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Essential Self-Adjointness of Powers of Generators of Hyperbolic Mixed Problems

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The theory of hyperbolic mixed initial-boundary value problems is used to prove the essential self-adjointness of certain differential operators associated with formally symmetric boundary value problems.

The framework of this note is the same as that of [1] with the exception that the manifold, M, is allowed to have a boundary. For simplicity, we restrict attention to the case $M = \bar{\mathcal{O}}$ where $\mathcal{O}$ is an open set in $\mathbb{R}^n$ with smooth compact boundary and the bundle ξ, is the trivial bundle $\mathbb{C}^k \times M$, with the standard inner product $\langle , \rangle$, on $\mathbb{C}^k$.

We consider the symmetric hyperbolic system

$$\frac{\partial u}{\partial t} = Lu = \sum_{i=1}^{n} A_i(x) \frac{\partial u}{\partial x_i} + B(x)u \tag{1}$$

supplemented by the homogenous boundary conditions

$$u(x) \in N(x) \qquad \text{for} \quad x \in \partial\mathcal{O} \tag{2}$$

where $N(x)$ is a smoothly varying subspace of $\mathbb{C}^k$ defined for $x \in \partial\mathcal{O}$.

For $x \in \partial\mathcal{O}$ let $\nu(x) = (\nu_1(x),..., \nu_n(x))$ be the outward unit normal to $\bar{\mathcal{O}}$ at x and let $A_\nu(x) = \sum_{i=1}^{n} A_i(x)\, \nu_i(x)$. We assume that *$\partial\mathcal{O}$ is noncharacteristic* for L, i.e., A_ν is nonsingular for $x \in \partial\mathcal{O}$. It follows that A_ν has only real, nonzero eigenvalues. The boundary condition is called formally self-adjoint if for each $x \in \partial\mathcal{O}$

$$A_\nu[N(x)] = N(x)^\perp \tag{3}$$

This identity insures that $(Lu, v) = -(u, Lv)$ for all smooth u, v

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which satisfy (2) and vanish for $|x|$ large. In addition, it follows that N is maximal with respect to this property.

For $x \in \mathcal{O}$ let $c_x = \sup_i \| A_i(x)\|$. For $r > 0$ let $\mathcal{O}_r = \{x \in \bar{\mathcal{O}} \mid |x| \leqslant r\}$, then for r large $\mathcal{O}_r$ is nonempty and we define $c(r) = \sup_{\mathcal{O}_r} c_x$.

THEOREM. *If* $\int^\infty 1/c(r)\, dr = \infty$ *then* $(iL)^k$ *with domain*

$$\mathscr{D} = \{u \in C^\infty_{(0)}(\bar{\mathcal{O}}) \mid (L^j u)(x) \in N(x) \textit{ for all } x \in \partial\mathcal{O} \textit{ and } j = 0, 1, 2,\ldots\}$$

is an essentially self-adjoint operator on $\mathscr{L}_2(\mathcal{O})$.

Proof. Under the stated hypothesis the mixed problem

$$(\partial u/\partial t) - Lu = 0$$
$$u(0) = f \in \mathscr{D}$$
$$u(t, x) \in N(x) \qquad \text{for} \quad x \in \partial\mathcal{O}$$

has a unique solution $u(t, x) \in C^\infty(\mathbb{R} \times \bar{\mathcal{O}})$ [2, Theorem 3.1] and

$$\int_{\mathcal{O}} \| u(t, x)\|^2\, dx = \int_{\mathcal{O}} \| u(0, x)\|^2\, dx$$

for all t. It follows that the operators $P_t\colon u(0) \to u(t)$ extend by continuity to a strongly continuous unitary group on $\mathscr{L}_2(\mathcal{O})$ with generator G, extending $iL \restriction \mathscr{D}$. Since L maps $\mathscr{D}$ into itself we also have $\mathscr{D} \subset D(G^k)$ and $G^k \restriction \mathscr{D} = (iL)^k \restriction \mathscr{D}$.

Without loss of generality we may suppose $0 \in \mathcal{O}$. The classical domain of dependence results show that if $u(0) = 0$ for $|x| \geqslant R$ then $u(t) = 0$ for $|x| > R'$ provided

$$\int_R^{R'} \frac{dr}{c(r)} > |t|.$$

Thus, if $u(0) \in \mathscr{D}$, then $u(t)$ has compact support for all $t > 0$. In addition, the differentiability results of Massey and Rauch [2] imply that if $u(0) \in \mathscr{D}$ then $u(t) \in \mathscr{D}$ for all t. Therefore $\mathscr{D}$ is invariant under the group P_t. By Lemma 2.1 of [1] it follows that all powers of the generator G of P_t are essentially self-adjoint on $\mathscr{D}$. Since $G^k \restriction \mathscr{D} = (iL)^k \restriction \mathscr{D}$ the proof is complete.

References

1. P. Chernoff, Essential Self-Adjointness of Powers of Generators of Hyperbolic Equations, *J. Functional Analysis* (1973), to appear.
2. F. Massey and J. Rauch, Differentiability of Solutions to Hyperbolic Initial-Boundary Value Problems, *Trans. Amer. Math. Soc.*, to appear.