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Blocks of Defect Zero of Split (B, N) Pairs*

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I. INTRODUCTION

In [1] Curtis showed that certain types of finite Lie type groups had a unique block of defect 0. Blocks of defect 0 were studied later in [7] and [2] in the apparently more general context of a finite group with a split (B, N)-pair. This note completes the discussion started in [7] and [2] by determining the number of blocks of defect 0 in a group with a split (B, N)-pair. The more difficult question of determining all blocks has been successfully attacked by Dagger [3] and Humphreys [6] in the contexts of finite Chevalley groups and finite Lie type groups, respectively. Some of their techniques seem difficult to use with only the split (B, N)-pair axioms, i.e., without appealing to classification theorems which say that a given split (B, N) pair is really a Lie type group.

The finite groups with a split (B, N) pair have been classified by Tits [9], Fong and Seitz [4], and Hering, Kantor and Seitz [5], and so the theorem of this note is a theorem about "known" groups. Nevertheless, the proof covers all these groups simultaneously and is much more elementary than the classification theorems.

We assume familiarity with either [2] or [7]; however, most of II can be read with only a familiarity with the first facts about groups with (B, N) pairs as presented in [8], for example. The notation is standard. $\langle \cdots \rangle$ is the subgroup generated by $\cdots$, and $X^g = g^{-1}Xg$.

II. A SUBGROUP OF A MINIMAL PARABOLIC SUBGROUP

Let G be a group with subgroups B, N which give G a (B, N) pair. Let $\{s_1, s_2, \ldots, s_n\}$ be the generating involutions of the Weyl group $W = N/H$, $H = B \cap N$. Assume G is saturated, that is $H = \bigcap \{B^n : n \in N\}$. (In the

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terminology of Tits' buildings [9], this is the same as assuming that N is the *full* set-wise stabilizer of an apartment.) For $w \in W$, let $l(w)$ be the smallest integer n such that w can be represented as a word of length n in the s_i's. There is a unique element $w_0 \in W$ such that $l(w_0)$ is maximal. $w_0^2 = 1$. Let $B_i = B_{s_i}^- = B \cap B^{w_0 s_i}$ for $1 \leqslant i \leqslant n$. Recall that $B \cup Bs_iB$ is a subgroup of G.

PROPOSITION 1. *For each* i, $1 \leqslant i \leqslant n$, *let* $w_0 s_i w_0 = s_j$ *and* $P_i = B \cup Bs_iB$. *Then* $P_i \cap P_j^{w_0} = B_i \cup B_i s_i B_i$.

In the context of Tits' buildings, we are considering a face A of codimension one in the chamber stabilized by B and the face A' in the opposite chamber (opposite with respect to the apartment stabilized by N) of the same type. The subgroup of the proposition is the stabilizer of A and A', and the proposition asserts that this subgroup is 2-transitive on minimal galleries connecting A and A'.

Proof. That $w_0 s_i w_0 = s_j$ for some j is a fact about root systems. (See, e.g., [2, 1.8 viii].)

Let $P_i \cap P_j^{w_0}$ act by conjugation on the set of conjugates of B which are contained in P_i. It suffices to show that this action is 2-transitive and that the stabilizer of B in $P_i \cap P_j^{w_0}$ is B_i.

Every parabolic subgroup, B included, is self-normalizing [8, Théorème 3]. Thus the stabilizer of B in $P_i \cap P_j^{w_0}$ is $B \cap P_i \cap P_j^{w_0}$. But $B \cap P_i = B \cap (B \cup Bs_iB) = B$ by the Bruhat Theorem ([8, Théorème 1 (ii)], which says if $w, w' \in W$ and $BwB = Bw'B$, then $w = w'$.) $s_j \in P_j$; so $B \cap P_j^{w_0} = B \cap P_j^{s_j w_0} = B \cap P_j^{w_0 s_i}$. $B \cap B^{w_0 s_i} = B_i$ by definition, and so, to show that the stabilizer of B in $P_i \cap P_j^{w_0}$ is B_i, it suffices to show that $B \cap (Bs_jB)^{w_0 s_i} = \phi$. Using the axioms for a BN pair, one can easily show that $s_i w_0 Bs_j B w_0 s_i \subseteq \cup\, BwB$, where w ranges over certain elements of W with the property that $l(w) \geqslant 1$. Knowing this, the Bruhat Theorem implies that $B \cap (Bs_jB)^{w_0 s_i} = \phi$. This proves that B_i is the stabilizer of B in $P_i \cap P_j^{w_0}$.

Théorème 3 of [8] says that two conjugate parabolic subgroups contained in a common parabolic subgroup P are conjugate in P. Thus the set on which $P_i \cap P_j^{w_0}$ acts is $\{B^g : g \in P_i\}$. But $P_i = B \cup Bs_iB_i$ (2.11 of [2]). Thus the set being permuted is $\{B\} \cup \{B^{s_i b} : b \in B_i\}$. B_i, the stabilizer of B in $P_i \cap P_j^{w_0}$, is certainly transitive on the second set in the union. Moreover, $w_0 s_i w_0 = s_j \in P_j$ says that $s_i \in P_i \cap P_j^{w_0}$, and so $P_i \cap P_j^{w_0}$ is transitive on the entire set. Thus it is 2-transitive, and the proof is complete.

III. SPLIT (B, N)-PAIRS AND BLOCKS OF DEFECT ZERO

The notation is the same as in Section II. In addition, we assume that G is finite and that its (B, N) pair is split and has characteristic p for some prime p.

This means that B has a normal p-subgroup U which complements H and that H is Abelian and has order coprime to p. (See Section 3 of [2] or [7]. In [7], U is called X, V is called Y, but otherwise the notation in [2] and [7] is the same as here.)

We collect some facts about G which will be needed below. For $w \in W$, define $U_w = U \cap U^w$ and $U_w^- = U \cap U^{w_0 w}$. (U_w is well defined in spite of the fact that w is a coset of H in N and not an element of N because U is normalized by H, and so any coset representative of w will give the same U_w.) Let $U_i = U_{s_i}^-$, $V_i = U_i^{s_i}$ and note that $U_i \cdot H = B_i$.

(A) Let $w \in W$. $l(ws_i) > l(w)$ implies $U_{ws_i}^- = U_i(U_w^-)^{s_i}$ and $U_i \cap (U_w^-)^{s_i} = \{1\}$.

$l(ws_i) < l(w)$ implies $U_w^- = U_i(U_{ws_i}^-)^{s_i}$ and $U_i \cap (U_{ws_i}^-)^{s_i} = \{1\}$.

(B) For $w \in W$, $U = U_w^- U_w$.

(C) $G = \bigcup_{w \in W} U_w^-(w)^{-1}B$ and $U^{w_0} \cap B = \{1\}$, where (w) is any coset representative of wH, and $\bigcup_{w \in W} U_w^-(w)^{-1}$ is a set of coset representatives of B in G.

Let (s_i), $1 \leqslant i \leqslant n$ be coset representatives in N of the s_i.

(D) If $u \in U_i$, $u \neq 1$, then $(s_i)^{-1}u(s_i) = f_i(u)\, h_i(u)(s_i)\, g_i(u)$, where $f_i(u) \in U_i$, $f_i(u) \neq 1$, $h_i(u) \in H$, and $g_i(u) \in U$, $g_i(u) \neq 1$.

These are 3.3, 3.4, 3.5 and 3.7 in [7] or 3.3 and 4.4 in [2].

From Proposition 1 we see that $U_iH \cup U_iH(s_i)\, U_i$ is a group. It contains the $(s_i)^{-1}u(s_i)$, $f_i(u)$, $h_i(u)$ and (s_i) of (D), and so $g_i(u)$ belongs to the intersection of U and this group which is just U_i. Thus $h_i(u)(s_i) = f_i(u)^{-1}(u)^{(s_i)}g_i(u)^{-1} \in U_iV_iU_i$. We have proved

PROPOSITION 2. *(s_i) may be selected from $U_iV_iU_i \cap s_iH$.*

In the terminology of [2], Proposition 2 says that all split (B, N) pairs are restricted (3.9 of [2]). This fact is useful for the construction of characteristic-p representations of G. (See 5.7 of [2] or 3.17 (c) of [7].)

From now on, assume that the (s_i) are chosen according to Proposition 2.

LEMMA. *$\langle U_i, V_i\rangle = U_iH_i \cup U_iH_i(s_i)\, U_i$, where $H_i = H \cap \langle U_i, V_i\rangle$.*

Proof. The inclusion $\supseteq$ is clear, since $(s_i) \in U_iV_iU_i$. To prove $\subseteq$, it suffices to show that $U_iH_i \cup U_iH_i(s_i)\, U_i$ is a group because it certainly contains U_i and $U_i^{s_i} = V_i$. Checking that $U_iH_i \cup U_iH_i(s_i)U_i$ is a group is straightforward using (D) and remembering that $g_i(u) \in U_i$, that (s_i) normalizes H_i and that H_i normalizes U_i. This proves the lemma.

Now let $H_0 = \langle H_i^w : w \in W,\ 1 \leqslant i \leqslant n\rangle$. ($H_i^w$ is well defined as H is Abelian.) By 3.28 of [7] or 3.10 of [2], a coset representative

(w) of each $w \in W$ may be chosen in such a way that for any $w, w' \in W$, $(w)(w')(ww')^{-1} \in H_0$. We choose the (w)'s with this property.

PROPOSITION 3. *Let G_0 be the subgroup of G generated by the p-subgroups of G. Then $G_0 = \bigcup_{w \in W} U_w^-(w)^{-1}H_0U$. G_0 has a split (B, N) pair with $B_0 = H_0U$, $N_0 = N \cap G_0$ and with Weyl group generated by $(s_1)H_0, \ldots, (s_n)H_0$. $G_0 \lhd G$ and $G_0H = G$.*

Proof. Once the second sentence is checked, every other assertion is easily verified. We first check $\supseteq$. G_0 contains U and each H_i by the Lemma. G_0 is normal in G and so G_0 contains H_0. By Proposition 2, $(s_i) \in G_0$, and since modulo H_0 each (w) is a product of the (s_i)'s, each $(w) \in G_0$. This is sufficient to give the inclusion $\supseteq$. To check the other inclusion we show that the set $\bigcup_{w \in W} U_w^-(w)^{-1}H_0U$, is a group which contains all p-Sylow subgroups of G. Call this set X.

U is p-Sylow in G by (C) and the fact that $U_w^- = \{1\}$ implies that $w = 1$. Hence an arbitrary p-Sylow subgroup of G is of the form $U^{(w)u}$, $w \in W$, $u \in U$ by (C). Thus if X is a group, then it surely contains all p-Sylow subgroups.

We check that X is a group. Let $w, w' \in W$. Then $U_{w'}^-(w')^{-1}H_0U\, U_w^-(w)^{-1}H_0U = U_{w'}^-(w')^{-1}U_w^-(w)^{-1}H_0U$, using (B) and remembering that H_0 normalizes any U_w and any (w) normalizes H_0. Using induction on $l(w')$, (B), and the fact that $(w_1)(w_2) = (w_1w_2)(\mathrm{mod}\, H_0)$ for all $w_1, w_2 \in W$, we see that to prove that this complex is contained in X it suffices to check that for all i $(s_i)^{-1}U_w^-(w)^{-1} \subseteq X$.

If $l(ws_i) > l(w)$, then

$$(s_i)^{-1}U_w^-(w)^{-1} = (U_w^-)^{(s_i)}(s_i)^{-1}(w)^{-1} \subseteq U_{ws_i}^-(ws_i)^{-1}H_0 \subseteq X \text{ by (A).}$$

Now suppose $l(ws_i) < l(w)$. Then $(s_i)^{-1}U_w^-(w)^{-1} = U_i^{(s_i)}(s_i)^{-2}U_{ws_i}^-(s_i)(w)^{-1}$ by (A). But $U_i^{(s_i)} \subseteq \{1\} \cup U_iH_i(s_i)\, U_i$ by (D). Thus $(s_i)^{-1}U_w^-(w)^{-1} \subseteq U_{ws_i}^-(ws_i)^{-1}H_0 \cup U_iH_i(s_i)\, U_i(s_i)^{-2}U_{ws_i}(s_i)(w)^{-1}$. The first set in the union is contained in X and the latter is just $U_i(s_i)\, U_iU_{ws_i}^-(s_i)^{-1}(w)^{-1}H_0 \cdot ((s_i)^2 \in H_0$, and so normalizes U_{ws_i}.) But this is contained in $U_i(s_i)\, U_{ws_i}^-(s_i)^{-1}(w)^{-1}H_0U$ by (B) which is equal to $U_w^-(w)^{-1}H_0U$ by (A). (That $(U_{ws_i}^-)^{(s_i)} = (U_{ws_i}^-)^{(s_i)^{-1}}$ is easily checked, since $s_i^2 = 1$ and since H normalizes $U_{ws_i}^-$.) But this set is contained in X and the proof is complete.

THEOREM. *Suppose G, a finite group, has a saturated split (B, N) pair of characteristic p and that G_0 is the subgroup of G generated by the p-subgroups of G. The number of p blocks of G of defect zero is $| G : G_0 |$.*

Proof. Corollary 5.12 of [2] and Proposition 3 say that G_0 has a unique p block of defect 0 (A hypothesis was omitted in 5.12 and 5.11 of [2]. You must assume that $H_R = H$, where H_R (what we have called H_0) is defined in 3.10 of [2]. The argument used to prove Theorem 4 of [1] may be used with facts from [7] to show that G_0 has a unique p block of defect 0, also.) Let ζ be the ordinary character of G_0 in this block. Then, $\zeta(1) = | U |$ by 5.11 of [2].

By Theorem 2 of [1], G has an ordinary character of degree $| B : H | = | U |$ which must then belong to a block of defect 0. In fact, the ordinary character χ in any block of defect 0 has degree $| U |$ because $| U | \mid \chi(1)$ and the irreducible modular representation afforded by χ has degree less than or equal to $| U |$ (4.3(b) of [2] or 3.9(b) of [7].)

Now Clifford's Theorem implies that the restriction of any such χ to G_0 is just ζ since $(| G : G_0 |, p) = 1$. Thus to prove the theorem it suffices to show that there are $| G : G_0 |$ extensions of ζ to a character of G. There are at least $| G : G_0 |$ extensions which may be obtained by multiplying a fixed extension by the various (linear) characters of G which contain G_0 in their kernels. But there are no more than $| G : G_0 |$ extensions as each is a constituent of ζ^G of degree $| U |$ and as $\zeta^G(1) = | U | \, | G : G_0 |$. The proof is complete.

This theorem corrects 3.30 [7] and extends 5.12 [2] (where, as pointed out in the first paragraph of the proof, a hypothesis about H was omitted.)

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