An Explicit Closed-Form Solution to the Limited-Angle Discrete Tomography Problem for Finite-Support Objects

Andrew E. Yagle

Department of Electrical Engineering and Computer Science, University of Michigan, Ann Arbor, MI 48109-2122

ABSTRACT: An explicit formula is presented for reconstructing a finite-support object defined on a lattice of points and taking on integer values from a finite number of its discrete projections over a limited range of angles. Extensive use is made of the discrete Fourier transform in doing so. The approach in this article computes the object sample values directly as a linear combination of the projections sample values. The well-known ill-posedness of the limited angle tomography problem manifests itself in some very large coefficients in these linear combinations; these coefficients (which are computed off-line) provide a direct sensitivity measure of the reconstruction samples to the projections samples. The discrete nature of the problem implies that the projections must also take on integer values; this means noise can be rejected. This makes the formula practical. © 1998 John Wiley & Sons, Inc. Int J Imaging Syst Technol, 9, 174–180, 1998

I. INTRODUCTION

The limited angle tomography problem of reconstructing an object from its projections (Radon transform) over a limited range of angles has applications in medical imaging [1] and industry [2]. Without some a priori information about the object, it cannot be reconstructed uniquely from a limited angular range of projections [3]. A priori information about the object that has been used to achieve a unique reconstruction includes finite support, upper and lower bounds on pixel values, and closeness to a reference function [4–7]. Many of these methods use an iterative algorithm of alternately projecting onto the spaces defined by the image a priori information and the limited-angle projection information. Since these spaces are convex, such a projection onto convex sets (POCS) algorithm is guaranteed to converge. However, POCS algorithms for this problem require reprojection of the reconstructed object at each iteration, and this can lead to consistency problems unless the reprojection part of the algorithm exactly matches the actual projection operation.

In the discrete tomography problem, we have an object defined on a discrete lattice of points and taking on only discrete values. We also have projection data at only a finite number of angles, corresponding to various sums of the discrete values. In the problem considered here, the object is assumed to have support only at integer-valued coordinates (discrete lattice) and the object values are restricted to integers. Then the projections become various sums of these integers, and they are thus integers themselves. This is a valuable property, since additive noise in the projections can be eliminated if the noise is known to be <0.5 in absolute value. Here, ''integer'' may, of course, be scaled to integer multiples of any small number.

This article derives an explicit formula for the integer values of a finite support object, defined on a lattice, from its projections (Radon transform) at a finite number of angles over a limited range. It does not require the solution of a linear system of equations, and it is not iterative. The formula expresses the object values as linear combinations of the projection values (all integer-valued) using precomputed coefficients; nowhere in the procedure is even a division required. This provides the following advantages over previous approaches:

1. The solution of an large ill-posed linear system of equations is avoided, eliminating error due to computational noise (roundoff error);
2. There are no iterative algorithms, whose requirement of reprojection at each iteration can lead to consistency problems in the reconstruction;
3. A greatly reduced computational load results, which can be reduced even further very simply by parallelization (iterative algorithms cannot be parallelized over iteration number);
4. Direct measures of the sensitivity of the reconstruction to the projection data are made, in the form of the coefficients in the linear combinations;
5. Since the object can only take on a discrete set of values (integers), the projections can also only take on integer values, permitting elimination of small amounts of noise. This is important for the reason described next.

The limited angle tomography problem is known to be ill-conditioned, i.e., a small perturbation of the data can produce a large change in the reconstructed object. This is manifested in our formula by the large values of some of the coefficients in the linear combinations, which provide a direct sensitivity measure. It also means that any noise in the projection data will result in a wrong (possibly very wrong) reconstruction. But since the object is defined on an integer lattice and the object values are restricted to integers, projections are also restricted to integers. Thus, small amounts of noise can be eliminated in the projections by rounding, and the formula can be used with confidence.

This article is organized as follows. Section II reviews the limited angle tomography problem and formulates the discrete tomography problem solved here. Section III reviews quickly the explicit formula for bandwidth extrapolation we have derived previously and which we apply to the limited angle discrete prob-

 CCC 0899–9457/020174-07

lem here. Section IV applies this formula to the limited angle discrete problem and derives the closed-form solution. Section V provides a simple illustrative numerical example which demonstrates how the algorithm works. Reconstruction of each Fourier value is presented in detail, illustrating both the operation of the algorithm and its veracity. All calculations were performed by hand, so that the operation can be followed in detail. Section VI concludes with a summary.

II. LIMITED ANGLE AND DISCRETE TOMOGRAPHY

A. Limited Angle Tomography. The limited angle tomography problem is defined as the reconstruction of an object $f(x, y)$ from its Radon transform (projections) $p(t, \theta)$, defined as

$$p(t, \theta) = \iint f(x, y)\delta(t - x\cos\theta - y\sin\theta)\, dx\, dy \quad (1)$$

where we are given the projections $p(t, \theta)$ over only a finite range of θ. This means that we cannot used filtered backprojection [8], the usual procedure for reconstruction from projections.

Using the projection-slice theorem [8], the one-dimensional (1D) Fourier transform $P(k, \theta) = \mathcal{F}\{p(t, \theta)\}$ of the projections is equal to the 2D Fourier transform $F(k_x, k_y) = \mathcal{FF}\{f(x, y)\}$ of the object, along a slice in the Fourier plane (k_x, k_y) passing through the origin at angle θ to the k_x axis. Hence, in the limited-angle tomography problem we know $F(k_x, k_y)$ in a ''bowtie'' region, and the limited angle problem is really a 2D extrapolation problem in the Fourier domain. To perform the extrapolation, we need a priori information about the object. One obvious approach is to simply set the unknown values of $F(k_x, k_y)$ (those outside the bowtie region) to zero. This is clearly the minimum energy solution, and it is also [9] equivalent to the ''squashing'' algorithm of [10]. Another approach is to interpolate the unknown values of $F(k_x, k_y)$ from the unknown values [11], but this does not seem to improve over squashing [11].

B. Discrete Tomography Formulation (DTF). We now make the following two assumptions about $f(x, y)$. First, we assume it can be written as a finite sum of weighted impulses with singularities at integer-valued coordinates: $f(x, y) = \Sigma\Sigma f(i, j)\delta(x - i)\delta(y - j)$. Note the usual Radon transform results (projection-slice theorem) still hold. Second, we assume that $f(x, y)$ can only take on integer values, so the line integral (1) becomes sums of various values of $f(i, j)$ and also becomes integer-valued. Of course, ''integer'' can be replaced by ''integer multiple of any small number e.'' We can still use the DFT by taking periodic extensions, as usual. In fact, the Fourier transform becomes the discrete-time Fourier transform (DTFT): $F(k_x, k_y) = DTFT[f(i, j)]$, which in turn becomes the DFT when periodic extensions are taken.

If the slope of the projections is an irrational number, then each $f(i, j)$ appears separately in the projections, making the reconstruction trivial. This is clearly not in the spirit of discrete tomography, so slopes are restricted to $M + 1$ rational values so each projection is a sum of several values of $f(i, j)$. Here, $M = MIN[M_1, M_2]$, where $f(i, j)$ has finite $M_1 \times M_2$-point support. The problem is then to reconstruct $f(i, j)$ from various sums of its values.

The observation noise in the projections is assumed to be <0.5 in absolute value. This allows immediate error correction in the projections by rounding, and thus removes the problems caused by the poor conditioning of the problem.

We note in passing that limited angle discrete tomography problems have been considered elsewhere, e.g., [14–16]. However, these references have treated only the binary (or at most a finite number of possible values) problem, whereas this article allows each pixel of the object to take on any integer value. Also, most previous references treat the existence problem of whether a unique solution exists, rather than deriving simple algorithms for computing it. And the algorithm proposed here seems to be much simpler than those proposed elsewhere.

It is clear that we can set up a linear system of M_1M_2 or more equations in M_1M_2 unknowns whose solution is $f(i, j)$. The problem is that this linear system of equations is: (a) very large (order of several thousand), (b) not sparse, and (c) ill-conditioned (even apart from being very large) owing to the close spacing of the angles. While we could still try to solve this linear system of equations, a closed-form solution would save a tremendous amount of storage and computation, and would also avoid computational roundoff error incurred in solving a large system of equations. We now show how to obtain such a closed-form solution.

III. AN EXPLICIT FORMULA FOR BANDWIDTH EXTRAPOLATION

A. Basic Idea. We now quickly summarize the results of [12] (see also [13]), which present a fast algorithm for exact extrapolation of a discrete-time periodic band-limited signal from its known values in an interval having the same length as the bandwidth of the signal. The procedure is a simple autoregression on the time-domain values of the signal, and it is much simpler than previous algorithms for discrete–discrete extrapolation, which required computation of large pseudo-inverses. The procedure is highly parallelizable, and the computational savings are especially significant for the 2D extrapolation we require for tomography. This method is related to other methods for bandwidth extrapolation in [12].

We consider the discrete–discrete band-limited extrapolation problem: Given $2M + 1$ consecutive values of a discrete-time periodic sequence $x(n)$ with period N whose DFT $X(k)$ is known to be zero for $M < |k| \le N/2$, determine the other values of $x(n)$. We then consider the 2D version of this problem, in which the 2D DFT is known to be band limited to some rectangle in the 2D frequency plane. Note that while simultaneous time limitation and band limitation is impossible for the continuous-time problem, it is entirely possible in the discrete–discrete problem, since periodic extensions are being made in both time and frequency.

Previous approaches have explicitly or implicitly written the band-limited constraint as follows. $x(n)$ is band limited if its N-point DFT $X(k)$ satisfies

$$X(k)S(k) = X(k); \quad S(k) = \begin{cases} 1, & \text{if } |k| \le M; \\ 0, & \text{if } |k| > M. \end{cases} \quad (2)$$

which in the time domain becomes

$$x(n) * s(n) = x(n), \tag{3}$$

where $s(n) = DFT^{-1}\{S(k)\}$ is a discrete sinc function and $*$ denotes cyclic convolution. The problem is that $s(n)$ does not have finite support; the ''tails'' of the sinc function go on forever.

Our approach is to replace Equation (2) with the equivalent but different condition

$$X(k)S(k) = X(k); \quad S(k) = \begin{cases} 1, & \text{if } |k| \le M; \\ \neq 1, & \text{if } |k| > M. \end{cases} \tag{4}$$

Note that any $S(k)$ satisfying Equation (4) implements the band-limited constraint just as well as the $S(k)$ in Equation (2); but the condition on $S(k)$ in Equation (4) allows greater flexibility in choosing $S(k)$, viz., we can impose time limitation in the time domain. That is, we may choose an $S(k)$ which satisfies Equation (4) but avoids the infinite-length tails of the discrete sinc function. Such an $s(n)$ turns the convolution (3) into an autoregression of order $2M + 1$ on $x(n)$; knowledge of any $2M + 1$ consecutive values of $x(n)$ allows the other values to be computed by running the autoregression (3).

B. Choice of $S(k)$. We make the following choice for $S(k)$:

$$S(k) = 1 + \prod_{i=-M}^{M} (Z_k - Z_i) = \sum_{n=1}^{2M+1} c(n) Z_k^n \tag{5a}$$

$$Z_k = e^{-j2\pi k/N}, \quad k = -N/2 + 1, \cdots N/2. \tag{5b}$$

The second equality in Equation (5a) defines the $\{c(n)\}$, and follows by multiplying and collecting coefficients of powers of Z_k. The significance of this is that it shows $s(n) = DFT^{-1}\{S(k)\}$ is time limited. In fact, from Equation (5) we have

$$s(n) = DFT^{-1}\{S(k)\} = DFT^{-1}\left\{1 + \prod_{i=-M}^{M} (Z_k - Z_i)\right\}$$

$$= \begin{cases} c(n), & \text{if } 1 \le n \le 2M + 1; \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

Since $S^*(k) = S(-k)$, the $c(n)$ are all real. It is straightforward to show that the $S(k)$ defined in Equation (5) also satisfies the condition in Equation (4).

With this choice of $S(k)$, Equation (3) becomes the autoregression

$$x(n) = \sum_{i=1}^{2M+1} c(i) x(n - i). \tag{7}$$

This shows that the unknown $x(n)$ can be computed from the $2M + 1$ consecutive known values of $x(n)$, without solving a system of equations, without even a division. The $c(n)$ can be computed ahead of time from Equation (5).

What if we are given nonconsecutive but equally spaced values of $x(n)$. For example, we might be given $2M + 1$ values $\{x(0), x(3), x(6), x(9) \cdots\}$. In this case, we simply modify Equation (5a) to $S(k) = 1 + \prod (Z_k^3 - Z_i^3) = \sum c(n) Z_k^{3n}$. Then, Equation (7) is clearly an autoregression that uses only every third value of $x(n)$, as desired. Note that now $S(k) = 1$ not only at $Z_k = Z_i$, but also at $Z_k = Z_i e^{\pm j2\pi/3}$, but this will not correspond to an *integer* k unless N is a multiple of 3.

For a 2D signal $x(n_1, n_2)$ band limited in frequencies k_1 and k_2 separately (as well as jointly), we may apply the 1D algorithm to extrapolate first in the n_1 direction, and then in the n_2 direction. It is clear that the bandwidth of $x(n_1, n_2)$ need not be the same in the n_1 and n_2 directions for this approach to work; this is important in the application to tomography. Note that the extrapolations over n_1 may all be done in parallel (separately, for each n_2), and then the extrapolations over n_2 may also all be done in parallel (separately, for each n_1). Hence, both stages of the extrapolation are parallelizable.

The well-known ill-posedness of the band-limited extrapolation problem manifests itself in the large values of the autoregression coefficients; these are a exact measure of the sensitivity of the extrapolation to the given $x(n)$. For example, $N = 64$ and $M = 4$ (nine-point support) produces extrapolation coefficients

$$[c(1), \ldots, c(9)] = [8.7136, -34.0200, 78.1091, -116.2225, 116.2225, -78.1091, 34.0200, -8.7136, 1.0000]$$

Note the wide variations in sensitivity to various given values of $x(n)$.

IV. APPLICATION TO LIMITED-ANGLE DISCRETE TOMOGRAPHY

A. Discussion of Application. We now show how this extrapolation formula applies to limited-angle discrete tomography. We assume we are given the projections at $M + 1$ discrete angles $\{arctan(k/L), |k| \le M/2\}$ for some L. Note that these angles are not exactly evenly spaced in θ (but they are close to evenly spaced if $L \gg M$). More important, the slopes are rational numbers, as required in the problem formulation. By the projection slice theorem, this means that we are given $F(k_x, k_y)$ along the slices $k_y = k_x(k/L)$ for $|k| \le M/2$. We assume for convenience of presentation that $f(i, j)$ has support $M_2 \times M$, where $M \le M_2$; otherwise, simply exchange i and j in the sequel.

Let $F(m, n)$ be $F(k_x, k_y)$ sampled on a concentric squares raster:

$$F(m, n) = F\left(k_x = 2\pi \frac{m}{N}, \quad k_y = k_x \frac{n}{L} = 2\pi \frac{mn}{NL}\right) \tag{8}$$

Note that the spacing between samples in k_y increases with $|k_x|$. The relevant portion of the concentric squares raster is shown in Figure 1.

It should be clear from the projection-slice theorem that the samples shown in Figure 1 can all be obtained from the given projections of $f(i, j)$. Note that the projections at angle $arctan(k/L)$ of $f(i, j)$ (defined on a rectangular lattice) will be nonzero only at integer multiples of some Δ, and that there are only a finite number of such nonzero values. This means that the DFT

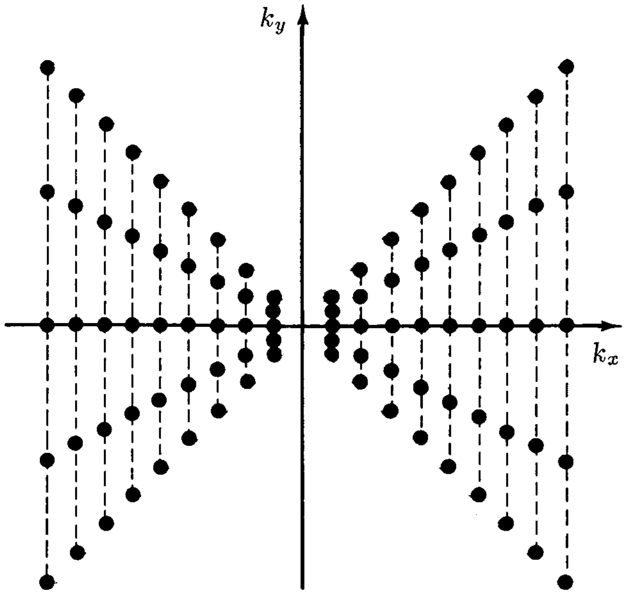

Figure 1. The bowtie section of the concentric squares raster in which the 2D discrete Fourier transform of the object is known directly from the projections.

can be used to compute the samples along each slice of Figure 1.

We are then faced with the problem of extrapolating the rest of $F(k_x, k_y)$ from the samples $F(m, n)$ shown in Figure 1. To do this, define $\hat{f}(m, j)$ as the N-point 1D DFT in i of the object $f(i, j)$, for each integer $0 \le j \le N - 1$ [note that $\hat{f}(m, j)$ is a ''half-2D-DFT'']. Then, $F(m, n)$ can be computed from $\hat{f}(m, j)$ by computing the 1D DFT of order NL/m in j of $\hat{f}(m, j)$. Note that the order of this transform varies with m.

Since the samples $f(i, j)$ of the object are nonzero only for $|j| < M/2$, $\hat{f}(m, j)$ is nonzero only for $|j| < M/2$. Now fix m. Since we have M samples in n of $F(m, n)$, the 1D DFT of $\hat{f}(m, j)$, we can extrapolate in n the samples of $F(m, n)$. We can repeat this for each m, and in so doing compute $F(k_x, k_y)$ everywhere on a sampled grid. This sampled grid will include as a subset a rectangular grid, from which $f(i, j)$ can be computed using a 2D DFT.

B. Comments. Since the spacing between the samples in k_y depends on k_x by $k_y = k_x(k/L)$, the DFT length used for extrapolation will also vary with m. This is not a problem, since the extrapolations are all performed in parallel. After the extrapolations are complete, we then need to upsample or downsample the variable spacing samples in k_y so that they all have the same length. This operation consists of an inverse DFT followed by a DFT of different order. Then a simple inverse DFT yields the samples $f(i, j)$.

The DFT length used for extrapolation in n of $F(m, n)$ is NL/m. If we choose NL to be the least common multiple of 1, $2 \cdots M/2$, then NL/m is always an integer. But this is unnecessarily restrictive. If NL/m is NOT an integer, we can still use the formulae in Section III and extrapolate the given values of $F(m, n)$. The only difference is that we need to extrapolate further. To see this, let $NL/m = N_1/N_2$ for some relatively prime integers N_1, N_2. A DFT of order N_1/N_2 can be viewed as a DFT of order N_1 of a signal interpolated by a factor N_2 (insert $N_2 - 1$ zeros between each sample value of the signal). Thus, we can handle noninteger NL/m by interpolating the signal and the extrapolation coefficients with zeros and then employing an integer-ordered DFT. However, a moment's thought shows that the operations will be the same as before.

This may seem confusing, but the numerical example below should clarify things.

C. Summary of Procedure.

1. OBJECT: $f(i, j)$ is
   a. defined for integer values of i, j;
   b. nonzero only for $|i| \le M_2/2$, $|j| \le M/2$;
   c. restricted to taking on integer values.
   DATA: Radon transform at limited range of $M + 1$ discrete angles
   $\{arctan(k/L), |k| \le M/2\}$ for some L chosen so that:
   d. These are sums of various values of $f(i, j)$; and are
   e. restricted to taking on integer values; so that
   f. any noise < 0.5 can be eliminated by rounding.
2. Compute DFT of projections, nonzero at integer multiples of some Δ.
   This yields $F(k_x, k_y)$ sampled on $k_y = k_x(k/L)$, $|k| \le M/2$. Designate the samples of this variably sampled $F(k_x, k_y)$ as $F(m, n)$.
3. If NL is the least common multiple of 1, $2 \cdots M/2$:
   For each m, extrapolate $F(m, n)$ in n from its given values $\{F(m, n), |n| \le M/2\}$ using 1D extrapolation coefficients based on an NL/m-order DFT.
   Then compute DFT^{-1} of order NL/m and DFT of order N (downsample) for each m. Then, compute 2D DFT^{-1} to obtain sampled $f(x, y)$.
4. If $NL/m = N_1/N_2$ is not an integer, extrapolate $F(m, n)$ in n as in Step 3.
   This will produce N_1 different values of $F(m, n)$ before they repeat.
   Compute DFT^{-1} of order N_1 and downsample the resulting zero interpolated half-transform $\hat{f}(m, j)$. Then, proceed as in Step 3.

Note that each step requires nothing more than a DFT or inverse DFT, which are well-conditioned unitary transforms, or an extrapolation (linear combination).

V. A SIMPLE ILLUSTRATIVE NUMERICAL EXAMPLE

We present a simple numerical example. This example is intended to be illustrative, demonstrating how the algorithm works and confirming that it does indeed reconstruct the object perfectly. The operation of the algorithm on larger-size objects should then be apparent. A few comments on numerical implementation for large objects are given later.

A. Problem Statement. The object and its projections are shown in Figure 2. The object is 3×5, and its discrete projections (sums of various pixels) are given for three different angles from the horizontal ($0°$ and $\pm 45°$). The goal is to reconstruct the pixel values from the projections at these three angles, which are also shown in Figure 2.

Since the pixel values are all integers, the discrete projections (which are various sums of the pixel values) are also all integers. Any additive noise < 0.5 can therefore be eliminated by rounding. We omit this here since it is quite obvious.

Note that for an object with rectangular support, the corner pixels can always be found directly from projections at any angle other than 0° or 90°, since the end points of the set of projections at any such angle pass only through a single corner pixel. Since this is misleading (it makes the problem look larger than it really is), we have set the four corner pixels of the object in Figure 2 to zero.

The problem could of course be solved directly by solving the linear system of equations

$$\begin{bmatrix} 12 \\ 15 \\ 18 \\ 4 \\ 6 \\ 15 \\ 14 \\ 16 \\ 10 \\ 8 \\ 15 \\ 12 \\ 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 7 \\ 8 \\ 9 \end{bmatrix} \tag{9}$$

which is 13 equations (the projections) in 11 unknowns (the pixel values). The method to follow is a closed-form solution to this linear system of equations.

B. Extrapolation Equations. We use an 8×8 2D DFT. Since $f(i, j)$ has support in $j\ |j| \leq 1$, we have $M = 1$. We will need the extrapolation coefficients for both $N = 8$ and $N = 4$. They turn out to be

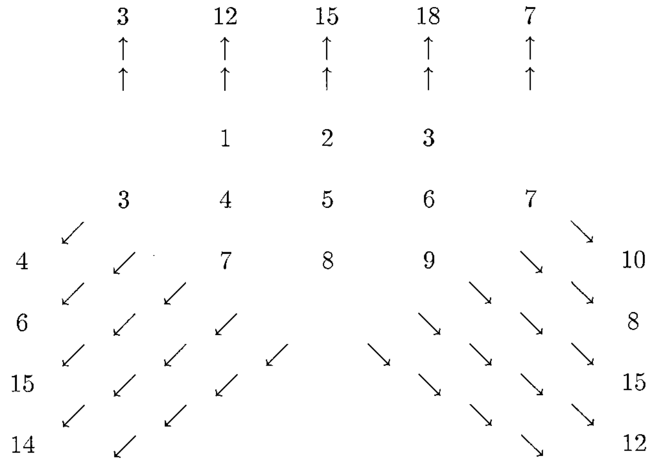

Figure 2. A simple numerical example of the procedure.

$$S(z) = 1 + \prod_{i=-1}^{1} (z - e^{-j2\pi i/4}) = 1 + (z - j)(z - 1)(z + j)$$
$$= z^3 - z^2 + z \tag{10a}$$

$$S(z) = 1 + \prod_{i=-1}^{1} (z - e^{-j2\pi i/8}) = 1 + (z - 1)(z^2 - \sqrt{2}z + 1)$$
$$= z^3 - (1 + \sqrt{2})z^2 + (1 + \sqrt{2})z \tag{10b}$$

$$S(z) = 1 + \prod_{i=-1}^{1} (z^3 - e^{-j2\pi 3i/8})$$
$$= 1 + (z^3 - 1)(z^6 + \sqrt{2}z^3 + 1) \tag{10c}$$
$$= z^9 + (\sqrt{2} - 1)z^6 - (\sqrt{2} - 1)z^3$$

The coefficients $c(i)$ can then be read off of these polynomials. The z-transform is easier to use here than the DFT, although the latter may, of course, be used.

We are given the projections shown in Figure 2. This means that by the projection slice theorem, we immediately know the 8×8-point DFT $F(m, n)$ of $f(i, j)$ at the points shown in Figure 1 (note $\Delta = 1/\sqrt{2}$). Specifically, we know the following values of $F(m, n)$:

$$\begin{array}{ccccccccc}
15 & * & * & * & * & * & * & * & 15 \\
* & \begin{smallmatrix} 15-10\sqrt{2} \\ -j2\sqrt{2} \end{smallmatrix} & * & * & * & * & * & \begin{smallmatrix} 15-10\sqrt{2} \\ +j(12-4\sqrt{2}) \end{smallmatrix} & * \\
* & * & -5-4j & * & * & * & -5-8j & * & * \\
* & * & * & \begin{smallmatrix} 15+10\sqrt{2} \\ -j2\sqrt{2} \end{smallmatrix} & * & \begin{smallmatrix} 15+10\sqrt{2} \\ -j(12+4\sqrt{2}) \end{smallmatrix} & * & * & * \\
-5 & \begin{smallmatrix} 15-15\sqrt{2} \\ -j(4-3\sqrt{2}) \end{smallmatrix} & 5+6j & \begin{smallmatrix} 15+15\sqrt{2} \\ +j(4+3\sqrt{2}) \end{smallmatrix} & 55 & \begin{smallmatrix} 15+15\sqrt{2} \\ j(4+3\sqrt{2}) \end{smallmatrix} & 5-6j & \begin{smallmatrix} 15-15\sqrt{2} \\ +j(4-3\sqrt{2}) \end{smallmatrix} & -5 \\
* & * & * & \begin{smallmatrix} 15+10\sqrt{2} \\ +j(12+4\sqrt{2}) \end{smallmatrix} & * & \begin{smallmatrix} 15+10\sqrt{2} \\ +j2\sqrt{2} \end{smallmatrix} & * & * & * \\
* & * & -5+8j & * & * & * & -5+4j & * & * \\
* & \begin{smallmatrix} 15-10\sqrt{2} \\ -j(12-4\sqrt{2}) \end{smallmatrix} & * & * & * & * & * & \begin{smallmatrix} 15-10\sqrt{2} \\ +j2\sqrt{2} \end{smallmatrix} & * \\
15 & * & * & * & * & * & * & * & 15
\end{array} \tag{11}$$

We now show how to extrapolate the unknown values of $F(m, n)$, denoted by $*$ in Equation (11).

C. Extrapolation of Unknown $F(m, n)$. Consider the column $m = 1$ of $F(m, n)$. We have the three consecutive values $F(1, 1)$, $F(1, 0)$, $F(1, -1)$ shown in Equation (11). Using Equation (10b), we can then compute

$$F(1, 2) = (1 + \sqrt{2})\begin{pmatrix} 15 + 10\sqrt{2} \\ -j(12 + 4\sqrt{2}) \end{pmatrix} - (1 + \sqrt{2})\begin{pmatrix} 15 + 15\sqrt{2} \\ -j(4 + 3\sqrt{2}) \end{pmatrix} + \begin{pmatrix} 15 + 10\sqrt{2} \\ +j2\sqrt{2} \end{pmatrix} = \begin{matrix} 5 + 5\sqrt{2} \\ -j(10 + 7\sqrt{2}) \end{matrix} \quad (12a)$$

$$F(1, 3) = (1 + \sqrt{2})\begin{pmatrix} 5 + 5\sqrt{2} \\ -j(10 + 7\sqrt{2}) \end{pmatrix} - (1 + \sqrt{2})\begin{pmatrix} 15 + 10\sqrt{2} \\ -j(12 + 4\sqrt{2}) \end{pmatrix} + \begin{pmatrix} 15 + 15\sqrt{2} \\ -j(4 + 3\sqrt{2}) \end{pmatrix} = \begin{matrix} -5 \\ -j(8 + 4\sqrt{2}) \end{matrix} \quad (12b)$$

$$F(1, 4) = (1 + \sqrt{2})\begin{pmatrix} -5 \\ -j(8 + 4\sqrt{2}) \end{pmatrix} - (1 + \sqrt{2})\begin{pmatrix} 5 + 5\sqrt{2} \\ -j(10 + 7\sqrt{2}) \end{pmatrix} + \begin{pmatrix} 15 + 10\sqrt{2} \\ -j(12 + 4\sqrt{2}) \end{pmatrix} = \begin{matrix} -5 - 5\sqrt{2} \\ +j(-4 + \sqrt{2}) \end{matrix} \quad (12c)$$

Now consider the column $m = 2$. Now we have only every other value of $F(2, n)$: $F(2, 2)$, $F(2, 0)$, $F(2, -2)$. But since we are using an 8×8-point 2D DFT and $8/2 = 4$ is an integer, we can simply use the $N = 4$ extrapolation coefficients from Equation (10a) instead of the $N = 8$ coefficients from Equation (10b). We then have

$$\begin{aligned} F(4, 2) &= F(2, 2) + (-1)F(2, 0) + F(2, -2) \\ &= (-5 - 8j) - (5 - 6j) + (-5 + 4j) \\ &= -15 + 2j. \end{aligned} \quad (13)$$

How do we get the other values? Take a four-point inverse 1D DFT of the four known values of $F(2, n)$. This amounts to downsampling in frequency, but owing to the zero padding of the original problem, the resulting $\hat{f}(2, j)$ will not be aliased (it will simply be repeated). Then, take $\hat{f}(2, j)$, discard the repetition, and take an eight-point 1D DFT, yielding $F(2, n)$.

Now consider the column $m = 3$. Now we have only every third value of $F(3, n)$: $F(3, 3)$, $F(3, 0)$, $F(3, -3)$. We can use the method discussed in Section IIIB for nonconsecutive values. Using the extrapolation coefficients from Equation (10c), we compute $F(3, -2) = F(3, 6)$ as

$$F(3, 6) = (1 - \sqrt{2})\begin{pmatrix} 15 - 10\sqrt{2} \\ j(12 - 4\sqrt{2}) \end{pmatrix} - (1 - \sqrt{2})\begin{pmatrix} 15 - 15\sqrt{2} \\ j(4 - 3\sqrt{2}) \end{pmatrix} + \begin{pmatrix} 15 - 10\sqrt{2} \\ j2\sqrt{2} \end{pmatrix} = \begin{matrix} 5 - 5\sqrt{2} \\ +j(10 - 7\sqrt{2}) \end{matrix}. \quad (14)$$

Other values of $F(3, n)$ can be computed similarly (note that they will not be computed in increasing order in n, but this hardly matters).

All of these DFT values can be confirmed to be correct by simply computing the 8×8 2D DFT of the image $f(i, j)$.

D. Application to Large Images. It is apparent that this procedure can be applied to arbitrarily large images. The parallelizability of the extrapolations becomes important in this case, since significant computation time can be saved if this can be done.

Although the algorithm is exact, roundoff error must be avoided. The extrapolation coefficients become very large for large problems, and even though $f(i, j)$ is known to take on only integer values, care must be taken that sufficient precision be retained so that multiplication of the known DFT values by the (large) extrapolation coefficients, followed by an inverse DFT, still yields numbers that are close to integers. This requires precise computation of the DFT values from the (given) integer-valued discrete projections. Fortunately, precise computation of the DFT using the fast Fourier transform is not a significant problem. Multiple precision may be necessary if MATLAB or a similar numerical algorithm package is employed.

VI. CONCLUSION

We have provided a closed-form solution of the limited-angle discrete tomography problem. This solution applies an explicit formula for bandwidth extrapolation to the limited-angle discrete tomography problem. It avoids the solution of an ill-conditioned system of equations (with its attendant roundoff error) and also avoids time-consuming iterative algorithms. It provides direct control over all variables in the problem, and shows explicitly the sensitivity of the solution to variations in the data.

By restricting the values of $f(x, y)$ to integers (discrete tomography), the projections are also restricted to integer values. This permits elimination of small amounts of additive noise in the projection data. Since the problem is very ill-conditioned, noise-free projection data are very important. The discrete tomography formulation allows the formula to be used with confidence.

It is interesting to note that the discrete nature of this problem is what makes this closed-form solution possible. Although N can be made arbitrarily large to simulate a continuous Radon transform, the discrete perspective is still needed to obtain the solution, showing the value of discrete tomography.

REFERENCES

1. J. S. Jaffe. ''Limited angle reconstruction using stabilized algorithms,'' *IEEE Trans. Med. Imaging* **MI-9,** 338–344 (1990).
2. K. C. Tam, J. W. Eberhard, and K. W. Mitchell. ''Incomplete-data image reconstruction in industrial applications,'' *IEEE Trans. Nucl. Sci.* **NS-37,** 1490–1499 (1990).
3. K. C. Tam and V. Perez-Mendez. ''Tomographical imaging with limited angle input,'' *J. Opt. Soc. Am. A.* **71,** 582–592 (1981).
4. P. Oskoui-Fard and H. Stark. ''Tomographic image reconstruction using the theory of convex projections,'' *IEEE Trans. Med. Imaging* **MI-7,** 45–59 (1988).
5. M. I. Sezan and H. Stark. ''Tomographic image reconstruction from incomplete data by convex projections and direct Fourier inversion,'' *IEEE Trans. Med. Imaging* **MI-3,** 91–98 (1984).
6. M. I. Sezan and H. Stark. ''Image restoration by the method of convex projections: Part II,'' *IEEE Trans. Med. Imaging* **MI-1,** 95–101, 1982.
7. D. C. Youla and H. Webb. ''Image restoration by the method of convex projections: Part I,'' *IEEE Trans. Med. Imaging* **MI-1,** 81–94 (1982).
8. L. A. Shepp and B. F. Logan. ''The Fourier reconstruction of a head section,'' *IEEE Trans. Nucl. Sci.* **NS-21,** 21–42 (1974).
9. T. Olson and J. S. Jaffe. ''An explanation of the effects of squashing in limited angle tomography,'' *IEEE Trans. Med. Imaging* **MI-9,** 242–246 (1990).
10. J. A. Reeds and L. A. Shepp. ''Limited angle reconstruction in tomography via squashing,'' *IEEE Trans. Med. Imaging* **MI-6,** 89–97 (1987).
11. P. Oskoui and H. Stark. ''A comparative study of three reconstruction methods for a limited-view computer tomography problem,'' *IEEE Trans. Med. Imaging* **MI-8,** 43–49 (1989).
12. H. Soltanian-Zadeh and A. E. Yagle. ''A fast algorithm for extrapolation of discrete-time periodic band-limited signals,'' *Signal Process.* **33,** 183–196 (1993).
13. H. Soltanian-Zadeh and A. E. Yagle, ''A fast algorithm for extrapolation of discrete band-limited signals,'' in *Proceedings of ICASSP 1993,* Minneapolis, MN, 1993, pp. III-591–III-594.
14. E. Bolker. ''The finite Radon-transform,'' *Integral Geom. Contemp. Math.* **63,** 27–50 (1987).
15. E. L. Grinberg. ''The admissibility theorem for the hyperplane transform over a finite field,'' *J. Combin. Theoret. A* **53,** 316–320 (1990).
16. P. M. Salzberg. ''Binary tomography on lattices,'' *Congress. Numer.* **111,** 185–192 (1995).