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On the Pointwise Completeness of Differential-Difference Equations*

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1. INTRODUCTION

In order to be able to discuss the question of Euclidean-space null controllability for differential-difference equations it is necessary to introduce the concept of pointwise completeness. In this work we present criteria for pointwise completeness of a class of differential-difference equations. The notion of pointwise completeness was first introduced by Weiss [10] for differential-difference equations. He was able to show that not all non-autonomous differential-difference equations were pointwise complete, but was unable to give a definitive answer for such equations with constant coefficients. Subsequently Popov [8] discovered an example of a third-order linear constant coefficient differential-difference equation which disproved a conjecture of Weiss that all such equations were pointwise complete.

For the equation

$$\dot{x}(t) = Ax(t) + Bx(t-1), \tag{1}$$

where $x(t) \in R^n$, and A and B are constant $n \times n$ matrices, Popov [8] has shown that it is pointwise complete if B has rank one. Similarly Brooks and Schmitt [3] have shown (1) is pointwise complete if $AB = BA$. More

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recently Popov [9] has discovered algebraic necessary and sufficient conditions for pointwise completeness of (1). His results bear some resemblance to some of the results to be presented here, but are obtained by entirely different methods.

In the second section we present several definitions as well as stating an important result on the representation of solutions of differential-difference equations. In the next section we present the main result for the pointwise completeness of nonautonomous differential-difference equations. Section 4 presents two forms of the solution of a linear constant coefficient differential-difference equation. The first form is nothing more than the "method of steps" described by Elsgoltz [4], but the second form is believed to be a new form for the solution of these equations. Finally in Section 5 we present various algebraic criteria for the pointwise completeness of time-invariant differential-difference equations. We complete this section with an application of our results to Popov's [8] third-order counterexample.

2. Definitions and Preliminary Results

In this section we introduce the basic system that will be considered in this paper, including appropriate assumptions; and then we present a result on the representation of solutions of these equations.

The system to be considered in this paper is the following nonautonomous differential-difference equation

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,x(t-1) \qquad \text{for} \quad t \in (0, T], \tag{2}$$

$$x(t) = \phi(t) \qquad \text{for} \quad t \in [-1, 0]. \tag{3}$$

In what follows we shall use matrix notation, where A^T denotes the transpose of A, and $\| A \|$ denotes the Euclidean norm of A. In (2) and (3) above, $x(t)$ is an n vector, and $A(t)$ and $B(t)$ are $n \times n$ matrix functions measurable in t, satisfying $\| A(t)\| \leqslant m(t)$, $\| B(t)\| \leqslant m(t)$, where $m \in L^2(0, T)$. We shall assume that the initial function $\phi(t)$ is continuous on $[-1, 0]$, that is, $\phi \in C([-1, 0]; R^n)$.

THEOREM 1. *With the above assumptions there exists a unique solution to (2), where the solution is continuous on $[-1, 0]$, and is absolutely continuous on $(0, T]$. Further, the solution at time T is*

$$x(T, \phi) = X(T, 0)\,\phi(0) + \int_{-1}^{0} \left[d_s \int_{0}^{s+1} X(T, \alpha)\, B(\alpha)\, d\alpha \right] \phi(s). \tag{4}$$

$X(t, s)$ is a unique $n \times n$ matrix solution, defined on $[-1, T] \times [0, T]$ of

$$(\partial/\partial t)\, X(t, s) = A(t)\, X(t, s) + B(t)\, X(t-1, s) \qquad (5)$$
$$\text{for} \quad (t, s) \in [s, T] \times [0, T]$$

and

$$X(t, s) = \begin{cases} I & (n \times n \text{ identity matrix}) \text{ for } t = s \\ 0 & \text{for } (t, s) \in [-1, s) \times [0, T]. \end{cases} \qquad (6)$$

The proof of the existence of the solution is given in [1]. The above variation-of-parameters formula is obtained by an appropriate substitution in the results of [2] or [6].

We will call $X(t, s)$ the *fundamental solution* of (2). When (2) is autonomous we can write the fundamental solution $X(t, s)$ as $X(t - s)$ without any loss of generality.

3. Main Result

In this section we present the main result which is a necessary and sufficient condition for pointwise completeness. In addition we present an alternate statement of this result in terms of quadratic forms.

Definition 1. The system (2), (3) is said to be *pointwise complete* at time T if for every $x_1 \in R^n$ there exists a $\phi \in C([-1, 0]; R^n)$ such that $x(T, \phi) = x_1$.

It is clear that (4) can be rewritten as

$$x(T, \phi) = \int_{-1}^{0} \left[d_s \left(\int_0^{s+1} X(T, \alpha)\, B(\alpha)\, d\alpha + X(T, 0)\, H(s)\right)\right] \phi(s), \qquad (7)$$

where

$$H(s) = \begin{cases} 1 & s = 0 \\ 0 & s \in [-1, 0). \end{cases}$$

To simplify the notation let us define

$$U(T, s) = \int_0^{s+1} X(T, \alpha)\, B(\alpha)\, d\alpha + X(T, 0)\, H(s), \qquad (8)$$

where we note $U(T, -1) = 0$, and $U(T, s)$ is a function of bounded variation in s. Hence (7) becomes

$$x(T, \phi) = \int_{-1}^{0} [d_s U(T, s)]\, \phi(s). \qquad (9)$$

We note (9) is a linear operator mapping the space of continuous functions $C([-1, 0]; R^n)$ into the Euclidean space R^n.

DEFINITION 2. The range of the operator (9), which will be termed the *pointwise reachable set* $\mathscr{P}(T)$, is defined by

$$\mathscr{P}(T) = \{x \in R^n \mid x = x(T, \phi), \phi \in C([-1, 0]; R^n)\}.$$

It can easily be seen that $\mathscr{P}(T)$ is a linear subspace of R^n.

THEOREM 2. *A necessary and sufficient condition for* (2) *to be pointwise complete for* $t = T$, *is that for every nonzero* $\eta \in R^n$,

(i) *there exists a set* $S \subset [0, 1]$, *of nonzero measure, such that*

$$\eta^T X(T, \alpha) B(\alpha) \neq 0 \qquad \text{for} \quad \alpha \in S,$$

or

(ii) $\eta^T X(T, 0) \neq 0$.

Further if (2) *is autonomous, where* $A(t) \equiv A$ *and* $B(t) \equiv B$ *for all* $t \in [0, \infty)$, *then a necessary and sufficient condition for it to be pointwise complete for* $t = T$ *is that for every nonzero* $\eta \in R^n$ *there exists a* $t \geqslant T$ *such that* $\eta^T X(t) \neq 0$.

Proof. Let us suppose (2) is not pointwise complete for $t = T$, then since the range of (9) is a linear subspace of R^n there exists a nonzero $\eta \in R^n$ such that $\eta^T x = 0$ for every $x \in \mathscr{P}(T)$. Hence

$$\int_{-1}^{0} [d_s \eta^T U(T, s)] \phi(s) = 0 \tag{10}$$

for every $\phi \in C([-1, 0]; R^n)$. We note that (10) is a bounded linear functional mapping $C([-1, 0]; R^n)$ into R. By the Riesz representation theorem [7] such bounded linear functionals are uniquely represented by Lebesque–Stieltjes integrals of the form of (10). Since the functional (10) equals zero for every continuous function ϕ, we conclude from the uniqueness of representation and the fact that $U(T, -1) = 0$ that

$$\eta^T U(T, s) = 0 \tag{11}$$

for $s \in [-1, 0]$. Substituting (8) into (11) we obtain

$$\eta^T \int_0^{s+1} X(T, \alpha) B(\alpha) d\alpha + \eta^T X(T, 0) H(s) = 0$$

for $s \in [-1, 0]$. Hence we find $\eta^T X(T, \alpha) B(\alpha) = 0$ for a.e. $\alpha \in [0, 1]$ and $\eta^T X(T, 0) = 0$, which contradicts the theorem.

The necessity of this result can be shown by assuming there exists a nonzero $\eta \in R^n$ such that $\eta^T X(T, \alpha) B(\alpha) = 0$ for a.e. $\alpha \in [-1, 0]$ and $\eta^T X(T, 0) = 0$, and simply reversing the arguments given above. It is clear that

$$\eta^T U(T, s) = 0 \tag{12}$$

for $s \in [0, 1]$. From (12) we see that (10) is true for every $\phi \in C([-1, 0]; R^n)$ which implies $\mathscr{P}(T) \neq R^n$ and so (2) is not pointwise complete.

To prove the second part of the theorem, let us suppose (2) is not pointwise complete for $t = T$. Then because (2) is autonomous

$$\eta^T x(T + \tau, \phi) = \eta^T X(T)\tilde{\phi}(\tau) + \int_{\tau-1}^{\tau} \eta^T X(T + \tau - \alpha - 1) B\tilde{\phi}(\alpha)\, d\alpha \tag{13}$$

for every $\tau \geqslant 0$; where

$$\tilde{\phi}(\tau) = X(\tau)\,\phi(0) + \int_{-1}^{0} X(\tau - \beta - 1)\, B\phi(\beta)\, d\beta.$$

From (i) and (ii) proved above, and (13), we see

$$\eta^T x(T + \tau, \phi) = 0$$

for all $\phi \in C([-1, 0]; R^n)$, and all $\tau > 0$. That is, (2) is not pointwise complete for all $t \geqslant T$. Hence from (ii) above there exists a nonzero $\eta \in R^n$ such that

$$\eta^T X(t) = 0$$

for all $t \geqslant T$.

Let us now suppose there exists a nonzero $\eta \in R^n$ such that $\eta^T X(t) = 0$ for all $t \geqslant T$, then

$$\eta^T X(T + \tau) = \eta^T X(T)\, X(\tau) + \int_{\tau-1}^{\tau} \eta^T X(T + \tau - \alpha - 1)\, BX(\alpha)\, d\alpha$$

for $\tau \geqslant 0$. Hence

$$\int_{0}^{\tau} \eta^T X(T + \alpha - 1)\, BX(\tau - \alpha)\, d\alpha = 0 \tag{14}$$

for all $\tau \geqslant 0$. For $\tau \in [0, 1]$ it can easily be seen from (5) and (6) that $X(\tau) = e^{A\tau}$, and consequently from (14) we have

$$\eta^T X(T - \alpha)B = 0$$

for $\alpha \in [0, 1]$. Hence, from (i) and (ii) above, (2) is not pointwise complete.

An equivalent statement for pointwise completeness to that given above is presented in the following corollary.

COROLLARY 1. *A necessary and sufficient condition for* (2) *to be pointwise complete for $t = T$ is that the matrix*

$$N(T, 0) = \int_0^1 X(T, \alpha)\, B(\alpha)\, B^T(\alpha)\, X^T(T, \alpha)\, d\alpha + X(T, 0)\, X^T(T, 0)$$

be positive definite.

Proof. Since $\| B(\alpha)\| < m(t)$, $m \in L^2(0, T)$, we see that $N(T, 0)$ is well defined. The rest of the proof follows immediately from Theorem 1, and the fact that $N(T, 0)$ is positive semidefinite.

It will be noted that this condition for pointwise completeness bears a strong resemblance to the Kalman [5] condition for controllability. Of course this resemblance is seen to be more than coincidence when one examines the form of (4).

4. THE REPRESENTATION OF THE FUNDAMENTAL SOLUTION OF A DIFFERENTIAL-DIFFERENCE EQUATION

In this section we present two explicit forms for the fundamental solution of (1). Since (1) is autonomous, instead of denoting its fundamental solution by $X(t, s)$ we can simply write it as $X(t)$.

The first approach we will consider is what Elsgoltz [4] termed the "method of steps." Let us now consider how this method is applied. On the interval $t \in (0, 1]$ the term $X(t - 1)$ in (5) is defined by (6) and hence we may solve (5) as an ordinary differential equation on this interval. The solution is

$$X(t) = e^{At} \qquad \text{for} \quad t \in (0, 1]. \tag{15}$$

On the interval $t \in (1, 2]$ we again have an ordinary differential equation, as $X(t - 1)$ is simply the solution of (5) over the interval $t \in (0, 1]$ and the initial condition is $X(1) = e^A$. By induction, on the interval $t \in (k, k + 1]$, $k = 0, 1, \ldots$, we find

$$X(t) = e^{At} + \int_1^t e^{A(t-s_1)} B e^{A(s_1-1)}\, ds_1 + \cdots$$
$$+ \int_k^t e^{A(t-s_k)} B ds_k \cdots \int_1^{s_1} e^{A(s_2-s_1)} B e^{A(s_1-1)}\, ds_1 . \tag{16}$$

The form of the fundamental solution given by (15) and (16) will prove useful in a few special cases in our later discussion. However, it gives very little insight into what properties the fundamental solution $X(t)$ may have. We now present a new form for the fundamental solution $X(t)$ which will prove of greater utility in our future discussion.

Let us now consider the matrix differential equation (5) and (6), where we assume the coefficient matrices $A(t)$ and $B(t)$ are constant and equal to A and B of (1), respectively. We introduce the following notation, by defining $X_k(\tau) = X(\tau + k)$ for $\tau \in [0, 1]$ and $k = 0, 1,\dots$. By direct substitution in (5) we obtain

$$\begin{aligned} &(d/d\tau)\, X_0(\tau) = AX_0(\tau) && X_0(0) = I \\ &(d/d\tau)\, X_1(\tau) = BX_0(\tau) + AX_1(\tau) && X_1(0) = X_0(1) \\ &\qquad \vdots && \qquad \vdots \\ &(d/d\tau)\, X_k(\tau) = BX_{k-1}(\tau) + A_k X(\tau) && X_k(0) = X_{k-1}(1), \end{aligned} \tag{17}$$

so that the solution of (5) and (6) over the interval $t \in [k, k + 1]$ is given by $X(t) = X_k(t - k)$.

Letting $Z_k(\tau) = [X_0^T(\tau),\dots, X_k^T(\tau)]^T$, (17) can be written more concisely as

$$(d/d\tau)\, Z_k(\tau) = A_k Z_k(\tau) \qquad \text{for} \quad \tau \in [0, 1], \tag{18}$$

$$X_k(\tau) = E_k Z_k(\tau), \tag{19}$$

where $Z_k(\tau)$ is an $n(k + 1) \times n$ matrix, and

$$A_k = \begin{bmatrix} A & 0 & 0 & \cdots & 0 & 0 \\ B & A & 0 & \cdots & 0 & 0 \\ 0 & B & A & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & B & A \end{bmatrix}, \qquad E_k = [0,\dots, 0, I].$$

A_k and E_k are $n(k + 1) \times n(k + 1)$ and $n \times n(k + 1)$ matrices, respectively.

As is well known, the unique solution of (18) is given by

$$Z_k(\tau) = e^{A_k \tau} Z_k(0), \tag{20}$$

and hence

$$X_k(\tau) = E_k e^{A_k \tau} Z_k(0). \tag{21}$$

It is clear from (17) and the definition of $Z_k(\tau)$ that

$$Z_0(0) = I. \tag{22}$$

It is easily shown by induction that

$$Z_k(0) = \begin{bmatrix} I \\ \cdots\cdots\cdots \\ e^{A_{k-1}} Z_{k-1}(0) \end{bmatrix} \quad \text{for } k = 1, 2, \ldots . \tag{23}$$

From the statement following (17) we can write an explicit relation for $X(t)$, $t \in [k, k+1]$, namely,

$$X(t) = X_k(t-k) = E_k e^{A_k(t-k)} Z_k(0). \tag{24}$$

It should be observed that the fundamental matrix solution $X(t)$ may be singular for some t, in contradistinction to the case for ordinary differential equations. The following example exhibits this property quite clearly.

EXAMPLE 1. Consider the following differential-difference equation studied by Popov [8]:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1(t-1) \\ x_2(t-1) \\ x_3(t-1) \end{bmatrix}. \tag{25}$$

From (24) and some simple calculations we see

$$X(2) = E_2 Z_2(0) = \begin{bmatrix} 2 & 4 & -4 \\ 1 & 1 & -2 \\ 0 & 2 & 0 \end{bmatrix},$$

which is a singular matrix.

5. ALGEBRAIC CRITERIA FOR POINTWISE COMPLETENESS OF AUTONOMOUS DIFFERENTIAL-DIFFERENCE EQUATIONS

In this section we apply the results of Sections 3 and 4 to obtain algebraic criteria for (1) to be pointwise complete.

COROLLARY 2. *Every system* (2), (3) *with the coefficients* $A(t)$ *and* $B(t)$ *equal to constant matrices* A *and* B, *respectively, is pointwise complete for all* $T \in [0, 2)$.

Proof. For $T \in [0, 1]$ we have from (15) that $X(T) = e^{AT}$. Hence $X(T)$ has rank n, so for every nonzero $\eta \in R^n$ we have $\eta^T X(T) \neq 0$, and thus Theorem 2 implies (2), (3) is pointwise complete.

Let us now consider the case $T \in [1, 2)$. Suppose (2), (3) is not pointwise complete, then there exists a nonzero $\eta \in R^n$ such that $\eta^T X(T - \alpha)B = 0$ for $\alpha \in [0, 1]$ and $\eta^T X(T) = 0$. Therefore

$$\eta^T X(T - \alpha)B = 0 \tag{26}$$

for $T - 1 < \alpha < 1$, and substituting for $X(T - \alpha)$ from (16) we have

$$\eta^T e^{A(T-\alpha)} B = 0 \tag{27}$$

for $T - 1 < \alpha < 1$. Since e^{At} is an analytic function of t we see $\eta^T e^{At} B = 0$ for every t. We now examine condition (ii) of Theorem 2. Substituting (16) for $X(T)$, $1 < T \leqslant 2$ we obtain

$$\eta^T \left[e^{AT} + \int_1^T e^{A(T-s_1)} B e^{A(s_1 - 1)} \, ds_1 \right] = 0. \tag{28}$$

Since $\eta^T e^{A(T-s_1)} B \equiv 0$ for all s_1, (28) implies $\eta^T e^{AT} = 0$, and so we conclude $\eta = 0$, which contradicts our assumption.

In the proof of Corollary 2 we have used the form of the fundamental solution $X(t)$ given by (16). It could equally well have been proved by using (21), but it is slightly more convenient to use (17).

The recent result of Brooks and Schmitt [3], that if (2) is autonomous and $AB = BA$, where $A(t) \equiv A$ and $B(t) \equiv B$ for $t \in [0, \infty)$, then it is pointwise complete for all $T \in [0, \infty)$, also follows easily from Theorem 2.

We will now obtain an algebraic necessary and sufficient condition for output pointwise completeness of the system (2), (3).

THEOREM 3. *Suppose the system* (2) *is autonomous with* $A(t)$ *and* $B(t)$ *identically equal to* A *and* B, *respectively. Then it is pointwise complete for* $T \in [k, k + 1)$, $k = 0, 1, 2,\ldots$ *if and only if for every nonzero* $\eta \in R^n$

(i) $\eta^T[E_{k-1}F_{k-1}, \ldots, E_{k-1}A_{k-1}^{nk-1}F_{k-1}] \neq 0$

or

(ii) $\eta^T E_k Z_k(0) \neq 0$,

where $F_k = Z_k(0)B$.

Proof. Suppose (2), (3) is not pointwise complete at time $T \in [k, k + 1)$, $k = 0, 1,\ldots$, then from Theorem 2 there exists a nonzero $\eta \in R^n$ such that

$$\eta^T X(t) = 0 \tag{29}$$

for all $t \geqslant T$. For $t \in [k, k+1)$ the fundamental solution $X(t)$ is given by (24) and so (29) becomes

$$\eta^T E_k e^{A_k(t-k)} Z_k(0) = 0 \tag{30}$$

for $t \in (T, k+1)$. Since the term on the left of (30) is an analytic function we see that

$$\eta^T X(k) = 0.$$

In addition since (2), (3) is not pointwise complete we have, from Theorem 2,

$$\eta^T X(t) B = 0 \tag{31}$$

for $t \in (T-1, k)$. Again since $X(t)$ is analytic on the interval $t \in (k-1, k)$ we conclude that

$$\eta^T X(k-\alpha) B = 0 \tag{32}$$

for $\alpha \in [0, 1]$. From Theorem 2 we see that (2), (3) is not pointwise complete for $T = k$. Again from Theorem 2 we see that if (2), (3) is not pointwise complete for $T = k$ then it is not pointwise complete for $T \in [k, k+1)$. Hence we have shown that (2), (3) is pointwise complete for $T \in [k, k+1)$ if and only if it is pointwise complete for $T = k$. Consequently we need only examine whether (2), (3) is pointwise complete for $T = k$.

Suppose (2), (3) is not pointwise complete for $T = k$, then from Theorem 2 there exists a nonzero $\eta \in R^n$ such that

$$\eta^T X(k-\alpha) B = 0 \tag{33}$$

for $\alpha \in [0, 1]$, and

$$\eta^T X(k) = 0. \tag{34}$$

Substituting (24) in (33) and (34) we obtain

$$\eta^T E_k e^{A_{k-1}(1-\alpha)} Z_k(0)\, B = 0, \tag{35}$$

for $\alpha \in [0, 1]$, and

$$\eta^T E_k Z_k(0) = 0. \tag{36}$$

By successive differentiation of (35) and setting $\alpha = 1$, we obtain

$$\eta^T E_{k-1} A_{k-1}^i F_{k-1} = 0 \tag{37}$$

for $i = 0, 1, \ldots, nk-1$. Hence (37) and (36) lead to a contradiction which completes the proof of sufficiency.

To prove necessity suppose there exists a nonzero $\eta \in R^n$ such that

$$\eta^T[E_{k-1}F_{k-1}, \ldots, E_{k-1}A_{k-1}^{nk-1}F_k] = 0 \tag{38}$$

and

$$\eta^T E_k Z_k(0) = 0. \tag{39}$$

From the Cayley–Hamilton theorem and (38) we obtain

$$\eta^T E_{k-1}A_{k-1}^{nk}F_{k-1} = 0. \tag{40}$$

Then by induction

$$\eta^T E_{k-1}A_{k-1}^{nk+l}F_{k-1} = 0 \tag{41}$$

for $l = 0, 1, \ldots$. Using the power-series expansion of the exponential matrix we find

$$\eta^T E_{k-1}e^{A_{k-1}(1-\alpha)}F_{k-1} = 0 \tag{42}$$

for $\alpha \in [0, 1]$. Hence from (39) and (42) we have

$$\eta^T X(k - \alpha)B = 0$$

for $\alpha \in [0, 1]$ and

$$\eta^T X(k) = 0,$$

which is a contradiction.

COROLLARY 3. *Suppose* (2) *is autonomous with* $A(t)$ *and* $B(t)$ *equal to* A *and* B, *respectively. Then a necessary and sufficient condition for* (2), (3) *to be pointwise complete for* $T \in [k, k + 1)$ *is that the matrix*

$$M(T) = [E_{k-1}F_{k-1}, \ldots, E_{k-1}A_{k-1}^{nk-1}F_{k-1}, E_k Z_k(0)]$$

has rank n.

Proof. The proof is an immediate consequence of Theorem 3.

As was mentioned previously it is not at all clear that there exist systems (2), (3) which are not pointwise complete. Popov recently gave the following example and we will use it to demonstrate in fact that not all systems (2), (3) are pointwise complete, as well as demonstrating an application of Theorem 3.

EXAMPLE 2. Let us consider the differential-difference equation given in

Example 1. From Corollary 3 and some straightforward calculations we find, for $T = 2$,

$$M(2) = \begin{bmatrix} 2 & -2 & 0 & 2 & -4 & 0 & 0 & -4 & 0 \\ 1 & -2 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 & 0 & 0 & -4 & 0 \end{bmatrix}.$$

It can easily be seen that this matrix has rank less than three since for $\eta^T = [1 \ -2 \ -1]$

$$\eta^T M(2) = 0.$$

Hence this system is not pointwise complete for $T \geqslant 2$.

To complete our discussion we present an algebraic characterization of the pointwise reachable set $\mathscr{P}(T)$.

THEOREM 4. *The pointwise reachable set* $\mathscr{P}(T)$ *equals the range of the matrix* $M(T)\,M^T(T)$ *for* $T \in [k, k+1)$, $k = 0, 1,\ldots$.

Proof. By examining the proof of Theorem 3 we see that the orthogonal complement, $\mathscr{P}^{\perp}(T)$, of $\mathscr{P}(T)$ is given by

$$\mathscr{P}^{\perp}(T) = \{\eta \in R^n \mid \eta^T M(T) = 0\} \tag{43}$$

for $T \in [k, k+1)$. It is clear that (43) can be rewritten as

$$\begin{aligned} \mathscr{P}^{\perp}(T) &= \{\eta \in R^n \mid M^T(T)\eta = 0\} = \{\eta \in R^n \mid M(T)\,M^T(T)\eta = 0\} \\ &= \text{null}(M(T)\,M^T(T)) = \text{range}(M(T)\,M^T(T))^{\perp}. \end{aligned}$$

The last two equalities follow from the basic properties of linear transformations, where null $(M(T)\,M^T(T))$ denotes the null space of $M(T)\,M^T(T)$ and range $(M(T)\,M^T(T))$ denotes the range of $M(T)\,M^T(T)$. Hence $\mathscr{P}(T) =$ range $(M(T)\,M^T(T))$.

We have treated the case of pointwise completeness for differential-difference equations with a single delayed term. It seems it would be quite easy to extend these results for the case of multiple delay terms. If we make appropriate continuity assumptions about the coefficients $A(t)$ and $B(t)$ in (2) then we can extend our algebraic results to this case also.

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