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Proof of A Conjecture by A.W. Burks and H.Wang: Some Relations between Net Cycles and States Cycles*

BERNARD P. ZEIGLER

Department of Computer and Communication Sciences, Logic of Computers Group, University of Michigan, Ann Arbor, Michigan 48104

In a foundational paper on the theory of automata A. W. Burks and H. Wang (1957) conjectured that a certain complexity measure involving the size of the strong components of a logical net formed a hierarchy for net behavior. This conjecture was established by Rhodes and Krohn. In this paper a strengthened version of the conjecture is proved by establishing that any logical net can be interpreted as a series-parallel composition of nets associated with its strong components. Some properties of the periodic behavior of machines, shown to be preserved under simulation and composition operations, are used to complete the proof. The relationship of this approach to algebraic proofs of series-parallel irreducibility is discussed.

I. INTRODUCTION

In their 1957 paper, Burks and Wang advanced a conjecture concerning the strong components of digraphs representing logical nets. These strong components were called cycles and the degree of a cycle was defined as the number of points (hence delays) contained in it. A net was said to be of degree d if it had at least one cycle of degree d and no cycles of higher degree.

Burks and Wang made the conjecture: For any degree d, there is some transformation not realized by any net of degree d.

In our current terminology "transformation" means transition function and the realization in question is isomorphic state-behavior realization (e.g., Hartmanis and Stearns, 1966).

The conjecture was established in essence by Rhodes and Krohn (1965). This paper will present a detailed proof of a generalization of that conjecture:

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namely, the word "realized" is replaced by "simulated" in its statement. In other words, I shall show that there is no upper bound on the highest degree needed so that all nets of this or lower degree can simulate any finite transition function. That this is the case is indeed surprising since arbitrary memory expansion and slowing of computation rate is allowed by the simulation concept. It is also notable that this "invariance" under memory and time-scale expansion is not characteristic of other interesting measures of feedback complexity (Zeigler, 1969).

The proof consists, in the first place, of establishing a correspondence between the strong components of a net and the component machines of a series-parallel (cascade) composition. Once this is done, one alternative is to invoke certain little-stressed results of the decomposition theorems of Krohn and Rhodes (1965) to complete the proof. In this alternative, one relies essentially on the irreducibility of simple groups. Instead, I shall present a direct proof of the main result which uses entirely machine (rather than semigroup) concepts. In this respect, there is a similarity with the proof of irreducibility given by Ginzburg (1968). However, even though his proof is machine oriented, it still makes heavy use of basic group-theory ideas. The present proof is of additional interest because of the attention given to the properties of state cycles in the transition diagram and their behavior under simulation and composition operations. Studies of the relation between net structure and the cyclic properties of net behavior have been initiated in connection with neural and genetic network models (e.g., Kaufman, 1969; Walker and Ashby, 1966). From the semigroup point of view, I shall be concerned in effect with the cyclic (singly generated) subgroups of the machine semigroup and the irreducibility of the cyclic groups of prime order. The restriction to this subclass may explain the ability to carry through fully machine-oriented proofs.

II. Basic Concepts

A *machine* (*automaton*, *transducer*) is a quintuple $A = \langle S, Q, O, M, N \rangle$ where S (input symbols), Q (states), O (output symbols) are finite sets, $M : Q \times S \rightarrow Q$ is the transition function and $N : Q \rightarrow O$ is the output function. Given a transition function M we are interested in machines which can simulate M via their input–output relations. To do this we need only consider the semiautomaton $A = \langle S_A, Q_A, M_A \rangle$.

$A = \langle S_A, Q_A, M_A \rangle$ *simulates* $M : Q \times S \rightarrow Q$ if there exists $Q' \subseteq Q_A$ and maps $g : S \rightarrow S_A^*$ (the free semigroup generated by S_A), $h : Q' \rightarrow Q$

(onto) such that Q' is closed under $g(S)$ and for all $q \in Q'$, $s \in S$, $h(\tilde{M}_A(q, g(s)) = M(h(q), s)$; ($\tilde{M}_A : Q_A \times S_A^* \to Q_A$ is the usual extension to S_A^* of M_A).

Here g represents the input encoding and h the output map of the simulation. When $g : S \to S_A$ we say that A *homomorphically* realizes M, and further when h is one–one A *isomorphically* realizes M.

DEFINITION 1. A *composition* over a finite set of machines

$$\{A_\alpha = \langle S_\alpha, Q_\alpha, M_\alpha \rangle \mid \alpha \in D\}$$

is specified by a set S, a family of subsets $\{I_\alpha \subseteq D \mid \alpha \in D\}$ and a family of maps $\{Z_\alpha \mid \alpha \in D\}$, where $Z_\alpha : \times_{\beta \in I_\alpha} Q_\beta \times S \to S_\alpha$.

This structural description uniquely defines a machine $A = \langle S, Q, M \rangle$ where $Q = \times_{\alpha \in D} Q_\alpha$, and for all $q \in Q$, $s \in S$, $\alpha \in D$,

$$\text{proj}_\alpha(M(q, s)) = M_\alpha(\text{proj}_\alpha(q), Z_\alpha(\text{proj}_{I_\alpha}(q), s)).$$

Here $\text{proj}_{D'}(q)$ is the projection of q on the coordinate subset $D' \subseteq D$. Interpretively, I_α is the set of machines directly influencing A_α and Z_α is the connecting map specifying the next input to A_α in terms of the next external input S and the present states of the machines indexed by I_α. In particular, a *logical net* (digital network, sequential machine realization) is a composition over a set of 2-state delay elements (cf. Hartmanis and Stearns, 1966).

The *digraph* (directed graph) $D(A)$ representing a composition A has as points the set D, and there is a line from point α to point β just in case $\beta \in I_\alpha$. [The graph-theory terminology and concepts used here are taken from Harary, Norman, and Cartwright (1965).]

A composition over $\{M_\alpha \mid \alpha \in D\}$ is said to be *series parallel* if there exists a linear order on D $\alpha_1, \alpha_2, \alpha_3, \ldots$ such that $I_{\alpha_1} = \phi$ and for each integer n, $I_{\alpha_n} \subseteq \{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\}$.

The digraph of a series-parallel composition then, assumes a simple one-dimensional form in which any line directed to α_n must come from some point α_m, $m < n$, for $n \geqslant 1$ and α_1 has no incident lines.

For any point $\alpha \in D(A)$, let C_α denote the *strong component* containing α i.e., $C_\alpha = \{\beta \mid$ there is a path (of possibly 0 length) from α to β and back in $D(A)\}$. It is well know that the set of strong components $\{C_\alpha \mid \alpha \in D\}$ is a partition of the points of $D(A)$. The *condensation* of $D(A)$ is the digraph $D^*(A)$ with points $\{C_\alpha \mid \alpha \in D\}$ and there is a line from C_α to C_β iff (if and only if) this is the case for some points $\alpha' \in C_\alpha$, $\beta' \in C_\beta$ (and $C_\alpha \neq C_\beta$).

A digraph D has an (*ascending*) *level assignment* if to each $\alpha \in D$ there is an integer n_α (called its *level*) such that for each line (α, β) in D $n_\alpha < n_\beta$.

Theorem 10.2 of Harary *et al.* (1965) states, in effect, that the following is a level assignment for an acyclic graph D: Let U be the set of transmitters of D and assign the integer 0 to each point in U. To every point α not in U assign n_α, where n_α is the *length of the longest path* to α from any point in U. We call this the *longest path length assignment.*

III. Logical Nets and Series-Parallel Compositions

The proof of the Burks–Wang conjecture begins by showing how a logical net A can be considered to be a series-parallel composition of component nets associated with the strong components of $D(A)$.

Let A be any logical net and $D(A)$ its representing digraph. It is well known that the condensation of $D(A)$ is acyclic and therefore has an ascending level assignment, which we take to be the longest path-length level assignment.

For any digraph D with level assignment let L_m be the subset of points of D having level m. Clearly, $\{L_m \mid 0 \leqslant m \leqslant \bar{m}\}$ (where $\bar{m}$ is the highest level) is a partition of D.

We shall employ the following:

Lemma 1. *Let D be an acyclic digraph with levels assigned according to the longest path-length assignment. For any level $m \geqslant 0$,*

(a) *there are no lines joining any two points in L_m,*

(b) *no point in L_m has any incoming lines from points in $L_{m'}$, $m' \geqslant m + 1$, and*

(c) *every point in L_{m+1} has at least one incoming line from some point in L_m.*

Proof. (a) and (b) follow directly from the definition of level assignment. To prove (c) let L be the longest path from the transmitters to a point P in L_{m+1}. We show that point Q immediately preceding P in L has level m. Since Q is in L there is a path of length at least m from the transmitters to Q. The level of Q is then at least m. Suppose it exceeds m. Then there is a path from the transmitters to P (running through Q) with length exceeding $m + 1$. This contradicts the fact that the level of P is $m + 1$ and thus Q has exactly level m. Q.E.D.

Corollary 1. *Every finite acyclic digraph can be put in the form of a digraph of a series-parallel composition.*

Proof. For each of the sets L_m of Lemma 1, let

$$P_{(m,1)}, P_{(m,2)}, P_{(m,3)}, \ldots, P_{(m,n_m)}$$

be an enumeration of its points (where n_m is the cardinality of L_m). The digraph represents a series-parallel composition for, according to the lemma, the ordering of the points

$$P_{(1,1)}, P_{(1,2)}, \ldots, P_{(1,n_1)}, P_{(2,1)}, P_{(2,2)}, \ldots, P_{(\bar{m},n_{\bar{m}})}$$

is such that if there is a line from (i, j) to (k, l) and $k \geqslant 1$, then $i < k$, in fact $i = k - 1$, and by definition of L_0 no lines are incident on any point with $k = 1$.

We need to show that a logical net can be interpreted as a composition over logical nets associated with its strong components. This demonstration, while conceptually straightforward, involves a degree of notational difficulty. It consists of (1) defining the logical net $A(C_\alpha)$ associated with a strong component C_α of $D(A)$, (2) defining a composition A^* over the $\{A(C_\alpha) \mid \alpha \in D(A)\}$ whose digraph is $D^*(A)$, and (3) verifying that A and A^* are isomorphic.

(1) $A(C_\alpha)$ will be a composition over the 2-state delay elements in C_α. The external input to an element α in C_α will consist of the external input to A together with all delay wires incident on C_α not originating within it. In other words, let $I^*_{C_\alpha} = \{C_\beta \mid \text{there is a line from } C_\beta \text{ to } C_\alpha \text{ in } D^*(A)\}$ and let $(I^*_{C_\alpha})$ denote the union of the sets C_β in $I^*_{C_\alpha}$. Then the external input,

$$S_{C_\alpha} = \mathop{\times}_{\beta \in (I^*_{C_\alpha})} Q_\beta \times S.$$

The set of elements within C_α directly influencing α is $I'_\alpha = I_\alpha \cap C_\alpha$. The new connecting map Z'_α is the old connecting map Z_α reinterpreted accordingly, i.e.,

$$Z'_\alpha(\text{proj}_{I'_\alpha}(q), (\text{proj}_{(I^*_{C_\alpha})}(q), s)) = Z_\alpha(\text{proj}_{I_\alpha}(q), s).$$

This defines a machine $A(C_\alpha) = \langle S_{C_\alpha}, Q_{C_\alpha}, M_{C_\alpha}\rangle$, where $Q_{C_\alpha} = \times_{\beta \in C_\alpha} Q_\beta$ and M_{C_α} are determined according to Definition 1.

(2) The composition A^* over $\{A(C_\alpha)\}$ will have external input S. The

set component machines influencing $A(C_\alpha)$ is given by $I^*_{C_\alpha}$. The connecting map $Z^*_{C_\alpha} : \times_{C_\beta \in I^*_{C_\alpha}} Q_{C_\beta} \times S \to S_{C_\alpha}$ is just the identity mapping.

Note that the digraph $D(A^*)$ is isomorphic with the condensation $D^*(A)$ since it is generated by the sets $I^*_{C_\alpha}$.

(3) It is now routine to verify that the transition function defined by the composition A^* is isomorphic with that of A.

In sum, we have shown that a logical net A can be interpreted as a composition over the nets associated with the strong components. The digraph of this composition of $D(A)$ is just the condensation of the original digraph and so is acyclic. Corollary 1 then allows us to conclude that this composition is a series-parallel composition. Thus, we have proved

THEOREM 1. *A logical net A is (isomorphic to) a series-parallel composition over the set of logical nets associated with the strong components of D(A).*

IV. State Cycles and Series-Parallel Compositions

Let $M : Q \times S \to Q$ be any transition function and $\tilde{M} : Q \times S^* \to Q$ its extension to S^*. M contains a *cycle* if there is a $q \in Q$ such that

$$q = qx^k (= \tilde{M}(q, x^k)) \tag{1}$$

for some $x \in S^*$ and positive integer k. For any fixed q and x, let k be the least positive integer for which (1) is true. Let the sequence

$$Z_1, Z_2, Z_3, \ldots, Z_{kl(x)}$$

be the sequence of initial substrings of x^k, where Z_1 is the first symbol of x^k and $Z_{kl(x)} = x^k$. [Here $l(x)$ denotes the length of x.]

The sequence of states

$$qZ_1, qZ_2, qZ_3, \ldots, qZ_{kl(x)}$$

is called the *cycle of x* or x cycle and consists of the states encountered in journey from q back to q in the order of encounter. The number $kl(x)$ is its *period T*. The *x period* T_x is the number of states in the subsequence

$$qx^1, qx^2, qx^3, \ldots, qx^k.$$

We note that each of these states must be distinct [since k was the least integer for which (1) held], so that

$$T_x = k \tag{2}$$

and hence

$$T = T_x l(x) \tag{3}$$

[$l(x)$ may be referred to as the input period].

We remark that the cycle of x need not form a cycle in the state diagram of M in the graph theoretic sense, i.e., not all qZ_i need be distinct (although all qx^i are distinct).

We say that M contains a *string cycle* of *string period* p if it contains a cycle of x for some $x \in S^*$ which has x period, $T_x = p$.

The following theorem is proved in Zeigler (1968).

THEOREM 2. *Let $M_i : Q_i \times S_i \to Q_i$ be finite transition functions such that M_1 simulates M_2, with maps $h : Q_1' \to Q_2$, and $g : S_2 \to S_1^*$. If for some $x \in S_2^*$,*

$$q_1', q_2', \ldots, q'_{ml(\tilde{g}(x))} = q' \in Q_1'$$

is a $\tilde{g}(x)$ cycle of M_1 with $\tilde{g}(x)$ period m, then

$$h(q_1'), h(q_2'), \ldots, h(q')$$

is an x cycle of M_2 with x period k dividing m. (Here $\tilde{g} : S_2^* \to S_1^*$ is the unique extension of g to a homomorphism.)

Conversely, if

$$q_1, q_2, \ldots, q_{kl(x)} = q$$

is an x cycle of M_2 with x period k then there exists a $\tilde{g}(x)$ cycle in

$$h^{-1}(q_1) \cup h^{-1}(q_2) \cdots \cup h^{-1}(q) \text{ in } M_1$$

with $\tilde{g}(x)$ period m a positive multiple of k.

Since homomorphism is a special case of simulation we can state

COROLLARY 2. *For finite transition functions M_1, M_2, if M_2 is a homomorphic image of M_1 then the string period of any string cycle in M_1 is a nonzero multiple of the string period of its homomorphic image. Every string cycle in M_2 is the homomorphic image of a string cycle in M_1.*

We shall be considering a series-parallel composition A, of arbitrary finite machines A_α, A_β. Let $M : Q_\alpha \times Q_\beta \times S \to Q_\alpha \times Q_\beta$ be the transition function of A and $M_\alpha : Q_\alpha \times S \to Q_\alpha$, $M_\beta : Q_\beta \times (Q_\alpha \times S) \to Q_\beta$ its components. Except for possibly a relabeling of the input alphabet these are the transition functions of A_α and A_β, respectively, and we need not make any distinction between them.

THEOREM 3. *Let A be a series-parallel composition of finite machines A_α, A_β. Let M contain a cycle of $x \in S^*$ with x period m. Let the homomorphic projection of the cycle of x on M_α have x period k. Then there is a string cycle in M_β with string period m/k.*

Proof. We must first justify the assumptions made in the statement of the theorem.

It is well known that the projection $\text{proj}_\alpha : Q \to Q_\alpha$ is a homomorphism from M to M_α. Moreover, the corresponding partition Π_α on Q has SP (substitution property), i.e., for all $q, q' \in Q$, $s \in S$, $q\Pi_\alpha q'$ implies $M(q, s)\, \Pi_\alpha M(q', s)$.

Thus there is indeed a homomorphic image of the cycle of x lying in M_α which by assumption has x period k. By Corollary 2, there is a positive integer n such that $m = nk$.

Let the sequence $Z_1, Z_2, \ldots, Z_{ml(x)}$ be the initial substrings of x^m, where Z_1 is the initial symbol of x^m and $Z_{ml(x)} = x^m$.

Let the cycle of x in M be $qZ_1, qZ_2, \ldots, qZ_{ml(x)} = q$.

The homomorphic image in M_α is then

$$q'Z_1, q'Z_2, \ldots, q'Z_{kl(x)} = q',$$

where q' is the image of q under the homomorphism and $Z_{kl(x)} = x^k$.

Every state $q \in Q$ has the form $q = ([q]_\alpha, [q]_\beta)$, where $[q]_\alpha$ is the block of Π_α containing q. Since Π_α has SP and since $qx^k\Pi_\alpha q$ we have $qZ_i\Pi_\alpha qZ_{jkl(x)+i}$ for all $0 \leqslant j < n$ and $0 \leqslant i < kl(x)$. We denote $[i]_\alpha = [qZ_i]_\alpha$. Also since the Z_i are initial substrings of x^k we have

$$Z_{jkl(x)+i} = x^{jk}Z_i$$

for all $0 \leqslant j < n$, $0 \leqslant i < kl(x)$. The sequence

$$qZ_1, qZ_2, \ldots, qZ_{kl(x)-1}, qZ_{kl(x)}, qZ_{kl(x)+1}, \ldots$$

then becomes

$$([1]_\alpha, [qZ_1]_\beta), ([2]_\alpha, [qZ_2]_\beta), \ldots$$
$$([kl(x) - 1]_\alpha, [qZ_{kl(x)-1}]_\beta)([0]_\alpha, [qx^k]_\beta), ([1]_\alpha, qx^kZ_1), \ldots .$$

Let $Z_i = a_1 a_2 \cdots a_i$, $0 < i \leqslant kl(x)$. Using the notation $q_1 \xrightarrow{(q,s)} q_2$ for $M_\beta(q_1, (q, s)) = q_2$ we obtain the following transition sequence in M_β starting at $[q]_\beta$:

$$[q]_\beta \xrightarrow{([0]_\alpha, a_1)} [qZ_1]_\beta \xrightarrow{([1]_\alpha, a_2)} [qZ_2]_\beta \longrightarrow \cdots$$

$$[qZ_{kl(x)-1}]_\beta \xrightarrow{([kl(x)-1]_\alpha, a_{kl(x)})} [qx^k]_\beta \xrightarrow{([0]_\alpha, a_1)} [qx^k Z_1]_\beta \cdots .$$

This sequence is a cycle in M_β of the string $y \in (Q \mid \Pi_\alpha \times S)^*$, where

$$y = ([0]_\alpha, a_1)([1]_\alpha, a_2) \cdots ([kl(x) - 1]_\alpha, a_{kl(x)}).$$

Now qx^{jk}, $0 \leqslant j < n$ are all distinct states and recalling that

$$qx^{jk} = ([0]_\alpha, [qx^{jk}]_\beta)$$

(i.e., these states all have the same α component) it must be that $[qx^{jk}]_\beta$, $0 \leqslant j < n$ are all distinct. Thus the existence of the subsequence of the y cycle

$$[qx^k]_\beta, [qx^{2k}]_\beta, \ldots, [qx^{nk}]_\beta = [qx^m]_\beta = [q]_\beta$$

having n distinct states proves that there is a y cycle of y period $n = m/k$ in M_β. Note that $l(y) = kl(x)$ and $T_y = n$ implies $T = nkl(x) = ml(x)$ which agrees with the period of the x cycle in M.

The basic theorem enabling us to prove the Burks–Wang extended conjecture can be stated:

THEOREM 4. *Let $M' : Q' \times S' \to Q'$ contain a cycle of $x \in S^*$ with x period p, a prime number. Assume that M' can be simulated by a series-parallel composition of A_α, A_β. Then at least one of M_α, M_β contains a string cycle of string period a nonzero multiple of p.*

Proof. Let M be the transition function of the composition. Since M' can be simulated by M and given the x cycle in M', we know by Theorem 2 that there is a string cycle in M of string period m a multiple of p, $m = np$, $n > 0$. By Theorem 3, M_α has a projection of this cycle with string period $k \geqslant 1$ and M_β has a string cycle with string period m/k. But since p is a prime either k divides n, in which case M_β has a string cycle of period a nonzero multiple of p, or k is a nonzero multiple of p, in which case M_α has a string cycle a nonzero multiple of p.

Corollary 3. *Let $M' : Q' \times S' \to Q'$ contain a cycle of $x \in S^*$, x period p a prime number. Assume that M' can be simulated by a series-parallel composition, A of $A_{\alpha_1}, A_{\alpha_2}, \ldots, A_{\alpha_n}$ with transition functions*

$$M_{\alpha_i} : Q_{\alpha_i} \times \prod_{j<i} Q_{\alpha_j} \times S \to Q_{\alpha_i}.$$

Then at least one of M_{α_i} has a string cycle with string period a nonzero multiple of p.

Proof. The proof proceeds by induction. The series-parallel composition A may be regarded as a series-parallel composition of A_{α_1} and a series-parallel composition of the A_{α_i}, $i > 1$. Applying Theorem 3 to this situation, the induction may now proceed. Q.E.D.

For series-parallel composition A of $A_{\alpha_i}, A_{\alpha_2}, \ldots, A_{\alpha_n}$ we define *size* (A) to be the number of states in the largest component machine, i.e.,

$$\text{size}(A) = \max_{\alpha_i} |Q_{\alpha_i}|,$$

where Q_{α_i} is the state set of A_{α_i}.

Theorem 5. *For any integer n, there is a finite transition function which cannot be simulated by a series-parallel composition A, with* size$(A) < n$.

Proof. Consider the set of primes which as is known has no greatest member. For any prime p, there is a mod p counter, namely, a machine which counts modulo p occurrences of a symbol, e.g., ..*a*.. in an input string. The transition function of this machine has an a cycle of a period p. By Corollary 3 any series-parallel composition which can simulate this transition function must have at least one component A_α whose transition function has a string cycle of string period a nonzero multiple of p. But this means that $|Q_\alpha| \geqslant p$ since there are at least p distinct states in such a cycle. Thus size$(A) \geqslant p$ for any series-parallel composition which can simulate a mod p counter. For any integer n, we can choose a prime $p \geqslant n$ and so for each n there is a finite transition function which cannot be simulated by a series-parallel composition A with size$(A) < n$.

We note that in particular, for each integer n, there is a finite transition function which cannot be isomorphically realized by any series-parallel composition A with size$(A) < s$. (Thus true because any isomorphic realization is in particular a simulation.)

We have seen (Theorem 1) that a logical net A is a series-parallel composition of the logical nets associated with the strong components of

$D(A)$. The *degree* of a logical net is the number of delays in the largest component, i.e., degree $(A) = \max_{\alpha \in D} | C_\alpha |$. A net of degree d has at least one component having 2^d states and no strong component having 2^{d+i} states where $i > 0$.

Thus size$(A) \leqslant 2^{\text{degree}(A)}$. The strengthened version of the Burks–Wang conjecture follows:

THEOREM 6. *For every integer d, there is a transition function which cannot be simulated by any logical net of degree d.*

CONCLUSION

A strengthened version of the Burks–Wang conjecture has been shown to be true. The proof relied heavily on the properties of state transition cycles exhibited in Theorems 2, 3, and 4.

These theorems can be readily interpreted as constituting a proof of the irreducibility of prime cyclic groups which uses only machine (rather than semigroup) ideas.

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