EQUIPMENT REPLACEMENT UNDER TECHNOLOGICAL CHANGE

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ABSTRACT

For infinite horizon replacement economy problems it is common practice to truncate the problem at some finite horizon. We develop bounds on the error due to such a truncation. These bounds differ from previous results in that they include both revenues and costs. Bounds are illustrated through a numerical example from a real case in vehicle replacement.

1. Introduction

The solution to a deterministic replacement economy problem is a sequence of assets that promises to provide the most economical service over an indefinite time horizon. The most important decision is the choice between keeping the existing asset (the defender) and replacing it by one of the current challengers. The anticipated future replacements are important primarily as they affect the current choice.

Sethi and Chand [1979]; Chand and Sethi [1982]; and Bean, Lohmann, and Smith [1985], present forward dynamic programming algorithms that can identify the optimal current decision for the infinite horizon replacement economy problem whenever a solution horizon exists. A *solution horizon* is a time, H_*, such that the optimal current decision remains the same for any $H \geq H_*$ and agrees with the initial solution of the infinite horizon problem. Associated with H_* is a *forecast horizon*, H^*, such that the forecasts

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of cash flows for challengers beyond H^* are unnecessary to determine the optimal current replacement decision. Equivalently, only information up to H^* is relevant to the current decision. Under certain conditions, an upper bound for H^* is $H_* + \bar{n}$, where $\bar{n}$ is the maximum physical life over all future challengers.

The algorithms in all papers cited above determine H_* and H^* a posteriori. That is, the procedure will identify H_* and H^* only if first provided with cash flow forecasts for all challengers covering at least H^* periods. Since H^* is not known a priori, the number of periods of forecasted data required may be arbitrarily large. Although Bean, Lohmann, and Smith show in a set of simulation experiments that H_* regularly occurs just beyond the economic service life of the optimal current asset choice, there are cases where H^* can be significantly larger. Given the difficulty of forecasting the economic effects of distant technological change, the optimality of the current choice becomes questionable. It is also possible that H_* may not exist, in which case the algorithm will never terminate.

In this paper we provide techniques for calculating a horizon time, H_ϵ, that is sufficiently large to guarantee the decision maker a near optimal current decision. More precisely, the optimal current decision for a horizon time of H_ϵ periods is guaranteed to be the first in a sequence of replacements that attains an infinite horizon cost within ϵ of the infinite horizon optimal value. We call H_ϵ an *error bounded horizon time*. An important characteristic of H_ϵ is that its value is largely independent of cash flows beyond $H_\epsilon + \bar{n}$. Thus, forecasted cash flows for challengers are only required for this time frame, which is known a priori.

Previous papers considering error bounded horizons include Bean and Smith [1985], Lee and Denardo [1986], Denardo and Lee [1986], Bean, Birge and Smith [1987], Bean, Smith and Yano [1987] and Chen and Lee [1991]. These papers consider models allowing cash flows in only one direction, that is, all costs or all revenues. Alden and Smith [1992] allow arbitrary cash flows, but consider a rolling horizon procedure. In this paper we consider a model with both costs and revenues. Traditional bounding approaches result in loose bounds for this case. A new approach based on equivalent annual values is developed here.

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In Section 2 we formulate the model and state the problem addressed. In Section 3 we describe a method to determine the maximum error, $\epsilon(T)$, associated with solving a finite horizon problem of T periods. This relationship can be inverted numerically to obtain H_ϵ. We also review the forward algorithm described in Bean, Lohmann, and Smith for solving the corresponding H_ϵ problem to find the first decision. Section 4 shows the error bound for real cases in vehicle replacement. Section 5 contains a summary and conclusions.

2. Problem Statement

Assets $j = 1, 2, \ldots, m$ correspond to the challengers. Let n be the life of the asset where $1 \le n \le \bar{n}^j$ and $\bar{n}^j$ represents the physical life of asset j. Asset $j = 0$ corresponds to the defender which, by assumption, cannot be repeated. We can enforce this by assuring that for $t > 1$ cash flows for asset 0 will prevent its selection. For notational convenience, assume that no asset is in place, but that the defender (if there is one) may be acquired at the beginning of period 1 at no cost. Let $\bar{n}^0$ represent its remaining physical life. Define a *replacement scenario* as the triple $\{j, t, n\}$ representing installation of asset j at the beginning of period t, keeping it in service for n periods. For an asset, j, acquired at the beginning of period t, let the net cash flow (receipts minus disbursements) for each of n periods of service be given by the column vector $b_n^j(t)$, including salvage. Models of technological change that relate the cash flows of future challengers to those of current challengers are discussed at length in Bean, Lohmann, and Smith; and Oakford, Lohmann, and Salazar [1984]. We assume that $b_n^j(t)$ is given for all t. Let $d = \frac{1}{1+r} > 0$ be the discount factor.

The problem is to find a sequence of asset choices and associated service lives to maximize the net present value of the service provided by those assets. The optimal service lives are referred to as their *economic lives*. To express the problem mathematically we present additional notation. Let $\alpha_n(t)$ be the row vector whose k^{th} element is given by d^{k+t-2} where $1 \le k \le n$. Then the present value of asset j acquired at the beginning of period t and kept in service for n periods is given by $\alpha_n(t)b_n^j(t)$. It follows that the net present value, $\bar{V}(x)$, associated with the replacement sequence $x = \{(j_1, n_1), (j_2, n_2), \ldots\}$

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is given by

$$\bar{V}(x) = \sum_{i=1}^{\infty} \alpha_{n_i}(t_i) b_{n_i}^{j_i}(t_i)$$

where $t_i = \sum_{l=1}^{i} n_{l-1}$ and $n_0 \equiv 1$.

The infinite horizon economic replacement problem seeks a replacement sequence, x^*, such that $V^* = \bar{V}(x^*) = \max_{x \in X} \bar{V}(x)$, where X is the set of replacement sequences, $\{(j_1, n_1), (j_2, n_2), \ldots\}$, such that $0 \leq j_i \leq m$ and $1 \leq n_i \leq \bar{n}^{j_i}$. It can be shown that x^* and V^* exist for sufficiently small discount factors if cumulative cash flows are uniformly eventually bounded by a geometric growth in time (Bean and Smith [1984]).

3. Error Bounded Horizon Times

To solve the infinite horizon economic replacement problem, note that the time epochs immediately following the retirement of an asset, and before its replacement, constitute regeneration points. That is, no information about the past assets is relevant for future decisions except the current time. For finite horizon versions of the economic replacement problem, this observation leads to an efficient finite state dynamic programming formulation of the problem. Specifically, let $V(t, t')$ be the maximum present worth value of assets beginning period t with no asset and ending period t' with no asset. Then we obtain the following functional equation for $V(t, t')$ which can be efficiently solved by forward reaching (Denardo [1982]). For $t' = 1, 2, 3, \ldots$; $1 \leq t \leq t'$,

$$V(t, t') = \max\{\alpha_n(t) b_n^j(t) + V(t + n, t') | 0 \leq j \leq m; \ 1 \leq n \leq \min(t' - t, \bar{n}^j)\}.$$

If $t > t'$ then $V(t, t') \equiv 0$. Note that V^* as defined earlier is equal to $V(1, \infty) = \lim_{t' \to \infty} V(1, t')$. We define the *optimal value for a T horizon problem* as the best value of any sequence of assets that has its last installation strictly before the beginning of period T. That is, $V^*(T)$ is the maximum reward *through* the beginning of period T;

$$V^*(T) = \max\{V(1, t - 1) + \alpha_n(t) b_n^j(t) : t < T, t + n \geq T; 0 \leq j \leq m; 1 \leq n \leq \bar{n}^j\}.$$

Note that the assets corresponding to $V^*(T)$ may optimally provide service for periods beyond T. This differs from traditional definitions of finite horizon problems which truncate

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cash flows at time T. Inclusion of these additional cash flows is necessary for the following bounding theory.

Bean, Lohmann, and Smith provide a procedure that determines a solution horizon, H_*, and forecast horizon, H^*, when they exist. Therefore, under very general conditions, the optimal first asset choice and economic life for a H_*-horizon problem are also optimal for $T \geq H_*$.

As mentioned earlier, H_* and H^* are only determined after the future asset cash flows, $b_n^j(t)$, have been forecasted. We will now establish an upper bound on error associated with solving the finite horizon problem to T as a surrogate for solving the infinite horizon problem.

We begin with more definitions. Let

$n^*(T)$ = remaining economic life, at the beginning of period T, of an infinite horizon optimal asset in service during period $T - 1$.

$n'(T)$ = remaining economic life, at the beginning of period T, of a finite (T) horizon optimal asset in service during period $T - 1$. If there are multiple optima, choose that with smallest remaining economic life.

$\bar{n} = \max_{1 \leq j \leq m} \bar{n}^j < \infty$, the maximum physical life over all challengers.

The fact that we allow both positive and negative cash flows complicates the derivation of bounds. Traditional methods would calculate upper bounds from pure revenues and lower bounds from pure costs. Such approaches result in loose bounds which give little information. Below we transform the problem to one that has the same optimal solutions. The optimal values of the transformed problem differ form the original by a constant. The transformed problem will have all nonpositive cash flows beyond the horizon time, T, and allow tight bounds.

If we add an appropriate constant to the purchase cost of each asset, the net present value of each replacement scenario can be made negative. However, it is possible that we have altered the optimal solutions to the problem. There is an important subtlety here. It is not known a priori how many replacements will occur over any fixed time horizon. The

comparison, over that horizon, of a scenario with a single replacement to a scenario with two replacements is changed by the addition of a single constant to the first scenario and two constants to the second.

To make the appropriate cash flows nonpositive without altering the optimal solutions we add a constant to each cashflow rather than just the purchase costs. Lemma 2 will show how large the constant must be. Theorem 1 will show that the tightness of error bounds that we can derive is directly related to the size of this constant. Hence, we want the smallest constant that achieves the desired nonpositivity.

To accomplish these dual goals we employ the concept of an equivalent annual value. This idea is developed in Shapiro and Wagner [1967] and has been used for turnpike results in capacity expansion (Smith [1979]) and equipment replacement (Ganesh [1987]).

Define $\gamma_n^j(t)$ as the equivalent annual value associated with the scenario of installing asset j at time t and keeping it in place for n periods. It is the unique value such that if the cash flow were exactly $\gamma_n^j(t)$ at the beginning of each period $\tau \in \{t, t+1, \ldots, t+n-1\}$, the net present value at the beginning of period 1 would be exactly $\alpha_n(t) b_n^j(t)$.

Definition: $\gamma_n^j(t) = d^{-(t-1)} \frac{1-d}{1-d^n} \alpha_n(t) b_n^j(t)$.

Definition: For $\tau \geq T$, let $\gamma^*(\tau) = \max\{\gamma_n^j(t) : T \leq t \leq \tau; 0 \leq j \leq m; 1 \leq n \leq \bar{n}^j\}$. That is, $\gamma^*(\tau)$ is the maximum equivalent annual value for any scenario with installation between T and τ, inclusive.

Construct the transformed problem as follows. For every replacement scenario, $\{j, t, n\}$, with $t + n > T$, subtract cash flow $\gamma^*(\tau)$ at the beginning of each period τ, $T \leq \tau < t+n$. Denote the cash flows of the transformed problem as $\tilde{b}_n^j(t)$. Denote the optimal values of the transformed problem with a tilde, e.g., $\tilde{V}(1, t)$ and $\tilde{V}^*(T)$.

Lemma 1: The infinite horizon problem with cash flows $\tilde{b}_n^j(t)$ has the same optimal strategies as the original problem with cash flows $b_n^j(t)$.

Proof: Any infinite horizon strategy has precisely one asset in place during each period $\tau \geq T$. Hence, they all incur exactly the same discounted cash flow adjustment,

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namely,

$$\sum_{\tau=T}^{\infty} -\gamma^*(\tau)d^{\tau-1}.$$

Since this constant is independent of any specific strategy information, the relative value of any two strategies is unchanged. Hence, the optimal strategies are equivalent in the two problems. ∎

This transformation assures nonpositivity of the net present values of scenarios beginning at $t \geq T$. Lemma 2 proves this assertion.

Lemma 2: For $t \geq T$, $\alpha_n(t)\tilde{b}_n^j(t) \leq 0$.

Proof: For $t \geq T$,

$$\begin{aligned}
\alpha_n(t)\tilde{b}_n^j(t) &= \alpha_n(t)b_n^j(t) - \sum_{\tau=t}^{t+n-1} d^{\tau-1}\gamma^*(\tau) \\
&= \sum_{\tau=t}^{t+n-1} d^{\tau-1}\gamma_n^j(t) - \sum_{\tau=t}^{t+n-1} d^{\tau-1}\gamma^*(\tau) \\
&= \sum_{\tau=t}^{t+n-1} d^{\tau-1}[\gamma_n^j(t) - \gamma^*(\tau)] \leq 0.
\end{aligned}$$

The last inequality follows from the fact that $\gamma^*(\tau) \geq \gamma_n^j(t)$ for $\tau \geq t \geq T$ by construction. ∎

The key to the bound derivation is the relationship between a finite horizon optimal solution through time T and the leading segment of an infinite horizon solution through the asset in place at the end of period $T - 1$. By definition of $n^*(T)$, the latter is disposed of during period $T + n^*(T) - 1$. Let $T + n'(T) = t + n$ for the last asset in the finite horizon optimal solution through T. We would like to show that $T + n'(T) \leq T + n^*(T)$. That is, the asset in service at T for a finite horizon optimal solution through T terminates no later than the corresponding asset in an infinite horizon optimal solution. Figure 1 shows the relationships between $T, T + n'(T)$ and $T + n^*(T)$.

To prove this lemma we require the main assumption of the paper, that beyond T the transformed problem allows no speculative motive. That is, there is no economic

FIGURE 1: Diagram of $T, T + n'(T)$ and $T + n^*(T)$
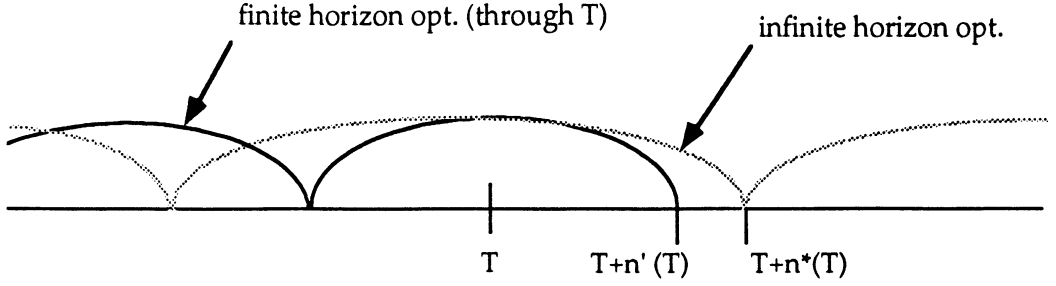
motivation to keep an asset in hopes that the salvage value will increase due to degrading technology.

Assumption 1: For any $T \le t \le t' < \tau$, $\tilde{V}(t, \tau) \le \tilde{V}(t', \tau)$.

Assumption 1 holds in any problem with stationary or improving technology. For example, in the stationary case, after transforming the problem, by Lemma 2 all present values beyond T are nonpositive. The same assets that were optimal for $\tilde{V}(t, \tau)$ are feasible for $\tilde{V}(t', \tau)$, but discounted additionally. Discounting nonpositive values makes them greater. If technology is improving, Assumption 1 is valid by a similar argument, though at time t' better alternatives may be available. Assumption 1 is only restrictive when technology is degrading, and then, only when it degrades at a rate faster than the discount rate.

Lemma 3: Under Assumption 1, $n'(T) \le n^*(T)$.

Proof: By optimality in the finite horizon problem

$$\tilde{V}(1, T + n'(T) - 1) \ge \tilde{V}(1, T + n^*(T) - 1). \tag{1}$$

By optimality in the infinite horizon problem

$$\tilde{V}(1, T + n'(T) - 1) + \tilde{V}(T + n'(T), \infty) \le \tilde{V}(1, T + n^*(T) - 1) + \tilde{V}(T + n^*(T), \infty). \tag{2}$$

Subtracting (1) from (2) gives

$$\tilde{V}(T + n'(T), \infty) \le \tilde{V}(T + n^*(T), \infty). \tag{3}$$

If (3) holds as a strict equality and $n'(T) > n^*(T)$, Assumption 1 is violated. If (3) holds as an equality then (2) reduces to the reverse inequality of (1). Then (1) must

hold at equality. In this case the optimal solution to $T + n^*(T) - 1$ is an alternative optimal solution to the finite horizon problem through T. Since $n'(T)$ is defined as the smallest of such times, $n'(T) \leq n^*(T)$. Hence, the result is proved. ∎

To construct a bound on error from solving the finite horizon problem to T in lieu of the infinite horizon problem, consider the following feasible strategy for the transformed problem. We will refer to it as the *bounding strategy*.

1. Solve optimally for the finite horizon transformed problem *through* the beginning of period T. Install the optimal solution for periods $\{1, 2, \ldots, T + n'(T) - 1\}$

2. Install any asset, j', following the solution in 1. (at the beginning of period $T + n'(T)$) that can serve to the end of period $T + n^*(T) - 1$. Retire it at the end of period $T + n^*(T) - 1$. This is possible by Lemma 3.

3. Append an optimal strategy from the beginning of period $T + n^*(T)$ forward.

The value of the bounding strategy is

$$\tilde{V}(1, T + n'(T) - 1) + \alpha_{n^*(T)-n'(T)}(T + n'(T))\tilde{b}^{j'}_{n^*(T)-n'(T)}(T + n'(T)) + \tilde{V}(T + n^*(T), \infty).$$

Definition: $\epsilon(T) = \max\{-\alpha_n(t)\tilde{b}^j_n(t) : T \leq t; t + n \leq T + \bar{n}; 0 \leq j \leq m; 1 \leq n \leq \bar{n}^j\}$.

We now return to the original problem and use the machinery above to bound error. By the principle of optimality, the best possible extension of the finite horizon optimal solution through T is $V(1, T + n'(T) - 1) + V(T + n'(T), \infty)$. This value is not greater than the infinite horizon optimal value, $V(1, \infty)$. Theorem 1, below, bounds the loss from forcing a replacement at $T + n'(T)$.

Theorem 1: $V(1, \infty) - [V(1, T + n'(T) - 1) + V(T + n'(T), \infty)] \leq \epsilon(T)$.

Proof: We begin by proving the result for the transformed problem. Since the bounding strategy is feasible for the infinite horizon problem over all strategies with a replacement at the beginning of period $T + n'(T)$,

$$\tilde{V}(1, T + n'(T) - 1) + \alpha_{n^*(T)-n'(T)}(T + n'(T))\tilde{b}^{j'}_{n^*(T)-n'(T)}(T + n'(T)) + \tilde{V}(T + n^*(T), \infty)$$

$$\leq \tilde{V}(1, T + n'(T) - 1) + \tilde{V}(T + n'(T), \infty).$$

Hence, the error associated with forcing a replacement at time $T + n'(T)$,

$$\tilde{V}(1, \infty) - [\tilde{V}(1, T + n'(T) - 1) + \tilde{V}(T + n'(T), \infty)],$$

is bounded above by

$$\tilde{V}(1, \infty) - [\tilde{V}(1, T + n'(T) - 1)$$

$$+ \alpha_{n^*(T) - n'(T)}(T + n'(T)) \tilde{b}^{j'}_{n^*(T) - n'(T)}(T + n'(T)) + \tilde{V}(T + n^*(T), \infty)]. \qquad (4)$$

By construction of $n^*(T)$,

$$\tilde{V}(1, \infty) = \tilde{V}(1, T + n^*(T) - 1) + \tilde{V}(T + n^*(T), \infty). \qquad (5)$$

Since the assets that optimize $\tilde{V}(1, T + n^*(T) - 1)$ were considered in the finite horizon problem through T, $\tilde{V}(1, T + n'(T) - 1) \geq \tilde{V}(1, T + n^*(T) - 1)$. Hence, from (4) and (5),

$$\tilde{V}(1, \infty) - [\tilde{V}(1, T + n'(T) - 1) + \tilde{V}(T + n'(T), \infty)]$$

$$\leq -\alpha_{n^*(T) - n'(T)}(T + n'(T)) \tilde{b}^{j'}_{n^*(T) - n'(T)}(T + n'(T)).$$

The remaining issue is that we do not know $n^*(T)$. However, we do know that

$$-\alpha_{n^*(T) - n'(T)}(T + n'(T)) \tilde{b}^{j'}_{n^*(T) - n'(T)}(T + n'(T)) \leq \epsilon(T).$$

Hence the bound holds for the transformed problem.

Since the value for each infinite horizon strategy in the transformed problem differs from its value in the original problem by a constant independent of the strategy, the difference in values of the infinite horizon optimal strategy and bounding strategy in the original problem are the same as in the transformed problem. The constants simply cancel. Hence the result also holds for the original problem. ∎

Given an acceptable value for ϵ, we would like to determine the horizon necessary to achieve an error not greater than this value. To accomplish this, evaluate the above error term for increasing values of T until the resulting $\epsilon(T) \leq \epsilon$. Then T is such a horizon. H_ϵ.

For this procedure to work we must know that $\epsilon(T) \to 0$ as $T \to \infty$. A sufficient condition for such convergence is that the problem is sufficiently discounted. See Bean and Smith [1984] for a full discussion of this topic. From a practical perspective, the interest rate for discounting must be larger than the rate of increase of costs or revenues. Such is the case in the real data analyzed below.

4. Application in Vehicle Replacement

The Dynamic Infinite Horizon Replacement Model (see Bean, Lohmann and Smith) was applied to a fleet vehicle replacement problem for a large utility company in Michigan. The company annually faced replacement decisions ranging from executive vehicles to vans and heavy utility trucks. In 1988, the company provided detailed data on replacement decisions for executive vehicles. Each replacement involved an existing vehicle and three alternatives. Executives 1 and 2 currently owned 1985 Chrysler 5th Avenues, Executives 3 and 4 owned 1985 Bonneville Broughams, and Executive 5 owned a 1985 Oldsmobile Cutlass. Executives 1 and 2 currently had a choice of either a 1988 GM 98 Regency, Ford Grand Marquis, or a Chrysler 5th Avenue, and their successors in future model years. Executives 3, 4 and 5 currently had a choice of either a 1988 GM Buick Century, Ford Crown Victory, or Chrysler Dynasty, and their successors in future model years. An illustration of the error-bound calculations for the replacement problem for Executive 1 is illustrated below. Similar calculations were performed for the other four vehicle replacement problems.

The three manufacturer's options are identified only as vehicles 1, 2 and 3 below to avoid implying that one car maker is superior to the other two as a consequence of this single example. In fact, different manufacturers were optimal for the two vehicle classes above. However, the data presented is real.

A summary of the costs associated with the current vehicle alternatives for Executive 1 is shown in Table 1. The data supporting these figures were derived from an in-depth analysis of accounting records, manufacturer estimates, trade publications, government indices and discussions with company personnel.

TABLE 1: Raw Data

	Defender	Challenger 1	Challenger 2	Challenger 3
Purchase Cost	$14,802	$15,750	$15,700	$17,100
Maximum Life	5	5	5	5
Remaining Life	2	5	5	5
MPG	16	16	16	18
Salvage Value % of Purchase Cost				
1985	N/A	68.8%	66.1%	63.3%
1986	N/A	57.4%	54.2%	52.7%
1987	47.0%	50.1%	47.0%	48.3%
1988	38.5%	41.1%	38.5%	38.7%
1989	28.3%	33.2%	28.3%	29.1%
Routine Maintenance Costs				
1985	N/A	-187.20	-293.80	-455.00
1986	N/A	-289.80	-479.39	-495.64
1987	N/A	-674.31	-617.88	-313.17
1988	-444.60	-408.52	-499.63	-255.69
1989	-463.14	-425.56	-520.47	-425.56
Major Maintenance Costs				
1985	N/A	-83.00	-53.00	-73.00
1986	N/A	-509.39	-966.70	-466.68
1987	N/A	-306.01	-314.69	-683.64
1988	-700.00	-452.16	-791.27	-540.33
1989	-937.53	-706.52	-1,059.77	-706.52
Fuel Costs ($1.10 per gallon, 26,000 miles per year)				
1985	N/A	-1,787.50	-1,787.50	-1,588.89
1986	N/A	-1,914.95	-1,914.95	-1,702.18
1987	N/A	-2,051.48	-2,051.48	-1,823.54
1988	-1,787.50	-2,197.76	-2,197.76	-1,953.56
1989	-1,914.95	-2,354.46	-2,354.46	-2,092.85

The company assumed that cost of fuel would inflate at the rate of 7.13% per year and that all other costs for future model years would inflate at the rate of 4.17% per year. They used an effective after-tax nominal dollar discount rate of 9.8% and assumed a Federal tax rate of 34% and a Michigan tax rate of 2.35%. Tables of all cash flow data are available from the authors.

Table 2 works through an example calculation for Executive 1 for a single challenger and with horizon before period 1. Note that a horizon at the beginning of period 1 means

TABLE 2: Example Calculation

$T=$	1				
Discount rate =	0.098				
$n=$	1	2	3	4	5
$\alpha_n(1)b_n^1(1)$	-5038.17	-8253.11	-10847.70	-13233.20	-15421.35
$\gamma_n^1(1)$	-5038.17	-4319.31	-3958.72	-3785.64	-3686.11
$\alpha_n(1)\tilde{b}_n^1(1)$	-1352.05	-1511.00	-1590.60	-1925.03	-2459.44

that we force replacement of the defender at the beginning of the problem. Error is possible since this may not be optimal. The first row of numbers are present values of the scenarios $\{j = 1, t = 1, n = 1\}$ through $\{j = 1, t = 1, n = 5\}$ since $\bar{n} = 5$. The second row are the $\gamma_n^j(t)$ for each scenario. The third row contains $\alpha_n(t)\tilde{b}_n^j(t)$. Note that only the first value in the third row can be calculated from the information contained in the table. In the third row, the value -1352.05 equals the value from the first row of that column, -5038.17, less the largest value in the second row, -3686.11. Other values in the third row involve similar computations, but require data such as $\gamma_5^1(2)$ which are not given here.

Table 3 shows the maximum percent error caused by solving for each horizon one through four years. Each column represents a different executive. Variations are due to alternate defenders, driving habits and replacement alternatives.

TABLE 3: Percent Error

	Executive				
T	1	2	3	4	5
1	3.76	4.04	5.12	5.09	5.09
2	3.58	3.85	4.90	4.88	4.87
3	3.41	3.66	4.70	4.67	4.65
4	3.25	3.49	4.50	4.47	4.44

The errors are relatively low for all combinations of executive and horizon, including period 1, and decrease as the horizon increases. The numbers in Table 3 suggest that for this class of vehicle replacement problems, providing the equivalent value transformation is used before comparing scenarios, the study horizon may be largely irrelevant. It should be chosen according to considerations other than loss due to assuming a finite horizon in

an infinite horizon problem.

5. Summary

In this paper we formulate an equipment replacement problem with technological change. We establish an error bounded horizon time that allows known error solution of the infinite horizon problem by solving that finite horizon problem. The analysis allows for the existence of both revenues and costs. An example shows calculation of the bounds for a real vehicle replacement problem in which errors range from three to five percent.

Such results are of value because of the high cost of gathering information for decision problems of this type. If a small number of years of data is available, and no forecast horizon is found within that time frame, the results of this paper give the decision-maker the power to evaluate the cost of not planning further in time.

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