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# FINAL REPORT

# ON SURFACE WAVES GENERATED BY TRAVELLING DISTURBANCES WITH CIRCULAR SYMMETRY

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## 1. Introduction

The ship-wave problem treated by Kelvin[1] in 1887 has recently received added attention in the literature. The most interesting papers in this connection are those by Peters[2] and Dean[3]. The former deals with the surface wave created by a concentrated pressure which moves on the surface of a body of water that is infinitely deep, and the latter treats the two-dimensional analoque of the same problem.

This paper deals with the more general problem of the surface wave generated by an arbitrary distribution of pressure moving with constant velocity on the surface of the infinitely deep body of water. It is shown that the amplitude of the surface wave can be obtained directly in terms of the applied pressure as the solution of an integrodifferential equation. The particular case in which the pressure distribution has circular symmetry is treated in some detail.* It is shown that in this case the surface wave is uniquely determined by requiring that the amplitude of the wave vanish at remote distances ahead of and on either side of the region over which the pressure is prescribed. Finally, in the last section, the surface wave generated by a pressure point is interpreted in terms of the results obtained for the symmetric pressure distribution. This is defined in the usual way as the limiting case of the wave corresponding to a pressure distribution that is constant over a circle and zero elsewhere as the radius of this circle tends to zero, the total force remaining constant. It is shown that this definition is consistent relative to a certain class of distributions which might equally well be employed in the definition.

It is believed that the method employed in the treatment of the ship-wave problem in this paper is new. Although the results are not

*However, the method is not restricted to this special case. A more careful study of the case with arbitrary pressure distributions is now in progress.

altogether novel, nevertheless the simplicity of the formalism and the mathematical precision afforded by the method in dealing with the cases involving more general pressure distributions may deserve some attention.

## 2. Formulation of the Linearized Problem for the Surface Wave

Let the undisturbed water fill the space $z_1 \leqq 0$, and let it move relative to a fixed $(x_1, y_1, z_1)$ coordinate system with constant speed c in the direction of the negative $x_1$-axis. The problem then is to determine the shape of the free surface corresponding to an arbitrary system of forces acting over this surface. In particular, let the applied surface pressure $p_1 = p_1(x_1, y_1)$ be independent of the time, so that the shape of the free surface relative to the fixed system of axes is also independent of the time. The equation of the free surface which must be determined can then be expressed in the form $z_1 = \eta_1(x_1, y_1)$. It is also supposed that the motion of the water is everywhere irrotational, that viscosity is negligible, and that the density $\delta$ is constant.

For convenience, the problem for the surface wave is formulated in terms of a set of nondimensional quantities. Let the components of the fluid velocity in the directions of the $x_1, y_1$, and $z_1$ coordinate axes be denoted by $u_1 - c$, $v_1$, and $w_1$, respectively, so that $u_1$, $v_1$, and $w_1$ are the components of a vector $\vec{q}_1$ representing the velocity of the fluid relative to its undisturbed motion. A set of nondimensional quantities may then be introduced as follows:

$$(2.1) \qquad x = \kappa x_1, \quad y = \kappa y_1, \quad z = \kappa z_1, \quad \eta = \kappa \eta_1,$$
$$\vec{q} = \vec{q}_1/c, \qquad p = p_1/c^2\delta,$$

where

$$\kappa = g/c^2$$

and g is the acceleration due to gravity. The components of the nondimensional velocity vector $\vec{q}$ in the directions of the x,y, and z coordinate axes will be denoted by u,v, and w, respectively.

Let it be supposed that the disturbance produced by the applied surface forces is negligible at large distances upstream $(x > 0)$, at great depths, and at large distances on either side of the (x,z)-plane. In particular, let the reference point on the pressure scale be selected so that on the free surface $p \to 0$ as $|x| \to \infty$ or $|y| \to \infty$. Then the boundary-value problem for the velocity vector $\vec{q}$ will, in accordance with the linear theory of surface waves, take the form:*

(2.3) $\quad \text{curl}\ \vec{q} = \text{div}\ \vec{q} = 0 \qquad \text{for} \quad z < 0,$

(2.4) $\quad u + p + \eta = 0 \qquad \text{for} \quad z = 0,$

(2.5) $\quad \dfrac{\partial \eta}{\partial x} + w = 0 \qquad \text{for} \quad z = 0,$

(2.6) $\quad u = v = w = 0 \qquad \text{for} \quad x = + \infty, \quad y = \pm \infty \quad \text{or} \quad z = -\infty.$

The assumptions which are made in the formulation of the linearized problem are already well known** and will not be discussed further.

It is convenient to make the additional assumption that the pressure $p(x,y)$ representing the distribution of the forces applied on the surface is normalized, so that

(2.7) $$\int\int_{-\infty}^{+\infty} p(x,y)\ dxdy = 1.$$

This assumes, of course, that the total force acting over the surface is bounded.

## 3. Formulation of an Integrodifferential Equation for the Surface Elevation

The equation of the free surface which corresponds to the solution of the boundary-value problem for the surface wave is of primary interest. It is therefore desirable to solve directly for the amplitude $\eta(x,y)$ of the wave. It is shown in this section that $\eta(x,y)$ can be expressed in terms of the prescribed pressure $p(x,y)$ by means of an integrodifferential equation.

According to the theory of the vector potential, the vector $\vec{q}$ for which (2.3) holds can be represented for $z < 0$ by the integral formula***

(3.1) $$\vec{q}\ (x,y,z) = \frac{1}{2n} \int\int_{-\infty}^{+\infty} \vec{q}\ (x',y',0) \ \text{x}\ (\vec{n} \times \nabla \frac{1}{R'})\ dx'dy',$$

*See Peters[2], pp. 126-127

**See Lamb[4], pp. 363-364

***See v. Mises[5], p. 604. The Cauchy formula (26) given by v. Mises is easily expressed in the formula (3.1) with the aid of Poisson's integral formula for the plane.

where $R' = [(x'-x)^2 + (y'-y)^2 + z^2]^{1/2}$, $\vec{n}$ is the unit vector in the direction of increasing z, and x denotes vector product. Consequently, the component u is represented in the space $z < 0$ by the integral formula

$$(3.2) \qquad u(x,y,z) = \frac{1}{2\pi} \iint_{-\infty}^{+\infty} \frac{x'-x}{[(x'-x)^2 + (y'-y)^2 + z^2]^{3/2}} w(x',y',0)dx'dy' .$$

After integrating by parts and setting $z = 0$, this yields an integral relation between the components u and w on the surface $z = 0$, namely,

$$(3.3) \qquad u(x,y,0) = \frac{1}{2\pi} \iint_{-\infty}^{+\infty} \frac{1}{[(x'-x)^2 + (y'-y)^2]^{1/2}} \frac{\partial}{\partial x'} w(x',y',0)\, dx'dy' .$$

This relation is particularly suited for the problem of the surface wave. For, in this case, u and w satisfy the conditions (2.4) and (2.5) on the surface $z = 0$. Therefore the components u and w can be eliminated from (2.4), (2.5), and (3.3) to obtain an integrodifferential equation for the amplitude $\eta(x,y)$ of the surface wave, namely,

$$(3.4) \quad \eta(x,y) + p(x,y) + \frac{1}{2\pi} \iint_{-\infty}^{+\infty} \frac{1}{[(x'-x)^2 + (y'-y)^2]^{1/2}} \frac{\partial^2}{\partial x'^2} \eta(x',y')dx'xy' = 0.$$

As a consequence of (2.4), (2.6), and (2.7) the boundary conditions that are satisfied by the function $\eta(x,y)$ can be written in the form

$$(3.5) \qquad \eta(x,y) = 0 \qquad \text{for} \quad x = +\infty \qquad \text{and} \quad y = \pm\infty.$$

It is clear from the above that in those cases in which the problem consists merely of determining the shape of the surface wave, the boundary-value problem of the preceding section for the velocity vector $\vec{q}$ can be replaced by one in the wave amplitude $\eta(x,y)$ consisting of the equation (3.4) and boundary conditions (3.5). It is this form of the problem which is treated in the subsequent sections. No attempt is made here to establish the complete equivalence of these two forms. A solution is obtained for the case in which the pressure distribution has circular symmetry with respect to the origin.

If the Fourier transforms of the functions $\eta(x,y)$, $\frac{\partial^2}{\partial x^2} \eta(x,y)$, and $p(x,y)$ with respect to y exist, and if

$$f(x,\nu) = \int_{-\infty}^{+\infty} \eta(x,y)\, e^{i\nu y}\, dy,$$

(3.6) $$f''(x,\nu) \equiv \frac{\partial^2}{\partial x^2} f(x,\nu) = \int_{-\infty}^{+\infty} \frac{\partial^2}{\partial x^2} \eta(x,y)\, e^{i\nu y}\, dy,$$

$$g(x,\nu) = \int_{-\infty}^{+\infty} p(x,y)\, e^{i\nu y}\, dy,$$

then the equation (3.4) takes the form

(3.7) $$f(x,\nu) + g(x,\nu) + \int_{-\infty}^{+\infty} \kappa(x-x',\nu)\, f''(x',\nu)\, dx' = 0,$$

where

(3.8) $$\kappa(x,\nu) = \frac{1}{\pi} K_0\,(|\nu x|),$$

and $K_0(x)$ is the modified Bessel function of order zero. Moreover, if the Fourier transform of $\eta(x,y)$ in (3.6) converges uniformly for $x > 0$, then the condition (3.5) implies that

(3.9) $$f(x,\nu) = 0 \quad \text{for} \quad x = +\infty.$$

## 4. Solution of the Integrodifferential Equation for $f(x,\nu)$

A formal solution of equation (3.7) can be obtained very simply with the aid of the two-sided Laplace transformation or the Fourier transform in the complex plane.* Let

$$F(s,\nu) = \int_{-\infty}^{+\infty} f(x,\nu)\, e^{-sx}\, dx,$$

(4.1) $$G(s,\nu) = \int_{-\infty}^{+\infty} g(x,\nu)\, e^{-sx}\, dx,$$

$$K(s,\nu) = \int_{-\infty}^{+\infty} \kappa(x,\nu)\, e^{-sx}\, dx,$$

*For the properties of the two-sided Laplace transformation see Doetsch[6]; for a discussion of the Fourier transform in the complex plane see Titchmarsh[7] and the introduction of Paley and Wiener[8].

where $s = \sigma + it$. Then,* for $0 < |\sigma| < |\nu|$,

$$(4.2) \qquad K(s,\nu) = \frac{1}{\pi} \int_{-\infty}^{+\infty} K_0(|\nu x|)\, e^{-sx}\, dx = (\nu^2 - s^2)^{-1/2},$$

where $(\nu^2-s^2)^{-1/2}$ denotes the single-valued branch of the corresponding algebraic function that is real and positive for $\sigma = 0$. For reasons that are made clear later, the right-hand member of (4.2) will be taken as the definition of $K(s,\nu)$ in the cut s-plane formed by deleting the points of the real axis for which $|\sigma| \geqq |\nu|$ (see Fig. 1). Also, in accord with the formal properties of the Fourier transform,

$$(4.3) \qquad \int_{-\infty}^{+\infty} f''(x,\nu)\, e^{-sx}\, dx = s^2 F(s,\nu),$$

$$(4.4) \qquad \int_{-\infty}^{+\infty} e^{-sx} \int_{-\infty}^{+\infty} \kappa(x-x',\nu)\, f''(x',\nu)\, dx'\, dx = s^2 F(s,\nu)\, K(s,\nu),$$

where the latter represents the transform of the Faltung of $\kappa(x,\nu)$ and $f''(x,\nu)$. Consequently, by (3.7),

$$(4.5) \qquad F(s,\nu) = -\frac{G(s,\nu)}{1 + s^2 K(s,\nu)} \quad .$$

Hence, making use of the complex inversion integral in the theory of the Laplace transform**, a solution of (3.7) can be written in the form:

$$(4.6) \qquad f(x,\nu) = -\frac{1}{2\pi i} \int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \frac{G(s,\nu)}{1 + s^2 K(s,\nu)}\, e^{xs}\, ds,$$

where the integration is taken along the vertical line $R(s) = \sigma_1$.

The validity of formula (4.6) as a solution of equation (3.7) can be established under very general conditions on the function $g(x,\nu)$.*** It is sufficient to suppose, for example,**** that $g(x,\nu)$ and its derivative with respect to x are sectionally continuous in every finite interval of x, and that these functions are $O(e^{-|\nu x|})$ for some $\nu \neq 0$ as $|x| \to \infty$.

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*Magnus and Oberhettinger[9], p. 116.

**See Doetsch[6], p. 210.

***See Titchmarsh[7], p. 305.

****Conditions of the pressure distribution $p(x,y)$ that are sufficient to establish these conditions on $g(x,\nu)$ are given in the next section.

Under these conditions $G(s,\nu)$ is analytic in the vertical strip $-|\nu| < \sigma < |\nu|$ and the integral in (4.6), with $K(s,\nu)$ defined as in (4.2), converges absolutely for $0 < |\sigma_1| < |\nu|$.

The function defined by (4.6) does not, however, represent the only solution of equation (3.7). In other words, the homogeneous equation obtained from (3.7) by setting $g(x,\nu) \equiv 0$ has nontrivial solutions. For it is observed that the characteristic function $[1 + s^2 K(s,\nu)] = 1 + s^2 (\nu^2-s^2)^{-1/2}$ has two simple zeros on the imaginary axis. Setting $2\tau = \sinh^{-1} 2\nu$, these zeros correspond to $(\nu^2-s^2)^{1/2} = \cosh^2\tau$ or $s = \pm it_0$, where

$$t_0 = \cosh\tau. \tag{4.7}$$

It is then readily verified by direct substitution that the function

$$\psi(x,\nu) = c_1 e^{it_0 x} + c_2 e^{-it_0 x}$$

is a solution of the homogeneous equation independent of the constants $c_1$ and $c_2$. More generally, it can be shown* that the family of functions $\psi(x,\nu)$ contains all the solutions of the homogeneous equation corresponding to (3.7) and (3.8) which are twice differentiable for all x and which are $O(e^{|\nu x|})$ in $(-\infty,\infty)$. Consequently the most general solution of the non-homogeneous equation (3.7), which is of the latter class of functions, is given by

$$f(x,\nu) = -\frac{1}{2\pi i}\int_{\sigma_1-i\infty}^{\sigma_1+i\infty} \frac{G(s,\nu)}{1 + s^2 K(s,\nu)} e^{xs}\,ds + c_1 e^{it_0 x} + c_2 e^{-it_0 x},$$

with $0 < |\sigma_1| < |\nu|$. Under the supposed conditions on the function $g(x,\nu)$, the solution can also be written in the form

$$f(x,\nu) = -g(x,\nu) + \frac{1}{2\pi i}\int_{\sigma_1-i\infty}^{\sigma_1+i\infty} G(s,\nu)\frac{s^2 K(s,\nu)}{1 + s^2 K(s,\nu)} e^{xs}\,ds + c_1 e^{it_0 x} + c_2 e^{-it_0 x}. \tag{4.8}$$

## 5. The Surface Wave for a Pressure Distribution with Circular Symmetry

Let the prescribed pressure distribution over the surface of the water have circular symmetry with respect to the origin. That is, let $p(x,y) = p(r)$ be a function of $r = (x^2 + y^2)^{1/2}$ alone. Then, under the assumption that the integral in (2.7) converges absolutely, the function

*The proof is suggested by Titchmarsh[7], pp. 306-307.

G(s,ν) in (4.1) can, for s = it (t real), be expressed in the form*

$$(5.1)\qquad G(s,\nu) = \iint_{-\infty}^{+\infty} p(x,y)\, e^{-i(tx-\nu y)}\, dxdy = \int_0^{\infty}\int_{-\pi}^{\pi} rp(r)\, e^{ir\zeta\cos\alpha}\, d\alpha dr$$
$$= 2\pi \int_0^{\infty} rp(r)\, J_0\,(\zeta r)\, dr,$$

where $J_0\,(\zeta r)$ is the Bessel function of the first kind of order zero and $\zeta = (\nu^2 + t^2)^{1/2} = (\nu^2 - s^2)^{1/2}$. Thus G(s,ν) is a Hankel transform of p(r). In the subsequent extension of the definition of G(s,ν) to all points of the s( = σ + it)-plane, the complex parameter ζ = λ + iμ will be defined over the cut s-plane by the single-valued branch of the function $(\nu^2 - s^2)^{1/2}$ which has already been introduced in (4.2).

Let the pressure distribution p(r) be further restricted by requiring that there exist a positive number a such that**

$$(5.2)\qquad p(r) \equiv 0 \quad \text{for } r > a$$

and such that p(r) has a bounded derivative for 0 < r < a. Then, since $J_0\,(\zeta r)$ is an entire function of s, G(s,ν) is also an entire function of s. Moreover, the growth of the function G(s,ν) for large |s| is then restricted in accordance with the following inequalities:

$$(5.3)\qquad |G(s,\nu)| \leqq M\, e^{a|s|} \quad \text{for all } s,$$

$$(5.4)\qquad |G(s,\nu)| \leqq \frac{M'}{|\zeta|}\, e^{a|\mu|} \leqq \frac{M'}{|s|}\, e^{a|\sigma|} \quad \text{whenever } |t| \geqq |\sigma| > 0,$$

where M and M' are constants independent of ν. These expressions are readily derived from (5.1) and (5.2) with the aid of the inequality***

$$|J_n(\alpha + i\beta)| \leqq e^{|\beta|} \quad \text{for all } \alpha \text{ and } \beta, \qquad (n = 0,1,2,\ldots),$$

*For an extensive discussion on the correspondence between the two-dimensional Fourier transform and the Hankel transform see, for example, Sneddon[10], p. 62.

**This corresponds to the practical situation in which the finite region over which p ≠ 0 is that which is covered by a ship. It is evident that condition (5.2) can be considerable weakened. For example, it is sufficient to require that $p(r) = O(e^{-\beta r})$ as $r \to \infty$ for every positive number β.

***This inequality readily follows from Bessel's integral representation for the function $J_n(z)$: Watson[11], pp. 19-21.

and the following properties of the relation between the complex variables $s = \sigma + it$ and $\zeta = \lambda + i\mu$ :

$$|\mu| \leqq |\sigma| \leqq |s| \quad \text{for all } s$$

and

$$|\zeta| \geqq |s| \quad \text{whenever} \quad |t| \geqq |\sigma|.$$

Finally, it should be observed that, as a consequence of (2.7),

$$(5.5) \qquad G(\pm \nu,\nu) = 2\pi \int_0^\infty rp(r)\, dr = 1.$$

Under the foregoing assumptions, the function

$$(5.6) \qquad L(s,\nu) \equiv \frac{s^2 K(s,\nu)}{1 + s^2 K(s,\nu)} G(s,\nu) = \frac{s^2}{s^2 + \zeta} G(s,\nu)$$

that appears in the integrand of the integral in (4.8) is analytic everywhere in the cut s-plane except for the simple poles at the points $s = \pm it_o$. The residues of $L(s,\nu)e^{xs}$ at these poles are respectively

$$(5.7) \qquad \pm i \frac{\cosh^3\tau}{\cosh 2\tau} P(\tau)e^{\pm it_o x},$$

where by (4.7)

$$(5.8) \qquad P(\tau) = G(t_o,\nu) = 2\pi \int_0^\infty rp(r)\, J_o\,(r \cosh^2\tau)\, dr.$$

Also, as a consequence of (5.5), (5.3), and (5.4), the function $L(s,\nu)$ can be shown to have the following properties:

$$(5.9) \qquad \lim_{s\to\pm|\nu|} L(s,\nu) = 1.$$

$$(5.10) \qquad |L(s,\nu)| = O(e^{a|s|}) \quad \text{as } |s| \to \infty.$$

$$(5.11) \qquad |L(s,\nu)| = O(\frac{1}{|s|} e^{a|\sigma|}) \quad \text{as } |s| \to \infty \quad \text{with } |t| \geqq |\sigma|.$$

Therefore the vertical path $R(s) = \sigma_1$ of integration for the integral in (4.8) can be replaced by either one of the hairpin paths $C_1$ or $C_2$ with the directions indicated in Fig. 1. Making use of the expressions in (5.7) for

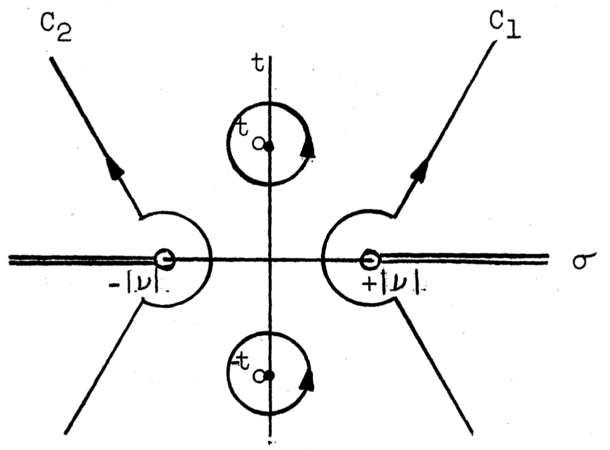


Fig. 1 The cut s-plane

the residues of $L(s,\nu)e^{xs}$ at its poles, it is possible to write

$$f_2(x,\nu) - f_1(x,\nu) = \frac{1}{2\pi i}\int_{C_2} L(s,\nu)\, e^{xs}ds - \frac{1}{2\pi i}\int_{C_1} L(s,\nu)\, e^{xs}ds$$

(5.12)

$$= f_o(x,\nu),$$

where $f_1(x,\nu)$ and $f_2(x,\nu)$ represent the integrals along $C_1$ and $C_2$, respectively, and

(5.13)
$$f_o(x,\nu) = 2\,\frac{\cosh^3\tau}{\cosh 2\tau}\,P(\tau)\sin xt_o.$$

Moreover, as a consequence of (5.9) and (5.10), the integrals $f_1(x,\nu)$ and $f_2(x,\nu)$ have the following asymptotic behavior:*

*See the treatment of the Abelian theorems for the complex inversion integral of Doetsch[6], chapter 15. The identity

$$L(s,\nu) \equiv -\sum_{\kappa=1}^{n} (-1)^{\kappa}\, G(s,\nu)\left(\frac{s^2}{\zeta}\right)^{\kappa} + (-1)^n\,[G(s,\nu) - L(s,\nu)]\left(\frac{s^2}{\zeta}\right)^{n+1}$$

is the starting point for the derivation of an asymptotic expansion of the functions $f_1(x,\nu)$ and $f_2(x,\nu)$. Following Doetsch, it is readily show that $\frac{1}{2\pi i}\int_{C\alpha}\left(\frac{s^2}{\zeta}\right)^n G(s,\nu)e^{|x|s}ds \sim \frac{|\nu x|^{\frac{3n}{2}}}{\Gamma\left(\frac{n}{2}\right)|x|^{n+1}}e^{-|\nu x|}$ as $|x|\to\infty$, where $\alpha = 1$ or 2.

$$(5.14)\qquad \begin{cases} f_1(x,\nu) = 0\left(\dfrac{e^{-|\nu x|}}{|x|}\right) & \text{as } x \to -\infty. \\ f_2(x,\nu) = 0\left(\dfrac{e^{-|\nu x|}}{|x|}\right) & \text{as } x \to +\infty. \end{cases}$$

Hence the solution of equation (3.7) that satisfies (3.9) is the unique* member of the family (4.8) obtained by setting $c_1 = c_2 = 0$ and

$$f(x,\nu) = -g(x,\nu) + f_2(x,\nu)$$

or, by virtue of (5.12),

$$f(x,\nu) = -g(x,\nu) + f_o(x,\nu) + f_1(x,\nu).$$

The expression for the amplitude $\eta(x,y)$ of the surface wave corresponding to the symmetric pressure distribution can be obtained from $f(x,\nu)$ by applying the inverse of the Fourier transform defined in (3.6). Under the foregoing conditions on the function $p(r)$, the amplitude can be expressed either as

$$(5.15)\qquad \eta(x,y) = -p(x,y) + \frac{1}{2\pi}\int_{-\infty}^{+\infty} f_2(x,\nu)\, e^{-iy\nu}\, d\nu,$$

or

$$(5.16)\qquad \eta(x,y) = -p(x,y) + \frac{1}{2\pi}\int_{-\infty}^{+\infty} f_o(x,\nu)e^{-iy\nu}d\nu + \frac{1}{2\pi}\int_{-\infty}^{+\infty} f_1(x,\nu)e^{-iy\nu}d\nu.$$

Since the asymptotic behavior of $f_1(x,\nu)$ and $f_2(x,\nu)$ given in (5.4) holds uniformly with respect to $\nu$ in every finite interval, and since**

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*It should be observed that the term $c_1 e^{it_o x} + c_2 e^{-it_o x}$ in (4.8), which represents the solution of the homogeneous equation associated with (3.7), determines the so-called free waves. These are the steady-state waves that can subsist on the surface of a stream of an inviscid fluid in the presence of no disturbing surface forces. To avoid the indeterminateness which prevails if this term is present, Lord Rayliegh employed the artifice of modifying the equations of motion by including the effects of small dissipative forces; see Lamb[4], p.399. However, as shown above, the natural boundary condition (3.5), of which (3.9) is a consequence, is sufficient to insure a unique solution of the surface-wave problem.

**Magnus and Oberhettinger[9], p.116.

$$\frac{1}{2}\int_{-\infty}^{+\infty}\frac{e^{-|\nu x|}}{|x|}e^{-iy\nu}\,d\nu = (x^2+y^2)^{-1} = r^{-2}, \qquad x \neq 0,$$

it is evident that the contribution to the amplitude $\eta(x,y)$ of the wave by the terms in (5.15) and (5.16) involving $f_1(x,\nu)$ and $f_2(x,\nu)$ behave asymptotically for large r as $O(r^{-2})$.* Since $p(r) \equiv 0$ for $r > a$, it is therefore possible to conclude that the disturbance produced on the water surface at large distances ahead of $(x > a > 0)$ a ship (corresponding to the applied surface forces) is negligible, and that at large distances behind $(x < -a)$ the ship the surface wave is predominately determined by the integral**

$$(5.17) \qquad \eta_o(x,y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f_o(x,\nu)e^{-iy\nu}d\nu = \int_{-\infty}^{+\infty} P(\tau)\,\kappa(x,y;\tau)\,d\tau,$$

where

$$(5.18) \qquad \kappa(x,y;\tau) = \frac{2}{\pi}\cosh^3\tau\,\sin(x\cosh\tau)\cos\left(\frac{1}{2}y\sinh 2\tau\right).$$

## 6. The Surface Wave Produced by a Moving Point of Disturbance

Because of its classical importance, the surface wave generated by a travelling pressure point will be interpreted in terms of the foregoing results. To this end, consider first the wave corresponding to the very special pressure distribution over a circle of radius $a > 0$:

$$(6.1) \qquad p = 1/\pi a^2 \quad \text{for } a < 0 \quad \text{and} \quad p \equiv 0 \quad \text{for } a > 0.$$

Since (2.7) is satisfied for all $a > 0$, the total applied force acting over the surface is independent of the radius a. According to (5.17), the amplitude $N_o(x,y;a)$ of the predominant part of the disturbance generated behind $(x < -a < 0)$ the circle is given by

*Pursuing further the suggested asymptotic expansion in the footnote on page 10, it is possible to show that the first term in the expansion representing the contribution of the terms in $f_1(x,\nu)$ and $f_2(x,\nu)$ to the surface elevation is $O(r^{-3})$ as $r \to \infty$. This same result is obtained by Peters[2], p. 142, for the case in which $G(s,\nu) \equiv 1$.

**The resolution of the surface wave into expressions representing respectively the disturbance that is produced in front of and behind a moving source appears to have been given for the first time by Peters[2]. Peters expresses the disturbance behind the ship by an integral that can be obtained by setting $P(\tau) \equiv 1$ in (5.17) above: see Peters[2], p. 142. However this integral diverges for all $x \neq 0$ and $y \neq 0$.

(6.2)
$$N_0(x,y;a) = \int_0^\infty P(a;\tau)\,\kappa(x,y;\tau)\,d\tau,$$

where by (5.8)

(6.3)
$$P(a;\tau) = 2\,\frac{J_1(a \cosh^2\tau)}{a \cosh^2\tau}.$$

Then, following the well-known procedure, the amplitude $\hat{N}_0(x,y)$ of the surface wave generated by a pressure point is defined by the limit

(6.4)
$$\hat{N}_0(x,y) = \lim_{a\to 0} N_0(x,y;a) = \lim_{a\to 0}\int_0^\infty P(a;\tau)\,\kappa(x,y;\tau)\,d\tau.$$

It can be shown that the limit in (6.4) exists for $x < 0$ and $y \neq 0$.* However, this limit cannot be obtained by merely interchanging the order of the integration and limit processes. For, it is evident that

(6.5)
$$\lim_{a\to 0} P(a;\tau) = 1, \quad 0 \leqq \tau < \infty.$$

and that the integral $\int_0^\infty \kappa(x,y;\tau)\,d\tau$ does not exist in the ordinary sense.

On the other hand, the limit (6.4) may be regarded as an evaluation of this divergent integral by a method of summation** with respect to the kernel $P(a;\tau)$.

It is not difficult to show that the method of summability with respect to $P(a;\tau)$ is regular in the sense that it sums every convergent integral and assigns to it the same value as defined in the ordinary sense. However, since the integral $\int_0^\infty \kappa(x,y;\tau)\,d\tau$ does not converge, the question regarding the consistency of the result of summing this integral by any regular process immediately arises. In particular, is the definition of the wave amplitude generated by a pressure point strongly dependent on the form

*It has been shown by the junior author that the limit does not, however, exist when $x < 0$ and $y = 0$. This implies that the amplitude of the surface wave is unbounded along the path of the pressure point. This result agrees with that indicated by Kelvin[1], but is contrary to the assertion by Peters[2] that the amplitude of the wave generated by a pressure point is finite along its path.

**For a detailed discussion of the summation of divergent integrals, see Hobson[12], pp. 384-388.

of the distribution chosen in (6.1)? Does there exist a second distribution also depending on a characteristic parameter a and describing, in the limit as $a \to 0$, a pressure point for which the corresponding limiting wave amplitude $\hat{N}_o(x,y)$ in (6.4) is distinctly different from the first? In still another form, the question may be phrased thusly: Total forces being equal, does the amplitude of the wave at large distances behind the source of the disturbance depend on the form of the pressure distribution in this region? A satisfactory answer to these questions can be given for a certain class of pressure distributions, which is introduced below.

In addition to (5.5), let the pressure distribution $p(a;r)$ also satisfy the following conditions:*

(6.6)

(i) $p(a;r)$ is continuous, and $p'(a;r)$, $p''(a;r)$ are sectionally continuous for $r \geqq 0$ and $a > 0$ (primes denote derivatives with respect to r).

(ii) $p(a;r)$, $rp'(a;r)$ and $p''(a;r)$ are $O(r^{-\alpha})$, with $\alpha > 2$, as $r \to \infty$.

(iii) $\lim_{a\to 0} p''(a;r) = 0$ uniformly with respect to r in every finite interval with $r \neq 0$.

After substituting (5.8) into (5.17), integrating by parts, and interchanging the orders of integration, the function $\eta_o(x,y;a)$ defined by the integral in (5.17) can be expressed in the form

$$\eta_o(x,y;a) = \int_0^\infty \phi(a;\rho)\, N_o(x,y;\rho)\, d\rho, \tag{6.7}$$

where $N_o(x,y;\rho)$ is defined by (6.2) and

$$\phi(a;r) = -\pi r^2 p'(a;r) = 2\pi r p(a;r) - \pi[r^2 p(a;r)]'.$$

As a consequence of (5.5) and (6.6), the function $\phi(a;r)$ has the following properties:

(i) $\int_0^\infty \phi(a;r)\, dr = 2\pi \int_0^\infty rp(a;r)\, dr = 1 \quad$ for $a > 0$.

(ii) $\int_0^\infty |\phi(a;r)|\, dr < \infty \quad$ for $a > 0$.

*No attempt is made here to describe the largest class of functions for which the definition in (6.4) is consistent.

(iii) $\lim_{a \to 0} \phi(a;r) = 0$ uniformly with respect to r in every finite interval with $r \neq 0$.

It therefore follows from (6.7) and these properties of $\phi(a;r)$ that*

(6.8) $$\lim_{a \to 0} \eta_0(x,y;a) = \lim_{\rho \to 0} N_0(x,y;\rho) = \hat{N}_0(x,y).$$

Hence the definition in (6.4) for the amplitude of the surface wave generated by a pressure point is consistent with respect to the class of pressure distributions described by (6.6).

As an example, take $p(r;a) = 2\pi a(a^2+r^2)^{-3/2}$. Then,** from (5.1), $G(s,\nu) = e^{-a\zeta}$, and the kernel $P(a;\tau)$ in (6.4) becomes

$$P(a;\tau) = \exp[-a \cosh^2\tau].$$

Therefore the amplitude of the wave generated by a pressure point can also be defined by the limit

(6.9) $$\hat{N}_0(x,y) = \lim_{a \to 0} \frac{2}{\pi} \int_0^\infty e^{-a \cosh^2\tau} \cosh^3\tau \sin(x \cosh \tau) \cos(\frac{y}{2} \sinh 2\tau)\, d\tau.$$

This is not unlike the Abel-Poisson summation method for infinite integrals.*** The last expression serves to give a partial justification for the artifice employed by Lamb, Havelock and others **** of introducing an exponential factor in the integrals representing the amplitude of the wave generated by a pressure point so as to insure the necessary convergence of the integrals.*****

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*The limit (6.8) follows from the theory of singular integrals; see for example Titchmarsh[7], p.28, or Hobson[12], pp. 446-456.

**Sneddon[10], p. 528.

***See Hardy[13], p. 11.

****See Lamb[4], p. 413; Havelock[14].

*****See the paper of Peters[2] for a discussion of the history of the problem. Peters also gives an excellent treatment of the asymptotic expansion of the integrals of the type in (6.9) by Debye's method of steepest descent.

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