# PIONIZATION STRUCTURE FUNCTION *

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We analyze the properties of a pionization structure function. Making some general assumptions about the analyticity structure of amplitudes describing inclusive processes (which are valid in a large class of models and are expected to be valid for the true amplitudes) we derive a general form of this structure function.

Recently, a large amount of interest has been centered on the studies of inclusive reactions, both from the experimental and theoretical point of view [1]. Mueller's generalized optical theorem [2], connecting a one particle inclusive cross section with an unphysical limit of a forward $3\to 3$ amplitude has proved a very stimulating idea and has given a new tool for the theoretical studies of inclusive cross sections. A one particle inclusive cross section is particularly simple to describe in the pionization limit i.e., limit defined as

$$t\to -\infty,\ u\to -\infty,\ s/tu \approx M^2/tu = \eta^{-1} \text{ fixed},\ \eta = p_{c\perp}{}^2 + M_c^2 \tag{1}$$

where (see fig. 1)

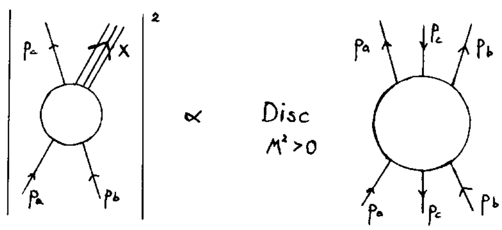


Fig. 1.

$t=(p_a-p_c)^2,\ u=(p_b-p_c)^2,\ s=(p_a+p_b)^2,\ X^2=M^2=s+t+u-M_b^2-M_c^2-M_a^2.$

In the pionization limit the leading contribution to the cross section comes from the two Reggeon exchanges (Fig. 2)

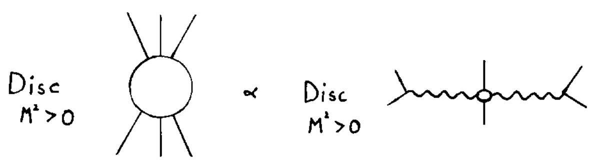


Fig. 2.

and so the cross section is given by

$$d\sigma/dt dM^2 \approx (\eta/tu) \operatorname*{Disc}_{M^2>0} (t^{\alpha(0)} u^{\alpha(0)} \beta(0)^2 f(\eta)) \tag{2}$$

where $f(\eta)$ is a 2 particle 2 Reggeon (of zero mass) coupling function. As $M^2$ enters the expression only through

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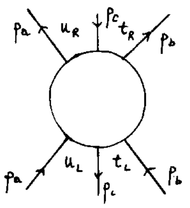

Fig. 3.

the dependence on $\eta$ the discontinuity with respect to $M^2 > 0$ is equivalent to the discontinuity with respect to $\eta > 0$.* For the Pomeron $\alpha(0) = 1$ (we assume that the Pomeron trajectory can be approximated by a Regge pole) and so asymptotically

$$\mathrm{d}\sigma/\mathrm{d}t\mathrm{d}M^2 \sim \eta \operatorname*{Disc}_{\eta>0} f(\eta) = g(\eta) \tag{3}$$

and we see that $\operatorname*{Disc}_{\eta>0} f(\eta)$ completely determines the shape of the pionization spectrum.

Experimentally $g(\eta) = h(p_\perp{}^2)$ shows (for small values of $p_\perp{}^2$) a very steep decrease (as $p_\perp{}^2$ increases) which at somewhat larger $p_\perp{}^2$ takes the form of

$$h(p_\perp{}^2) \sim A \exp(-ap_\perp{}^2) \tag{4}$$

where $a \sim 3$–$4\ \mathrm{GeV}^{-2}$.** This behavior was obtained in the Veneziano model in which Pomeron was treated as an ordinary trajectory***. The question then arises whether this behavior is specific to the Veneziano model or whether it should be expected in general. A similar question was studied for the 2 Reggeo-particle coupling function $f_{\alpha_1\alpha_2}(\eta)$ in ref. [5], its general form and its properties derived and discussed.

The pionization structure function $g(\eta)$ is very closely related to $\operatorname*{Disc}_{\eta>0} f_{11}(\eta)$ and the general forms of these two functions are essentially the same [6] (the difference being hidden in an undetermined function $H(z)$ which enters the final expression see formula (18) below). Using this fact we investigate the general form of $g(\eta)$ closely following ref. [5].

To study the inclusive reaction cross section we have to know this part of the $3 \to 3$ amplitude (fig. 1) which has the non zero discontinuity with respect to $M^2 > 0$. Let us call this part of the amplitude $A$. As we are interested in a strictly forward limit of the $3 \to 3$ amplitude (see fig. 1) $A$ is a function of only 3 variables for which we choose $t$, $u$ and $M^2$ (in the pionization limit they behave as in eq. (1)). We shall argue now that, at least in this limit, the part of the amplitude which gives the leading contribution can be written as

$$\widetilde{A}(t,u,M^2) = \int_0^\infty \exp(-\mathrm{i}\lambda_1 t) \int_0^\infty \exp(-\mathrm{i}\lambda_2 u) \int_0^\infty \exp(-\mathrm{i}\lambda M^2) f(\lambda_1\lambda_1, \lambda_2)\, \mathrm{d}\lambda_1\, \mathrm{d}\lambda_2\, \mathrm{d}\lambda \tag{5}$$

and $A \sim \widetilde{A}$ in the pionization limit.

For the general values of $u$, $t$, $M^2$ the amplitude is expected to have a very complicated singularity structure (including complex singularities) [7] and so this representation would not be valid.

If we consider writing a multiple Mandelstam-like representation (see fig. 3) of $A$

* In the pionization limit $s \sim M^2$ and so we have to calculate only this part of the $\eta > 0$ discontinuity which comes from the discontinuity $M^2 > 0$ neglecting the contribution from $s > 0$. As we argue further on (formula (7) below) we expect the leading term in the pionization limit not to have any $s$ discontinuity; if we were interested also in the nonleading terms a more careful analysis would be necessary. This could be done along the lines indicated by Botke (see his paper in ref. [1]).

** Strictly speaking the pionization region probably has not been reached; it is very likely, however, because of the experimental scaling, that this behavior will be true in the pionization region.

*** Problems with existence of Pomeron poles in the 6-point function are not important in the pionization limit and so this approximation might be not too bad in this limit.

$$\widetilde{A} = \int \frac{\mathrm{d}u_L}{u_L - u + \mathrm{i}\epsilon} \int \frac{\mathrm{d}u_R}{u_R - u + \mathrm{i}\epsilon} \int \frac{\mathrm{d}t_L}{t_L - t + \mathrm{i}\epsilon} \int \frac{\mathrm{d}t_R}{t_R - t + \mathrm{i}\epsilon} \int \frac{\mathrm{d}M'^2}{M'^2 - M^2 + \mathrm{i}\epsilon} \, \zeta(u_L, u_R, t_L, t_R, M'^2) + 17 \text{ other terms* [8]}. \tag{6}$$

Here $\widetilde{A}$ describes only that part of the amplitude which does not have complex singularities. Motivated by the success of the Veneziano and others dual models and some perturbation theory calculations we expect that these complex singularities will not contribute to the leading term in the pionization region and so can be neglected. What is the relative importance of the 18 terms in eq. (6)? This depends on the corresponding $\zeta$'s which are not known. We shall take the clue from the dual models with Mandelstam analyticity [8]. Here, in the pionization limit, except for the first term (the one shown in eq. (6)) all other terms vanish exponentially and so their contribution can be neglected. This vanishing parallels the exponential vanishing of the "third term" in the Veneziano model (i.e., $(s,u)$ term in the $|s| \to \infty$, $t$ fixed limit). Of course, we do not know whether to expect these terms to vanish exponentially in the proper theory – they might be responsible for Regge cuts, etc. and to provide an important though a non-leading behavior. Assuming that these terms do not contribute to the leading behavior, we see that this part of A which gives a leading behavior in the pionization limit has the following Mandelstam like representation

$$\widetilde{A} = \int \frac{\mathrm{d}u_L \, \mathrm{d}u_R \, \mathrm{d}t_L \, \mathrm{d}t_R \, \mathrm{d}M'^2 \zeta(u_L, u_R, t_L, t_R, M'^2)}{(u_L - u + \mathrm{i}\epsilon)\ (u_R - u + \mathrm{i}\epsilon)\ (t_L - t + \mathrm{i}\epsilon)\ (t_R - t + \mathrm{i}\epsilon)\ (M'^2 - M^2 + \mathrm{i}\epsilon)} \,. \tag{7}$$

The i$\epsilon$'s tell us that the singularities in $M^2, t, u > 0$ should be approached from above. In inclusive reactions $t$ and $u$ are negative, we are in the region outside the cuts and so the direction of approach does not matter.

Next we write

$$\frac{1}{u_{\genfrac{}{}{0pt}{}{L}{R}} - u + \mathrm{i}\epsilon} = \mathrm{i} \int_0^\infty \exp\{\mathrm{i}\lambda_{\genfrac{}{}{0pt}{}{L}{R}}(u_{\genfrac{}{}{0pt}{}{L}{R}} - u + \mathrm{i}\epsilon)\} \, \mathrm{d}\lambda_{\genfrac{}{}{0pt}{}{L}{R}}, \quad \frac{1}{t_{\genfrac{}{}{0pt}{}{L}{R}} - t + \mathrm{i}\epsilon} = \mathrm{i} \int_0^\infty \exp\{\mathrm{i}\mu_{\genfrac{}{}{0pt}{}{L}{R}}(t_{\genfrac{}{}{0pt}{}{L}{R}} - t + \mathrm{i}\epsilon)\} \mathrm{d}\mu_{\genfrac{}{}{0pt}{}{L}{R}},$$

$$\frac{1}{M'^2 - M^2 + \mathrm{i}\epsilon} = \mathrm{i} \int_0^\infty \exp\{\mathrm{i}\rho(M'^2 - M^2 + \mathrm{i}\epsilon)\} \, \mathrm{d}\rho \tag{8}$$

and see that

$$\widetilde{A} \propto \iiiint\!\!\int \mathrm{d}\lambda_R \, \mathrm{d}\lambda_L \, \mathrm{d}\mu_L \, \mathrm{d}\mu_R \, \mathrm{d}\rho \, \exp\{-\mathrm{i}(\lambda_R + \lambda_L)(u - \mathrm{i}\epsilon)\} \exp\{-\mathrm{i}(t - \mathrm{i}\epsilon)(\mu_L + \mu_R) - \mathrm{i}\rho(M^2 - \mathrm{i}\epsilon)\} f(\lambda_R, \lambda_L, \mu_L, \mu_R, \rho) \tag{9}$$

where

$$f(\lambda_R, \lambda_L, \mu_L, \mu_R, \rho) = \iiiint\!\!\int \exp\{\mathrm{i}\lambda_L u_L + \mathrm{i}\lambda_R u_R + \mathrm{i}\mu_L t_L + \mathrm{i}\mu_R t_R + \mathrm{i}\rho M'^2\} \, \xi(u_L, u_R, t_L, t_R, M'^2) \, \mathrm{d}u_L \, \mathrm{d}u_R \, \mathrm{d}t_L \, \mathrm{d}t_R \, \mathrm{d}M'^2.$$

Changing the variables $\lambda_R + \lambda_L = \lambda, \mu_L + \mu_R = \mu, \lambda_R - \lambda_L = K, \mu_L - \mu_R = \chi$

we obtain

$$\widetilde{A} \propto \int_0^\infty \mathrm{d}\lambda \int_0^\infty \mathrm{d}\mu \int_0^\infty \mathrm{d}\rho \, \exp\{-\mathrm{i}\lambda(u - \mathrm{i}\epsilon) - \mathrm{i}\mu(t - \mathrm{i}\epsilon) - \mathrm{i}\rho(M^2 - \mathrm{i}\epsilon)\} \, \xi(\lambda, \mu, \rho). \tag{10}$$

* We have already performed the integrations over $(p_a - p_a)^2$, $(p_b - p_b)^2$, $(p_a - p_a - p_c)^2$ and $(p_a - p_a + p_a)^2$ channels in the most general 9 fold represensation.

Using the identity $\int_{-\infty}^{\infty} \mathrm{d}\mu \exp\{\mathrm{i}\mu(b-a)\} \propto \delta(b-a)$, $\zeta(\lambda,\mu,\rho)$ can be expressed as

$$\zeta(\lambda,\mu,\rho) = \iiint \mathrm{d}u_{\mathrm{L}}\,\mathrm{d}t_{\mathrm{L}}\,\mathrm{d}M'^2 \exp \mathrm{i}(\lambda u_{\mathrm{L}} + \mu t_{\mathrm{L}} + \rho M'^2)\, \xi(u_{\mathrm{L}},u_{\mathrm{L}},t_{\mathrm{L}},t_{\mathrm{L}},M'^2) \tag{11}$$

we see that we have obtained (5) with

$$f(\lambda_1,\lambda_2,\lambda) = \iiint \mathrm{d}u_{\mathrm{L}}\,\mathrm{d}t_{\mathrm{L}}\,\mathrm{d}M'^2\, \xi(u_{\mathrm{L}},u_{\mathrm{L}},t_{\mathrm{L}},t_{\mathrm{L}},M'^2) \exp\{\mathrm{i}\lambda_1 u_{\mathrm{L}} + \mathrm{i}\lambda_2 t_{\mathrm{L}} + \mathrm{i}\lambda M'^2\}\,.$$

The asympototic behavior as $|t|$ and $|u|$ get large of $\widetilde{A}$ at fixed $\eta$ is controlled by the exponent $-\mathrm{i}\chi = \mathrm{i}(\lambda_1 t + \lambda_2 u + \lambda M^2)$ not becoming large.

We find the conditions on $\lambda_1, \lambda_2, \lambda$ by: $\chi = \lambda_1 t + \lambda_2 u + \lambda tu/\eta$ therefore $\lambda_1 = \mathrm{O}(1/t), \lambda_2 = \mathrm{O}(1/u), \lambda = \mathrm{O}(1/ut)$. This leads us to define new variables $x_1 = -\lambda_1 t$, $x_2 = -\lambda_2 u$, $x = \lambda/\lambda_1\lambda_2$ and we see that $x_1, x_2$ and $x$ remain finite as $t, u \to -\infty$ thus we must examine the behavior of $f(\lambda_1,\lambda_2,x\lambda_1\lambda_2)$ for small $\lambda_1,\lambda_2$ at fixed $x$. The expected Regge behavior (fig. 2) is obtained by choosing

$$f(\lambda_1,\lambda_2,x\lambda_1\lambda_2) \sim g(x)\lambda_1^{-\alpha_1-2}\lambda_2^{-\alpha_2-2} \text{ for } \lambda_1,\lambda_2 \sim 0 \tag{13}$$

then we see that

$$\widetilde{A} \propto (-t)^{\alpha_1}(-u)^{\alpha_2} \int_0^\infty \int_0^\infty \int_0^\infty \mathrm{d}x\,\mathrm{d}x_1\,\mathrm{d}x_2 \exp\left\{\mathrm{i}\left(x_1 + x_2 - \frac{x x_1 x_2}{\eta}\right)\right\} g(x) x_1^{-\alpha_1-1} x_2^{-\alpha_2-1}$$

$$= (-t)^{\alpha_1}(-u)^{\alpha_2} \int_0^\infty \mathrm{d}x\, g(x)\, \Phi(\alpha_1,\alpha_2,x/\eta) \tag{14}$$

where

$$\Phi(\alpha_1,\alpha_2,z) = \int_0^\infty \exp(\mathrm{i}x_1) x_1^{-\alpha_1-1} \int_0^\infty \exp(\mathrm{i}x_2) x_2^{-\alpha_2-1} \exp(\mathrm{i}z x_1 x_2)\, \mathrm{d}x_1\,\mathrm{d}x_2$$

and for the Pomeron $\alpha_1 = \alpha_2 = \alpha(0) = 1$. To obtain the cross section we have to take the discontinuity $\eta > 0$ of

$$f(\eta) = \int_0^\infty g(x)\, \Phi(1,1,x/\eta)\,\mathrm{d}x. \tag{15}$$

However, $\Phi(\alpha_1,\alpha_2,z)$ is undefined for $\alpha_1,\alpha_2 > 0$. The proper analytic continuation is

$$\operatorname*{Disc}_{\eta>0} f(\eta) = \lim_{\substack{\alpha_1\to 1\\ \alpha_2\to 1}} \int_0^\infty g(x) \operatorname*{Disc}_{\eta>0} \Phi(\alpha_1,\alpha_2,x/\eta)\,\mathrm{d}x.$$

$$\operatorname*{Disc}_{\eta>0} \Phi = \int_0^\infty dx_1 \int_0^\infty dx_2 x_1^{-\alpha_1-1} x_2^{-\alpha_2-1} \exp\left\{ i\left(x_1x_2 - x\frac{x_1x_2}{\eta}\right)\right\} =$$

$$= \Gamma(-\alpha_2)(-i)^{\alpha_2} \operatorname*{Disc}_{\eta>0} \int_0^\infty dx_1 x_1^{-\alpha_1-1} \exp(ix_1)\left(1-\frac{xx_1}{\eta}\right)^{\alpha_2} \propto \frac{1}{\Gamma(1+\alpha_2)} \int_{\eta/x}^\infty dx_1 x_1^{-\alpha_1-1}\exp(ix_1)\left[\frac{xx_1}{\eta}-1\right]^{\alpha_2} =$$

$$= \frac{1}{\Gamma(1+\alpha_2)} (\eta/x)^{-\alpha_1} \exp(i\eta/x) \int_0^\infty du\, u^{\alpha_2} \exp(i\eta \tfrac{u}{x})(1+u)^{-\alpha_1-1} =$$

$$= \psi(\alpha_2+1, 1+\alpha_2-\alpha_1, -i\eta/x)(\eta/x)^{-\alpha_1}\exp(i\eta/x) \tag{16}$$

where $\psi$ is the confluent hypergeometrical function [10].

So

$$\operatorname*{Disc}_{\eta>0} f(\eta) \propto \int_0^\infty g(x)(\eta/x)^{-\alpha_1} \exp(i\eta/x)\, \psi(2,1,-i\eta/x)\, dx$$

$$= \int_0^\infty h(z)(\eta z)^{-1} \exp(i\eta z)\, \psi(2,1,-i\eta z)\, dz \propto \int_0^{-i\infty} h,(z)(\eta z)^{-1} \exp(-\eta z)\, \psi(2,1,\eta z)\, dz \tag{17}$$

We rotate the integration contour bringing it to the final form

$$\operatorname*{Disc}_{\eta>0} f(\eta) = \int_0^\infty H(z)(\eta)^{-1} \exp(-\eta z)\, \psi(2,1,\eta z)\, dz \tag{18}$$

where $H(z)$ is an unknown function. To justify rotation of the contour we have to know $\xi(u_L, u_L, t_L, t_L, M'^2)$ in (11). However, motivated by the experience with the dual and perturbation theory models and arguments based on polynominal boundness of the amplitude we expect this rotation to be allowed and no singularities encountered (we known that $\operatorname*{disc}_{\eta>0} f(\eta)$ is real). Our result (expression (18)) is a convenient representation of the most general form of the pionization structure function. This generality is contained in the unknown function $H(z)$. Pionization structure function has been obtained in the Veneziano model [1] and in a field theory model [11]. The results of these calculations are consistent with our expression (18) except that these models predict particular forms of $H(z)$. Our result can be looked at as a fairly model independent generalization of these results and so provides a general form of this structure function.

To study the properties of $g(\eta) = \eta \operatorname*{disc}_{\eta>0} f(\eta)$ (3) we look at the properties of $\exp(-\eta z)\,\psi(2,1,\eta z)$.

For $\eta \to 0$ (small $q_\perp^2$) we find $\exp(\eta z)\,\psi(2,1,\eta z) \to \ln(\eta z)$

while for $\eta \to \infty$ $\exp(-\eta z)\,\psi(2,1,\eta z) \to \exp(-\eta z)/(\eta z)^2$. (19)

Of course, depending on the form of $H(z)$ the $z$ integration in general can modify this behavior. If however, for some dynamical reasons, $H(z)$ is strongly peaked around some value of $z = z_0$ (or as in the Veneziano model $H(z) \sim \theta(z-4) f(z)$ where $f(z) \xrightarrow[z\to\infty]{} 0$), then

$$g(\eta) \sim \exp(-\eta z_0)\,\psi(2,1,\eta z_0). \tag{20}$$

The explanation why experimentally $z_0 \approx 3.5$ clearly lies byond the scope of this work – and comes from more dynamics than we have put in. Experiment, then, clearly tells us something about the physical $\xi(u_L, u_L, t_L, t_L, M'^2)$ – but this information is difficult to analyze.

We have analyzed the pionization structure function making several assumptions which we expect to be valid for the true amplitudes and derived a general form of this function. This form contains the integration with an unknowm function $H(z)$ and we observed that the experimental data suggest that this function $H(z)$ is strongly peaked at some value $z \approx z_0 = 3.5$. It would be very interesting to understand why $H(z)$ has this property and what this property means in terms of the spectral functions, $\zeta$.

Perhaps a clue should be taken from the Veneziano model. In this model $H(z) \sim \theta(z-4)g(z)$. This form arises from the full duality restrictions which are imposed on the original 6 point function and the equal slope requirement (which guarantees Regge behavior in all limits). This problem is currently under investigation.

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