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ON THE TRANSITION FROM SHELL STRUCTURE TO COLLECTIVE BEHAVIOR

A simplified shell-model study

K. T. HECHT
Physics Department, University of Michigan, Ann Arbor, Michigan 48104 †
and
Oak Ridge National Laboratory
and
J. B. McGRORY
Oak Ridge National Laboratory ††, *Oak Ridge, Tennessee 37830*
and
J. P. DRAAYER
University of Michigan, Ann Arbor, Michigan 48104

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Abstract: To study the feasibility of carrying out shell-model calculations in nuclei with active protons and neutrons in different major shells, the following simple idealized model has been studied: (i) Proton and neutron configurations are chosen to be $(f_{\frac{7}{2}}p_{\frac{3}{2}}p_{\frac{1}{2}})^{n_p}$ and $(g_{\frac{7}{2}}d_{\frac{5}{2}}d_{\frac{3}{2}}s_{\frac{1}{2}})^{n_n}$, so that results for the separate proton and neutron basis states to be used in any approximation scheme can be compared with the results for exact shell-model calculations. (ii) The proton and neutron single-particle energies for these active shells are separately taken to be degenerate. (iii) The two-body interaction is approximated by the simple surface delta interaction (SDI). To effect the severe truncation of the full shell-model space needed to make such a shell-model study possible the separate proton and neutron parts of the shell-model basis are built from a superposition of the favored pair states of the SDI (with $J \neq 0$, as well as $J = 0$). In the neutron configuration $(g_{\frac{7}{2}}d_{\frac{5}{2}}d_{\frac{3}{2}}s_{\frac{1}{2}})^{n_n=4}$, for example only three of the 94 shell-model states with $J_n = 2$ are retained in the truncation scheme. In this highly truncated basis both the energies and the strong $B(E2)$ values for the transitions from these states to similar favored states with other J-values are within a few percent (or better) of the results of exact shell-model calculations. A truncation of the shell-model space based on such superpositions of favored pair states leads to a manageable shell-model basis (dimensions $\leqq 200$). (a) The number of states in the separate proton and neutron parts of the basis are small enough (8–13 for the proton space, 15–30 for the neutron space). They are also the key states in the following sense. (b) They include the low-lying energy eigenstates of the separate p-p and n-n parts of the interaction. (c) They contain most of the collective coherence of the separate proton and neutron configurations. (d) The matrix elements of the n-p part of the interaction between the favored states are in general very large compared with the matrix elements between a favored and an excluded state. The latter effect is studied from several aspects, in particular in terms of sum rules for the matrix elements of the surface multipole operators from which the n-p part of the SDI is built. For most of the low-lying favored states the sum over *all* favored states gives more than 90 % of the total sum rule for the squares of matrix elements of the surface multipole operators. The results of shell-model calculations in this truncation scheme, with $n_p = 4$ or 6, and $n_n = 4$, show many of the features of a quadrupole vibrational spectrum. The presence and exact nature of a 0^+ member of the 0^+, 2^+, 4^+ "two-phonon triplet" is dependent on the inclusion of the key favored states with seniorities of 6.

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1. Introduction

Nature has presented us with many striking examples of series of nuclei which exhibit the transition from shell structure to collective behavior. Nuclei with neutron numbers of 50 and 82, for example, are "good shell-model nuclei", described well in terms of the properties of a few protons distributed over a few active shell-model orbits [1,2]). As neutrons are added to these nuclei, however, the spectra take on a distinctly vibrational character, and, if a sufficient number of neutrons are added, the spectra may even become rotational. At first thought it seems impossible that the transitional region could be understood in terms of a conventional shell-model description since the dimensionality of the shell-model space becomes prohibitively large with even a small number of both protons and neutrons distributed over relatively few single-particle orbits. Any description in terms of a microscopic model, however, must somehow be based on a shell-model framework. The question therefore arises: Can the shell-model space be truncated sufficiently so that the shell-model calculations for such nuclei are technically feasible and, more important, are simple enough to lead to an understanding of the transition from shell structure to collective behavior?

For a nucleus in which protons and neutrons are filling different major shells, such a truncation scheme must be built from a relatively small number of key many-particle states of the separate proton and neutron parts of the configuration, where these key proton and neutron states must satisfy the following requirements:

(i) They must include the low-lying energy eigenstates (α) of the proton part of the configuration, where these must have built into them a major part of the collective coherence associated with the separate proton part of the configuration. Similar requirements hold for the key neutron states (β).

(ii) The n-p interaction must act mainly within the subspaces (α) and (β); that is, the matrix elements of the n-p interaction between two key states must be large compared with the matrix elements between a key state and one built from proton and neutron states excluded from the sets (α) and (β).

(iii) The total number of key states in the sets (α) and (β) must be small. Even though the requirement of good total angular momentum and parity somewhat restricts the way in which the states from the sets (α) and (β) can combine, if the dimension of the full shell-model matrix is to be of the order of 100–200, the number of key states from each set cannot be more than 10–20.

It is the purpose of this investigation to show by means of a simple, idealized model that these requirements can be met, and hence demonstrate the feasibility of carrying out shell-model studies of nuclei in which protons and neutrons are filling different major shells.

2. The model

It is the ultimate aim of this investigation to follow a sequence of isotopes such as the Ru or Pd family as neutron numbers are increased, starting with $N = 50$. Such

nuclei involve the proton configurations $(p_{\frac{3}{2}} p_{\frac{1}{2}} f_{\frac{5}{2}} g_{\frac{9}{2}})^{n_p}$ and neutron configurations $(g_{\frac{7}{2}} d_{\frac{5}{2}} d_{\frac{3}{2}} s_{\frac{1}{2}} h_{\frac{11}{2}})^{n_n}$. In the simplified model these will be replaced by $(p_{\frac{3}{2}} p_{\frac{1}{2}} f_{\frac{5}{2}})^{n_p}$ and $(g_{\frac{7}{2}} d_{\frac{5}{2}} d_{\frac{3}{2}} s_{\frac{1}{2}})^{n_n}$ with even numbers of protons, n_p and neutrons, n_n. Obvious simplifications result from the exclusion of the single-particle states of opposite parity. The primary motivation for eliminating them, however, is so that results for any truncation scheme can be compared with more detailed exact shell-model calculations. The model Hamiltonian to be studied will incorporate the following additional simplifying assumptions:

(i) The single-particle energies of the proton configuration are assumed to be degenerate: similarly for the neutron configuration. Since the interest is in proton and neutron configurations with fairly large numbers of particles this may not be too drastic a zeroth approximation. The qualitative features of spectra with nucleon numbers near the half-full shell mark (unlike the spectra for very small numbers of particles or holes) are not influenced very markedly by the exact nature of the single-particle spectra.

(ii) The effective two-body interaction is to be approximated by the simple surface delta interaction (SDI) [ref. [3])]. This interaction has served as a remarkably good effective interaction in many regions of the periodic table. Moreover, its key low-lying states have a remarkably large overlap with the corresponding state vectors calculated with more realistic interactions [see, e.g. table 4 of ref. [7])]. Finally, its matrix elements are relatively simple functions of the quantum numbers, making it possible to study many approximations in analytical form (e.g. requirement (ii) on the n-p interaction can be studied in terms of sum rules for the matrix elements of the multipole components of the interaction).

3. Properties of the model

The properties of the SDI are well known [4, 5]). It may, however, be useful to review some features which are important for our model.

(a) In the limit in which the n-p part of the interaction can be imagined to be turned off, the total seniority numbers for the separate proton and neutron parts of the configuration are good quantum numbers. The spectrum for a fixed number of particles n (n_p or n_n) therefore contains the spectrum for $n-2$ particles (which is repeated with no change in spacing of the levels), and a set of additional levels corresponding to seniority $v = n$ ($v_p = n_p$ or $v_n = n_n$). In trying to identify the key states (low energy, high collectivity) of the separate proton and neutron configurations, it is thus possible to examine in succession the spectra for 2, 4, 6, . . . particles (protons or neutrons). The goodness of the seniority quantum numbers also simplifies calculations since it is possible to determine the n-dependence of matrix elements by quasi-spin techniques [6]), where the quasi-spin raising and lowering operators are related to the usual $J = 0$ coupled-pair creation and annihilation operators of ordinary pairing theory.

(b) Matrix elements of the SDI are functions only of j, provided the relative parities of the single-particle states of the configuration are the same. In a configuration $(f_{\frac{7}{2}} f_{\frac{5}{2}} p_{\frac{3}{2}} p_{\frac{1}{2}})$ of identical particles the SDI acts only on two-particle states coupled to a spin $S = 0$. For configurations of the type $(g_{\frac{7}{2}} d_{\frac{5}{2}} d_{\frac{3}{2}} s_{\frac{1}{2}})$, it is possible to assign to a single particle in the doublet $g_{\frac{7}{2}} d_{\frac{5}{2}}$ pseudo-orbital and pseudo-spin [7-9]) angular momenta $\tilde{l} = 3$ and $\tilde{s} = \frac{1}{2}$, and to a single particle in the doublet $d_{\frac{3}{2}} s_{\frac{1}{2}}$ pseudo-orbital and spin angular momenta $\tilde{l} = 1$, and $\tilde{s} = \frac{1}{2}$, with $\tilde{j} = \tilde{l} + \tilde{s}$. In our model, in which the pseudo-spin orbit coupling in the configuration $(g_{\frac{7}{2}} d_{\frac{5}{2}} d_{\frac{3}{2}} s_{\frac{1}{2}})$ is neglected (that is, the $g_{\frac{7}{2}} d_{\frac{5}{2}}$ splitting and the $d_{\frac{3}{2}} s_{\frac{1}{2}}$ splitting are neglected), the total many-particle pseudo-spin $\tilde{S}$ is therefore a good quantum number. For any seniority the states with $\tilde{S} = 0$ (even v) lie lowest in energy. Candidates for the key states of the separate proton and neutron configurations should therefore be found among the states with $\tilde{S} = 0$. In the spectrum of the configuration $(g_{\frac{7}{2}} d_{\frac{5}{2}} d_{\frac{3}{2}} s_{\frac{1}{2}})^4$, for example, there are 94 states with $J = 2$. Of these there are only 18 with $\tilde{S} = 0$, $\tilde{L} = J = 2$. Although this is a considerable saving, a truncation in terms of $\tilde{S}$ alone is not nearly severe enough for our present purposes, and it is necessary to search for additional criteria or quantum numbers to isolate from among the $\tilde{S} = 0$ states the key proton and neutron states from which the shell-model space is to be built. The additional criterion is to be found in the notion of the favored pair [7]).

(c) One of the characteristic features of the SDI when acting in configurations of identical particles is that it favors one specific superposition of two-particle states for each value of J (J = even only, if the single-particle states of the configuration all have the same parity, as in the present model). In the two-particle spectrum only a single one of the several possible states for each J-value (with $J = \tilde{L}$ = even, $\tilde{S} = 0$), is depressed in energy; all others have eigenvalues of zero. Moreover, the favored pair state with $J \neq 0$ exhausts entirely the sum rule for a 2^J pole transition connecting it to the $J = 0$, $v = 0$ ground state (in the approximation in which the radial matrix elements of the 2^J pole operator can be replaced by a constant, the so-called surface multipole approximation [4,8])). In the two-particle spectrum therefore the favored $J \neq 0$ pairs satisfy the two key requirements of low energy and high collectivity. In the spectra with $v \geqq 4$ the key proton and neutron states will be built from superpositions of such favored pair states. Before proceeding, it will be useful to re-examine some of the mathematics of the favored pair and surface multipole operators.

4. Favored pair and surface multipole operators

In the l–s (or the analogous pseudo $\tilde{l}$–$\tilde{s}$) coupling scheme, the favored pair operators, $\mathscr{A}^+(JM)$, can be constructed from identical particle pair creation operators, coupled to $S = 0$, $L = J =$ even

$$[a_l^+ \times a_{l'}^+]^{L=JM_J;\,S=M_S=0} = \sum_{mm_s} \langle lml'm'|JM_J\rangle \langle \tfrac{1}{2}m_s\tfrac{1}{2} - m_s|00\rangle a^+_{lmm_s} a^+_{l'm'-m_s}, \quad \text{(1a)}$$

by

$$\mathscr{A}^+(JM) = \sum_{ll'} q^{(0)}(ll'J)[a_l^+ \times a_{l'}^+]^{JM;\,00}, \tag{1b}$$

where

$$q^{(0)}(ll'J) = \left[\frac{(2l+1)(2l'+1)}{2J+1}\right]^{\frac{1}{2}} \langle l0l'0|J0\rangle. \tag{1c}$$

Although our configurations are made up of pseudo $\tilde{l}$–$\tilde{s}$ doublets, tildes will be omitted to simplify the notation. It will be quietly understood that all l-values are to be interpreted as pseudo-orbital angular momenta $\tilde{l}$; e.g. $\tilde{l} = 3; 1$ in the $g_{\frac{7}{2}} d_{\frac{5}{2}}; d_{\frac{3}{2}} s_{\frac{1}{2}}$ configuration; and $\tilde{l} = 2; 0$ in the $f_{\frac{5}{2}} p_{\frac{3}{2}}; p_{\frac{1}{2}}$ configuration. The l-values are those for a major oscillator (or rather pseudo-oscillator) shell, with l (or $\tilde{l}$) $= l_m, l_m-2, \ldots 0$(or 1) where l_m denotes the maximum l-value of the shell. The superscript (0) on q denotes the favored pair combination. The companion favored pair annihilation operators are defined by

$$\mathscr{A}(JM) = (\mathscr{A}^+(JM))^+, \tag{2}$$

while the one-body operators which are the natural partners of these pair operators are given by

$$Q(JM) = \sum_{ll'} (-1)^l q^{(0)}(ll'J)\sqrt{2}[a_l^+ \times a_{l'}]^{L=JM;\,S=M_S=0}, \tag{3a}$$

with

$$\sqrt{2}[a_l^+ \times a_{l'}]^{JM;\,00} = \sum_{m(m')} \langle lml'-m'|JM\rangle \sum_{m_s} a^+_{lmm_s} a_{l'm'm_s}(-1)^{l'-m'}. \tag{3b}$$

Alternatively

$$Q(JM) = \sum_i \sqrt{4\pi} Y_{JM}(\theta_i \varphi_i), \tag{3c}$$

where the dependence on the coordinates of the ith particle is on the angular coordinates of the spherical harmonics only. Hence $Q(JM)$ has been named a surface multipole operator [8]); it is the 2^J pole operator in which the radial dependence has been replaced by the constant $\sqrt{4\pi}$.

The pair operators with $J = 0$ play a special role. They can be identified with the quasi-spin raising and lowering operators. Specifically

$$\mathscr{S}_+ = \frac{1}{\sqrt{2}} \mathscr{A}^+(00), \qquad \mathscr{S}_- = \frac{1}{\sqrt{2}} \mathscr{A}(00),$$
$$\mathscr{S}_0 = \tfrac{1}{2}(Q(00)-\Omega) = \tfrac{1}{2}(N_{\mathrm{op}}-\Omega), \tag{4}$$

where Ω is the pair degeneracy number: $\Omega = \sum(2l+1) = \sum(j+\frac{1}{2})$; and where $\mathscr{S}_+$, $\mathscr{S}_-$, $\mathscr{S}_0$ satisfy the usual angular momentum commutation rules. The quantum number $\mathscr{S}$ associated with the operator $\mathscr{S}^2$ is related to the seniority quantum number v by $\mathscr{S} = \frac{1}{2}(\Omega-v)$, while $M_{\mathscr{S}} = \frac{1}{2}(n-\Omega)$. (Note that both the terms "quasi-spin" and "pseudo-spin" have been taken from earlier work; there should be no confusion between the quasi-spin $\mathscr{S}$ and the pseudo-spin $\tilde{S}$.) From the commutation properties of the components of $\mathscr{S}$ with the operators $\mathscr{A}^+$, $\mathscr{A}$ and Q, it can be seen that these

are the three components of a rank-1 tensor in quasi-spin space. Specifically, the tensor character of the operators is:

$$\mathscr{A}^{+}(JM) = T^{J}_{M,+1}{}^{1}, \qquad -Q(JM) = T^{J}_{M,0}{}^{1}, \qquad -(-1)^{J-M}\mathscr{A}(J-M) = T^{J}_{M,-1}{}^{1}, \quad (5)$$

where $T^{J}_{M,M_{\mathscr{S}}}{}^{\mathscr{S}}$ denotes an operator which is a spherical tensor of rank J and spherical component M in ordinary three-space, and rank $\mathscr{S}$ and spherical component $M_{\mathscr{S}}$ in quasi-spin space, so that

$$[\mathscr{S}_{\pm}, T^{J}_{M,M_{\mathscr{S}}}{}^{\mathscr{S}}] = [(\mathscr{S}\mp M_{\mathscr{S}})(\mathscr{S}\pm M_{\mathscr{S}}+1)]^{\frac{1}{2}} T^{J}_{M,(M_{\mathscr{S}}\pm 1)}{}^{\mathscr{S}}. \quad (6)$$

This latter tensor property can be used together with the Wigner-Eckart theorem in quasi-spin space to factor out the n-dependence of any matrix element in the seniority scheme [6]).

In our model involving active protons and neutrons in different major shells it is best to split the two-body interaction into p-p, n-n, and n-p parts

$$H = H_{\mathrm{pp}} + H_{\mathrm{nn}} + H_{\mathrm{np}}, \quad (7)$$

where the different parts of the interaction can be written entirely in terms of favored pair and surface multipole operators

$$H_{\mathrm{pp}}(\text{or } H_{\mathrm{nn}}) = -\tfrac{1}{2}G \sum_{JM} \mathscr{A}^{+}(JM)\mathscr{A}(JM), \quad (8)$$

where the favored pair operators are defined in terms of proton or neutron pairs, respectively while

$$H_{\mathrm{np}} = -G \sum_{kq} (-1)^{q} Q^{\mathrm{p}}(kq) Q^{\mathrm{n}}(k-q), \quad (9)$$

where G is the common strength factor of the simple SDI. It is to be noted that the n-p interaction involving particles in different major shells has the simple form (9) only if the interaction is the simple SDI with equal strengths in the two-particle states coupled to isospin $T = 0$ and $T = 1$. (Note that this requirement has been relaxed somewhat in the so-called modified surface delta interaction of Glaudemans *et al.* [10]).) Since the surface multipole operators $Q(kq)$, like the favored pair operators, are coupled to spin $S = 0$ (or rather $\tilde{S} = 0$), the n-p interaction of the simple form (9) will, like the interactions H_{pp} and H_{nn}, preserve the total pseudo-spins $\tilde{S}_{\mathrm{p}}$ and $\tilde{S}_{\mathrm{n}}$ of the separate proton and neutron configurations. However, it is possible to modify the strength factors of eqs. (8) and (9), using different values G_{pp}, G_{nn}, G_{np} (or even introduce a k-dependence into the coefficients G of eq. (9)), without changing the basic symmetry which implies the goodness of $\tilde{S}_{\mathrm{p}}$ and $\tilde{S}_{\mathrm{n}}$. More realistic forms of the n-p interaction will in general include pseudo-spin breaking terms. Since the final states with different values of $\tilde{S}_{\mathrm{p}}$ or $\tilde{S}_{\mathrm{n}}$ will be well separated in energy, the inclusion of such pseudo-spin breaking terms will not change the essential thesis of this investigation, provided their coefficients are relatively small compared with the coefficients of the surface multipole terms of eq. (9).

Since the two-body interaction of our simplified model is built entirely from favored pair and surface multipole operators, it is natural to ask whether these operators form a familty closed under the commutation process. For example

$$[Q(kq), \mathscr{A}^+(JM)] = \sum_{J'} 2\left[\frac{(2J+1)(2k+1)}{2J'+1}\right]^{\frac{1}{2}} \langle JMkq|J'M'\rangle$$

$$\times \sum_{l_1 l_2} [a^+_{l_1} \times a^+_{l_2}]^{J'M';00} \left\{ \sum_l (-1)^l q^{(0)}(ll_1 k) q^{(0)}(ll_2 J) \begin{Bmatrix} J & k & J' \\ l_1 & l_2 & l \end{Bmatrix} [2J'+1]^{\frac{1}{2}} \right\}. \quad (10a)$$

For values of l_1 and l_2 small enough such that $l_1 + k \leqq l_m$, $l_2 + J \leqq l_m$, where l_m is the maximum possible l-value in the shell under consideration ($l = l_m, l_m - 2, \ldots, 0$ (or 1)), the sum over l in the curly bracket can be carried out,

$$\{\sum_l \ldots\} = \langle J0k0|J'0\rangle q^{(0)}(l_1 l_2 J'), \quad (10b)$$

and the pair built from $a^+_{l_1}$ and $a^+_{l_2}$ is multiplied with the appropriate factor $q^{(0)}(l_1 l_2 J')$ needed to make the favored pair combination. With $J = 0$ (or $k = 0$) this condition is met for *all* values of l_1 and l_2. The commutator of $Q(kq)$ with $\mathscr{A}^+(00)$ gives only the *favored* pair $\mathscr{A}^+(kq)$, a result which has already been expressed by one of the eqs. (6). With both $J \neq 0$, $k \neq 0$, however, and large values of l_1 and l_2, there will be missing terms in the l-sum in the curly bracket (the product of one Racah and two Wigner coefficients cannot be summed to yield the two simple Wigner coefficients needed for the right-hand side of eq. (10b)), and the resultant pair operators will involve contributions from pair combinations other than the favored ones. With $k = 2$, for example, if the missing terms with $l = l_m + 2$ are added and subtracted in the coefficient for the pair creation operators with $l_1 = l_m$, the above commutator can be expressed in the form

$$[Q(2q), \mathscr{A}^+(JM)] = 2\sum_{J'} \langle JM2q|J'M'\rangle \left\{ q^{(0)}(J2J')\mathscr{A}^+(J'M') - \sum_{l_2} [a^+_{l_m} \times a^+_{l_2}]^{J'M';00} \right.$$

$$\left. \times [5(2J+1)]^{\frac{1}{2}} q^{(0)}(l_m, l_m+2, 2) q^{(0)}(l_m+2, l_2, J) \begin{Bmatrix} J & 2 & J' \\ l_m & l_2 & l_m+2 \end{Bmatrix} \right\}, \quad (11)$$

that is, the commutator gives a combination of favored pairs $\mathscr{A}^+(J'M')$ with $J' = J+2$, J and $J-2$, except for a few additional terms involving only pairs built from single-particle states with $l_1 = l_m$ and $l_2 = l_m, \ldots |l_m - J'|$. In a heavy shell (corresponding to a large value of l_m), in particular, these additional terms are small compared with the dominant favored pair terms, so that the favored operators form a family which to some degree of approximation is (almost) closed under the commutation process. Similarly, the commutator of favored pair operators $\mathscr{A}$ with $\mathscr{A}^+$,

which is

$$[\mathscr{A}(J_2 M_2), \mathscr{A}^+(J_1 M_1)] = 2\delta_{J_1 J_2}\delta_{M_1 M_2}\sum_{ll'}[q^{(0)}(ll'J_1)]^2$$
$$-2\sum_k \langle J_1 M_1 J_2 - M_2 | kq\rangle (-1)^{J_2 - M_2}\sum_{l_1 l_2}\sqrt{2}[a^+_{l_1}\times a_{l_2}]^{kq;\,00}$$
$$\times\left\{\sum_l q^{(0)}(ll_2 J_2)q^{(0)}(ll_1 J_1)\begin{Bmatrix} J_1 & J_2 & k\\ l_2 & l_1 & l\end{Bmatrix}[(2J_1+1)(2J_2+1)]^{\frac{1}{2}}\right\}, \qquad (12)$$

can be written

$$[\mathscr{A}(J_2 M_2), \mathscr{A}^+(J_1 M_1)] = 2\delta_{J_1 J_2}\delta_{M_1 M_2}\sum_{ll'}[q^{(0)}(ll'J_1)]^2$$
$$-2\sum_k \langle J_1 M_1 J_2 - M_2 | kq\rangle (-1)^{J_2 - M_2}\left\{q^{(0)}(J_1 J_2 k)Q(kq) - \sum_{l_1 l_2}\sqrt{2}[a^+_{l_1}\times a_{l_2}]^{kq;\,0}\right.$$
$$\left.\times\sum_{l>l_m} q^{(0)}(ll_2 J_2)q^{(0)}(ll_1 J_1)\begin{Bmatrix} J_1 & J_2 & k\\ l_2 & l_1 & l\end{Bmatrix}[(2J_1+1)(2J_2+1)]^{\frac{1}{2}}\right\}, \qquad (13)$$

that is, the commutator of a favored pair annihilation operator with a favored pair creation operator, yields, besides a constant term, a linear combination of surface multipole operators, and a few additional terms which are one-body operators built only from single-particle creation and annihilation operators with $l_m \geqq l_1 \geqq |l_m + 2 - J_1|$ and $l_m \geqq l_2 \geqq |l_m + 2 - J_2|$. These additional terms can again be expected to become relatively unimportant compared with the favored surface multipole operators in the limit in which l_m becomes large and J_1 and J_2 are relatively small.

5. The key states of the truncation scheme

The key states of the separate proton and neutron configurations should ideally satisfy the following criteria. They should be limited to a relatively small set of states $\alpha_1, \alpha_2, \ldots, \alpha_m$ and $\beta_1, \beta_2, \ldots, \beta_k$ which are eigenstates of H_{pp} and H_{nn} respectively, with the following properties: They must have (i) low energy and (ii) the matrix elements for any surface multipole operators should satisfy the condition

$$\langle \gamma_j^p || Q^p(k) || \alpha_i\rangle = 0, \qquad \langle \gamma_j^n || Q^n(k) || \beta_i\rangle = 0 \qquad (14)$$

for any proton state γ excluded from the set (α), and any neutron state γ excluded from the set (β). If satisfied exactly, condition (2) would imply exact truncation of the shell-model space for our *simplified* model Hamiltonian. If satisfied only approximately, conditions (2) will nevertheless lead to a very good approximate truncation of the shell-model space if the states γ_j lie high in energy compared to the states in the sets (α) and (β).

5.1. STATES WITH $v \leqq 2$

Of the states with $v \leqq 2$, only the favored pair states (with $J = 0, 2, 4, \ldots 2l_m$), are to be included in the sets (α) or (β). They are the only states depressed in energy. The remaining two-particle states (all $v = 2$ states with $\tilde{S} = 1$, and all $\tilde{S} = 0$ states *other* than the favored pair combinations) are not acted upon at all by the SDI and form a degenerate sea above the favored pair states. These are the states to be excluded from the sets of key states (α) or (β). Relative to these excluded $v = 2$ states, the favored pair states have energies [see eqs. (8) and (12)]

$$E_J = -G \sum_{ll'} [q^{(0)}(ll'J)]^2. \tag{15}$$

For major oscillator, rather pseudo-oscillator shells, with $\tilde{l} = l_m, l_m-2, \ldots, 0$ (or 1) such as those of our model, these energies can be expressed in terms of l_m by

$$\begin{aligned} E_0 &= -G\Omega, \\ E_2 &= -G\Omega \left[1 - \frac{3}{2l_m+3}\right], \\ E_4 &= -G\Omega \left[1 - \frac{15}{2} \frac{(3l_m^2+9l_m+2)}{(2l_m+5)(2l_m+3)(2l_m+1)}\right], \end{aligned} \tag{16}$$

where the pair degeneracy number Ω has the value $\frac{1}{2}(l_m+1)(l_m+2)$ in this case.

The favored pair state with $J = 0$ ($v = 0$) satisfies condition (2) above exactly. In terms of the normalization coefficients N_J for the favored pair states

$$|n = 2 \text{ fav. } JM\rangle = N_J \mathscr{A}^+(JM0|)\rangle,$$

with

$$N_J = [2 \sum_{ll'} [q^{(0)}(ll'J)]^2]^{-\frac{1}{2}}, \tag{17}$$

the reduced matrix elements of the surface multipole operators are

$$\langle n\, v = 2 \text{ fav. } k||Q(k)||n\, v = 0\, J = 0\rangle = \left[\frac{n(2\Omega-n)}{2(2\Omega-2)}\right]^{\frac{1}{2}} \frac{2N_0(2k+1)^{\frac{1}{2}}}{N_k}, \tag{18}$$

where the n-dependent factor follows at once from a quasi-spin Wigner coefficient [6]). Reduced matrix elements connecting favored pairs with $J \neq 0$ can also be written in terms of l_m. For example,

$$\langle n\, v = 2 \text{ fav. } J = 2||Q(2)||n\, v = 2 \text{ fav. } J=2\rangle = -10 \sqrt{\frac{2}{7}} \left[1 - \frac{3}{2(2l_m+3)}\right] \left[\frac{\Omega-n}{\Omega-2}\right], \tag{19a}$$

$$\langle n\, v = 2 \text{ fav. } J = 4||Q(2)||n\, v = 2 \text{ fav. } J=2\rangle = 6 \sqrt{\frac{10}{7}} \left[1 - \frac{15(3l_m+5)}{8(2l_m+3)(2l_m+1)}\right] \left[\frac{1 - \frac{3}{2l_m+3}}{1 - \frac{15(3l_m^2+9l_m+2)}{2(2l_m+5)(2l_m+3)(2l_m+1)}}\right]^{\frac{1}{2}} \left[\frac{\Omega-n}{\Omega-2}\right]. \tag{19b}$$

Matrix elements of the $Q(kq)$ between a favored $J \neq 0$ pair state and an excluded state with $\tilde{S} = 1$ are exactly equal to zero; while matrix elements between a favored $J \neq 0$ pair state and an excluded state with $\tilde{S} = 0$ are in general small compared with the matrix elements connecting two favored pair states; and very small in the limit of rich configurations (large l_m). This can be seen in several ways.

In the limit of large l_m, the commutator of a Q with a favored pair operator $\mathscr{A}^+$ will yield mainly favored pair operators, eq. (11), and the few additional terms will have a small overlap with any one $\tilde{S} = 0$ state of the unfavored (excluded) variety.

To study the relative importance of matrix elements of Q it is useful to derive sum rules for the reduced matrix elements, such as

$$\sum_{v'\alpha'J'} \langle n = 2\, v'\alpha'J' || Q(k) || n = v = 2 \text{ fav. } J \rangle^2.$$

Such sum rules will be discussed in detail in sect. 6. For the special case of the quadrupole operator and the favored $v = 2$, $J = 2$ state, they have been studied in detail by Arvieu and Moszkowski [5]). (The present investigation was inspired in part by the work of Arvieu and Moszkowski. Eqs. (18) and (19a) can be found in somewhat different form in ref. [5]); except for a disagreement in sign in the case of eq. (19a).) For the favored state with $n = 2$, $J = 2$, the quadrupole sum over J' states of the *favored* variety *only* has a value of $100[1 - (\text{a number which tends to zero as } l_m \to \infty)]$; see eqs. (18), (19) and (16), particularly for the nature of the l_m dependent terms. The same sum over all states v', α', J' *other* than favored pair states is a complicated function of l_m, but a number which tends to zero as $l_m \to \infty$. For $l_m = 2$, 3 and 4 it has the values 3.12, 3.48 and 3.33, respectively. Although it goes to zero slowly as $l_m \to \infty$ (for $l_m = 10$ it has the value 2.07), it is small compared with the factor of 100 which appears in the sum over favored states. Even for $l_m = 2$ or 3 (the cases needed in our model), therefore, the square of a matrix element of Q connecting a favored $v = 2$ state to any excluded $v = 2$ state must be expected to be only a few percent of the square of one of the larger matrix elements, connecting a favored state to a favored state.

Finally, for the $v = 2$ states it is of course easy to calculate *all* such matrix elements for the specific configurations of interest. For the $(g_{\frac{7}{2}}\, d_{\frac{5}{2}}\, d_{\frac{3}{2}}\, s_{\frac{1}{2}})$ configuration, for example, there are, besides the favored pair states with $J = 0$, 2, 4 and 6, additional $v = 2$ states with $\tilde{S} = 0$ and the following J-values: 0, 2, 2, 3 and 4. Let such excluded pair states be expressed in terms of pair creation operators $A^{(a)+}_{JM}$ with superscript $(a) = 1, 2, \ldots$ used to distinguish multiple occurrences of a given J,

$$|\text{excluded state } n = v = 2, \tilde{S} = 0, JM\rangle = N_J A^{(a)}(JM)^+ |0\rangle$$
$$= N_J \sum_{ll'} q^{(a)}(ll'J)[a^+_l \times a^+_{l'}]^{JM;\,00}|0\rangle. \quad (20)$$

Note that $a \neq 0$ indicates an excluded pair state $(A^{(a)+})$ while $a = 0$ indicates the favored pair state $A^{(0)+} \equiv \mathscr{A}^+$; compare with eq. (1). For J-values for which there is more than one pair state with $a \neq 0$ the choice of $q^{(a)}$ with $a \neq 0$ is made arbitrarily.

However, the q must satisfy orthogonality conditions

$$\sum_{l_1 l_2} q^{(a)}(l_1 l_2 J) q^{(a')}(l_1 l_2 J) = \delta_{aa'} \frac{1}{2N_J^2}, \qquad a(\text{or } a') = 0, 1, 2, \ldots, \tag{21}$$

where N_J, eq. (17), is independent of a. The inverse of eqs. (17) and (20) is also useful:

$$[a_l^+ \times a_{l'}^+]^{JM;\,00} = 2N_J^2 \{q^{(0)}(ll'J)\mathscr{A}^+(JM) + \sum_{a\neq 0} q^{(a)}(ll'J) A^{(a)}(JM)^+\}. \tag{22}$$

In terms of this notation the reduced matrix elements of the surface multipole operators between two $v = 2$ states are

$$\langle n\, v' = 2\, J' a' \| Q(k) \| n\, v = 2\, J a\rangle = \frac{\Omega - n}{\Omega - 2} \left[\frac{4(2J'+1)(2J+1)(2k+1)}{(\sum_{ll'} [q^{(0)}(ll'J)]^2)(\sum_{ll'} [q^{(0)}(ll'J')]^2)}\right]^{\frac{1}{2}}$$

$$\times \sum_{ll_1 l_2} (-1)^l q^{(0)}(ll_1 k) q^{(a)}(l_1 l_2 J) q^{(a')}(ll_2 J') \begin{Bmatrix} J & k & J' \\ l & l_2 & l_1 \end{Bmatrix}. \tag{23}$$

For a matrix element connecting two favored pair states, $a = a' = 0$, the phases of the $q^{(0)}$ [viz.$(-1)^{\frac{1}{2}(l+l'+J)}$], are such that the three $q^{(0)}$ factors in eq. (23), together with the $(-1)^l$ factor, can be replaced by the absolute values of the $q^{(0)}$ factors, provided the matrix element is multiplied by an overall phase factor $(-1)^{\frac{1}{2}(J+J'+k)}$. Moreover, the magnitudes of the factors $q^{(0)}(ll'J)$ with $J \neq 0$ are very mild functions of l, l' and J. (For the pseudo f-p shell with $l_m = 3$, e.g., they vary between 1.095 and 1.366). Finally almost all of the 6-j symbols needed in the l-sums of eq. (23) are positive. The matrix elements connecting two favored pair states, $a = a' = 0$, are thus built from coherent superpositions of terms of comparable magnitudes, and hence such matrix elements are large. With $a = 0$ and $a' \neq 0$, however, the orthogonality requirement (21) insures that the l-sums now give an incoherent superposition of terms of comparable magnitudes, so that the cancellation of positive and negative terms leads to the small values of the matrix elements connecting a favored state ($a = 0$) to an excluded state ($a' \neq 0$).

5.2. THE $v = 4$ STATES

In the case of the states with $v = 4$ (the next set to be considered) the separation into the two sets, favored and excluded, is less obvious. The large number of states with $\tilde{S} = 1$ and $\tilde{S} = 2$ must clearly be excluded (they lie high in energy and have no connections to favored $v = 2$ states via matrix elements of the surface multipole operators). Since there are a large number of states with $v = 4$ and $\tilde{S} = 0$, however, the additional criteria of low energy and high collectivity must be used to isolate the key favored states. For the $(f_{\frac{7}{2}} p_{\frac{3}{2}} p_{\frac{1}{2}})$ proton configuration of our model there are 12 $v = 4$ states with $\tilde{S} = 0$, with J-values: 0, 1, 2^3, 3^2, 4^3, 5 and 6. For the $(g_{\frac{7}{2}} d_{\frac{5}{2}} d_{\frac{3}{2}} s_{\frac{1}{2}})$ neutron configuration on the other hand, there are 86 $v = 4$ states with $\tilde{S} = 0$, with J-values: 0^5, 1^6, 2^{15}, 3^{11}, 4^{16}, 5^{11}, 6^{10}, 7^5, 8^5, 9 and 10. In these cases it is in principle still possible to diagonalize the full Hamiltonian and calculate all matrix elements of

surface multipole operators to choose the key states from among the many $v = 4$, $\tilde{S} = 0$ states by a process of elimination. The full spectrum of $v = 4$ states for the $(\frac{5}{2}\,\frac{3}{2}\,\frac{1}{2})$ configuration is shown as part of fig. 1. For the $(\frac{7}{2}\,\frac{5}{2}\,\frac{3}{2}\,\frac{1}{2})$ configuration the low-lying $v = 4$ states are shown as part of fig. 2. The exact eigenvalues of the Hamiltonian (8) were calculated with the Oak-Ridge-Rochester shell-model code [11]). (For the $(\frac{5}{2}\,\frac{3}{2}\,\frac{1}{2})$ configuration they have already been given by Arvieu and Moszkowski [5]).) Clearly such a process of trial and error will no longer be feasible in the case of richer

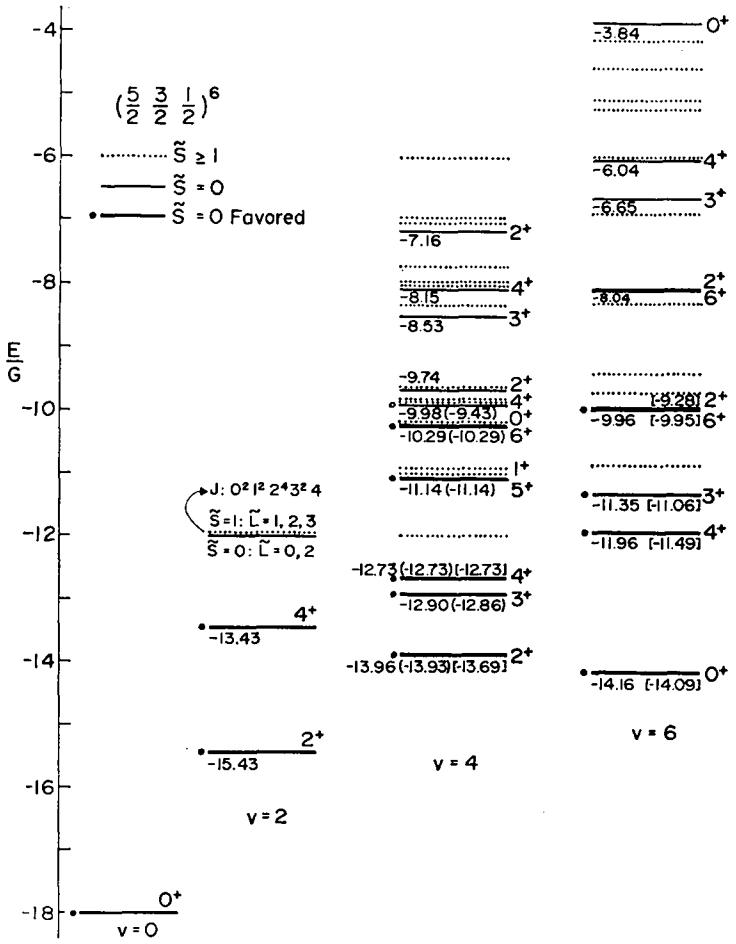

Fig. 1. The spectrum for the configuration $(f_{\frac{5}{2}}p_{\frac{3}{2}}p_{\frac{1}{2}})^6$. Levels with $\tilde{S} \geqq 1$ are indicated by dotted lines, levels with $\tilde{S} = 0$ by solid lines. Favored states to be retained in the truncated shell-model space are indicated by an additional dot. Numbers below the levels are exact eigenvalues. Numbers in parentheses are eigenvalues of H in the truncated subspace of favored $v = 4$ states constructed from favored $J \neq 0$ pairs according to eqs. (24)–(26) (see table 4; note that $E(n = 6, v = 4) = [E(n = v = 4) - 6]$). Numbers in square brackets are expectation values of H in states obtained by seniority projection techniques from single states built from the lowest favored $J \neq 0$ pairs, as discussed in sect. 7.

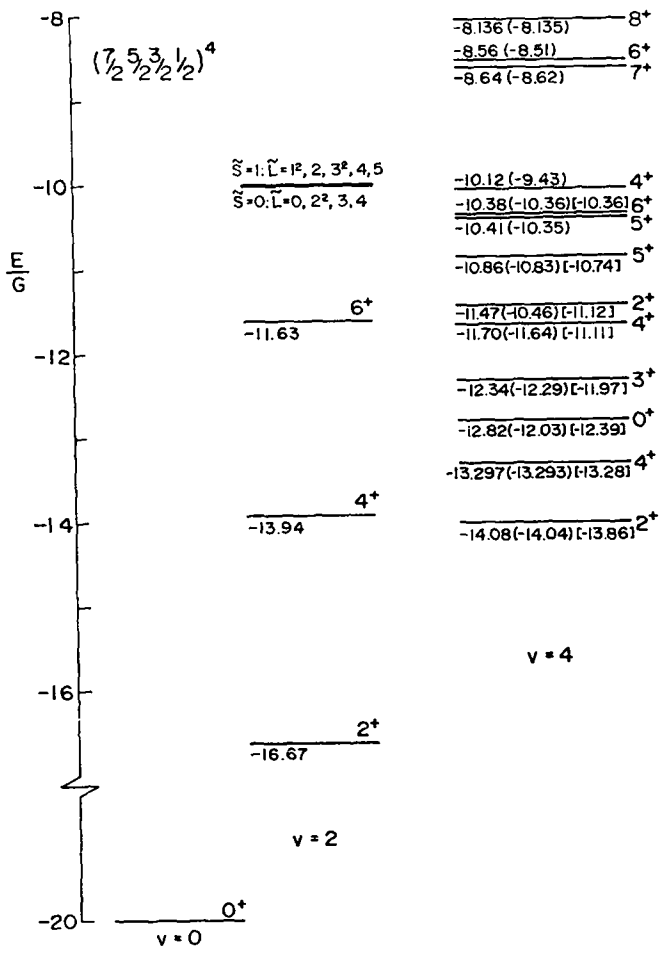

Fig. 2. The spectrum for the configuration $(g_{\frac{7}{2}}d_{\frac{5}{2}}d_{\frac{3}{2}}s_{\frac{1}{2}})^4$. Only the lowest $v = 4$ states are shown. Numbers below the levels are the exact eigenvalues. Numbers in parenthesis are eigenvalues of H in the truncated subspace of favored $v = 4$ states constructed from favored $J \neq 0$ pairs according to eqs. (24)–(26) (see table 3). Numbers in square brackets are expectation values of H in states obtained by seniority projection techniques from single states built from the lowest favored $J \neq 0$ pairs, as discussed in sect. 7.

configurations or larger numbers of particles (higher seniorities) so that good approximation techniques must be developed to isolate and construct the key states to be selected for our truncation scheme. The importance of favored $J \neq 0$ pair states will again be used. It will be shown that the key $v = 4$ states can be constructed to very good approximation from superpositions of favored $J \neq 0$ pair states. Let $|[J_1 \times J_2]$ $IM_I\rangle$ be a normalized four-particle state constructed by coupling two favored $J \neq 0$ pair states:

$$|[J_1 \times J_2]IM_I\rangle = N(J_1 J_2 I) \sum_{M_1 M_2} \langle J_1 M_1 J_2 M_2 | IM_I\rangle \mathscr{A}^+(J_1 M_1)\mathscr{A}^+(J_2 M_2)|0\rangle, \quad (24)$$

where $N(J_1 J_2 I)$ is a normalization factor. Such a state will be a superposition of $v = 4$, $v = 2$ and (for $I = 0$) $v = 0$ states. However, it is possible to construct pure $v = 4$ states from linear combinations of such states

$$|v = 4\ bIM_I\rangle = \sum_{J_1 J_2} c^{(b)}(J_1 J_2 I)|[J_1 \times J_2]IM_I\rangle, \tag{25}$$

if the coefficients $c^{(b)}(J_1 J_2 I)$, $b = 1, 2, \ldots$, are chosen such that the resultant four-particle states are free of favored pair states coupled to $J = 0$; that is

$$\mathscr{A}(00)\sum_{J_1 J_2} c^{(b)}(J_1 J_2 I)|[J_1 \times J_2]IM_I\rangle = 0. \tag{26}$$

The result of acting with $\mathscr{A}(00)$ gives a combination of the various two-particle states with $\tilde{S} = 0$ and $J = I$; so that the requirement (26) gives a number of conditions on the $c(J_1 J_2 I)$ equal to the number of independent two-particle states with $\tilde{S} = 0$ and $\tilde{L} = J = I$. In the $(\frac{7}{2}\ \frac{5}{2}\ \frac{3}{2}\ \frac{1}{2})$ configuration, e.g., the favored $J \neq 0$ pair states have J_1 (or J_2) of 2, 4 and 6. The vector coupling triangle conditions $\triangle(J_1 J_2 I)$ restrict the possible number of combinations. In addition, states (24) with $J_1 = J_2$ and I odd would be identically zero. For $I = 3$, e.g., only the combinations $[J_1 \times J_2] = [2 \times 4]$ and $[4 \times 6]$ are possible, while the relation (26) leads to a single condition on the coefficients $c(J_1 J_2 I)$ since there is only a single two-particle state with $\tilde{S} = 0$, $J = 3$ in this case. Thus there is a single favored $v = 4$ state with $I = 3$ which can be built from superpositions of favored $J \neq 0$ pairs. Table 1 shows the arithmetic for the construction process for the $(\frac{7}{2}\ \frac{5}{2}\ \frac{3}{2}\ \frac{1}{2})$ configuration and gives the number of favored $v = 4$ states for each of the possible I-values. Table 2 gives similar results for the $(\frac{5}{2}\ \frac{3}{2}\ \frac{1}{2})$ configuration. The favored $v = 4$ states built through this process must of course be linearly independent. For very high I, in particular, the requirement of

TABLE 1

Favored $v = 4$ states for the $(\frac{7}{2}\ \frac{5}{2}\ \frac{3}{2}\ \frac{1}{2})^4$ configuration

I	Number of $[J_1 \times J_2]$ combinations	Number of conditions	Number of favored $v = 4$ states
0	3	2	1
2	5	3	2
3	2	1	1
4	6	2	4
5	3	0	3
6	5	1	4
7	2	0	2
8	4	1 [a])	3
9	1	0	1
10	2	1 [a])	1
			—
			22

[a]) From the requirement of linear independence.

TABLE 2
Favored $v = 4$ states for the $(\frac{7}{2}\ \frac{5}{2}\ \frac{3}{2}\ \frac{1}{2})^4$ configuration

I	Number of $[J_1 \times J_2]$ combinations	Number of conditions	Number of favored $v = 4$ state
0	2	2	0
2	3	2	1
3	1	0	1
4	3	1	2
5	1	0	1
6	2	1 [a])	1

[a]) From the requirement of linear independence.

independence may add conditions over and above those of eqs. (26) (see table 1). For the $(\frac{7}{2}\ \frac{5}{2}\ \frac{3}{2}\ \frac{1}{2})$ configuration this construction process for the favored $v = 4$ states leads to a single state with $I = 0$, none with $I = 1$, two with $I = 2$, one with $I = 3$, four with $I = 4$, etc. It is interesting to note that among all of the low-lying states with $v = 4$ (fig. 2), there is a single one with $I = 0$, two with $I = 2$, one with $I = 3$, three with $I = 4$ (the fourth one lies somewhat above the highest $v = 4$ states shown in fig. 2) etc. To see whether the favored $v = 4$ states built from a superposition of favored $J \neq 0$ pairs are indeed good approximations to the exact low-lying eigenstates of the Hamiltonian (8), this Hamiltonian has been diagonalized in the subspace of favored $v = 4$ states. Note that this is a highly truncated subspace. In the configuration $(\frac{7}{2}\ \frac{5}{2}\ \frac{3}{2}\ \frac{1}{2})^4$ there are, e.g., 94 states with $I = 2$. Of these 86 are states with $v = 4$. Of these only 15 have $\tilde{S} = 0$ and finally, of these only *two* can be built from superpositions of favored $J \neq 0$ pairs and are to be retained in the truncation scheme. The eigenvalues of H in the truncated subspace are shown in figs. 1 and 2 in parentheses alongside the exact eigenvalues (from a diagonalization of H in the full space). The agreement between the two is very good and spectacular in many cases, particularly for the lowest eigenvalues of a given I. It must therefore be expected that the $v = 4$ states built from superpositions of favored $J = 0$ pairs according to the prescription (24)–(26) are very good approximations to the exact eigenvectors for the low-energy $v = 4$ states. The eigenvectors of the favored $v = 4$ states in the truncated subspace are shown in tables 3 and 4 for the $(\frac{7}{2}\ \frac{5}{2}\ \frac{3}{2}\ \frac{1}{2})$ and $(\frac{5}{2}\ \frac{3}{2}\ \frac{1}{2})$ configurations, respectively. These eigenvectors form an orthonormal set (they are eigenvectors of the Hermitian operator H); although state vectors $|[J_1 \times J_2]IM_I\rangle$ and $|[J_1' \times J_2']IM_I\rangle$ are not orthogonal to each other and frequently have large overlaps. (Since these overlaps are also related to the $4 \rightarrow 2 \times 2$ particle fractional parentage coefficients, they are numbers basic to the technique of calculation used in this investigation; see appendix A.) To further test the goodness of the eigenvectors of tables 3 and 4 it might be interesting to calculate their overlaps with the eigenvectors of the full shell-model space. Since matrix elements of the surface multipole operators are the crucial numbers of our

model, we have chosen instead to compare the reduced matrix elements for $Q(k)$ as calculated with our approximate state vectors (eigenvectors of H in the truncated subspace of favored $v = 4$ states) with those calculated with the exact eigenvectors (eigenvectors of H in the full space). Some sample comparisons are given for the two

TABLE 3

Favored $v = 4$ states for the configuration $(\frac{7}{2}\ \frac{5}{2}\ \frac{3}{2}\ \frac{1}{2})^4$

I	b [a])	E_{Ib}	J_1J_2	22	24	44	26	46	66
0^+	1	−12.03		0.4971		−0.6008			0.1358
2^+	1	−14.04		0.8052	0.2976	0.0357		0.1691	0.0280
2^+	2	−10.46		0.1898	−0.2874	−0.8452		0.0360	0.1918
3^+	1	−12.29			0.9374			0.2166	
4^+	1	-13.29_3		0.8381	0.1417	−0.1049	−0.2362	0.1165	−0.0517
4^+	2	−11.64		0.0576	0.5865	−0.1246	0.6253	−0.0688	−0.0164
4^+	3	− 9.43		0.0146	0.3779	0.7797	−0.2527	−0.1886	−0.0883
4^+	4	− 5.34		0.0329	0.0647	0.3638	0.1523	0.7418	0.6015
5^+	1	−10.83			0.9068		−0.4028	0.1120	
5^+	2	−10.35			0.4221		0.8545	−0.2215	
5^+	3	− 6.07			0.0217		0.3424	0.9741	
6^+	1	−10.36			0.6427	0.2071	0.3180	−0.2523	0.0982
6^+	2	− 8.51			−0.0343	0.7945	−0.5055	0.0109	0.0438
6^+	3	− 6.92			0.0421	0.2906	0.4887	0.7987	−0.1176
6^+	4	− 4.27			0.0681	0.1659	0.1253	−0.2075	−0.9776
7^+	1	− 8.62					1.0015	−0.0968	
7^+	2	− 6.62					0.0328	0.9974	
8^+	1	− 8.13				0.3597	−0.7705		−0.0373
8^+	2	− 7.44				1.1221	0.9040		0.0533
8^+	3	− 3.95				0.5301	0.4635		1.1383

[a]) The label b orders the energies: $b = 1, 2, \ldots$ for the lowest, next lowest, ... eigenvalues, E_{Ib}, of H in the subspace of favored $v = 4$ states.

The table gives the coefficients $c^{(b)}(J_1J_2I)$ for:

$$|v = 4; IM_I b\rangle = \sum_{J_1J_2} c^{(b)}(J_1J_2I)|[J_1 \times J_2]IM_I\rangle.$$

TABLE 4

Favored $v = 4$ states for the configuration $(\frac{5}{2}\ \frac{3}{2}\ \frac{1}{2})^4$

I	b	E_{Ib}	J_1J_2	22	24	44
2	1	−7.93		0.6889	0.4823	0.0265
3	1	−6.86			1	
4	1	−6.73		0.6958	0.3630	−0.2029
4	2	−3.47		0.0890	0.5160	0.9187
5	1	-5.14_3			1	
6	1	-4.28_6			1 (or 0)	0 (or 1)

The table gives the coefficients $c^{(b)}(J_1J_2I)$ for:

$$|v = 4; IM_I b\rangle = \sum_{J_1J_2} c^{(b)}(J_1J_2I)|[J_1 \times J_2]IM_I\rangle.$$

configurations of our model in tables 5 and 6. It can be seen that the two sets of numbers are in very good agreement, particularly for the larger values of the reduced matrix elements, for which the numbers calculated with the approximate eigenvectors of tables 3 and 4 are always within a few percent of the exact values. The technique of choosing the favored $v = 4$ states in terms of appropriate superpositions of favored $J \neq 0$ pairs thus not only gives a ready prescription for isolating the low-energy $v = 4$ states but gives us a simple and excellent approximation scheme. A slightly different but closely related technique based on seniority projection from a *single* state $|[J_1 \times J_2]IM_I\rangle$ will be discussed in sect. 7. The question remains whether the favored $v = 4$ states of tables 3 and 4 satisfy the crucial condition (2) of eq. (14). Are these favored states the only ones with important connections to the favored $v = 2$ states via large matrix elements of the surface multipole operators? The $v = 4$ states excluded from the favored class lie higher in energy; but do they have only weak surface multipole matrix connections to the key states retained in our truncation scheme? These conditions are indeed satisfied. For the richer configurations (large l_m), this can be seen qualitatively from the general commutation properties of the favored pair and surface multipole operators, eq. (10). A more quantitative estimate of the relative magnitudes of favored → favored *versus* favored → excluded matrix elements can be given in terms of multipole sum rules to be presented in sect. 6.

TABLE 5

Reduced matrix elements $\langle v'I'||Q(k)||vI\rangle$ for the $(\frac{5}{2}\ \frac{3}{2}\ \frac{1}{2})^4$ configuration

Sample comparisons between exact and approximate calculations

k	v'	v	I'	I	exact	approx.	k	v'	v	I'	I	exact	approx.
2	2	0	2	0	4.276	4.276	4	2	0	4	0	3.703	3.703
	4	2	2	2	4.82	4.89		4	2	2	2	2.02	1.99
			3	2	0	0				3	2	3.52	3.53
			4	2	5.80	5.81				4	2	2.21	2.21
										5	2	3.71	3.71
			2	4	3.13	3.09				6	2	1.82	1.82
			3	4	−5.45	−5.48							
			4	4	3.42	3.43				2	4	0.73	0.55
			5	4	−5.74	−5.74				4	4	−2.45	−2.43
			6	4	2.82	2.82				6	4	6.31	6.31
	4	4	3	2	6.60	6.47		4	4	3	2	−1.22	−1.28
			4	2	−1.41	−1.47				4	2	4.06	4.13
										5	2	0.81	0.84
			4	3	2.63	2.51				6	2	−1.82	−1.73
			5	3	−4.47	−4.41				4	3	2.37	2.39
										5	3	−0.93	−0.85
			5	4	2.19	2.25				6	3	1.62	1.80
			6	4	5.57	5.55				5	4	−5.02	−4.99
			6	5	−6.03	−6.03				6	4	0.23	0.31

The calculations marked "approximate" have used the $v = 4$ state vectors of table 4. Only $v = 4$ states with $b = 1$ are included in the table.

TABLE 6

Reduced matrix elements $\langle v'I'b'||Q(k)||vIb\rangle$ for the $(\frac{7}{2}\ \frac{5}{2}\ \frac{3}{2}\ \frac{1}{2})^4$ configuration

Sample comparisons between exact and approximate calculations

k	v'	v	I'	b'	I	b	exact	approx.	k	v'	v	I'	b'	I	b	exact	approx
2	2	0	2		0		4.869	4.869	4	2	0	4		0		5.021	5.021
2	4	2	0	1	2		1.73	1.68	4	4	2	2	1	2		1.37	1.43
			2	1	2		5.18	5.17				2	2	2		−1.30	−1.18
			2	2	2		0.91	1.05				3	1	2		−3.68	−3.73
			3	1	2		0	0				4	1	2		0.93	0.95
			4	1	2		6.78	6.79				4	2	2		2.90	2.95
			4	2	2		0.62	0.58				4	3	2		1.59	1.67
			4	3	2		0.01	−0.06				5	1	2		−3.98	−4.11
												5	2	2		−2.18	−1.95
2	4	2	2	1	4		1.78	1.86				6	1	2		3.63	3.64
			2	2	4		−1.69	−1.54				6	2	2		0.22	0.22
			3	1	4		4.79	4.86									
			4	1	4		1.21	1.23	4	4	2	2	1	4		0.24	0.32
			4	2	4		3.77	3.84				2	2	4		−4.10	−4.35
			4	3	4		2.07	2.17				3	1	4		0	0
			5	1	4		5.18	5.35				4	1	4		−1.23	−1.21
			5	2	4		2.84	2.54				4	2	4		−0.83	−0.88
			6	1	4		4.72	4.73				4	3	4		4.10	5.12
			6	2	4		0.30	0.28									
									4	4	4	4	1	0	1	1.95	2.00
2	4	4	0	1	2	1	−2.76	−2.55				4	2	0	1	1.92	1.86
			2	2	2	1	−1.88	−1.80				4	3	0	1	−2.68	−2.79
			3	1	2	1	6.52	6.70									
			4	1	2	1	−3.10	−3.06				2	2	2	1	1.78	1.66
			4	2	2	1	4.15	4.10				3	1	2	1	−0.62	−0.63
			4	3	2	1	1.29	1.32				4	1	2	1	5.33	5.43
												4	2	2	1	4.06	3.89
2	4	4	3	1	4	1	−0.46	−0.37				4	3	2	1	−0.99	−0.67
			4	2	4	1	1.72	1.75				5	1	2	1	−0.79	−0.55
			4	3	4	1	0.56	0.80				5	2	2	1	5.29	5.19
			5	1	4	1	3.37	3.47				6	1	2	1	−0.47	−0.35
			5	2	4	1	1.19	0.99				6	2	2	1	−2.60	−2.58
			6	1	4	1	8.51	8.50									
			6	2	4	1	0.20	0.18									

The calculations marked "approx." have used the $v = 4$ state vectors of table 3.

Finally, the total number of favored $v = 4$ states to be retained in the truncation scheme is not unreasonably large. In the $(\frac{7}{2}\ \frac{5}{2}\ \frac{3}{2}\ \frac{1}{2})$ shell there are altogether 22 $v = 4$ states which can be constructed from superpositions of favored $J \neq 0$ pairs (table 1). A few of these, particularly a few of those with large I, lie relatively high in energy and have no important surface multipole connections to low-energy favored states so that they can also be excluded. In the final calculations only the 13 $v = 4$ states shown in fig. 2 have been retained in the truncation scheme.

6. Multipole sum rules

To estimate the relative importance of matrix elements of the surface multipole operators connecting a favored state to an excluded state *versus* those connecting favored states to favored states, it is useful to calculate sum rules for these reduced matrix elements, such as

$$\sum_{\alpha' J'} \langle nv'\alpha' J' || Q(k) || nv \text{ fav. } J\rangle^2 = S(v'vJ), \tag{27}$$

where the states in the right-hand side of the matrix element is a particular state of the favored variety, while the sum is over *all* states J', α' for a *fixed* v'. Such sums are calculated from the relation

$$\sum_{v'\alpha' J'} \langle nv'\alpha' J' || Q(k) || nv \text{ fav. } J\rangle^2 = (2J+1)\langle nv \text{ fav. } J | \sum_q (-1)^q Q(kq) Q(k-q) | nv \text{ fav. } J\rangle. \tag{28}$$

The sum can then be restricted to particular values of v' by successive choices of n and with the use of quasi-spin techniques. For the special case $k = 2$, $v = 2$, $J = 2$ (favored state) these sums have been given by Arvieu and Moszkowski [5]) using the above technique.

Since the n-p interaction of the SDI involves all possible surface multipole operators with equal weight, it may be most instructive to give a sum rule of the above type which involves a sum over the multipole order k as well. Such sum rules are particularly simple. Since the operator Q with $k = 0$ is just the number operator (with diagonal matrix elements only), it will be excluded in order to obtain the most critical estimate of the relative importance of favored → favored *versus* favored → excluded matrix elements. For the favored states with $v = 2$ the sum rules are,

for $v = 2 \to v = 2$ connections:

$$\sum_{k\neq 0} \sum_{\alpha' J'} \langle n\, v' = 2\, \alpha' J' || Q(k) || n\, v = 2 \text{ fav. } J\rangle^2 = \left(\frac{\Omega - n}{\Omega - 2}\right)^2 2(2J+1)(\Omega-2)\left\{1 + \frac{1}{\Omega}\sum_{ll'} [q^{(0)}(ll'J)]^2\right\}, \tag{29}$$

for $v = 2 \to v = 4$ connections:

$$\sum_{k\neq 0} \sum_{\alpha' J'} \langle n\, v' = 4\, \alpha' J' || Q(k) || n\, v = 2 \text{ fav. } J\rangle^2 = \left[\frac{(n-2)(2\Omega-n-2)}{2(2\Omega-6)}\right] 4(2J+1)\frac{\Omega-3}{\Omega-2}\left\{\Omega + \frac{1}{\Omega-1}\sum_{ll'} [q^{(0)}(ll'J)]^2\right\}. \tag{30}$$

Here, Ω is again the pair degeneracy number, $\Omega = \sum(2l+1)$. The n-dependent factors follow from quasi-spin Wigner coefficients [6]) and show the increased importance of the $v = 2 \to v = 4$ relative to the $v = 2 \to v = 2$ connections as n approaches the middle of a shell, $n \to \Omega$.

Sum rules for a fixed multipole order k can also be given although they are much more complicated than the simple results of eqs. (29) and (30). The case $k = 2$ may be of greatest interest, since the quadrupole operators may be the most important (the real n-p interaction may have more quadrupole content than indicated by the SDI which gives equal weight to all multipole orders). For the special case of a major oscillator (or pseudo-oscillator) shell with l-values: l_m, l_m-2, . . ., 0 (or 1), sum rules for the lower favored states with $v = 2$ are:

(i) $J = 2$; $v = 2 \to v = 2$ connections:

$$\sum_{\alpha' J'} \langle n\, v' = 2\, \alpha' J' || Q(2) || n\, v = 2 \text{ fav. } J = 2 \rangle^2$$

$$= \left(\frac{\Omega - n}{\Omega - 2}\right)^2 5 \left[16 - \frac{15(5l_m+7)}{2(2l_m+1)(2l_m+3)} + \frac{30 l_m}{(2l_m+1)(2l_m+3)^2} - \frac{30}{2l_m+1} + \frac{12}{2l_m+3}\right], \tag{31}$$

(ii) $J = 2$; $v = 2 \to v = 4$ connections:

$$\sum_{\alpha' J'} \langle n\, v' = 4\, \alpha' J' || Q(2) || n\, v = 2 \text{ fav. } J = 2 \rangle^2$$

$$= 5 \left[\frac{(n-2)(2\Omega - n - 2)}{2(2\Omega - 6)}\right] \left\{ \frac{8\Omega(3\Omega^2 - 9\Omega + 7)}{(\Omega-1)(\Omega-2)^2} \left[1 - \frac{3}{2l_m+3}\right] \right.$$

$$- \frac{40(\Omega-1)}{(\Omega-2)^2} \left[1 - \frac{3(5l_m+7)}{4(2l_m+1)(2l_m+3)}\right]$$

$$\left. - \frac{40}{(\Omega-2)^2} \left[1 - \frac{3}{2l_m+1} + \frac{3l_m}{(2l_m+1)(2l_m+3)^2}\right] \right\}, \tag{32}$$

(iii) $J = 4$; $v = 2 \to v = 2$ connections:

$$\sum_{\alpha' J'} \langle n\, v' = 2\, \alpha' J' || Q(2) || n\, v = 2 \text{ fav. } J = 4 \rangle^2 = 9 \left(\frac{\Omega - n}{\Omega - 2}\right)^2 [20 - 30 F(l_m)], \tag{33a}$$

where $F(l_m)$ has been introduced for short-hand notation,

$$F(l_m) = \frac{2(2l_m+5)(2l_m+1)}{[2(2l_m+5)(2l_m+3)(2l_m+1) - 15(3l_m^2 + 9l_m + 2)]}$$

$$\times \left\{ 2 - \frac{[5\Omega(22l_m^3 + 207l_m^2 + 395l_m - 168)(2l_m+3) - 900 l_m(l_m+3)]}{4(2l_m+7)(2l_m+5)(2l_m+3)^2(2l_m+1)(2l_m-1)} \right\}, \tag{33b}$$

(iv) $J = 4$; $v = 2 \to v = 4$ connections:

$$\sum_{\alpha' J'} \langle n\, v' = 4\, \alpha' J' || Q(2) || n\, v = 2 \text{ fav. } J = 4 \rangle^2$$

$$= 9 \left[\frac{(n-2)(2\Omega - n - 2)}{2(2\Omega - 6)}\right] \left\{ \frac{20\Omega}{\Omega - 2} \left[1 - \frac{3}{2l_m+3}\right] \right.$$

$$\left. - \frac{40}{\Omega - 2} G(l_m) - \frac{40}{(\Omega-2)^2} [2 - 3F(l_m)] \right\}, \tag{34a}$$

TABLE 7
Sum rules [a])

case \ Sum rule	$v' = 2$ $\Sigma k \neq 0$	$v' = 4$ $\Sigma k \neq 0$	$v' = 2$ $k = 2$ only	$v' = 4$ $k = 2$ only
$j = \frac{1}{2}\,\frac{3}{2}\,\frac{5}{2}$ shell				
$J = 2$	94.1 %	97.9 %	92.5 %	96.8 %
$J = 4$	77.6 %	94.8 %	97.3 %	99.4 %
$j = \frac{1}{2}\,\frac{3}{2}\,\frac{5}{2}\,\frac{7}{2}$ shell				
$J = 2$	93.9 %	95.5 %	93.4 %	98.8 %
$J = 4$	88.0 %	94.7 %	93.6 %	95.6 %
$J = 6$	66.0 %	92.2 %	96.0 %	

[a]) Defined as

$$\frac{\sum_{\text{fav. states only}} \langle {}^{n,v'}_{\alpha'J'} || Q(k) ||^{n,v=2}_{\text{fav.}\,J} \rangle^2}{\sum_{\text{all } \alpha'J'} \langle {}^{n,v'}_{\alpha'J'} || (Qk) ||^{n,v=2}_{\text{fav.}\,J} \rangle^2} .$$

where $F(l_m)$ is given by eq. (33b), and

$$G(l_m) = \left[1 - \frac{15(3l_m^2+9l_m+2)}{2(2l_m+5)(2l_m+3)(2l_m+1)}\right]^{-1} \times \left\{1 - \frac{3}{2l_m+3} - \frac{15l_m(37l_m^3+130l_m^2+43l_m-98)}{8(2l_m+5)(2l_m+3)^2(2l_m+1)(2l_m-1)}\right\}. \quad (34b)$$

Eqs. (31) and (32) have been given in somewhat different form by Arvieu and Moszkowski [5]). Part of the derivation of eqs. (29)–(34) is sketched in appendix B insofar as the techniques used are different from those of ref. [5]), and may shed some light on the structure of the commutator algebra for the favored pair and surface multipole operators.

Since the reduced matrix elements of $Q(k)$ connecting any favored state to any other favored state have been calculated explicitly, the sums over favored states only ($\alpha'J'$ = favored states only in eqs. (27)) can also be calculated. The full sum rules of eqs. (29)–(34) (where the sums over $\alpha'J'$ run over both favored and excluded states) can be used to estimate the orders of magnitude of the neglected matrix elements of $Q(k)$ connecting a favored state to an excluded state. The results are shown in table 7 in terms of percentage ratios of favored state sums relative to full sums. Results are given for the $J = 2$ and 4 favored $v = 2$ states of the $(\frac{5}{2}\,\frac{3}{2}\,\frac{1}{2})$ shell and the favored $J = 2$, 4 and 6 $v = 2$ states of the $(\frac{7}{2}\,\frac{5}{2}\,\frac{3}{2}\,\frac{1}{2})$ shell, both for the pure quadrupole sum rules ($k = 2$ only) and the sums over all multipole orders (other than $k = 0$). The percentage ratios are well over 90 % in almost all cases. The excluded states make important contributions to the sum rules only in the case of the favored pair states of the highest J and here largely through matrix elements connecting these to excluded states with $v' = 2$ through the highest possible multipoles ($k = 2l_m$). Thus the major

part of the missing 34 % of the $v' = 2$ sum rule for the $J = 6$ state of the $(\frac{7}{2}\ \frac{5}{2}\ \frac{3}{2}\ \frac{1}{2})$ neutron configuration comes via matrix elements of $Q(6)$. However, matrix elements with $k = 6$ do not enter into the calculation of the n-p interaction of our simplified model, since the $(\frac{5}{2}\ \frac{3}{2}\ \frac{1}{2})$ proton configuration of the model has a maximum k-value of 4. The results of table 7 show in particular the small contributions made by excluded states to matrix connections between the favored $v = 2$ states and the totality of $v = 4$ states. In this case the sum over favored states soaks up all but a few percent of the total sum rule. Since the number of excluded $v = 4$ states is much, much larger than the number of favored $v = 4$ states, any *one* matrix element connecting a favored state to a specific excluded state is likely to be very small and will in addition connect states which are separated by large energy differences. Our prescription for separating the $v = 4$ states into two sets, the favored states built from superpositions of favored $J \neq 0$ pairs, and the states to be excluded from the truncated shell-model space, should therefore be expected to be good.

Finally, some of the largest matrix elements of $Q(k)$ involve connections between favored $v = 2$ states and low-lying $v = 4$ states, particularly as n approaches Ω (half-full shell). It is therefore to be expected that there are equally important connections between favored $v = 4$ states and certain $v = 6$ states, particularly if some of these lie relatively low in energy.

7. States with $v \geqq 6$; seniority projection techniques

In a rich configuration the number of states with $v = 6$ and $\tilde{S} = 0$ is very large. In the $(\frac{7}{2}\ \frac{5}{2}\ \frac{3}{2}\ \frac{1}{2})$ configuration, for example, states with $v = 6$ and $\tilde{S} = 0$ include 18 states with $J = 0$ and 48 states with $J = 2$. Even if these were further restricted to $v = 6$ combinations made up only of superpositions of three favored $J \neq 0$ pair states, the number of $v = 6$ states would exceed the limits set by the requirements of the truncation scheme. For states with $v \geqq 6$ therefore very stringent requirements are needed to select from among the lowest energy states a very small number having large multipole connections to the favored states of lower v.

In this connection it is useful to note that Arvieu and Moszkowski have suggested that the lowest 2^+, 4^+, 0^+ "triplet" of $v = 4$ states might be approximated well by the $v = 4$ state projected from $|[2 \times 2]IM_I\rangle$, with $I = 0$, 2 and 4. An examination of the very lowest $v = 4$ states shows that they do indeed have very large overlaps with $v = 4$ states projected from single states of the type $|[J_1 \times J_2]IM_I\rangle$ involving only the very lowest favored $J \neq 0$ pairs, such as $|[2 \times 2]IM_I\rangle$ and $|[2 \times 4]IM_I\rangle$. The three lowest $v = 4$ states of the $(\frac{7}{2}\ \frac{5}{2}\ \frac{3}{2}\ \frac{1}{2})$ shell (fig. 2) are indeed 2^+, 4^+ and 0^+, with a center of gravity at -13.52 energy units (G), compared with -13.33 units, the double excitation of the favored $J = 2$ pair in the extreme "pairing vibration" limit [12]). In the $(\frac{5}{2}\ \frac{3}{2}\ \frac{1}{2})$ shell, 2^+ and 4^+ states are again among the lowest $v = 4$ states. The 0^+ "member of the triplet", however, lies at somewhat higher energy (fig. 1) and does not have particularly large $Q(k)$ matrix connections to $v = 2$ states. (Ac-

cording to our earlier prescription it has in fact not been included in the family of favored $v = 4$ states). Nevertheless, in the limit of large l_m the work of Arvieu and Moszkowski suggests that the lowest $v = 4$, 0^+, 2^+ and 4^+ states might be approximated well by the $v = 4$ states projected from $|[2\times 2]IM_I\rangle$.

The seniority projection can be carried out by means of projection operators. For example, the operator

$$P(v) = N\left\{1 - \frac{1}{2(\Omega-4)}\mathscr{A}^+(00)\mathscr{A}(00) + \frac{1}{8(\Omega-3)(\Omega-4)}[\mathscr{A}^+(00)]^2[\mathscr{A}(00)]^2 - \frac{1}{48(\Omega-2)(\Omega-3)(\Omega-4)}[\mathscr{A}^+(00)]^3[\mathscr{A}(00)]^3\right\}, \quad (35)$$

(N is a normalization factor) can be used to project a $v = 6$ state from an arbitrary state with $n = 6$. Even simpler (and more direct) techniques employ overlaps of the type $\langle n = 4\ vaIM_I|[J_1\times J_2]IM_I\rangle$ with $v < 4$ which follow directly from our method of calculation. Details are given in appendix A; eqs. (A.9) and (A.10) give general expressions for the normalized $v = 4$ vectors projected from a single state of the type $|[J_1\times J_2]IM_I\rangle$. Numerical results are shown in table 8 for both the configurations $(\frac{5}{2}\ \frac{3}{2}\ \frac{1}{2})$ and $(\frac{7}{2}\ \frac{5}{2}\ \frac{3}{2}\ \frac{1}{2})$ for the $v = 4$ state vectors with $I = 0$, 2 and 4 projected from $|[2\times 2]IM_I\rangle$. For the richer $(\frac{7}{2}\ \frac{5}{2}\ \frac{3}{2}\ \frac{1}{2})$ configurations, table 8 also shows the $v = 4$ state vectors for a 2^+, 3^+, 4^+, 5^+, 6^+ multiplet projected from $|[2\times 4]IM_I\rangle$. The amplitudes of the components with $v \leqq 2$, needed to subtract out the unwanted $v \leqq 2$ content of $|[J_1\times J_2]IM_I\rangle$, are in general quite small, though by no means negligible. The expectation values of the Hamiltonian (8) in these $v = 4$ projected states are also shown in table 8 and are in general again in very good agreement with the exact shell-model eigenvalues (cf. figs. 1 and 2). In two cases, the 0^+ and second 2^+ states of the $(\frac{7}{2}\ \frac{5}{2}\ \frac{3}{2}\ \frac{1}{2})$ shell, they seem to be in better agreement with the exact eigenvalues than our earlier approximation, although in general the eigenvectors of H in the truncated subspace of $v = 4$ states constructed according to eqs. (25)–(26) are in better agreement with the results of the exact calculation. To further compare the two types of approximations, table 8 shows the overlaps between the corresponding state vectors. In almost all cases these overlaps are very close to unity, showing that both approximation techniques would give acceptable state vectors for the favored $v = 4$ states to be retained in the approximation scheme. (It should perhaps be noted that the $v = 4$ state with $I = 2$, projected from $|[4\times 4]2M\rangle$, has an overlap of -0.971 with the second 2^+ $v = 4$ state of the $(\frac{7}{2}\ \frac{5}{2}\ \frac{3}{2}\ \frac{1}{2})$ configuration, compared with the overlap of -0.513 for the $v = 4$ state projected from $|[2\times 4]2M\rangle$.) Undoubtedly, the accuracy of these state vectors could be improved further by diagonalizing H in a subspace of $v = 4$ state vectors projected from a few states $|[J_1\times J_2]IM_I\rangle$; e.g. $|[2\times 2]IM_I\rangle$, $|[2\times 4]IM_I\rangle$ and $|[4\times 4]IM_I\rangle$. Although the accuracy of such state vectors would rival that of our earlier approximation and lead to overlaps closer to unity between the two types of vectors, the method of calculation becomes somewhat

TABLE 8

The $v = 4$ states projected from $|[J_1 \times J_2]IM_I\rangle$

Configuration	$[J_1 \times J_2]$	I	E [a]	c_0	$c_{v(a)I}$				Overlaps [b] $\langle$proj. $v = 4 \vert v = 4Ib\rangle$
					$v(a) = 2(0)$ [c]	$v(a) = 2(1)$	$v(a) = 2(2)$	$v(a) = 0(0)$	
$(\frac{5}{2}\ \frac{3}{2}\ \frac{1}{2})$	$[2\times2]$	2	−7.691	1.0943	−0.4228	0.1364			0.985
	$[2\times2]$	4	−6.730	1.0744	0.3929				0.99999
	$[2\times2]$	0	−4.286	1.5085		0.5292		0.9978	[d]
$(\frac{7}{2}\ \frac{5}{2}\ \frac{3}{2}\ \frac{1}{2})$	$[2\times2]$	2	−13.86	1.0240	−0.2098	−0.0496	0.0455		0.973
	$[2\times2]$	4	−13.28	1.0197	0.1997	0.0038			0.996
	$[2\times2]$	0	−12.39	1.0680		0.1652		0.3365	0.928
	$[2\times4]$	3	−11.97	1.0018		0.0598			0.979
	$[2\times4]$	2	−11.12	1.1092	0.4532	0.1531	0.0279		−0.513
	$[2\times4]$	4	−11.11	1.0479	−0.3059	−0.0664			0.824
	$[2\times4]$	5	−10.74	1					0.905
	$[2\times4]$	6	−10.36	1.0452	0.3042				0.998

[a]) $E \equiv \langle$proj. $v = 4$ state $|H|$ proj. $v = 4$ state$\rangle$.

[b]) The overlaps are between the projected $v = 4$ states shown and the favored $v = 4$ state b (built from superpositions of favored $J \neq 0$ pairs according to eqs. (24)–(26)), where the state b is that which most nearly matches the energies E_{Ib} to E (see figs. 1 and 2).

[c]) $(a) = (0)$ denotes the favored pair state.

[d]) There is only one $v = 4$ state with $I = 0$ in this configuration; hence this state has an overlap of 1 with the exact eigenstate.

$$|\text{proj. } v = 4 \text{ state}\rangle = c_0|[J_1 \times J_2]IM_I\rangle + \sum_{v \leqq 2,\, a} c_{v(a)I}|n = 4, v(a)IM_I\rangle.$$

more cumbersome, since the state vectors of the type shown in table 8 contain small pieces of the $v = 2$ states, with $a \neq 0$, which had previously been excluded from the shell-model space. For $v = 4$ therefore, the favored states built from linear combinations of states $|[J_1 \times J_2]IM_I\rangle$, (tables 3 and 4) are to be retained in our approximation scheme; but they are preferred on grounds of simplicity as much as on grounds of accuracy.

For $v = 6$, however, the seniority projection technique seems to afford the best chance of isolating the small number of key states to be retained in the truncation scheme. In a relatively rich shell the lowest $v = 4$ states are approximated well by the $v = 4$ 2^+, 4^+, 0^+ triplet projected from $|[2 \times 2]IM_I\rangle$. It might therefore be expected that the lowest $v = 6$ states could to a good approximation be projected from the vector $|[2 \times [2 \times 2]I_{12}]IM_I\rangle$, constructed from three identical favored pairs with $J = 2$, where

$$|[J_3 \times [J_1 \times J_2]I_{12}]IM_I\rangle = N([J_3 \times [J_1 \times J_2]I_{12}]I) \sum_{M_1 M_2 M_3} \langle J_1 M_1 J_2 M_2 | I_{12} M_{12}\rangle \times \langle J_3 M_3 I_{12} M_{12} | IM_I\rangle \mathscr{A}^+(J_3 M_3) \mathscr{A}^+(J_1 M_1) \mathscr{A}^+(J_2 M_2)|0\rangle. \quad (36)$$

With $J_1 = J_2 = J_3 = 2$, the possible values of I are restricted to 0, 2, 3, 4, and 6 (the three-quadrupole phonon spectrum). (With $J_1 = J_2 = J_3 = 2$, states with different values of I_{12} have overlaps of 1 (or -1); expansions in terms of fractional parents I_{12} are not needed since states (36) are automatically totally antisymmetric in nucleons but totally symmetric in the identical $J = 2$ pairs.)

For the $(\frac{5}{2}\ \frac{3}{2}\ \frac{1}{2})$ shell the full $v = 6$ spectrum is known (see fig. 1). It is interesting to note that the $\tilde{S} = 0$ part of the $v = 6$ spectrum contains each of the J-values 0, 2, 3, 4, 6 twice. No other J-values occur. The two 0, 4, 3, 6, 2 $\tilde{S} = 0$, $v = 6$ multiplets are each other's mirror image. The lower 0, 4, 3, 6, 2 multiplet lies in the region of

TABLE 9

The $v = 6$ $\tilde{S} = 0$ spectrum for the $(\frac{5}{2}\ \frac{3}{2}\ \frac{1}{2})^6$ configuration; comparison of exact energies with expectation values of H for $v = 6$ states projected from $|[2 \times [2 \times 2]\ I_{12}]I\rangle$

I	E (exact)	E (proj.) [a]
0	−14.161	−14.09
	− 3.893	
2	− 9.958	− 9.28
	− 8.042	
3	−11.347	(−11.06) [b]
	− 6.653	− 6.74
4	−11.962	−11.49
	− 6.038	
6	− 9.958	− 9.95
	− 8.042	

[a]) $E(\text{proj.}) \equiv \langle P(v = 6)[2 \times [2 \times 2]\ I_{12}]I | H | P(v = 6)\ [2 \times [2 \times 2]I_{12}]I\rangle$.
[b]) For this entry: $E(\text{proj.}) \equiv \langle P(v = 6)[4 \times [2 \times 2]\ 4]3 | H | P(v = 6)[4 \times [2 \times 2]4\ 3\rangle$.

TABLE 10

The $v = 6$ states projected from $|[2\times[2\times2]I_{12}]I\rangle$ for the $(\frac{5}{2}\,\frac{3}{2}\,\frac{1}{2})^6$ configuration

$$|P(v=6)\,[2\times[2\times2]2]0\rangle = 1.1407|[2\times[2\times2]2]0\rangle - 0.3798|v=4,\ I=0\rangle + 0.3071|v=2,\ I=0\rangle + 0.2502|v=0,\ I=0\rangle$$

$$\begin{aligned}|P(v=6)\,[2\times[2\times2]2]2\rangle = {} & 1.7916|[2\times[2\times2]2]2\rangle + 0.8112\{1.9866|P(v=4)\,[2\times4]2\rangle - 2.8602|P(v=4)\,[2\times2]2\rangle \\ & - 0.5037|P(v=4)\,[2\times2(a=1)]2\rangle + 1.3798|P(v=4)\,[2\times0(a=1)]2\rangle\} \\ & + 1.1304|v=2\ I=2\ a=0\rangle - 0.5234|v=2\ I=2\ (a=1)\rangle\end{aligned}$$

$$\begin{aligned}|P(v=6)\,[2\times[2\times2]4]4\rangle = {} & 1.3797|[2\times[2\times2]4]4\rangle + 0.7381\{0.9779|P(v=4)\,[2\times4]4\rangle - 1.7961|P(v=4)\,[2\times2]4\rangle \\ & - 0.4184|P(v=4)\,[2\times2(a=1)]4\rangle\} + 0.5090|v=2\ I=4\rangle\end{aligned}$$

$$|P(v=6)\,[2\times[2\times2]4]6\rangle = 1.4142|[2\times[2\times2]4]6\rangle + 1.0000|v=4[2\times4]6\rangle$$

$$|P(v=6)\,[2\times[2\times2]2]3\rangle = 3.9414|[2\times[2\times2]2]3\rangle - 3.8125\{0.7767|v=4[2\times4]3\rangle + 0.2865|v=4\,[2\times2(a=1)]3\rangle\}$$

$$|P(v=6)\,[4\times[2\times2]4]3\rangle = 1.1966|[4\times[2\times2]4]3\rangle + 0.6572\{1.1586|v=4[2\times4]3\rangle - 0.2457|v=4[2\times2(a=1)]3\rangle\}$$

The symbol $|P(v=x)\ldots\rangle$ denotes a *normalized* state with $v=x$ projected from $|\ldots\rangle$. The combinations in curly brackets are normalized $v=4$ states. In $|[J_1\times J_2(a=1)]I\rangle$ J_1 denotes the favored pair, $J_2(a=1)$ the pair labelled $a=1$.

favored $v = 4$ states. The lowest $v = 6\ 0^+$ state actually lies below the lowest $v = 4$ state. For the $(\frac{7}{2}\ \frac{5}{2}\ \frac{3}{2}\ \frac{1}{2})$ shell the exact eigenvalues of the full $v = 6$ spectrum have not been calculated. However, the exact diagonalization has been carried out in the full shell-model space for the 0^+ states. The lowest $v = 6\ 0^+$ state with $\tilde{S} = 0$ (at -22.895 energy units (G)) is almost degenerate with the favored $v = 4\ 0^+$ state (it lies 0.07 units of G below it); while the next $v = 6\ 0^+$ state lies above the main region of favored $v = 4$ states (actually just below the 7^+, 6^+ and 8^+ $v = 4$ states shown in fig. 2).

In the $(\frac{5}{2}\ \frac{3}{2}\ \frac{1}{2})$ configuration the $v = 6$ $\tilde{S} = 0$ spectrum, although quite simple, is nevertheless rich enough to test the hypothesis that the key low-energy $v = 6$ states should be given to good approximation by the $v = 6$ projections from $|[2\times[2\times 2]I_{12}]IM_I\rangle$. The details of the calculation for the seniority projection technique for $v = 6$ states are given in appendix A. Results for the energies are given in table 9, while the state vectors are shown in table 10. It can be seen that the expectation values of H for the $v = 6$ states projected from $|[2\times[2\times 2]I_{12}]I\rangle$ are in very good agreement with the lower of the two exact eigenvalues for $I = 0$, 2, 4 and 6. For $I = 3$, however, the $v = 6$ state projected from $|[2\times[2\times 2]2]3\rangle$ seems to approximate the higher of the two $I = 3$ states, at -6.653 in units of G, the wrong one from the point of view of our model. (The $I = 3$ state at -11.347 in units of G satisfies the criteria of (1) low energy and (2) important multipole connections to favored $v = 4$ states.) The difficulty may arise partly from the fact that the state $|[2\times[2\times 2]2]3\rangle$ has very little $v = 6$ content, compared with the corresponding states for $I = 0$, 2, 4 and 6. The $v = 6$ content of the various states can be read off from table 10. (It is given by the square of the inverse of the coefficient of the leading term.) Thus the states $|[2\times[2\times 2]I_{12}]I\rangle$ with $I = 0$, 4, 6 and 2 have $v = 6$ components of the following percentages, 76.9 %, 52.5 %, 50.0 % and 31.2 %, the remaining percentages being made up by the components with $v \leqq 4$. The state $|[2\times[2\times 2]2]I = 3\rangle$ on the other hand contains only a 6.4 % $v = 6$ piece. In general one might expect the $v = 6$ content in the states $|[2\times[2\times 2]I_{12}]I\rangle$ to increase in richer configurations (larger values of l_m). The four-particle states $|[2\times 2]IM_I\rangle$, for example, are more nearly pure $v = 4$ states in the richer configurations. The state $|[2\times 2]IM_I\rangle$ with $I = 2$ contains only a 4.6 % piece with $v = 2$ in the $(\frac{7}{2}\ \frac{5}{2}\ \frac{3}{2}\ \frac{1}{2})$ shell, compared with 16.5 % $v = 2$ content in the $(\frac{5}{2}\ \frac{3}{2}\ \frac{1}{2})$ shell. For $I = 4$, the corresponding numbers are 3.8 % and 13.4 % (see table 8). It is nevertheless possible to find an $I = 3$ state built from a superposition of three favored $J \neq 0$ pairs with a large $v = 6$ component. As an example, the state $|[4\times[2\times 2]4]3\rangle$ has a $v = 6$ content of 69.8 %. The full $v = 6$ state projected from this state is also shown in table 10. Moreover, the expectation value of H in this $v = 6$ state is in good agreement with the exact eigenvalue for the *lower* of the two $I = 3$ $v = 6$ states (table 9) and the state vector is a good approximation for the corresponding exact eigenvector. To test the goodness of the $v = 6$ state vectors projected from $|[J\times[2\times 2]I_{12}]I\rangle$ the matrix elements of the surface multipole operators have again been calculated both with the exact eigenvectors (from the diagonalization in the full shell-

TABLE 11

Reduced matrix elements $\langle v = 4\ I'||Q(k)||v = 6\ I\rangle$ for the $(\frac{5}{2}\ \frac{3}{2}\ \frac{1}{2})^6$ configuration; comparison between exact and approximate calculations

k	I'	I	exact	approx.	k	I'	I	exact	approx.
2	2	0	−4.74	−4.66	4	4	0	−2.27	−2.25
2	2	4	−5.99	−6.13	4	2	4	−2.21	−2.26
	3		−3.84	−3.81		3		4.66	4.64
	4		−6.38	−6.37		4		−4.57	−4.43
	5		−3.64	−3.42		5		−1.92	−1.73
	6		−6.05	−6.04		6		1.69	1.94
2	2	3	−0.24	−0.52	4	2	3	3.09	3.18
	3		6.72	7.06		3		−0.91	−0.84
	4		0.00	0.06		4		−3.71	−3.69
	5		−4.17	−4.15		5		2.18	2.28
						6		−1.50	−1.37

The matrix elements shown are those for the 3 lowest $v = 6$ states with $I = 0$, 4 and 3; and the 5 lowest favored $v = 4$ states, with $I' = 2$, 3, 4, 5 and 6. Calculations marked "approx." have used the $v = 4$ state vectors of table 4, with $b' = 1$; and the $v = 6$ state vectors of table 10, the latter being projected from $|[2\times[2\times2]I_{12}]I\rangle$ for $I = 0$, 4 and from $|[4\times[2\times2]4]3\rangle$ for $I = 3$.

model space) and with the $v = 6$ states constructed by means of the seniority projection technique. Some sample comparisons are shown in table 11. The good agreement between the two sets of numbers shows that our method of construction gives an excellent approximation scheme for the key states of the $v = 6$ $\tilde{S} = 0$ spectrum.

Experience with the simple $(\frac{5}{2}\ \frac{3}{2}\ \frac{1}{2})$ configuration shows that it may be necessary to resort to some process of trial and error in finding the $v = 6$ states corresponding to the key low-energy states of the spectrum. By projecting the $v = 6$ pieces out of a few states built from superpositions of favored $J \neq 0$ pairs, such as $|[2\times[2\times2]I_{12}]I\rangle$, $|[4\times[2\times2]I_{12}]I\rangle$ and possibly also $|[2\times[4\times4]I_{12}]I\rangle$, it is possible to find good approximations for the eigenvectors corresponding to the lower $v = 6$ eigenvalues for each I. The accuracy of these eigenvectors could be improved further by diagonalizing H in the 2×2 (or possibly 3×3) subspaces of the vectors $|P(v = 6)[2\times[2\times2]I_{12}]I\rangle$, $|P(v = 6)[4\times[2\times2]I_{12}]I\rangle$ (possibly including $|P(v = 6)[2\times[4\times4]I_{12}]I\rangle$). The details for the construction of high-seniority states for the richer configurations will be left to a subsequent investigation.

8. Shell-model calculations

Without considering the highest seniorities, it will be interesting to see whether the simplified model of this investigation throws some light on how a nucleus begins to make the transition from shell structure to collective behavior. For this purpose the Hamiltonian of the model, eqs. (7)–(9), has been diagonalized in the truncated shell-model space for the configurations $(f_{\frac{5}{2}}p_{\frac{3}{2}}p_{\frac{1}{2}})^{n_p}(g_{\frac{7}{2}}d_{\frac{5}{2}}d_{\frac{3}{2}}s_{\frac{1}{2}})^{n_n}$ for two cases; $n_p = n_n = 4$ and $n_p = 6$, $n_n = 4$. The truncated shell-model basis is made up of the favored $v = 4$ states of tables 3 and 4. In the $(\frac{5}{2}\ \frac{3}{2}\ \frac{1}{2})$ shell the highest $v = 4$ state of table 4, with

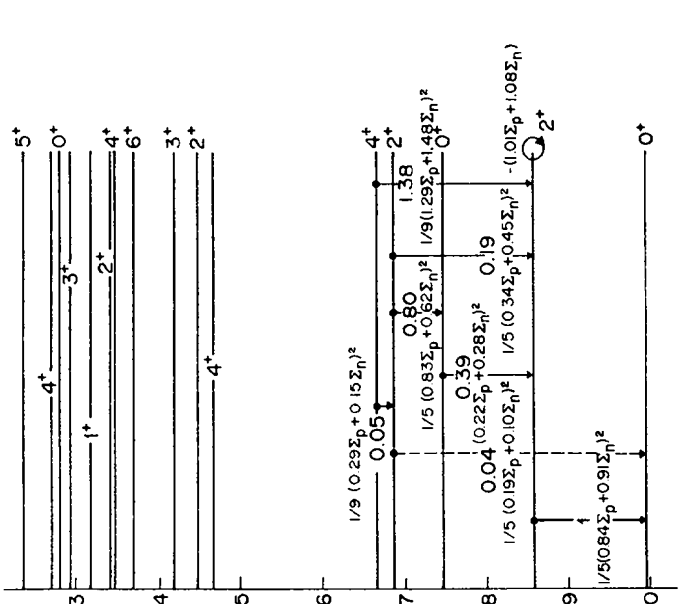

Fig. 4. The spectrum for the configuration $(f_{\frac{7}{2}}p_{\frac{3}{2}}p_{\frac{1}{2}})^6(g_{\frac{7}{2}}d_{\frac{5}{2}}d_{\frac{3}{2}}s_{\frac{1}{2}})^4$, with interaction strengths $G_{np} = G_{pp} = G_{nn}$ (see caption to fig. 3).

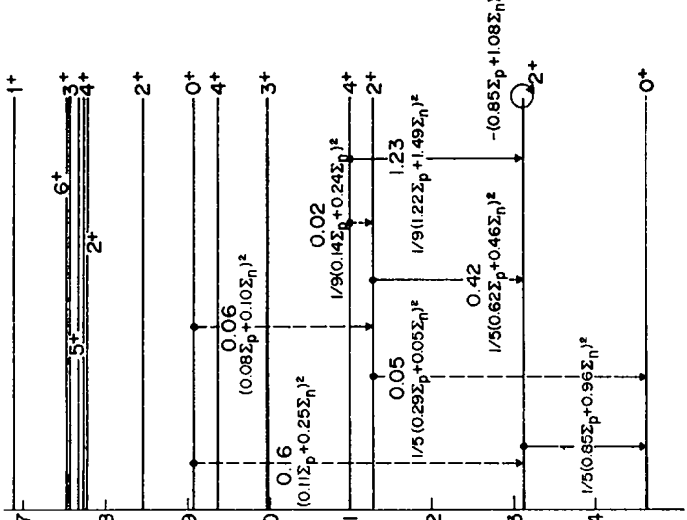

Fig. 3. The spectrum for the configuration $(f_{\frac{7}{2}}p_{\frac{3}{2}}p_{\frac{1}{2}})^4\ (g_{\frac{7}{2}}d_{\frac{5}{2}}d_{\frac{3}{2}}s_{\frac{1}{2}})^4$, with interaction strengths $G_{np} = G_{pp} - G_{nn}$. The pure numbers give the $B(E2)$ values for the transitions *relative* to that for $2_1^+ \to 0_1^+$, normalized at unity, with effective charges, $e_n = 0.7$, $e_p = 1.1 + 0.7$. The $B(E2)$ values are also shown in terms of the separate proton and neutron amplitudes Σ_p and Σ_n given by eq. (18).

$I = 4$, has no large surface multipole connections to the favored $J = 2$ pair state. (If this high $v = 4$, $I = 4$ state is excluded, the sums of table 7 to the favored $J = 2$ state are reduced by only about 2 % while the sum to the favored $J = 4$ state is reduced by 13 %.) For this reason the $v = 4$ proton part of the truncated shell-model basis has been restricted to the proton states with $b = 1$ only; i.e. the five lowest states of table 4 with $I_p = 2, 3, 4, 5$ and 6. For similar reasons the 22 favored $v = 4$ states of tables 1 and 3 have been restricted to the 13 states with $v_n = 4$, $I_n \leqq 8$, shown in fig. 2. For case (i), $n_p = n_n = 4$, the truncated shell model basis is therefore built from a total of 8 proton states $|v_p I_p(b_p)\rangle$ and 17 neutron states $|v_n I_n(b_n)\rangle$. With this truncation the full shell-model basis $|v_p I_p(b_p); v_n I_n(b_n); IM_I\rangle$, with $\boldsymbol{I} = \boldsymbol{I}_p + \boldsymbol{I}_n$, leads to dimensions of 22, 45, 81, 91, 108, 101, 99, 80 and 67 for the shell-model matrices for I-values of 0–8, respectively. For case (ii), $n_p = 6$, $n_n = 4$; the lower $v_p = 6$ multiplet with $I_p = 0, 4, 3, 2, 6$ (from table 10) has been added to the proton part of the basis, expanding the number of favored proton states from 8 to 13. In this case the dimension of the shell-model basis is again largest for $I = 4$, leading to a dimension of 171 for this shell-model matrix.

From the reduced matrix elements of the $Q^p(k)$ and $Q^n(k)$, calculated earlier, the full Hamiltonian matrix is constructed through straightforward application of the angular momentum coupling formalism. The results of the diagonalization are shown in figs. 3 and 4. For the configuration with $n_p = n_n = 4$ (fig. 3) the low-energy part of the spectrum has many of the features of a vibrational spectrum (except for the missing 0^+ member of a two-phonon 0^+, 2^+, 4^+ "triplet"). Fig. 3 shows the spectrum calculated for equal strengths of the n-p, p-p and n-n parts of the interaction ($G_{np} = G_{pp} = G_{nn}$) for the coefficients of eqs. (8) and (9). However, the qualitative features of the spectrum are not sensitive to the ratios of these strengths; for the range $G_{np}/G_{pp} = 0.5$ to 2.0, for example. In addition, a change in the relative strength of the $k = 2$ to the $k = 4$ components of H_{np} again does not change the qualitative features of the spectrum.

To try to gain a further understanding of this spectrum the eigenvectors for the four lowest levels of fig. 3 are given below expanded in terms of vectors $|v_p I_p; v_n I_n(b_n)\rangle$. (Note that the ordering index (b_n) is given only when needed; i.e. for neutron states with $v_n = 4$, $I_n = 2, 4, 5$ and 6. To save space the total angular momentum I is not written explicitly in each ket.) In this notation,

$$|0_1^+\rangle = 0.583|00;00\rangle + 0.629|22;22\rangle + 0.246|24;24\rangle + 0.229|42;42(1)\rangle + 0.275|44;44(1)\rangle + 0.104|46;46(1)\rangle + 0.147|00;40\rangle - 0.084|22;42(1)\rangle - 0.114|22;42(2)\rangle + \ldots,$$

$$|2_1^+\rangle = 0.465|00;22\rangle + 0.443|22;00\rangle - 0.289|22;22\rangle + 0.315|22;44(1)\rangle + 0.274|44;22\rangle + 0.185|22;42(1)\rangle + 0.209|42;22\rangle + 0.149|22;40\rangle + 0.148|22;24\rangle + 0.086|24;22\rangle - 0.141|00;42(1)\rangle - 0.073|00;42(2)\rangle - 0.093|42;00\rangle - 0.160|44;44(1)\rangle + 0.130|44;46(1)\rangle + 0.125|43;42(1)\rangle - 0.099|43;24(1)\rangle - 0.090|42;44(1)\rangle + \ldots,$$

$$
\begin{aligned}
|2_2^+\rangle = \; & 0.286|00;42(1)\rangle - 0.092|00;42(2)\rangle + 0.409|42;00\rangle + 0.332|22;22\rangle \\
& + 0.285|42;22\rangle + 0.084|22;42(1)\rangle - 0.121|22;42(2)\rangle - 0.363|43;22\rangle \\
& + 0.156|22;43\rangle + 0.240|24;44(1)\rangle + 0.169|44;24\rangle + 0.109|22;44(2)\rangle \\
& + 0.188|22;00\rangle + 0.064|00;22\rangle + 0.092|22;24\rangle + 0.190|24;22\rangle \\
& - 0.175|42;42(1)\rangle - 0.154|43;44(1)\rangle + 0.115|43;42(1)\rangle + 0.143|42;40\rangle \\
& + 0.097|44;44(1)\rangle + \ldots, \\
|4_1^+\rangle = \; & 0.396|00;44(1)\rangle - 0.085|00;44(2)\rangle + 0.285|44;00\rangle + 0.436|22;22\rangle \\
& - 0.228|44;22\rangle - 0.322|22;44\rangle + 0.221|22;46(1)\rangle + 0.114|46;22\rangle \\
& - 0.181|22;42(1)\rangle + 0.104|22;42(2)\rangle - 0.165|42;22\rangle + 0.138|24;42(1)\rangle \\
& - 0.076|24;42(2)\rangle + 0.129|00;24\rangle - 0.014|24;00\rangle + 0.106|44;40\rangle \\
& + 0.180|44;44(1)\rangle + 0.095|43;42(1)\rangle - 0.096|43;43\rangle + 0.117|42;42(1)\rangle \\
& - 0.115|44;46(10\rangle - 0.076|46;44(1)\rangle - 0.082|45;42(1)\rangle + \ldots . \qquad (37)
\end{aligned}
$$

Kets with amplitudes less than 0.08 are not shown. It can be seen that the significant amplitudes are spread among many different pieces. The results are therefore far from what might have been expected from an extreme weak-coupling approximation. However, the 0^+ ground state is made up predominantly (74 %) of two pieces, built from the coupling of favored proton and neutron pairs with $J_p = J_n = 0$ and $J_p = J_n = 2$. The largest pieces of the first excited 2_1^+ state (41 %) come from the coupling of the favored pair combinations $J_p = 0(v_p = 0)$, $J_n = 2(v_n = 2)$ and $J_p = 2(v_p = 2)$, $J_n = 0(v_n = 0)$. On the other hand the 2_2^+, 4_1^+ doublet, at roughly twice the excitation of the 2_1^+ state, has predominant pieces of seniority-4 states coupled to seniority-0 states; i.e., $v_p = 4$, $I_p = I$ coupled to $v_n = 0$, $I_n = 0$, and $v_p = 0$, $I_p = 0$ coupled to $v_n = 4$, $I_n = I$, which together with the coupling of two favored $J = 2$ pair states ($v_p = v_n = 2$), make up 37 % and 44 % of the 2_2^+ and 4_1^+ state vectors. The remaining (smaller) pieces of these two eigenvectors also have enough similarity that these two eigenvectors can be identified as two members of a doublet.

Fig. 3 also shows the relative B(E2) values for the transitions among the lower energy levels. For simplicity these have been calculated with the surface quadrupole operators of eq. (3) (corresponding to the approximation in which all radial parts of matrix elements of the real $Q(2)$ operator have been replaced by a single constant). However, this should be a very good approximation. For example, the proton matrix elements $v' = 4$, $I'_p(b'_p) \rightarrow v_p = 2$, $I_p = 2$, when compared to the matrix element for $v'_p = 2$, $I'_p = 2 \rightarrow v_p = 0$, $I_p = 0$, lead to relative B(E2) values of 1.29, 0.00025 and 1.03 for $I'_p = 2$, 3 and 4, respectively, as calculated with proper radial (harmonic oscillator) parts of the matrix elements for the $(f_{\frac{7}{2}} p_{\frac{3}{2}} p_{\frac{1}{2}})^4$ configuration [5]). In the surface multipole approximation, the corresponding numbers are 1.27, 0 and 1.03; where the B(E2) value for the transition $v'_p = 2$, $I'_p = 2 \rightarrow v_p = 0$, $I_p = 0$ has been normalized at 1.00 in both cases.

The pure numbers in fig. 3 show B(E2) values relative to the B(E2) value for the transition $2_1^+ \rightarrow 0_1^+$ which has been given a strength of 1 unit, where these numbers have been calculated by assigning effective charges $e_n = 0.7$ and $e_p = 1 + 0.7$ to the

neutron and proton E2 operators, respectively. (The magnitude 0.7 has been picked quite arbitrarily; but the relative numbers in fig. 3 are insensitive to small changes in this magnitude for the effective charge.) Fig. 3 also shows the B(E2) values in terms of separate proton and neutron amplitudes, Σ_p and Σ_n, given by the reduced matrix elements of $Q(2)$ connecting the states $v = 0$, $I = 0$ to $v = 2$, $I = 2$, as given by eqs. (18) or (A.23). These have been denoted $\Sigma_p(\Sigma_n)$ since they exhaust the full quadrupole sum rule for the transitions from the ground states to the favored $v=2$, $I=2$ states in the separate proton (or neutron) configurations (again, in the surface multipole approximation.) From the numbers in fig. 3 it can be seen that the proton and neutron matrix elements add coherently for all the strong transitions, leading to enhancements in the E2 rates. In the extreme weak-coupling limit with similar proton and neutron spaces, assuming a ground state of $|[v_p I_p; v_n I_n]I\rangle = |[00;00]0\rangle$ and a 2_1^+ state of $(1/\sqrt{2})(|[22;00]2\rangle + |[00;22]2\rangle)$ the B(E2) value for the transition $2_1 \to 0_1$ would be $\frac{1}{5}(0.707\Sigma_p + 0.707\Sigma_n)^2$. Relative to this value the E2 rate for the transition $2_1 \to 0_1$ of fig. 3 is enhanced by a factor of $\approx$ 1.4–1.6 depending somewhat on the effective charges assigned to the proton and neutron parts of the E2 operators. In the extreme quadrupole-phonon vibrational limit the B(E2) values for the transitions $4_1 \to 2_1$ and $2_2 \to 2_1$ would be enhanced by a factor of 2 relative to the transition $2_1 \to 0_1$. Although the B(E2) value for the transition $4_1 \to 2_1$ of fig. 3 is larger than that for the transition $2_1 \to 0_1$, the $2_2 \to 2_1$ transition is considerably too weak. It should be noted, however, that this particular B(E2) value is a fairly sensitive function of the n-p interaction strength. For interaction strength ratios $G_{n,p}/G_{p,p} = 0.5$, 1.0 and 2.0 (with $G_{p,p} = G_{n,n}$) the reduced matrix element for the $2_2 \to 2_1$ transition varies, from $(0.82\Sigma_p + 0.62\Sigma_n)$ to $(0.62\Sigma_p + 0.46\Sigma_n)$ to $(0.37\Sigma_p + 0.17\Sigma_n)$, respectively; whereas the corresponding numbers for the $2_1 \to 0_1$ transition are relatively more constant, viz. $(0.87\Sigma_p + 0.88\Sigma_n)$, $(0.85\Sigma_p + 0.96\Sigma_n)$ and $(0.79\Sigma_p + 0.98\Sigma_n)$; similarly for the $4_1 \to 2_1$ transition for which the corresponding numbers are $(1.34\Sigma_p + 1.46\Sigma_n)$, $(1.22\Sigma_p + 1.49\Sigma_n)$ and $(1.11\Sigma_p + 1.42\Sigma_n)$. For somewhat weaker n-p interactions therefore the $2_2 \to 2_1$ E2 transition rate is at least comparable with that for $2_1 \to 0_1$. The spectrum of fig. 3 thus shares the essential properties of a vibrational spectrum: strong E2 rates for the transitions $2_1 \to 0_1$, $4_1 \to 2_1$ and $2_2 \to 2_1$, particularly compared with the E2 rates for the cross-over transition $2_2 \to 0_1$, and a $4_1 \to 2_2$ transition involving no change in vibrational excitation. However, a 0^+ member of a 0^+, 2^+, 4^+ "two-phonon triplet" is missing. From the weak E2 rate for the transition $0_2 \to 2_1$, it can be seen that the second 0^+ state is not merely pushed into the wrong energy region of the spectrum.

To see whether the spectrum becomes more vibrational or collective, it is interesting to compare fig. 3 with the spectrum for the somewhat richer configuration, case (ii): $n_p = 6$, $n_n = 4$, shown in fig. 4. The main difference between the two spectra involves the appearance of a 0^+ state at about twice the excitation energy of the lowest 2^+ state, seemingly the missing member of the two-phonon triplet. The $v \leqq 4$ components of the eigenvectors for the 0_1, 2_1, 2_2 and 4_1 states show no qualitative changes

over those for the configuration with $n_p = n_n = 4$. These eigenvectors also have relatively small components of state vectors built from the seniority-6 part of the proton space, the percentages being 3.8 %, 10.2 %, 15.9 % and 14.7 %, respectively. The 0_2^+ state on the other hand has a fairly large piece built from $v_p = 6$ components (≈ 34 %); more significantly, it contains large pieces (≈ 50 %) built from $v_p = 4$, $v_n = 2$, $v_p = 2$, $v_n = 4$ and $I_p = I_n = 2$ states; which can lead to large E2 matrix elements to the 2_1 state (through the v_p, $v_n = 2,2$ or 2,4; 4,2 components of the latter). In the notation of eq. (37) the eigenvector for the second 0^+ state is

$$\begin{aligned}|0_2^+\rangle = \; & 0.228|00;00\rangle - 0.097|00;40\rangle + 0.106|22;22\rangle + 0.170|24;24\rangle \\ & + 0.612|42;22\rangle + 0.331|22;42(1)\rangle + 0.131|44;24\rangle + 0.127|24;44(1)\rangle \\ & + 0.139|44;44(2)\rangle - 0.511|60;00\rangle - 0.103|60;40\rangle + 0.108|64;24\rangle \\ & + 0.197|64;44\rangle + \ldots . \end{aligned} \tag{38}$$

The B(E2) values for the transitions among the lower-energy levels are shown in fig. 4 (with the notation used in fig. 3). The enhancements of the $2_1 \rightarrow 0_1$ and $4_1 \rightarrow 2_1$ transitions are similar to those for the case $n_p = n_n = 4$. The $2_2 \rightarrow 2_1$ transition is again much weaker than might be expected for a vibrational spectrum but is again, the one E2 rate which is a sensitive function of the ratio G_{np}/G_{pp} and again approaches the strength of the $2_1 \rightarrow 0_1$ transition for a somewhat smaller value of this ratio (≈ 0.4). The E2 rates for transitions involving the 0_2^+ state also are far off the extreme quadrupole-phonon limit. Although the $0_2 \rightarrow 2_1$ transition (corresponding to a change of one unit of vibrational excitation) has gained some strength compared with the configuration with $n_p = n_n = 4$, it is still weak and in fact weaker than the $2_2 \rightarrow 0_2$ transition (corresponding to no change in vibrational excitation) which should be very weak according to the predictions of the quadrupole-phonon model.

In summary, however, the spectra of figs. 3 and 4 do contain many of the essential features of a vibrational spectrum and are perhaps more reminiscent of the spectra of real doubly even "vibrational" nuclei which are frequently not in good quantitative agreement with the extreme harmonic quadrupole-phonon model.

In conclusion, it seems clear that still richer configurations are required to approach the extreme vibrational limit and, even more so, to gain the further collective enhancement needed to make the transition from vibrational to rotational character. However, the technique used in this investigation to effect the severe truncation of the shell-model space needed for shell-model studies of configurations with active proton and neutron numbers, $n \leqq 6$, can be generalized to even richer configurations. It should therefore be feasible to carry out more realistic shell-model studies for real doubly even nuclei in the transition region, around $A = 110$ or $A = 146$, for example. On the other hand it is perhaps somewhat disappointing that the state vectors in the $|[v_p I_p; v_n I_n] I M_I\rangle$ basis, such as those of eqs. (37) and (38), show fragmentation into such a large number of pieces. A description of the low-energy states of the spectrum in terms of such a basis may therefore not lead to a *simple* understanding of the transition from shell structure to collective behavior.

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Appendix A

CALCULATIONAL TECHNIQUE

Since all of the state vectors used in this investigation are built from two-particle operators coupled to $\tilde{S} = 0$, the fractional parentage techniques to be used will be based on the direct uncoupling of a full $\tilde{S} = 0$ pair (rather than the uncoupling of two single particles in succession). For the four-particle states the basic numbers follow from the scalar products

$$\langle 0|[A^{(a_3)+}(J_3)\times A^{(a_4)+}(J_4)]^+ IM_I|[A^{(a_1)+}(J_1)\times A^{(a_2)}(J_2)^+]IM_I|0\rangle, \qquad \text{(A.1)}$$

where $A^{(a)+}(JM)$ are the pair creation operators defined in eqs. (1) and (20). (Note that the superscript (0) denotes the favored pair combination $A^{(0)+}(JM) \equiv \mathscr{A}^+(JM)$.) The overlaps (A.1) are also related to reduced matrix elements of the operators $A^{(a)}(JM)$. Straightforward recoupling techniques give

$$\begin{aligned}
&\langle 0|[A^{(a_3)}(J_3)^+ \times A^{(a_4)}(J_4)^+]^+ IM_I|[A^{(a_1)}(J_1)^+ \times A^{(a_2)+}(J_2)]IM_I|0\rangle \\
&\quad = \langle A^{(a_4)}(J_4)||A^{(a_3)}(J_3)||[A^{(a_1)}(J_1)^+ \times A^{(a_2)}(J_2)^+]I\rangle[2I+1]^{-\frac{1}{2}} \\
&\quad = 4\{\sum_{ll'}[q^{(0)}(ll'J_1)]^2\}\{\sum_{ll'}[q^{(0)}(ll'J_2)]^2\}\{\delta_{J_1J_3}\delta_{a_1a_3}\delta_{J_2J_4}\delta_{a_2a_4} \\
&\quad + (-1)^{I-J_1-J_2}\delta_{J_1J_4}\delta_{a_1a_4}\delta_{J_2J_3}\delta_{a_2a_3}\} - 8\sum_{l l_0 l_1 l_2}(-1)^{I+J_1+J_2-J_3-J_4} q^{(a_1)}(l_0 l J_1) \\
&\qquad\qquad \times q^{(a_2)}(l_1 l_2 J_2) q^{(a_3)}(l_0 l_1 J_3) q^{(a_4)}(l l_2 J_4) \\
&\quad \times [(2J_1+1)(2J_2+1)(2J_3+1)(2J_4+1)]^{\frac{1}{2}} \begin{Bmatrix} J_1 & J_2 & I \\ l & l_2 & J_4 \\ l_0 & l_1 & J_3 \end{Bmatrix},
\end{aligned} \qquad \text{(A.2)}$$

with $J_3 a_3 = J_1 a_1$; $J_4 a_4 = J_2 a_2$, this also gives the normalization factor $N(J_1{}^{(a_1)} J_2{}^{(a_2)} I)$ for the four-particle states

$$|[J_1^{(a_1)} \times J_2^{(a_2)}]IM_I\rangle = N(J_1^{(a_1)} J_2^{(a_2)} I)[A^{(a_1)}(J_1)^+ \times A^{(a_2)}(J_2)^+]IM_I|0\rangle$$

and gives the reduced matrix element of the *normalized* two-particle annihilation operator $N_J A(JM)$ between *normalized* four- and two-particle states

$$\begin{aligned}
&\langle J_4^{(a_4)}||N_{J_3} A^{(a_3)}(J_3)||[J_1^{(a_1)} \times J_2^{(a_2)}]I\rangle[2I+1]^{-\frac{1}{2}} \\
&\quad = \frac{N_{J_3} N_{J_4} \langle 0|[A^{(a_3)}(J_3)^+ \times A^{(a_4)}(J_4)^+]^+ IM_I|[A^{(a_1)}(J_1)^+ \times A^{(a_2)}(J_2)^+]IM_I|0\rangle}{[\langle 0|[A^{(a_1)}(J_1)^+ \times A^{(a_2)}(J_2)^+]^+ IM_I|[A^{(a_1)}(J_1)^+ \times A^{(a_2)}(J_2)^+]IM_I|0\rangle]^{\frac{1}{2}}}.
\end{aligned} \qquad \text{(A.3)}$$

Except for an n-dependent factor, these are $4 \to 2\times 2$ particle fractional parentage coefficients. To save writing it will be convenient to rename these Z-coefficients:

$$Z(J_1 a_1, J_2 a_2; J_3 a_3, J_4 a_4; I) \equiv \langle J_4^{(a_4)} || N_{J_3} A^{(a_3)}(J_3) || [J_1^{(a_1)} \times J_2^{(a_2)}] I \rangle [2I+1]^{-\frac{1}{2}}$$
$$= [\tfrac{1}{2}n(n-1)]^{\frac{1}{2}} \langle [J_1^{(a_1)} \times J_2^{(a_2)}] I \{ | J_3(a_3); J_4(a_4) \rangle. \quad \text{(A.4)}$$

These Z-coefficients satisfy the sum rule

$$\sum_{J_3 a_3 J_4 a_4} [Z(J_1 a_1, J_2 a_2; J_3 a_3, J_4 a_4; I)]^2 = \tfrac{1}{2}n(n-1) \times \tfrac{1}{2} = 3. \quad \text{(A.5)}$$

The additional factor of $\frac{1}{2}$ (the dimension ratio of the irreducible representations [2] relative to [22] for the permutation groups of 2 and 4 particles) comes from the fact that the c.f.p. sum is not over all two-particle states – the sum in eq. (A. 5) is over $\tilde{S} = 0$ two-particle states only. With these Z-coefficients, matrix elements of H are calculated by conventional c.f.p. techniques.

In terms of these Z-coefficients, we also have

$$N(J_1^{(a_1)} J_2^{(a_2)} I) = N_{J_1} N_{J_2} [Z(J_1 a_1, J_2 a_2; J_1 a_1, J_2 a_2; I)]^{-1}, \quad \text{(A.6)}$$

and the overlap between two normalized four-particle states is:

$$\langle [J_3^{(a_3)} \times J_4^{(a_4)}] I M_I | [J_1^{(a_1)} \times J_2^{(a_2)}] I M_I \rangle$$
$$= \frac{Z(\underline{J_1 a_1}, \underline{J_2 a_2}; \underline{J_3 a_3}, \underline{J_4 a_4}; \underline{I})}{Z(J_3 a_3, J_4 a_4; J_3 a_3, J_4 a_4; I)} = \frac{Z(\underline{J_3 a_3}, \underline{J_4 a_4}; \underline{J_1 a_1}, \underline{J_2 a_2}; \underline{I})}{Z(J_1 a_1, J_2 a_2; J_1 a_1, J_2 a_2; I)}. \quad \text{(A.7)}$$

For the special case when one of the four-particle states is a normalized state with $v = 2$ or $v = 0$, these overlaps reduce to

$$\langle n = 4\, v = 2, J^{(a)} M | [J_1^{(a_1)} \times J_2^{(a_2)}] I M \rangle = \left[\frac{\Omega}{\Omega - 2}\right]^{\frac{1}{2}} Z(J_1 a_1, J_2 a_2; Ja, 00; I = J), \quad \text{(A.8)}$$

$$\langle n = 4\, v = 0, I = 0 | [J_1^{(a_1)} \times J_2^{(a_2)}] 00 \rangle = \left[\frac{\Omega}{2(\Omega - 1)}\right]^{\frac{1}{2}} Z(J_1 a_1, J_2 a_2; 00, 00; 0). \quad \text{(A.9)}$$

Knowing these overlaps it is then straightforward to project the $v = 4$ components out of the four-particle states $|[J_1 \times J_2] I M_I \rangle$, built from a superposition of two favored pair operators. For $I \neq 0$, the normalized $v = 4$ state vector is

$$|v = 4 \text{ proj. from } [J_1 \times J_2] I M_I \rangle = \left[1 - \sum_{a''} \frac{\Omega}{\Omega - 2} Z^2(J_1 0, J_2 0; Ia'', 00; I)\right]^{-\frac{1}{2}}$$
$$\times \left\{ |[J_1 \times J_2] I M_I \rangle - \sum_{a'} \left[\frac{\Omega}{\Omega - 2}\right]^{\frac{1}{2}} Z(J_1 0, J_2 0; Ia', 00; I) | n = 4\, v = 2\, I^{(a')} M_I \rangle \right\}. \quad \text{(A.10)}$$

For $I = 0$ the normalized $v = 4$ state vector is:

$$
\begin{aligned}
&|v = 4 \text{ proj. from } [J \times J]I = 0\rangle \\
&\quad = \left[1 - \frac{\Omega}{\Omega - 2} \sum_{a'' \neq 0} Z^2(J0, J0; 00, 0a''; 0) - \frac{\Omega}{2(\Omega-1)} Z^2(J0, J0; 00; 00; 0)\right]^{-\frac{1}{2}} \\
&\quad \times \left\{ |[J \times J]I = 0\rangle - \left[\frac{\Omega}{\Omega - 2}\right]^{\frac{1}{2}} \sum_{a' \neq 0} Z(J0, J0; 00, 0a'; 0) |n = 4\, v = 2\, I = 0(a')\rangle \right. \\
&\qquad \left. - \left[\frac{\Omega}{2(\Omega-1)}\right]^{\frac{1}{2}} Z(J0, J0; 00, 00; 0) |n = 4\, v = 0\, I = 0\rangle \right\}. \qquad \text{(A.11)}
\end{aligned}
$$

Finally, reduced matrix elements of $Q(k)$ connecting $v = 4$ to $v = 2$ states can also be related to the Z-coefficients, since such matrix elements can be related to scalar products of the type (A.1) by means of eqs. (5) and (6). For a $v = 4$ state of the type of eq. (25)

$$
\begin{aligned}
&\sum_{J_1 J_2} c^{(b)}(J_1 J_2 I) \langle n = 4\, v = 2\, I'(a') || Q(k) || [J_1 \times J_2] I \rangle \\
&\quad = \sum_{J_1 J_2} c^{(b)}(J_1 J_2 I) 2 Z(J_1 0, J_2 0; k0, I'a'; I) \left[\frac{(2I+1) \sum_{ll'} [q^{(0)}(ll'k)]^2}{\Omega - 2}\right]^{\frac{1}{2}}. \qquad \text{(A.12)}
\end{aligned}
$$

The matrix elements between two $v = 4$ states on the other hand can, by commutator techniques, be reduced to matrix elements of $Q(k)$ between two-particle states and scalar products of the type (A.1). Thus

$$
\begin{aligned}
&\langle n\, v = 4\, I'b' || Q(k) || n\, v = 4\, Ib \rangle \\
&\quad = \frac{\Omega - n}{\Omega - 4} \sum_{J_1 J_2 J'_1 J'_2} c^{(b)}(J_1 J_2 I) c^{(b')}(J'_1 J'_2 I') Q(k; J'_1 J'_2 I'; J_1 J_2 I), \qquad \text{(A.13a)}
\end{aligned}
$$

where

$$
\begin{aligned}
&Q(k; J'_1 J'_2 I'; J_1 J_2 I) = (-1)^{I + I' + k} [(2I+1)(2I'+1)]^{\frac{1}{2}} [Z(J_1 0, J_2 0; J_1 0, J_2 0; I)]^{-1} \\
&\quad \times \left\{ \sum_{J_3 a_3} Z(J'_1 0, J'_2 0; J_2 0, J_3 a_3; I') \begin{Bmatrix} I & J_2 & J_1 \\ J_3 & k & I' \end{Bmatrix} \right. \\
&\quad \times \langle n = v = 2\, J_3 a_3 || Q(k) || n = v = 2\, J_1 a_1 = 0 \rangle \\
&\quad + (-1)^{J_1 + J_2 + I} \sum_{J_3 a_3} Z(J'_1 0, J'_2 0; J_1 0, J_3 a_3; I') \begin{Bmatrix} I & J_1 & J_2 \\ J_3 & k & I' \end{Bmatrix} \\
&\qquad \left. \times \langle n = v = 2\, J_3 a_3 || Q(k) || n = v = 2\, J_2 a_2 = 0 \rangle \right\}, \qquad \text{(A.13b)}
\end{aligned}
$$

and where $\langle n = v = 2\, J'a' || Q(k) || n = v = 2Ja \rangle$ is given by eq. (23).

For the six-particle states of the type introduced in eq. (36), the uncoupling of one $\tilde{S} = 0$ pair is accomplished by the relation (illustrated for the most important case

$J_1 = J_2 = J_3 = 2$):

$$\mathscr{A}(JM)[\mathscr{A}^+(2)\times[\mathscr{A}^+(2)\times\mathscr{A}^+(2)]I_{12}]IM_I|0\rangle$$
$$= \sum_{J'a'I'}(-1)^{J-M}\langle IM_1J-M|I'M_I'\rangle\left[\frac{2I+1}{2I'+1}\right]^{\frac{1}{2}}\frac{N_{J'}}{N_JN_2^2}[\mathscr{A}^+(2)\times A^{(a')}(J')^+]I'M_I'|0\rangle$$
$$\times F([2\times[2\times2]I_{12}]I;\,J0[20\times J'a']I'), \qquad (A.14)$$

with

$$F([2\times[2\times2]I_{12}]I;\,J0[20\times J'a']I')$$
$$= \delta_{J2}\delta_{J'2}\delta_{a'0}\delta_{I'I_{12}} + \sum_i(-1)^{I'}[(2I'+1)(2I_{12}+1)]^{\frac{1}{2}}\begin{Bmatrix} I & 2 & i \\ J' & J & I'\end{Bmatrix}$$
$$\times Z(20,20;J0,J'a';i)Z(20,20;20,20;i)$$
$$\times\left\{\delta_{I_{12}i}+2(2i+1)\begin{Bmatrix}2 & 2 & I_{12}\\ 2 & I & i\end{Bmatrix}\left[1-\frac{[1+(-1)^i]\delta_{J2}\delta_{J'2}\delta_{a'2}}{[Z(20,20;20,20;i)]^2}\right]\right\}. \qquad (A.15)$$

In terms of these F-coefficients the normalization factor for the state $|[2\times[2\times2]I_{12}]I\rangle$ (see eq. (36)) is:

$$N([2\times[2\times2]I_{12}]I)$$
$$= N_2^3[\sum_{J'a'}F([2\times[2\times2]I_{12}]I;\,20[20\times J'a']I'=I_{12})Z(20,20;20,J'a';I_{12})$$
$$\times Z(20,20;20,20;I_{12})]^{-\frac{1}{2}}. \qquad (A.16)$$

Other quantities needed for the calculation are the scalar products of the states $|[2\times[2\times2]I_{12}]I\rangle$ and (normalized) six-particle states with $v<6$. With $I\neq0$, for example, using eq. (A.10),

$$\langle n=6\,P(v=4)[2\times J(a)]I|[2\times[2\times2]I_{12}]I\rangle = \frac{N([2\times[2\times2]I_{12}]I)}{N_2^3}\left[\frac{\Omega}{\Omega-4}\right]^{\frac{1}{2}}$$
$$\times\left[1-\frac{\Omega}{\Omega-2}\sum_{a''}Z^2(20,Ja;00,Ia'';I)\right]^{-\frac{1}{2}}$$
$$\times\left\{\sum_{J'a'}F([2\times[2\times2]I_{12}]I;\,00[20\times J'a']I)\left[Z(20,Ja;20,J'a';I)\right.\right.$$
$$\left.\left.-\frac{\Omega}{\Omega-2}\sum_{a''}Z(20,Ja;00,Ia'';I)Z(20,J'a';00,Ia'';I)Z(20,J'a';20,J'a';I)\right]\right\}, \qquad (A.17)$$

while

$$\langle n=6\,v=2\,I(a)|[2\times[2\times2]I_{12}]I\rangle = \frac{\Omega N([2\times[2\times2]I_{12}]I)}{N_2^3[2(\Omega-2)(\Omega-3)]^{\frac{1}{2}}}$$
$$\times\sum_{J'a'}F([2\times[2\times2]I_{12}]I;\,00[20\times J'a']I)Z(20,J'a';00,Ia;I)$$
$$\times Z(20,J'a';20,J'a';I). \qquad (A.18)$$

The states $|P(v=6)[2\times[2\times 2]I_{12}]I\rangle$ can be constructed, using such scalar products, or with the aid of the seniority projection operator (35). Finally, the expectation value of H for a state $|[2\times[2\times 2]I_{12}]I\rangle$ can be written in terms of F- and Z-coefficients

$$\langle[2\times[2\times 2]I_{12}]I|\frac{H}{G}|[2\times[2\times 2]I_{12}]I\rangle = -\tfrac{1}{2}\sum_{J}\frac{N^2([2\times[2\times 2]I_{12}]I)}{N_2^6 N_J^2}$$
$$\times\sum_{I'J'a'J''a''}F([2\times[2\times 2]I_{12}]I;J0[20\times J'a']I')F([2\times[2\times 2]I_{12}]I;J0[20\times J''a'']I')$$
$$\times Z(20,J'a';20,J'a';I')Z(20,J'a';20,J''a'';I'). \qquad \text{(A.19)}$$

Matrix elements of $Q(k)$ connecting $v=4$ states to states of the type $|P(v=6)\rangle$ can be reduced to combinations of eqs. (A.14), (A.13) and (A.12) by means of eqs. (5) and (6) of the text.

Appendix B

MULTIPOLE SUM RULES

The multipole sum rules of sect. 6 follow from eq. (28) and the calculation of the matrix element

$$\langle n\ v\ \text{fav.}\ J|Q(k)\cdot Q(k)|n\ v\ \text{fav.}\ J\rangle.$$

For $n=v=2$, this matrix element can be calculated by double application of the commutator, eq. (10a). By using well-known properties of the 6-j symbols, the sums over the J' introduced by eq. (10a) can be carried out. The matrix element then becomes

$$\langle n=v=2\ \text{fav.}\ J|Q(k)\cdot Q(k)|n=v=2\ \text{fav.}\ J\rangle$$
$$=4N_J^2(2k+1)\left[\sum_{ll_1l_2}\frac{1}{2l+1}[q^{(0)}(ll_1k)]^2[q^{(0)}(ll_2J)]^2\right.$$
$$\left.+\sum_{ll_0l_1l_2}q^{(0)}(ll_1k)q^{(0)}(l_0l_2k)q^{(0)}(ll_2J)q^{(0)}(l_0l_1J)\begin{Bmatrix}l & l_2 & J\\ l_0 & l_1 & k\end{Bmatrix}\right]. \qquad \text{(A.20)}$$

The simplest sum rules are those involving sums over all possible k-values, since sums over k of the products of Wigner and Racah coefficients implicit in eq. (A.20) can be performed to give

$$\langle n=v=2\ \text{fav.}\ J|\sum_k Q(k)\cdot Q(k)|n=v=2\ \text{fav.}\ J\rangle = 2\{\Omega+\sum_{ll'}[q^{(0)}(ll'J)]^2\}. \qquad \text{(A.21)}$$

Now, using

$$(2J+1)\langle n=v=2\ \text{fav.}\ J|\sum_k Q(k)\cdot Q(k)|n=v=2\ \text{fav.}\ J\rangle$$
$$=\sum_{v'=0,2}\sum_k\sum_{\alpha'J'}\langle n=2\ v'\alpha'J'||Q(k)||n=v=2\ \text{fav.}\ J\rangle^2, \qquad \text{(A.22)}$$

$$\langle n=2\ v'=0\ J'=0||Q(J)||n=v=2\ \text{fav.}\ J\rangle = \left[\frac{4(2J+1)}{\Omega}\sum_{ll'}[q^{(0)}(ll'J)]^2\right]^{\frac{1}{2}}, \qquad \text{(A.23)}$$

and the fact that $Q(00) = N_{op}$, with eigenvalue n, these equations lead to the sum rule (29) of the text. To obtain the sum rule for $v' = 4$, the matrix element (A.20) must be evaluated for $n = 4$. For this purpose it is useful to note that the Hamiltonian (8) can be put in the form

$$\frac{H}{G} = -\tfrac{1}{2}\sum_{JM} \mathscr{A}^+(JM)\mathscr{A}(JM) = -\tfrac{1}{4}\Omega n - \tfrac{1}{8}\sum_k Q(k)\cdot Q(k) + \tfrac{1}{4}\sum_k Q(k, k_s = 1) : Q(k, k_s = 1), \quad \text{(A.24)}$$

where $Q(kq, k_s q_s)$ is a one-body operator with the same orbital factors as $Q(k)$ but coupled to rank 1 in spin space:

$$Q(kq, k_s = 1\, q_s) = \sum_{ll'} (-1)^l q^{(0)}(ll'k)[a_l^+ \times a_{l'}]^{kq,\, k_s = 1\, q_s}, \quad \text{(A.25)}$$

where again, k = even only. (Note a difference of $\sqrt{2}$ in the definitions of eqs. (A.25) and (3a)). The double dot in (A.24) denotes scalar products in both orbital and spin space. The operators $Q(kq, k_s q_s)$ with $k + k_s$ = odd are scalars in quasi-spin space so that matrix elements of the last term in eq. (A.24) are independent of n and can therefore be evaluated for $n = v$. For $v = 2$, the eigenvalues of H are given by (see eq. (15)):

$$(1/G)E(n\, v = 2\ \text{fav.}\ J) = -\{\sum_{ll'} [q^{(0)}(ll'J)]^2 + \tfrac{1}{2}(n-2)\Omega\}. \quad \text{(A.26)}$$

Together with eqs. (A.21) and (A.24) this gives

$$\langle n = v = 2\ \text{fav.}\ J|\tfrac{1}{4}\sum_k Q(k, k_s = 1) : Q(k, k_s = 1)|n = v = 2\ \text{fav.}\ J\rangle = \tfrac{3}{4}\{\Omega - \sum_{ll'} [q^{(0)}(ll'J)]^2\}. \quad \text{(A.27)}$$

Since this result is independent of n, eq. (A.24) can be used once more to evaluate the matrix element of $\Sigma Q(k)\cdot Q(k)$ for $n = 4$:

$$\langle n = 4\ v = 2\ \text{fav.}\ J|\sum_k Q(k)\cdot Q(k)|n = 4\ v = 2\ \text{fav.}\ J\rangle = 6\Omega + 2\sum_{ll'} [q^{(0)}(ll'J)]^2. \quad \text{(A.28)}$$

Finally, for arbitrary n, eq. (28) of the text leads to

$$\begin{aligned}(2J+1)&\langle n\, v = 2\ \text{fav.}\ J|\sum_{k\neq 0} Q(k)\cdot Q(k)|n\, v = 2\ \text{fav.}\ J\rangle \\ &= \frac{n(2\Omega - n)}{2(2\Omega - 2)}\langle n = 2\, v' = 0\, J' = 0||Q(J)||n = v = 2\ \text{fav.}\ J\rangle^2 \\ &+ \left(\frac{\Omega - n}{\Omega - 2}\right)^2 \sum_{k\neq 0}\sum_{\alpha' J'} \langle n = v' = 2\, \alpha' J'||Q(k)||n = v = 2\ \text{fav.}\ J\rangle^2 \\ &+ \frac{(n-2)(2\Omega - n - 2)}{2(2\Omega - 6)} \sum_{k\neq 0}\sum_{\alpha' J'} \langle n = v' = 4\, \alpha' J'||Q(k)||n = 4\, v = 2\ \text{fav.}\ J\rangle^2.\end{aligned} \quad \text{(A.29)}$$

To obtain the n-dependent factors, we have used the fact that operators $Q(k)$ with $k \neq 0$ (even) are tensors of rank 1 in quasi-spin space. Matrix elements connecting states n, v to n, v' are thus proportional to Wigner coefficients

$$\langle \mathscr{S} M_{\mathscr{S}} 10 | \mathscr{S}' M_{\mathscr{S}} \rangle = \langle \tfrac{1}{2}(\Omega - v)\tfrac{1}{2}(n-\Omega) 10 | \tfrac{1}{2}(\Omega - v')\tfrac{1}{2}(n-\Omega) \rangle .$$

For $n = 4$ all but the last term of eq. (A.29) are now known (through eqs. (A.28), (A.23) and (29), combined with $Q(00) = N_{\text{op}}$, with eigenvalue n). Hence the last term can be evaluated leading to the sum rule of eq. (30) of the text.

So far these sum rules are completely general, valid for any configuration, that is any combination of l-values (or j-values if $\sum_{ll'} [q^{(0)}(ll'J)]^2$ is replaced by $|E(n = v = 2, \text{fav.}\ J)|1/G)$. The sum rules for specific multipole orders such as $k = 2$, on the other hand, have been evaluated only for major oscillator (or pseudo-oscillator) shells with $l = l_m, l_m - 2, \ldots 0$ (or 1). The technique used involves carrying out the sums over l in eq. (A.20) after adding and subtracting the terms with the missing l-values corresponding to $l > l_m$. For example

$$\sum_{l_1} \frac{1}{(2l+1)} [q^{(0)}(ll_1 k)]^2 = \sum_{l_1} \langle l0k0 | l_1 0 \rangle^2 = 1 - \sum_{l_1 > l_m} \langle l0k0 | l_1 0 \rangle^2, \qquad \text{(A.30)}$$

where, with $k = 2$, the sum over $l_1 > l_m$ has non-zero terms only for $l = l_m$, and then contains but a single term with $l_1 = l_m + 2$. Using such techniques

$$\begin{aligned} &\langle n = v = 2 \text{ fav. } J | Q(2) \cdot Q(2) | n = v = 2 \text{ fav. } J \rangle \\ &\quad = 20 - 20 N_J^2 (2l_m + 1) \langle l_m 020 | l_m + 20 \rangle^2 \Bigg[3 - 2 \sum_{l_2 > l_m} \langle l_m 0J0 | l_2 0 \rangle^2 \\ &\quad - \sum_{l_1 > l_m} \langle l_m + 20J0 | l_1 0 \rangle^2 - (2l_m + 1) \langle l_m 0J0 | l_m + 20 \rangle^2 \begin{Bmatrix} l_m & J & l_m + 2 \\ l_m & 2 & l_m + 2 \end{Bmatrix} \Bigg] . \end{aligned} \qquad \text{(A.31)}$$

To obtain the matrix element for $Q(2) \cdot Q(2)$ for $n = 4$, we follow ref. [5]) and introduce the operators $T^{\mathscr{S}=0}$ and $T^{\mathscr{S}=2}$, tensors of rank 0 and 2 in quasi-spin space:

$$T^{\mathscr{S}=0} = Q(k) \cdot Q(k) + \sum_M \mathscr{A}^+(kM)\mathscr{A}(kM) + \sum_M \mathscr{A}(kM)\mathscr{A}^+(kM), \qquad \text{(A.32)}$$

$$T^{\mathscr{S}=2} = 2Q(k) \cdot Q(k) - \sum_M \mathscr{A}^+(kM)\mathscr{A}(kM) - \sum_M \mathscr{A}(kM)\mathscr{A}^+(kM), \qquad \text{(A.33)}$$

so that, using the Wigner-Eckart theorem in quasi-spin space

$$\begin{aligned} &\langle n\, v = 2 \text{ fav. } J | Q(k) \cdot Q(k) | n\, v = 2 \text{ fav. } J \rangle \\ &\quad = \tfrac{1}{3} \langle n = v = 2 \text{ fav. } J | T^{\mathscr{S}=0} | n = v = 2 \text{ fav. } J \rangle \\ &\quad + \tfrac{1}{3} \left[\frac{2\Omega^2 + 2\Omega(1-3n) + 3n^2}{2(\Omega-2)(\Omega-3)} \right] \langle n = v = 2 \text{ fav. } J | T^{\mathscr{S}=2} | n = v = 2 \text{ fav. } J \rangle . \end{aligned} \qquad \text{(A.34)}$$

Using $\mathscr{A}^+\mathscr{A} + \mathscr{A}\mathscr{A}^+ = 2\mathscr{A}^+\mathscr{A} + [\mathscr{A}, \mathscr{A}^+]$, the matrix elements on the right-hand

side of (A.34) can be evaluated from

$$\langle n = v = 2 \text{ fav. } J | 2 \sum_M \mathscr{A}^+(kM)\mathscr{A}(kM) | n = v = 2 \text{ fav. } J \rangle = 4\delta_{kJ} \sum_{ll'} [q^{(0)}(ll'J)]^2, \quad \text{(A.35)}$$

$$\sum_M [\mathscr{A}(kM), \mathscr{A}^+(kM)] = 2(2k+1) \sum_{ll'} [q^{(0)}(ll'k)]^2 \left[1 - \frac{1}{2l+1} (N_{\text{op.}})_l \right], \quad \text{(A.36)}$$

where the expectation value for $(N_{\text{op}})_l = \sum_{mms} a^+_{lmms} a_{lmms}$ in a favored pair state is

$$\langle n = v = 2 \text{ fav. } J | \frac{(N_{\text{op.}})_l}{2l+1} | n = v = 2 \text{ fav. } J \rangle = 4N_J^2 \sum_{l''} \frac{[q^{(0)}(ll''J)]^2}{2l+1} = 4N_J^2 [1 - \sum_{l'' > l_m} \langle l0J0 | l''0 \rangle^2]. \quad \text{(A.37)}$$

By substituting explicit algebraic expressions for Wigner coefficients such as $\langle l_m 020 | | l_m+20 \rangle$, and by combining eqs. (A.31), (A.34) for $n = 4$, (A.35)–(A.37), the multipole sum rules of eqs. (31)–(34) of the text are obtained.

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