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METHODS OF ESTIMATING A POWER UTILITY FUNCTION

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ABSTRACT

An experimental procedure based on the normative theory of the subjective expected utility (SEU) model can be represented as a special case of the additive conjoint measurement model. The power utility function can be shown to be consistent with the strictly additive conjoint measurement model. This paper considers methods of estimating the power utility function using experimental data. It discusses the experimental procedure and the conjoint measurement model, then presents three different estimators. The first estimator, which has been reported previously [9], requires an extensive experimental design in order to yield satisfactory results. The second estimator is derived using a least squares method. The third estimator is obtained via the methods of maximum likelihood and requires that the model be expanded to include a conditional distribution on the subjective probability. This conditional distribution is selected to be consistent with known experimental results. The three estimators are compared using simulation, and the maximum likelihood estimator is shown to be the most accurate.

Introduction

Within the framework of decision theory it is necessary to specify a utility or loss function which is a mapping from the consequences into the reals. Different approaches to decision theory are developed using various forms of utility. For example, a squared error loss function is used predominantly in Bayesian estimation. Actual decision making, especially using monetary rewards, requires a utility function.

Since a utility function is very difficult to assess, experimental procedures for its evaluation have become necessary. A survey of some of the utility models and their associated experimental methods is given by Becker, DeGroot, and Marschak [2]. The specific experimental procedure with which this report is concerned is given in another paper by Becker, DeGroot, and Marschak [3]. In the experiment under consideration, each subject is presented with a sequence of n two-outcome gambles, denoted by (a, p'), such that if an event which has objective probability p' is realized, then the subject receives the outcome a. If the complement event occurs, the subject receives nothing. The subject is asked to state the smallest amount of money for which he would be willing to sell the right to the gamble, which will be denoted by M. In assessing the value of M, the subject will be assessing the subjective probability of the event that had objective probability p'. The symbol p, without the prime, will represent this subjective probability.

Hence, each wager in the sequence can be represented by (a_i, p_i', p_i, M_i); $i = 1, 2, \ldots, n$. The subjective probability, p, is not observed.

On the basis of the subjective expected utility model (SEU) Becker, DeGroot, and Marschak [3] have shown that a subject can do no better than state his true, lowest selling price (M). Tversky [9] has used this experimental procedure in an effort to determine whether the subjective expected utility model is descriptive in actual practice. Following the lead of Tversky [9] we describe the SEU model as a special case of the conjoint measurement model and then show that a specific form for the utility function is appropriate. For each experiment performed, an estimation procedure must be chosen in order to produce a representative utility function. This study will present and compare three methods of estimation.

The SEU Model As an Additive Conjoint Measurement Model

In the SEU model the utility of a wager is the sum of the utilities of the consequences weighted by the subjective probabilities of the events which lead to the various consequences. In general,

$$u[(a_1,p_1);\ (a_2,p_2);\ \ldots\ldots;\ (a_k,p_k)] = \sum_{i=1}^{k} p_i\, u(a_i) \tag{1}$$

where $u(\cdot)$ is the utility function, a_i, the actual value of the various consequences, and p_i, the subjective probability of the corresponding events. If we perform an experiment similar to the one described in the introduction, our purpose is to somehow measure both the utility function and the subjective probability. This is an example of a conjoint measurement problem as described in the psychological literature

[1], [8]. We are attempting to measure more than one variable based on their joint effects, under the assumption that a specified composition rule holds. Since the composition rule in this case is determined by the SEU model as given in equation (1), we say that the SEU model is an example of a conjoint measurement model.

A conjoint measurement model is called <u>additive</u> [8] if the data matrix D and the component matrices $C_1, C_2, \ldots, C_k$ which conjoin to form D ($D = C_1 \overset{o}{x} C_2 \overset{o}{x} \ldots \overset{o}{x} C_k$, where $\overset{o}{x}$ represents the specific combination rule) have the following properties:

(i) There exists real valued functions $\phi, f_1, f_2, \ldots, f_k$ on $D, C_1, C_2, \ldots, C_k$ respectively, such that for all

$c_1 \varepsilon C_1, c_2 \varepsilon C_2, \ldots, c_k \varepsilon C_k$

(2) $$\phi(c_1, c_2, \ldots, c_k) = f_1(c_1) + f_2(c_2) + \ldots + f_k(c_k)$$

and (ii) $\phi(c_1, c_2, \ldots, c_k) \geq \phi(c_1', c_2', \ldots, c_k')$ if, and only if, $(c_1, c_2, \ldots, c_k) \geq^* (c_1', c_2', \ldots, c_k')$ where $\geq^*$ denotes the order observed in the data matrix.

Furthermore, the model is termed <u>strictly additive</u> if ϕ can be chosen to be the identity, i.e.,

$$\phi(c_1, c_2, \ldots, c_k) = c_1 \overset{o}{x} c_2 \overset{o}{x} \ldots \overset{o}{x} c_k,$$

where $\overset{o}{x}$ again denotes the combination rules under consideration and $c_1, c_2, \ldots, c_k$ are specific values of the k variables.

Using the experiment described in the introduction, if we subscribe to the SEU model we have

(3) $$u(M) = p\ u(a) + (1-p)\ u(0).$$

Allowing $u(0) = 0$ and taking logarithms,

(4) $$\log u(M) = \log u(a) + \log p.$$

Therefore this experiment in the context of the SEU model is a specific case of the strictly additive conjoint measurement model with ϕ, f_1, and f_2 all being log functions.

Furthermore, if we assume that utility is a power function, i.e.,

(5) $$u(x) = x^{\theta} \quad ; x \geq 0,$$

then our SEU model yields

(6) $$M^{\theta} = a^{\theta} p,$$

and we see that $\log M = \log a + \theta^{-1} \log p$. Therefore, if we consider the logarithm of the bid to be the data matrix, we have a strictly additive conjoint measurement model.

Throughout this report we assume that equations (5) and (6) properly model the utility function and the experimental procedure under investigation. The assumption that utility of money is a power function of the actual money value has been suggested in the previous literature on utility theory (see [7] for example).

Estimating the Exponent

Based on the model given by equations (5) and (6), Tversky [9] suggested a method of estimating θ, the exponent in the power utility function. This method does not require that additional structure be added to the model; however, it does require that complementary events be used in the experimental procedure. That is, if we want to use the data that resulted from a wager that had objective probability p', then we also must have a wager in which the objective probability is 1-p'.

This estimation procedure says that $\tilde{\theta}_{p'}$ is an estimator of θ if it is the solution of

$$\text{(7)} \qquad \left[\frac{\sum_{p'} M_i}{\sum_{p'} a_i}\right]^{\tilde{\theta}_{p'}} + \left[\frac{\sum_{1-p'} M_i}{\sum_{1-p'} a_i}\right]^{\tilde{\theta}_{p'}} = 1$$

where $\sum_{p'}$ denotes the sum over all events with objective probability p'. This estimator seems reasonable since equation (6) can be given as

$$\text{(8)} \qquad p = \left[\frac{M}{a}\right]^{\theta}$$

and therefore

$$\left[\frac{\sum_{p'} M_i}{\sum_{p'} a_i}\right]^{\theta}$$

would be a reasonable estimate of p if θ were known. It should be noted, however, that adding the two terms to equal one in equation (7) assumes that the subjective probability assessment of complementary events sum to unity. Once a $\tilde{\theta}_{p'}$ is evaluated for each set of complementary events, the estimator $\tilde{\theta}$ is the average of these estimators, with each $\tilde{\theta}_{p'}$ receiving equal weight. Notice that in finding $\tilde{\theta}_{p'}$ this estimation procedure gives a different estimate of θ for each set of complementary events. Since we want the end product to be a utility function covering the entire range of consequences and valid for any probability, we propose two estimators that accomplish this.

The first alternative is a fitted estimator using the methods of least squares. If it is assumed that

(9) $$p = \lambda p'$$

in which λ is a random variable with $E[\ln\lambda] = 0$, then we can write equation (3) as

(10) $$\theta(\ln r_i) = \ln p_i' + \ln \lambda_i \qquad i = 1,2,\ldots,n$$

where

(11) $$r_i = M_i/a_i.$$

The assumption that $E[\ln\lambda] = 0$ implies that the support of λ has values both greater than and less than one. The validation of this type of assumption will be discussed after the density in equation (13) is given. The θ that minimizes

$$\sum_{i=1}^{n} (\theta \ln r_i - \ln p_i)^2 \text{ is given by}$$

(12) $$\theta^* = \frac{\sum_{i=1}^{n} (\ln r_i)(\ln p_i)}{\sum_{i=1}^{n} (\ln r_i)^2}.$$

θ^* will be referred to as the fitted estimator.

In order to obtain a maximum likelihood estimator of θ we will assume a conditional probability distribution on the subjective probability p, given the objective probability p':

(13) $$f(p|p') = Q(p')p^{\delta(p')} I_{B(p')}(p),$$

where $I_A(x)$ is the indicator function for the set A,

(14) $$B(p') = \{x: \ .5-|p'-.5| \le x \le .5+|p'-.5|\};$$

and

(15) $$\delta(p') = \begin{cases} -p' & \text{if } p' < .5 \\ 0 & \text{if } p' = .5 \\ 1/p' & \text{if } p' > .5 \end{cases} .$$

This conditional density for the subjective probability has the following properties:

(a) If $p' > .5$, the distribution indicates that the subjective assessment of probability will be less than the stated objective probability,

(b) If $p' < .5$, the distribution indicates that the subjective assessment of probability will be greater than the stated objective probability,

(c) If $p' = .5$, the subjective assessment of probability will be exact because the conditional distribution of p given $p' = .5$ is degenerate at .5, and

(d) The support of the distribution decreases as the objective probability approaches .5, which means that there is greater variation for very large or very small probabilities.

These properties were selected based on the reports of Preston and Baratta [6] and Mosteller and Nogee [4] in which experiments indicated that large probabilities are underestimated and small probabilities are overestimated.

Since we want to use this conditional distribution and the model (6) to estimate θ based on the observations (r_i, p_i'); $i = 1, 2, \ldots, n$, we will make the transformation

$$r = p^{1/\theta}.$$

This yields the conditional density of the bid ratio r,

$$(16) \qquad f(r|p') = Q(p')\theta r^{\theta\delta(p')+\theta-1} I_{R(p')}(r)$$

where

$$(17) \qquad R(p') = \{r: \; (.5-|p'-.5|)^{1/\theta} \le r \le (.5+|p'-.5|)^{1/\theta}\}.$$

The resulting maximum likelihood estimator of θ is

$$(18) \qquad \hat{\theta} = \begin{cases} \hat{\theta}_A \text{ if } \hat{\theta}_B < \hat{\theta}_A \\ \hat{\theta}_B \text{ if } \hat{\theta}_A \le \hat{\theta}_B \le \hat{\theta}_C \\ \hat{\theta}_C \text{ if } \hat{\theta}_C < \hat{\theta}_B \end{cases}$$

where

$$(19) \qquad \hat{\theta}_A = \max_i \left\{ \frac{\ln(.5+|p_i'-.5|)}{\ln r_i} \right\},$$

$$(20) \qquad \hat{\theta}_B = \frac{-n}{\sum_{i=1}^{n} (\delta(p_i)+1)\ln r_i},$$

$$(21) \qquad \hat{\theta}_C = \min_i \left\{ \frac{\ln(.5-|p_i'-.5|)}{\ln r_i} \right\}.$$

The immediate advantage this estimator has over the previous two estimators is that it enjoys the well-known properties associated with a maximum likelikhood estimator. For a comprehensive survey of these properties see Norden [5].

Comparison of Estimators

In order to compare the three estimation procedures introduced above, the following experimental procedure was employed: One replication consisted of offering an individual 40 wagers consisting of eight different objective probabilities (.1, .2, .3, .4, .6, .7, .8, .9) and five different values for a (10, 30, 50, 70, 90) which should be considered as some monetary unit. Actual execution of this experiment would be valuable for estimating θ; however this study investigates only a simulation of this experiment.

Five different values of θ were fixed in the simulation process: .5, .6, .7, .8, .9. Two different conditional distributions were employed in the generation of the subjective probability assessment. For a fixed p', the following distribution was used in addition to the distribution given by (13):

$$(22) \qquad f(p|p') = \begin{cases} 5 \text{ if } p' - .2 < p < p' \text{ if } p' > .5 \\ 5 \text{ if } p' < p < p' + .2 \text{ if } p' < .5. \end{cases}$$

Five hundred replications were executed for each fixed θ and specified conditional distribution of p. The data is summarized in Table 1. This table gives the mean of the 500 estimates generated with a plus or minus term, which is twice the estimate of the standard deviation of the estimate of θ, i.e., $2\hat{\sigma}_{\hat{\theta}}$ where the caret (^) should be replaced by the tilda (~) or star (*) when using one of the other two estimation procedures. The maximum likelihood estimator definitely estimates θ the best. The MLE always has a smaller $\hat{\sigma}_{\hat{\theta}}$. In only one case did $\tilde{\theta}$ estimate θ closer than $\hat{\theta}$, and in that instance, again, $\hat{\sigma}_{\tilde{\theta}} > \hat{\sigma}_{\hat{\theta}}$. Also, it should be noted that in some instances the fitted estimator θ^* was closer than $\tilde{\theta}$.

TABLE 1

Estimates of the Exponent in the Power Utility Function

Actual Value of θ		.5	.6	.7	.8	.9
Conditional Distribution (13)	$\hat{\theta}$	.4959 $\pm$.0342	.5951 $\pm$.0419	.6943 $\pm$.0496	.7935 $\pm$.0564	.8925 $\pm$.0624
	$\tilde{\theta}$	.5806 $\pm$.1016	.6852 $\pm$.1201	.7895 $\pm$.1381	.8938 $\pm$.1563	.9980 $\pm$.1747
	θ^*	.5789 $\pm$.1052	.6947 $\pm$.1266	.8105 $\pm$.1475	.9262 $\pm$.1684	1.0420 $\pm$.1895
Conditional Distribution (22)	$\hat{\theta}$	.5008 $\pm$.0220	.6009 $\pm$.0279	.7011 $\pm$.0337	.8012 $\pm$.0375	.9013 $\pm$.0419
	$\tilde{\theta}$	.5046 $\pm$.0362	.6035 $\pm$.0443	.7024 $\pm$.0523	.8014 $\pm$.0590	.9003 $\pm$.0662
	θ^*	.6087 $\pm$.0466	.7304 $\pm$.0564	.8521 $\pm$.0655	.9739 $\pm$.0747	1.0956 $\pm$.0838

The experiment as summarized by Table 1 was carefully designed so that complementary events were included repeatedly. This was done so that the estimator $\tilde{\theta}$ could be formed. In an actual situation it is doubtful that 40 different wagers would be practical. Also, complementary events might not be included in all cases which would mean that the evaluation of $\tilde{\theta}$ could not use all the data. Furthermore, even if complementary events exist, the number of wagers for each set of complementary events may not be the same. The evaluation of $\tilde{\theta}$ does not consider this possibility. All of these comments give more credence to the proposed maximum likelihood procedure.

REFERENCES

[1] Adams, E., and Fagot, R. "A Model of Riskless Choice." Behavior Science 4, No. 1 (Jan. 1959): 1-10.

[2] Becker, G. M., DeGroot, M. H., and Marschak, J. "Stochastic Models of Choice Behavior." Behavior Science 8, No. 1 (Jan. 1963): 41-55.

[3] Becker, G. M., DeGroot, M. H., and Marschak, J. "Measuring Utility by a Single-Response Sequential Method." Behavior Science 9, No. 3 (July 1964): 226-32.

[4] Mosteller, F., and Nogee, P. "An Experimental Measurement of Utility." Journal of Political Economy 59 (1951): 371-404.

[5] Norden, R. H. "A Survey of Maximum Likelihood Estimation, Part I and II." International Statistical Review 40, No. 3 (Dec. 1972): 329-54; and 41, No. 1 (Apr. 1973): 39-58.

[6] Preston, M. G., and Baratta, P. "An Experimental Study of the Auction-Value of an Uncertain Outcome." American Journal of Psychology 61 (1948): 183-93.

[7] Stevens, S. S. "Measurement, Psychophysics, and Utility." In Measurement: Definitions and Theories. Edited by C. W. Churchman and P. Ratoosh. New York: Wiley, 1959, pp. 18-63.

[8] Tversky, A. "A General Theory of Polynomial Conjoint Measurement." Journal of Mathematical Psychology 4 (1967): 1-20.

[9] Tversky, A. "Utility Theory and Additivity Analysis of Risky Choices." Journal of Experimental Psychology 75, No. 1 (1967): 27-36.