \mathbb{C}\backslash W^{\theta+r}$. In fact, if $0 < \arg z \leqslant \pi$, then $\pi \geqslant \arg nz/(n - z) \geqslant \arg z$, and if $\pi \leqslant \arg z < 2\pi$, then $\pi \leqslant \arg nz/(n - z) \leqslant \arg z$. The first resolvent formula and elementary manipulations show that

(iv) $$\frac{1}{z - T_n} = -\left(\frac{1}{n - z}\right) I + \frac{n^2}{(n - z)^2}\left[\frac{1}{nz/(n - z) - T}\right].$$

The formulas (ii) and (iv) and the inequalities on $\arg nz/(n - z)$ imply that

$$\|(z - T_n)^{-1}\| \leqslant (1/|z|)\,[(|z| + Mn)/|n - z|].$$

Since

$$M_1 = \sup_{n, z \notin W^{\theta+r}} \left(\frac{|z| + Mn}{|n - z|}\right) = \sup_{z \notin W^{\theta+r}} \frac{|z| + M}{|z - 1|} < \infty$$

we have proven (a).

If we now subtract $(z - T)^{-1}$ from (iv) and use the first resolvent identity, we obtain:

(v) $$\frac{1}{z - T_n} - \frac{1}{z - T} = \left(-\frac{1}{n - z}\right) I + \frac{n^2}{(n - z)^2}\left[\frac{1}{nz/(n - z) - T}\right] \times \left(\frac{1}{z - T}\right)\left(\frac{z^2}{n - z}\right) - \left(\frac{2nz - z^2}{(n - z)^2}\right)\left(\frac{1}{z - T}\right).$$

From (v) and (ii) we find that

$$n\left\|\frac{1}{z - T_n} - \frac{1}{z - T}\right\| \leqslant \frac{n}{|n - z|}\left[1 + \frac{Mn}{n - z}\left\|\frac{z}{z - T}\right\| + \frac{M\,|2n - z|}{|n - z|}\right]$$
$$\leqslant C_1 + C_2\left\|\frac{z}{z - T}\right\| \leqslant C_1 + C_2 M.$$

for some constants C_1, C_2 and all $z \in \mathbb{C}\backslash W^{\theta+r}$. Thus, if L is a line in $\mathbb{C}\backslash W^{\theta+r}$ (or if L is in $\overline{\mathbb{C}\backslash W^{\theta+r}}$ in the case where $0 \notin \sigma(T)$), we have $\|(z - T_n)^{-1} - (z - T)^{-1}\| \leqslant M_3/n$ for $z \in L$. But, using the estimate of part (a) and (ii) we have $\|(z - T_n)^{-1} - (z - T)^{-1}\| \leqslant M_3/|z|$ for all $z \in L$. Thus, given any $0 < \epsilon < 1$,

$$\|(z - T_n)^{-1} - (z - T)^{-1}\| \leqslant \frac{M_2 M_3}{n^\epsilon |z|^{1-\epsilon}}.$$

This proves (b). ∎

THEOREM 5 (Spectral mapping theorem for $A \otimes I + I \otimes B$). *Let A and B be generators of holomorphic semigroups on Banach spaces X and Y, respectively. Let α be a uniform cross-norm on $X \otimes Y$ and let P denote the operator $P(\varphi \otimes \psi) = A\varphi \otimes \psi + \varphi \otimes B\psi$ defined on D_0, the set of finite linear combinations of vectors of the form $\varphi \otimes \psi$ where $\varphi \in D(A)$, $\psi \in D(B)$. Then $\bar{P}$ is the generator of the holomorphic semigroup $e^{-tA} \otimes e^{-tB}$ on $X \hat{\otimes}_\alpha Y$ and $\sigma(\bar{P}) = \sigma(A) + \sigma(B)$.*

Proof. We define $A_n = A(I + A/n)^{-1}$ and $B_n = B(I + B/n)^{-1}$. By Lemma 7, A_n and B_n are resolvent approximates for A and B, respectively. Let $\lambda > 0$. We first need to establish the formula

$$\text{(vi)} \quad (A_n \otimes I + I \otimes B_n + 2\lambda)^{-1} = \frac{1}{2\pi} \int_{-\infty}^{\infty} (A_n \otimes I + \lambda + i\mu)^{-1} \otimes (I \otimes B_n + \lambda - i\mu)^{-1}\, d\mu$$

which is motivated by the Laplace transform formula

$$\frac{1}{x + y + 2\lambda} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{1}{x + \lambda + i\mu}\right) \left(\frac{1}{y + \lambda - i\mu}\right) d\mu$$

for x, y, and λ all in the right half-plane. The estimate in Lemma 7 shows that the integral on the right of (vi) is norm convergent for all $\lambda > 0$. For $\lambda > \|A_n\| + \|B_n\|$, the formula follows by expanding each of the three resolvents in its Neumann series, and integrating term by term using the Cauchy integral theorem. Since the three resolvents are analytic for $\lambda > 0$ and the integral is norm convergent, (vi) holds for all $\lambda > 0$.

We now define

$$\text{(vii)} \quad F = \frac{1}{2\pi} \int_{-\infty}^{\infty} (A + 1 + i\mu)^{-1} \otimes (B + 1 - i\mu)^{-1}\, d\mu.$$

Since A and B generate holomorphic semigroups, F is well-defined and the integral is norm convergent (properties (i) and (ii)). Furthermore, it follows immediately from (vi) and the estimate in Lemma 7b that

(viii) $\|(A_n \otimes I + I \otimes B_n + 2)^{-1} - F\| \xrightarrow[n\to\infty]{} 0.$

We will now show that F is the bounded inverse of $\bar{P} + 2I$. Since the integral in (vii) is norm convergent, we can find a sequence of Riemann sums, F_n, so that $\| F_n - F \| \to 0$. Suppose that $\varphi \otimes \psi \in D_0$; then $F_n(\varphi \otimes \psi) \in D_0$ and $PF_n(\varphi \otimes \psi) = F_n P(\varphi \otimes \psi)$. Since $\| F_n - F \| \to 0$, $F(\varphi \otimes \psi) \in D(\bar{P})$ and $\bar{P}F(\varphi \otimes \psi) = FP(\varphi \otimes \psi)$. Now, there exist $\tilde{\varphi} \in X$, $\tilde{\psi} \in Y$ so that $\varphi = (I + A)^{-1} \tilde{\varphi}$ and $(I + B)^{-1} \tilde{\psi}$. Define $\varphi_n = (I + A_n)^{-1} \tilde{\varphi}$ and $\psi_n = (I + B_n)^{-1} \tilde{\psi}$. Then $A_n\varphi_n \to A\varphi$ and $B_n\psi_n \to B\psi$. Thus, from (vi), (vii), and (viii),

$$\begin{aligned}(\bar{P} + 2) F(\varphi \otimes \psi) &= F[(A\varphi \otimes \psi + \varphi \otimes B\psi) + 2(\varphi \otimes \psi)] \\ &= \lim_{n\to\infty}(A_n \otimes I + I \otimes B_n + 2)^{-1} \\ &\quad \times [A_n\varphi_n \otimes \psi_n + \varphi_n \otimes B_n\psi_n + 2(\varphi_n \otimes \psi_n)] \\ &= \lim_{n\to\infty} \varphi_n \otimes \psi_n \\ &= \varphi \otimes \psi.\end{aligned}$$

Thus on D_0, $(\bar{P} + 2) F = I$. Since D_0 is dense in $X \hat{\otimes}_\alpha Y$, it follows immediately that $F: X \hat{\otimes}_\alpha Y \to D(\bar{P})$, F is one to one, $-2 \in \rho(\bar{P})$, and $F = (\bar{P} + 2)^{-1}$. Thus, $\|(A_n \otimes I + I \otimes B_n - \lambda)^{-1} - (\bar{P} + \lambda)^{-1}\| \to 0$ for all $\lambda \in \rho(\bar{P})$ since it is true for $\lambda = 2$. Therefore, by Theorem 4, $\sigma(\bar{P}) = \sigma(A) + \sigma(B)$.

Suppose $\sigma(A) \subset W^{\theta_1}$, $\sigma(B) \subset W^{\theta_2}$ and let $\theta = \max\{\theta_1, \theta_2\}$. Then $V(t) = e^{-tA} \otimes e^{-tB}$ is a holomorphic semigroup in the sector $|\arg t| < (\pi/2) - \theta$. Furthermore, $V(t)$ is strongly differentiable on D_0 and its generator C is equal to P on D_0. Thus, C extends $\bar{P}$. But, $\mathrm{Ran}(\bar{P} + 2) = X \hat{\otimes}_\alpha Y$, so since $C + 2$ is injective we must have $\bar{P} = C$. ∎

THEOREM 6 (Spectral mapping theorem for $A \otimes B$). *Let A and B be generators of holomorphic semigroups on Banach spaces X and Y, respectively, such that $\sigma(A) \subset W^{\theta_1}$, $\sigma(B) \subset W^{\theta_2}$ and $\theta_1 + \theta_2 < \pi/2$. Assume further that $0 \notin \sigma(A)$, $0 \notin \sigma(B)$. Let α be a uniform crossnorm on $X \otimes Y$ and let P denote the operator $P(\varphi \otimes \psi) = A\varphi \otimes B\psi$ with domain D_0 (defined in Theorem 5). Then $\bar{P}$ generates a bounded holomorphic semigroup and $\sigma(\bar{P}) = \sigma(A)\,\sigma(B)$.*

Proof. The first part of the proof is similar to the proof of Theorem 5, so we merely provide a sketch. Let r_1 and r_2 be positive numbers so that $\theta_1 + r_1 + \theta_2 + r_2 < \pi/2$ and denote by Γ_1 and Γ_2 the boundaries of $W^{\theta_1+r_1}$ and $W^{\theta_2+r_2}$, respectively. As before, define $A_n = A(I + A/n)^{-1}$, $B_n = B(I + B/n)^{-1}$. First, one uses the Cauchy integral formula and the estimate of Lemma 7a to show that

$$\text{(ix)}\quad (A_n \otimes B_n + I)^{-1} = \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_2}\int_{\Gamma_1} (z\zeta - 1)^{-1} \times \{(A_n \otimes I + z)^{-1} \otimes (I \otimes B_n + \zeta)^{-1}\}\, dz\, d\zeta.$$

In the proof of (ix) the hypothesis that $0 \notin \sigma(A)$, $\sigma \notin \sigma(B)$ is used. It permits us to choose Γ_1 and Γ_2 as paths of integration (rather than the boundaries of wedges whose vertices are to the left of the origin) which insures that $(z\zeta - 1)^{-1}$ has no poles for $z \in W^{\theta_1+r_1}$ and $\zeta \in W^{\theta_2+r_2}$. Then one defines

$$F = \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_2}\int_{\Gamma_1} (z\zeta - 1)^{-1}\{(A \otimes I + z)^{-1} \otimes (I \otimes B + \zeta)^{-1}\}\, dz\, d\zeta$$

and using the estimate of Lemma 7b proves that

$$\|(A_n \otimes B_n + 1)^{-1} - F\| \to 0,$$

$F\colon D_0 \to \mathscr{H}$, and $(\bar{P} + 1)F = I$ on D_0. As before this shows that F is the resolvent of $\bar{P}$ at -1 and $A_n \otimes B_n$ converges in norm resolvent sense to $\bar{P}$ [it is sufficient to prove convergence at one point of $\rho(\bar{P})$]. Thus, from Theorem 4, $\sigma(\bar{P}) = \sigma(A)\,\sigma(B)$.

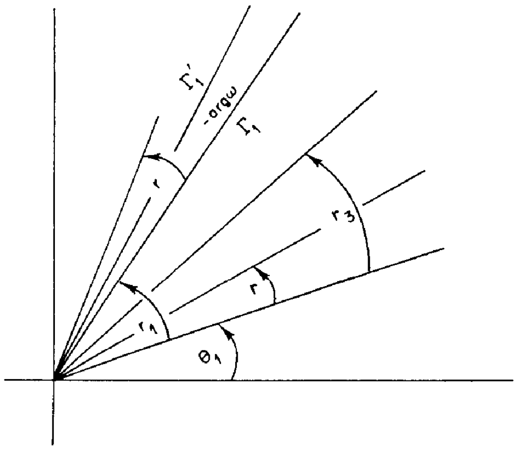
We need to work a little harder than in Theorem 5 in order to prove that $\bar{P}$ generates a bounded holomorphic semigroup. We know that $\sigma(\bar{P}) \subseteq W^{\theta_1} W^{\theta_2} = W^{\theta_1+\theta_2}$. What we must show is that an estimate of the form (ii) in sets of the form $\mathbb{C}\backslash W^{\theta_1+\theta_2+r}$. To do this, it is sufficient ([7], p. 490) to show that for each $0 < r < (\pi/2) - (\theta_1 + \theta_2)$, there is a constant $M(r)$ so that $\|(\bar{P} + \lambda)^{-1}\| \leqslant M(r)/|\lambda|$ for all $\lambda = \omega t$, where $t = |\lambda|$, $|\arg \omega| \leqslant r$. Without loss, we may assume $\theta_1 \leqslant \theta_2$. We first consider the case $-r \leqslant \arg \omega \leqslant 0$. Choose r_1 and r_2 so that

$$\theta_1 + \theta_2 + r_1 + r_2 < \pi/2,$$
$$\theta_1 + r_1 + r < \pi/2,$$
$$r < r_1.$$

Then, if Γ_1 is $\partial W^{\theta_1+r_1}$ and Γ_2 is $\partial W^{\theta_2+r_2}$,

$$(\bar{P}+t\omega)^{-1}=\left(\frac{1}{2\pi i}\right)^2\int_{\Gamma_2}\int_{\Gamma_1}\left(\frac{1}{z\zeta+t\omega}\right)\left\{\frac{1}{z-A}\otimes\frac{1}{\zeta-B}\right\}dz\,d\zeta$$
$$=\left(\frac{1}{\omega t}\right)\left(\frac{1}{2\pi i}\right)^2\int_{\Gamma_2}\int_{\Gamma_1'}\left(\frac{1}{z\zeta+1}\right)\left\{\frac{1}{z-A/\omega t}\otimes\frac{1}{\zeta-B}\right\}dz\,d\zeta$$

where $\Gamma_1'=\partial W^{\theta_1+r_1-\arg\omega}$. Thus we need only show that $\|[z-(A/\omega t)]^{-1}\|\leqslant M/|z|$ on Γ_1' (where M is independent of t).



Consider the part of Γ_1' in the upper half-plane. Let $r_3=(r_1+r)/2$ then $\theta_1+r<r_3<\theta_1+r_1$ so the spectrum of $A/\omega t$ is contained on the right side of the half-plane making an angle of θ_1+r_3 with the real axis. Thus, the usual formula expressing the resolvent of a semigroup in the complementary half-plane from the spectrum of the generator gives:

$$\left(z-\frac{A}{\omega t}\right)^{-1}=\int_{\Gamma_3}e^{z\zeta}e^{-(A/\omega t)\zeta}\,d\zeta$$

for z on the upper part of Γ_1' where Γ_3 is the ray $\{se^{i(\pi/2-r_3-\theta_1)}\mid s\geqslant 0\}$. Thus,

$$\left\|\left(z-\frac{A}{\omega t}\right)^{-1}\right\|\leqslant\int_0^\infty\exp[-|z||\zeta|\sin(r_1-r_3)]\,d|\zeta|$$
$$\leqslant\frac{C(r)}{|z|\sin(r_1-r_3)}$$

since $\|e^{-(A/\omega t)\zeta}\|\leqslant C(r)$ on Γ_3. The proof of the estimate for the lower part of Γ_1' and for $\arg\omega\geqslant 0$ is similar. ∎

We conclude with several remarks. First, formula (ix) was motivated by Ichinose's work [6]. Secondly, the spectral mapping theorem in Theorem 6 holds under the weaker hypothesis that $\theta_1 + \theta_2 < \pi$ but in that case $A \otimes B$ may not generate a semigroup. Thirdly, in both Theorems 5 and 6 we chose the operators $A_n = A(I + A/n)^{-1}$ and $B_n = B(I + B/n)^{-1}$ as resolvent approximates. It can be shown that any other choice of resolvent approximates yields the same definition of $A \otimes I + I \otimes B$ and $A \otimes B$. Finally, we note that a special case of Theorem 5 and Theorem 6 is the case when X and Y are Hilbert spaces and A and B are m-sectorial. For an application of Theorem 5 in this special case to quantum mechanics, see [1] and [12].

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