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# COMPLEXITY OF WINNING STRATEGIES

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**Abstract.** Rabin has given an example of a game with recursive rules but no recursive winning strategy. We show that such a game always has a hyperarithmetical winning strategy, but arbitrarily high levels of the hyperarithmetical hierarchy may be needed. We also exhibit a recursively enumerable game which has no hyperarithmetical winning strategy.

To each set $G$ of infinite sequences of natural numbers, we associate the infinite two-person game played as follows. The two players alternately and perpetually choose natural numbers. At each move, the players know all the previous moves (perfect information). If the sequence of their choices is in $G$, then the first player wins; otherwise, the second player wins. A winning strategy for the first player is a function $\sigma$ mapping finite sequences of natural numbers to natural numbers, and satisfying

$$(1) \qquad \forall f[\forall n(f(2n) = \sigma(\bar{f}(2n))) \to f \in G] ,$$

where $\bar{f}(k)$ is the sequence $\langle f(0), ..., f(k-1)\rangle$. This means that the first player is assured of winning the game if, at each of his moves, he chooses the number $\sigma(s)$, where $s$ is the sequence of previous moves. Similarly, a winning strategy for the second player is a function $\sigma$ satisfying

$$(2) \qquad \forall f[\forall n(f(2n+1) = \sigma(\bar{f}(2n+1))) \to f \notin G] .$$

Clearly, at most one of the players has a winning strategy, but it is possible that neither does. The problem of finding conditions on $G$ which guarantee the existence of a winning strategy has been studied by Gale and Stewart [2], Davis [1] and Martin [3,4].

We shall be concerned mainly with games for which $G$ is recursive. It is known (see [6, p. 352] and [2]) that then one of the players has a winning strategy. The question arises of how complex (in the sense of Turing degrees or hierarchies) such a strategy must be. (Here, and in the future, we assume that finite sequences of natural numbers have been identified with natural numbers in one of the standard recursive ways. Thus, strategies are functions from $\omega$ to $\omega$.)

If $G$ is recursive, then the formulas (1) and (2) are $\Pi^1_1$. As already remarked, one of them is satisfied by some $\sigma$. By the Kondo-Addison theorem [6, p. 430], one of them is satisfied by a $\Delta^1_2$ function $\sigma$. Thus, we have a bound for the complexity required of winning strategies for recursive games, but we have not used the full strength of the recursiveness of $G$. Indeed, the same argument shows that, if a player has a winning strategy for a hyperarithmetical game, then he has a $\Delta^1_2$ winning strategy. It is thus reasonable to expect (and we shall obtain) a better bound when $G$ is recursive.

Let $G$ be a recursive set of functions. Then there are recursive predicates, $P$ and $Q$, such that

$$(3) \qquad f \in G \leftrightarrow \exists x P(\bar{f}(x))$$

and

$$(4) \qquad f \notin G \leftrightarrow \exists x Q(\bar{f}(x)) .$$

Consider a particular play of the associated game. If the first player won, then, by (3), there is a number $x$ such that $P(s)$ holds, where $s$ is the sequence of the first $x$ moves. Anything the players did after the sequence $s$ was played had no effect on the outcome. Of course, similar remarks apply if the second player won. Thus, no matter how they play, the players will eventually produce a finite sequence $s$ such that $P(s) \vee Q(s)$; at this point, they may as well stop playing and declare the first (resp. second) player the winner if $P(s)$ (resp. $Q(s)$), for no further moves can alter this outcome. When viewed in the light of the preceding discussion, recursive games are seen to be essentially the same as the actual games defined by Rabin [5]. Rabin showed that such a game need not admit a recursive winning strategy.

**Theorem 1.** *If G is recursive, then one of the players has a hyperarithmetical winning strategy.*

**Proof.** Consider the function $v$ whose value at a finite sequence $s$ is 1 (resp. 2) if, after $s$ has been played, the first (resp. second) player has a winning strategy. This $v$ clearly has the following properties:

(5) $$v(s) = 1 \text{ or } 2\,,$$

(6) $$(\exists x \leqslant \text{length}\,(s))\, P(\bar{s}(x)) \to v(s) = 1\,,$$

(7) $$(\exists x \leqslant \text{length}\,(s))\, Q(\bar{s}(x)) \to v(s) = 2\,,$$

(8) $$\text{length}\,(s) \text{ is even} \to [v(s) = 1 \leftrightarrow (\exists n) v(s*n) = 1]\,,$$

(9) $$\text{length}\,(s) \text{ is odd} \to [v(s) = 2 \leftrightarrow (\exists n) v(s*n) = 2]\,;$$

here $\bar{s}(x)$ is the sequence of the first $x$ components of $s$, $s*n$ is the sequence obtained by adjoining $n$ to $s$ as a last term, and $P$ and $Q$ are recursive relations such that (3) and (4) hold. Statements (6) and (7) say that a player who "has already won" has a winning strategy. Statements (8) and (9) say that the player whose move it is has a winning strategy iff he can move so as to have a winning strategy afterward.

Formulas (5) through (9) completely characterize $v$. For if $v'$ were another such function and $v'(s) \neq v(s)$ for some $s$, then, by (8) and (9), there is an $n$ such that $v'(s*n) \neq v(s*n)$. Proceeding inductively, we obtain a function $f$ and a number $k$ (=length $(s)$) such that

$$(\forall x \geqslant k)\, v'(\bar{f}(x)) \neq v(\bar{f}(x))\,.$$

By (6) and (7), it follows that

$$\forall y\, [\neg P(\bar{f}(y)) \wedge \neg Q(\bar{f}(y))]\,,$$

which is impossible because of (3) and (4).

As the unique function satisfying the arithmetical conditions (5) through (9), $v$ is hyperarithmetical. Therefore, so is the function $\sigma$, recursive in $v$, defined by

(10) $\sigma(s)$ = the least $n$ such that $v(s) = v(s*n)$ ;

such an $n$ always exists by (8) and (9).

To complete the proof, we show that $\sigma$ is a winning strategy for one of the players. Suppose the empty sequence is mapped to 1 by $v$. Let $f$ be any function satisfying the hypothesis of (1):

(11) $(\forall n) f(2n) = \sigma(\bar{f}(2n))$ .

I claim that $(\forall k)\, v(\bar{f}(k)) = 1$. This is proved by induction on $k$. It is true by assumption if $k = 0$. If it is true for an even $k$, then it is true for $k$+1 by (10) and (11). If it is true for an odd $k$, then it is true for $k$+1 by (9). In view of (7),

$$(\forall k) \neg Q(\bar{f}(k)) ,$$

so, by (4), $f \in G$. Thus, $\sigma$ is a winning strategy for the first player. Similarly, if $v$ of the empty sequence is 2, then $\sigma$ is a winning strategy for the second player.

Having reduced the bound on complexity of winning strategies from $\Delta^1_2$ to $\Delta^1_1$, we may ask whether further reductions are possible. The following theorem shows that they are not. It also improves the result of Rabin [5].

**Theorem 2.** *Let A be any hyperarithmetical set. There is a recursive G such that A is recursive in every winning strategy for the associated game.*

**Proof.** As $A$ is $\Pi^1_1$, there is a recursive relation $P$ such that, for all $n$,

(12) $n \in A \leftrightarrow \{x \mid P(n, x)\}$ is well-ordered by the Kleene-Brouwer ordering $<^*$ .

(See [6, pp. 396–397].) Similarly, as $A$ is $\Sigma^1_1$, there is a recursive $Q$ such that

(13) $\quad n \notin A \leftrightarrow \{x | Q(n, x)\}$ is well-ordered by $<^*$ .

Consider the following game. The first player begins by proposing a number $n$. The second player replies by guessing whether or not $n \in A$. If he guesses "Yes", then he claims that $\{x | Q(n, x)\}$ has an infinite descending (with respect to $<^*$) sequence; on his remaining moves, he is to list, in order, the elements of such a sequence. Meanwhile, the first player is to use his remaining moves to list an infinite descending sequence in $\{x | P(n, x)\}$ . In view of (12) and (13), it is not possible for both players to succeed. The first to fail in his task—either by listing an element outside $\{x | Q(n, x)\}$ or $\{x | P(n, x)\}$, or by listing two elements in ascending Kleene-Brouwer order—is the loser. If the second player guessed that $n \notin A$, the remainder of the game is the same as above except that $P$ and $Q$ are interchanged. We leave to the reader the easy task of checking that this is the game associated to a certain recursive set $G$. (It is, of course, necessary to code the second player's yes-or-no answer as a number.)

The second player has a winning strategy for this game; namely: answer the question "Is $n \in A$?" correctly; thereafter, confronted with the task of producing a descending sequence in a certain set which is, in fact, not well-ordered, give the first such sequence in some fixed well-ordering of ${}^{\omega}\omega$. (In fact, the set in question has constructible descending sequences, so the axiom of choice can be avoided here.) Notice that *any* winning strategy for the second player requires him to answer "Is $n \in A$ ?" correctly. For, if he answers incorrectly, he cannot produce the descending sequence required of him, while his opponent can. Thus, $A$ is recursive in every winning strategy $\sigma$ for this game; $n \in A$ iff $\sigma(\langle n \rangle) =$ "Yes."

**Corollary.** *There is a recursive game with no arithmetical winning strategy.*

Our final theorem shows that the assumption in Theorem 1 that $G$ is recursive cannot be weakened.

**Theorem 3.** *There is a recursively enumerable $G$ whose associated game has no hyperarithmetical winning strategy.*

**Proof.** Let $P$ be a recursive set which is not well-ordered by the Kleene-Brouwer ordering $<^*$, but which has no infinite descending hyperarithmetical sequences. (See [6, pp. 396, 419].) Define $G$ by

$$f \in G \leftrightarrow \exists n(\neg P(f(2n+1)) \vee f(2n+1) \leqslant^* f(2n+3)) .$$

Clearly, $G$ is recursively enumerable. The associated game is won by the second player iff he writes (in order) an infinite descending sequence in $P$. As $P$ is not well-ordered, such a sequence exists, so the second player has a winning strategy.

Let $\sigma$ be any winning strategy for the second player. By (2), the function $g$ defined by

$$g(n) = \begin{cases} \sigma(\bar{g}(n)) & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

is not in $G$. Hence, the function $f$ such that

$$f(n) = g(2n+1)$$

is a descending sequence in $P$. By choice of $P$, $f$ is not hyperarithmetical. Since $g$ and $f$ are clearly recursive in $\sigma$, we conclude that $\sigma$ is not hyperarithmetical.

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