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EFFECT OF ELECTROMAGNETIC FIELDS ON GRAVITATIONAL STABILITY OF A VISCOUS FIELD

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SUMMARY

The effect of electromagnetic fields upon the stability of a conductive viscous fluid with a negative temperature gradient is investigated. The numerical relationship between the imposed electromagnetic field and the critical temperature-gradient has been found. In most cases considered, convection is inhibited by the electromagnetic field, but the effectiveness of the inhibition has been found to depend very markedly on the mode of convection.

CHAPTER I

PRELIMINARIES

1. Introduction

The stability of a horizontal layer of fluid under a negative vertical temperature gradient was studied by Rayleigh(1), Jefferys(2), Southwell and Pellew(3) and others. The stability of an infinitely high column of fluid with a negative vertical temperature gradient has been investigated by Hale(4), Taylor(5), and Yih(6). These authors relate the temperature gradient (β) to the dimensions of the fluid-container, the viscosity (ν), the thermal diffusivity (κ), and the expansivity (α) of the fluid, and to gravitational body-force per unit mass (g), at the point of instability. Chandrasekhar has found that the stability of an electrically conducting fluid can be affected by the presence of an electromagnetic field. In this work, the instability studied by Hale, influenced by imposed electromagnetic fields, is investigated. The relationship between the imposed magnetic field, the magnetic diffusivity (η) of the fluid, and the other pertinent variables affecting the stability of an infinitely high column of viscous fluid (of various cross-sections) is sought. For the fluid contained between two walls, three cases will be discussed, with the imposed magnetic field being either in the direction of the (vertical) temperature gradient, or normal to the walls, or perpendicular to both of these directions. For the fluid contained in a circular vertical tube, the imposed field is also vertical. For the fluid contained in a rectangular vertical tube, the effect of a vertical electric current on the stability of the fluid is investigated.

Instability of a quiescent fluid occurs, in general, if the loss of potential energy associated with a certain mode is larger than the energy-dissipation for this mode. The effect of viscosity is therefore to inhibit the instability of quiescent fluids. It will be shown in this work that an imposed magnetic field, which will give rise to electrical currents and therefore energy-dissipation in case convection takes place, also has a general stabilizing effect, which is different for different modes of convection.

2. Governing Equations

For stationary fluid, the equation governing the x_3- dependent imposed temperature (T) distribution is:

$$\nabla^2 T = 0 \tag{1}$$

in which

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

is the Laplacian, where Cartesian coordinates x_1, x_2, and x_3 are chosen, with x_3 axis being vertical. The equations of equilibrium for a stationary fluid under the action of an imposed current $\vec{J}$ and magnetic field $\vec{H}$ is:

$$0 = -\nabla p + \mu\vec{J} \times \vec{H} + \rho_0 \vec{g}\,(1-\alpha T) \tag{2}$$

where p is the pressure and ρ_0 is the density of the fluid at some reference temperature T_0.

The equation of magnetic diffusion [see Reference 7, Equations (1-10)] is:

$$\left(\frac{\partial}{\partial t} - \eta\nabla^2\right)\vec{H} = \nabla \times (\vec{u} \times \vec{H}) \tag{3}$$

where $\vec{u}$ is the velocity of the fluid. Therefore, in the case of no motion, Equation (3) reduces to:

$$(\frac{\partial}{\partial t} - \eta\nabla^2)\,\vec{H} = 0 \qquad (3)$$

If the perturbation scalar quantities are denoted by prime, p' and T', and perturbation vector quantities by small letters, $\vec{j}$, $\vec{h}$ and $\vec{u}$ (unprimed or capital letters denote the mean quantities p, T, $\vec{H}$, and $\vec{J}$), Equations (1) to (3) become:

$$\frac{D}{Dt}\,(T+T') = \kappa\,\nabla^2(T+T') \quad , \qquad (4)$$

$$\rho_o\,[\frac{\partial}{\partial t} + (\vec{u}\nabla) - \nu\nabla^2]\,\vec{u} = -\nabla(p+p') + \mu(\vec{J}+\vec{j})\times(\vec{H}+\vec{h}) + \rho_o\,\vec{g}\,[1 - \alpha(T+T')] \quad , \qquad (5)$$

and

$$(\frac{\partial}{\partial t} - \eta\nabla^2)(\vec{H}+\vec{h}) = \nabla\times\,[\vec{u}\times(\vec{H}+\vec{h})] \quad . \qquad (6)$$

Subtracting Equations (1) to (3) from Equations (4) to (6) and neglecting products of perturbation quantities and the effect of expansivity on inertia, one has $(\frac{\partial}{\partial t} - \kappa\nabla^2)\,T' = -\,\beta u_3$, (7)

$$\rho_o(\frac{\partial}{\partial t} - \nu\nabla^2)\,\vec{u} = -\nabla p' + \mu[\vec{J}\times\vec{h} + \vec{j}\times\vec{H}] - \rho_o\,\vec{g}\alpha T' \quad , \qquad (8)$$

and

$$(\frac{\partial}{\partial t} - \eta\nabla^2)\,\vec{h} = \nabla\times(\vec{u}\times\vec{H}) \quad , \qquad (9)$$

in which, since T has only a vertical gradient

$$\beta = \frac{\partial T}{\partial x_3} \text{ and } \frac{\partial T}{\partial x_1} = \frac{\partial T}{\partial x_2} = 0 \; .$$

The equations of continuity for velocity and magnetic fields are:

$$\nabla.\vec{u} = 0 \quad , \qquad (10)$$

$$\nabla.\vec{h} = 0 \quad , \qquad (11)$$

and the Maxwell equation relating current to the magnetic field is:

$$4\pi\vec{j} = \nabla \times \vec{h} \tag{12}$$

Equations (7) to (12) govern the phenomenon under investigation.

3. Boundary Conditions

In all cases considered in this work, fluid is assumed to be contained within solid boundaries. At such a boundary, the condition of no slip demands that all velocity components vanish.

Furthermore, continuity of heat flow, as well as temperature at the solid-fluid boundary requires (see Figure 1) that:

$$T' = T'^{*} \quad ,$$

$$k^{*} \frac{\partial T'^{*}}{\partial x_1} = k \frac{\partial T'}{\partial x_1} \quad ,$$

where the k's are the coefficients of thermal conductivity, and where asterisk denotes that the quantity involved is a function or a constant pertaining to the solid.

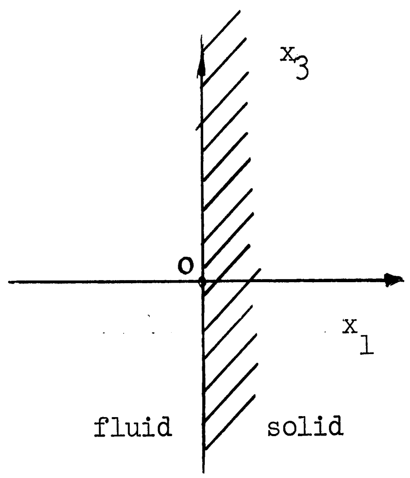

Figure 1. Solid Fluid Boundary.

If $k^* << k$,

$$\frac{\partial T'}{\partial x_1} = \frac{k^*}{k}\frac{\partial T'^*}{\partial x_1} = 0,$$

$\frac{\partial T'^*}{\partial x_1}$ being finite.

If $\kappa^* >> \kappa$,

$$T' = T'^* = 0,$$

as throughout the wall $T'^* = 0$. It is worth noting that the boundary conditions $T' = 0$ or $\frac{\partial T'}{\partial x_1} = 0$ can be imposed, in practice, not only by the choice of proper wall material, but also by varying the thickness of the wall.

The boundary conditions to be imposed upon electromagnetic quantities can be formulated by the use of Maxwell equations. Due to the continuity of magnetic flux,

$$\mu\, h_1 = \mu^*\, h_1^*,$$

on the boundary. Continuity of tangential components of the magnetic field demands that:

$$h_2 = h_2^*$$

and

$$h_3 = h_3^*$$

on the boundary. The last two conditions imply that:

$$4\pi j_1 = \left(\frac{\partial h_2}{\partial x_3} - \frac{\partial h_3}{\partial x_2}\right) = \left(\frac{\partial h_2^*}{\partial x_3} - \frac{\partial h_3^*}{\partial x_2}\right) = 4\pi j_1^*,$$

or continuity of current. Since the velocity vanishes on the boundary, continuity of the electric field demands that:

$$\frac{1}{\sigma}\, j_2 = \frac{1}{\sigma^*}\, j_2^*$$

and

$$\frac{1}{\sigma} j_3 = \frac{1}{\sigma^*} j_3^*$$

on the boundary.

Phenomena under consideration will be accompanied by the accumulation of surface charge on the boundary. Surface charge, as well as displacement current, can justifiably be neglected in this work, as it is electrodynamical in its nature and deals mainly in time independent cases.

Here again, if σ's and μ's are of comparable size, variations of electromagnetic quantities would penetrate into the wall. But if $\mu^* >> \mu$ flux continuity is maintained by $h_1^* << h_1$. It can be shown that generally the magnetic field h_1^* is not parallel to the boundary, and hence $h_1 >> h_2$ and $h_1 >> h_3$, or simply

$$h_2 = h_3 = 0$$

on the boundary. Also, $\sigma^* >> \sigma$ implies

$$j_2 = \frac{\sigma}{\sigma^*} j_2^* \quad \text{and} \quad j_3 = \frac{\sigma}{\sigma^*} j_3^*$$

at the boundary, while current continuity demands: $j_1 = j_1^*$ there. Again, if j_1^* is not parallel to the boundary:

$$j_2 = j_3 = 0$$

on the boundary.

CHAPTER II

VERTICAL MAGNETIC FIELD: FLUID BETWEEN WALLS

The effect of imposed, uniform vertical magnetic field (H_3) on the stability of a conductive fluid contained between two walls and heated from below will be investigated. The destabilizing factor, in this and other cases investigated in this work, is the negative temperature gradient.

4. Reduction to Ordinary Differential Equations

Since the imposed magnetic field is uniform, the mean current is zero. The equations of motion are therefore, in this case:

$$\rho_o \left(\frac{\partial}{\partial t} - \nu\nabla^2\right) u_i = - \frac{\partial}{\partial x_i}\left(p' + \mu \frac{H_3}{4\pi} h_3\right) + (0, 0, \rho_o g\alpha T') + \mu \frac{H_3}{4\pi} \frac{\partial h_i}{\partial x_3} , \qquad (1)-$$

and the equations of magnetic diffusion are:

$$\left(\frac{\partial}{\partial t} - \eta\nabla^2\right) h_i = H_3 \frac{\partial u_i}{\partial x_3} \qquad (4)-(6)$$

The equations of continuity are:

$$\frac{\partial u_i}{\partial x_i} = 0 , \qquad (7)$$

$$\frac{\partial h_i}{\partial x_i} = 0 , \qquad (8)$$

where i = 1, 2 and 3. The disturbance temperature is governed by

$$\left(\frac{\partial}{\partial t} - \kappa \nabla^2\right) T' = - \beta u_3 . \qquad (9)$$

The boundary conditions for the velocity at the solid wall are:

$$u_i = 0 \quad \text{at} \quad x_1 = \pm d .$$

Assuming the wall to be highly conductive electrically but insulative thermally,

$$j_2 = j_3 = 0 \quad \text{and} \quad \frac{\partial T'}{\partial x_1} = 0 \quad \text{at} \quad x_1 = \pm d.$$

This is a rather artificial case, as walls which are very conductive electrically are not normally insulating thermally. It must be remembered, however, that instability under consideration is gravitational in its nature, and gravitational instability can be induced by an adverse density gradient caused by a variation of concentration of solute. The mathematical formulation for the problem in that case remains the same, but an electrically conductive wall impermeable to mass diffusion is now not at all artificial. For convenience, density gradient shall be considered as effected by temperature variation, bearing in mind that any artificiality in the boundary conditions may disappear when the density gradient is induced otherwise.

A two-dimensional case will now be considered; it will be assumed that $u_2 = 0$ identically, and that all variations of pertinent quantities with respect to x_2 are zero.

The boundary condition on j_3 can thus be simplified to:

$$4\pi j_3 = 0 = \frac{\partial h_2}{\partial x_1} - \frac{\partial h_1}{\partial x_2} = \frac{\partial h_2}{\partial x_1} \doteq 0 \quad \text{at} \quad x_1 = \pm d \ .$$

The differential equation governing h_2 is:

$$\left(\frac{\partial}{\partial t} - \eta\nabla^2\right) h_2 = 0 \ .$$

A time-dependent solution for h_2 is in general non-trivial. However, if the solution for h_2 is of the form $h_2 = \widetilde{h_2}\,(x_1, x_3) \exp\,(\gamma t)$, γ will always be real and negative (see the more rigorous proof given later). Consequently, h_2 will be taken to be identically zero.

Cross-differentiation of Equations (1) and (3) produces:

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) = \frac{\mu H_3}{4\pi}\frac{\partial}{\partial x_3}\left(\frac{\partial h_1}{\partial x_3} - \frac{\partial h_3}{\partial x_1}\right)$$

$$- \rho_o g\alpha \frac{\partial T'}{\partial x_1} . \qquad (10)$$

Since the fluid flow and the magnetic field are both two-demensional, the equations of continuity (7) and (8) permit the use of the stream functions ψ and χ such that:

$$u_1 = \frac{\partial\psi}{\partial x_3},\ u_3 = -\frac{\partial\psi}{\partial x_1},\ \frac{h_1}{H_3} = \frac{\partial\chi}{\partial x_3} \quad \text{and} \quad \frac{h_3}{H_3} = -\frac{\partial\chi}{\partial x_1} .$$

Substituting these values of h_i and u_i into Equations (4) and (6), one obtains:

$$\left(\frac{\partial}{\partial t} - \eta\nabla^2\right)\frac{\partial\chi}{\partial x_3} = \frac{\partial}{\partial x_3}\left(\frac{\partial\psi}{\partial x_3}\right) \qquad (4')$$

and

$$\left(\frac{\partial}{\partial t} - \eta\nabla^2\right)\frac{\partial\chi}{\partial x_1} = \frac{\partial}{\partial x_3}\left(\frac{\partial\psi}{\partial x_1}\right) . \qquad (6')$$

Equations (4') and (6') can be integrated to yield:

$$\left(\frac{\partial}{\partial t} - \eta\nabla^2\right)\chi = \frac{\partial\psi}{\partial x_3} . \qquad (11)$$

In this operation, the constant of integration can be omitted without affecting u_i or h_i. Substituting the expressions for u_i and h_i in terms of stream functions into Equations (9) and (10), one obtains:

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\nabla^2\psi = \frac{\mu H_3^2}{4\pi\rho_o}\frac{\partial}{\partial x_3}(\nabla^2\chi) - \rho_o g\alpha\frac{\partial T'}{\partial x_1} \qquad (10')$$

and

$$\left(\frac{\partial}{\partial t} - \kappa\nabla^2\right) T' = \beta \frac{\partial\psi}{\partial x_1} \quad . \tag{9'}$$

Equations (9'), (10'), and (11) are the governing differential equations. These shall be non-dimensionalized with the aid of the following substitutions:

$$x = \frac{x_1}{d}, \quad z = \frac{x_3}{d} \quad \text{and} \quad \tau = (\kappa t)/d^2.$$

The transformed equations are:

$$\left(\frac{\kappa}{d^2} \frac{\partial}{\partial\tau} - \frac{\kappa}{d^2}\bar{\nabla}^2\right) T' = \frac{\beta}{d} \frac{\partial\psi}{\partial x}, \tag{9''}$$

$$\left(\frac{\kappa}{d^2} \frac{\partial}{\partial\tau} - \frac{\nu}{d^2}\bar{\nabla}^2\right)\left(\frac{1}{d^2}\bar{\nabla}^2\psi\right) = (4\pi) \frac{\mu^2\sigma H_3^2}{4\pi\rho_o} (\eta) \frac{1}{d^3} \frac{\partial}{\partial z} (\bar{\nabla}^2\chi)$$

$$- \frac{g\alpha}{d} \frac{\partial T'}{\partial x}, \tag{10''}$$

$$\left(\frac{\kappa}{d^2} \frac{\partial}{\partial\tau} - \frac{\eta}{d^2}\bar{\nabla}^2\right) \chi = \frac{1}{d} \frac{\partial\psi}{\partial z}, \tag{11''}$$

in which

$$\bar{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \quad .$$

In the following, one will assume the mode of convection to be:

$$\psi = \nu\, f(x) \cos(az) \exp(\gamma\tau),$$

$$T' = (\beta d)\, \theta(x) \cos(az) \exp(\gamma\tau),$$

$$\chi = d\, s(x) \sin(az) \exp(\gamma\tau).$$

The corresponding solution is quite general, as a large class of x_2-independent disturbances can be represented by:

$$\psi = \sum_i \sum_j \nu f_{ij}(x) \cos(a_i z) \exp(\gamma_j \tau)$$

and similar expressions for T' and χ. In view of the linearity of the differential equations, an analysis of a single mode can be applied to any disturbance expressible as a summation of the basic modes.

Substituting ψ, T', and χ into (9'') to (11'), one obtains:

$$[\gamma - (D^2 - a^2)]\ \theta(x) = P_r\ Df(x), \tag{12}$$

$$[\gamma - P_r(D^2-a^2)](D^2-a^2)f(x) = Q\ a\ (D^2-a^2)[\frac{\eta}{\kappa}\ s(x)] + RD\theta, \tag{13}$$

$$[\gamma - \frac{\eta}{\kappa}\ (D^2 - a^2)]\ s(x) = -\ a\ P_r\ f(x), \tag{14}$$

in which D denotes $\frac{d}{dx}$, P_r is Prandtl number $\frac{\nu}{\kappa}$, R is Rayleigh's number $-\ \frac{g\alpha\beta d^4}{\nu\kappa}$ and $Q = \frac{\mu^2\sigma(H_3)^2d^2}{\rho_o\nu}$.

The boundary conditions shall be expressed in terms of the newly-defined functions. Since $\frac{\partial T'}{\partial x_1} = 0$ at $x_1 = \pm\ d$, one obtains

$$D\theta = 0 \quad \text{at } x = \pm\ 1\ .$$

Conditions $u_1 = u_3 = 0$ at $x_1 = \pm\ d$, or $\frac{\partial\psi}{\partial x_1} = \frac{\partial\psi}{\partial x_3} = 0$ at $x_1 = \pm\ d$, become

$$Df = af = 0 \quad \text{at} \quad x = \pm\ 1\ .$$

From Equations (13) and (14), one notes that the magnetic field is unlinked with the differential system for f and θ if a, the wave number, is zero. Thus vertical magnetic field has no effect on a zero-wavenumber convection mode, which is the most critical one; it does have effect on other modes. Hence, assuming here $a \neq 0$, boundary conditions on f are:

$$f = Df = 0 \quad \text{at} \quad x = \pm\ 1$$

The condition $j_2 = 0$ at $x_1 = \pm d$ yields

$$\nabla^2 \chi = 0 \quad \text{at} \quad x_1 = \pm d \quad ,$$

or in terms of s(x):

$$(D^2 - a^2)\, s(x) = 0 \quad \text{at } x = \pm 1 \quad .$$

5. Solution

The value of γ in Equations (12) to (14) assumed complex ($\gamma = \gamma_r + i\gamma_i$, where γ_r and γ_i are real), indicates whether the mode tends to grow ($\gamma_r > 0$), decay ($\gamma_r < 0$), or oscillate ($\gamma_i \neq 0$). It can be proved that when the mode considered is neutrally stable $\gamma_i = 0$ (see Chapter IV). A non-trivial time-independent solution for Equations (12) to (14) and the corresponding relationship between a, Q and R will therefore be sought.

Setting $\gamma = 0$ and combining Equations (13) and (14), one obtains:

$$(D^2 - a^2)^2\, (P_r f) = -Q\, a^2 (P_r f) - RD\theta, \tag{15}$$

$$(D^2 - a^2)\, \theta(x) = -D(P_r f) \quad . \tag{12}$$

The boundary conditions are now:

$$f = Df = 0 \quad \text{at} \quad x = \pm 1 \; ,$$

$$D\theta = 0 \quad \text{at} \quad x = \pm 1 \quad .$$

The last condition can, with the aid of Equation (15) and the boundary condition imposed on f, be written as:

$$(D^2 - a^2)^2\, f = 0 \quad \text{at} \quad x = \pm 1 \quad .$$

In order to utilize a solution obtained by Yih[6], the operator L will be defined:

$$L \equiv D^2 - a^2 \; .$$

Combining Equations (12) and (15) and rewriting the boundary conditions governing f, one obtains:

$$[L^3 - (R - Qa^2)\,L - Ra^2]\,f = 0 \qquad (16)$$

$$f = Df = L^2 f = 0 \quad \text{at} \quad x = \pm 1 \quad .$$

The differential system for the eigenfunctions in[6] [Equation (46)] is:

$$(L^3 - RL - Ra^2)\,f = 0 \qquad (46)$$

$$f = Df = L^2 f = 0 \quad \text{at} \quad x = \pm 1 \quad .$$

In [6] the author deals with the case considered here, not accounting for electromagnetic effect. The symbols a and f have the same meanings as here. Designating quantities of the quoted paper by a bar, one is given $\bar{R} = \bar{R}(\bar{a})$ for non-trivial solutions of Equation (46) satisfying the boundary conditions. Any pair $(\bar{R}, \bar{a})$, when substituted in Equation (46), would correspond to an eigenfunction. The same eigenfunction is obtained from Equation (16) if one sets:

$$R - Qa^2 = \bar{R} \quad ,$$

$$\bar{R}\,\bar{a}^2 = R \quad .$$

Thus the relationship between Q, R, and a is readily obtained from $\bar{R} = \bar{R}(\bar{a})$ given by Yih. The results are plotted in Figure (2).

Evidently, for $a \neq 0$, Q raises the Rayleigh-number required for marginal stability. As mentioned before (Section 4), for a = 0 there is no electromagnetic interaction with the flow. Also, if one utilizes $\bar{R}\,(\bar{a})$ of the analogous system and lets a approach zero, one gets $R = \bar{R}$. Thus Q does not affect the critical (lowest) Rayleigh-number required for marginal stability corresponding to a mode independent of x_3.

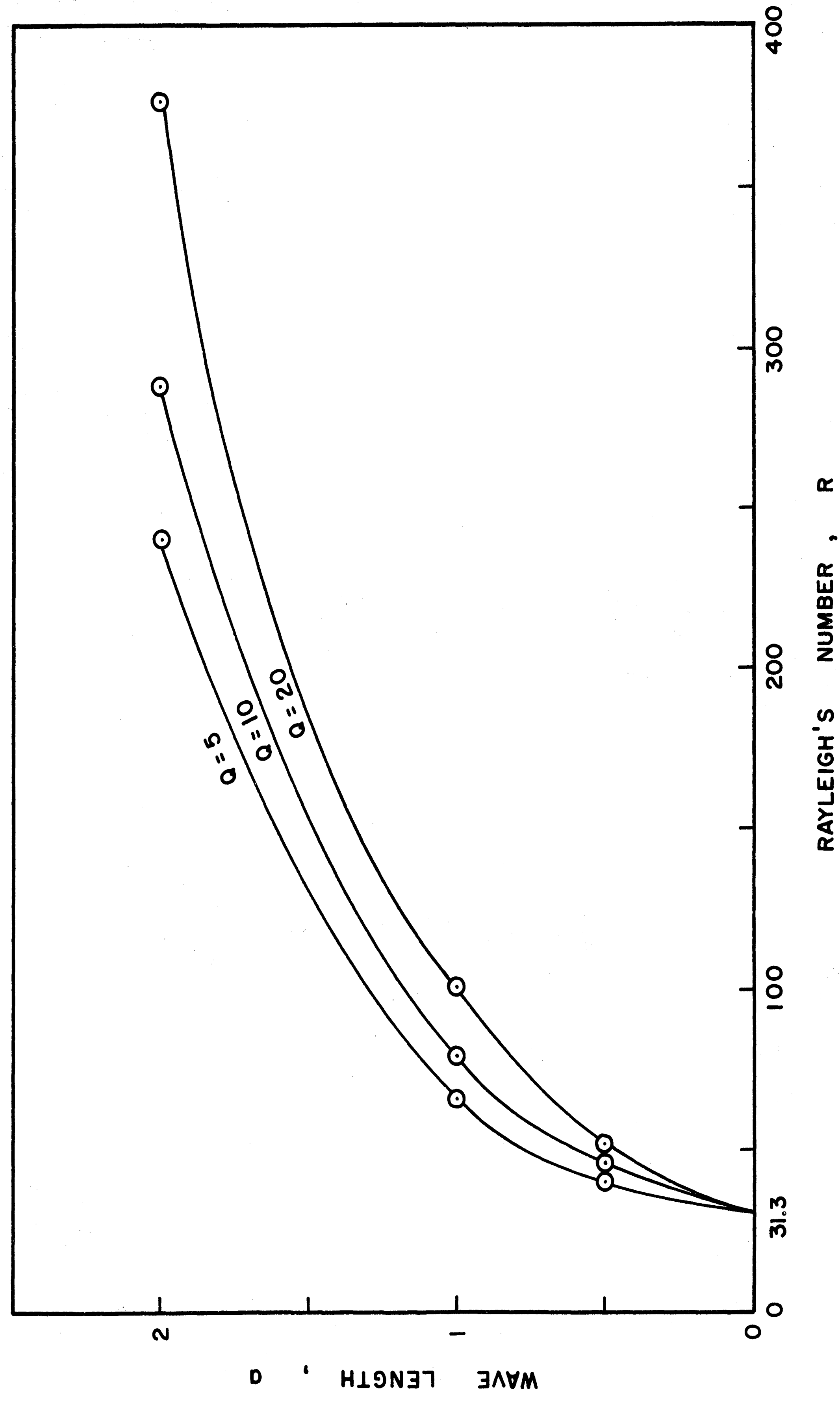

Figure 2. Q-R-a Relationship - H_3 Imposed.

CHAPTER III

HORIZONTAL MAGNETIC FIELD PARALLEL TO WALLS

The stability of a fluid contained between two walls and under a negative temperature gradient β, in the presence of a horizontal magnetic field parallel to the walls will be investigated.

6. Reduction to Ordinary Differential Equations

As the imposed field H_2 is uniform throughout the fluid, there are no imposed currents. The basic Equations (I.7) to (I.8) become in this case,

$$\rho_o\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)u_i = -\frac{\partial}{\partial x_i}\left(p' + \frac{\mu H_2 h_2}{4\pi}\right) + (0,0,\rho_o g\alpha T') + \frac{\mu H_2}{4\pi}\frac{\partial h_i}{\partial x_2}, \qquad (1) - (3)$$

$$\left(\frac{\partial}{\partial t} - \eta\nabla^2\right) h_i = H_2 \frac{\partial u_i}{\partial x_2}, \qquad (4) - (6)$$

$$\frac{\partial u_i}{\partial x_i} = 0, \qquad (7)$$

$$\frac{\partial h_i}{\partial x_i} = 0, \qquad (8)$$

and

$$\left(\frac{\partial}{\partial t} - \kappa\nabla^2\right) T' = -\beta u_3 \qquad (9)$$

The same boundary conditions as in Chapter II will be assumed:

$$\frac{\partial T'}{\partial x_1} = 0, \quad j_3 = j_2 = 0 \quad \text{at} \quad x_1 = \pm d .$$

In this chapter, it will be assumed that $u_2 = h_2 = 0$ identically. However, variations with respect to x_2 will now not be neglected, for otherwise the magnetic field would neither affect nor be affected by the velocity field.

Cross-differentiation of Equations (1) and (3) yields:

$$\rho_o(\frac{\partial}{\partial t} - \nu\nabla^2)\ (\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}) = \frac{\mu H_2}{4\pi}\ \frac{\partial}{\partial x_2}\ (\frac{\partial h_1}{\partial x_3} - \frac{\partial h_3}{\partial x_1})$$

$$- \rho_o g\alpha\ \frac{\partial T'}{\partial x_1}\ . \qquad (10)$$

Again, if $u_2 = h_2 = 0$, it is possible to utilize the stream functions χ and ψ:

$$u_1 = \frac{\partial\psi}{\partial x_3}\ ,\quad u_3 = -\frac{\partial\psi}{\partial x_1},\quad \frac{h_2}{H_2} = \frac{\partial\chi}{\partial x_3} \quad \text{and} \quad \frac{h_3}{H_2} = -\frac{\partial\chi}{\partial x_1}\ .$$

Rewriting Equation (10), one obtains:

$$(\frac{\partial}{\partial t} - \nu\nabla^2)\ \nabla_1^2\psi = \frac{\mu H_2^2}{4\pi\rho_o}\ \frac{\partial}{\partial x_2}\ (\nabla_1^2\chi) - g\alpha\ \frac{\partial T'}{\partial x_1}\ . \qquad (10')$$

Integration of Equations (4) and (6) yields:

$$(\frac{\partial}{\partial t} - \eta\nabla^2)\ \chi = \frac{\partial\psi}{\partial x_2}\quad , \qquad (11)$$

where

$$\nabla_1^2 = \frac{\partial^2}{\partial x_1^{\ 2}} + \frac{\partial^2}{\partial x_3^{\ 2}}\ .$$

Equations (9) to (11) with the boundary conditions, govern the phenomenon under consideration. Equations (5) and (2) are not implied in (9) to (11). However, Equation (5) is indentically satisfied. Equation (2) yields $\frac{\partial p'}{\partial x_2} = 0$, and therefore need not be included in the differential system, as no boundary conditions are given in terms of p'.

With the aid of the substitutions,

$$\frac{x_1}{d} = x, \quad \frac{x_2}{d} = y, \quad \frac{x_3}{d} = z, \quad \tau = \frac{\kappa t}{d^2},$$

$$\psi = \nu f(x) \sin(by) \cos(az) \exp(\gamma\tau),$$

$$\chi = d\ s(x) \cos(by) \cos(az) \exp(\gamma\tau)$$

and

$$T' = (\beta d)\ \theta(x) \sin(by) \cos(az) \exp(\gamma\tau),$$

Equations (9) to (11) become:

$$[\gamma - (D^2 - a^2 - b^2)]\ \theta(x) = P_r\ Df, \tag{12}$$

$$[\gamma - P_r(D^2 - a^2 - b^2)]\ (D^2 - a^2)\ f(x) = -\ Qb\ (D^2 - a^2)[\tfrac{\eta}{\kappa}\ s(x)] + RD\theta, \tag{13}$$

and

$$[\gamma - \tfrac{\eta}{\kappa}(D^2 - a^2 - b^2)]\ s(x) = b\ P_r f(x). \tag{14}$$

The boundary conditions imposed on T' and u_i now become:

$$D\theta = 0 \quad \text{and} \quad f = Df = 0 \quad \text{at } x = \pm 1 .$$

The requirement $j_3 = 0$ reduces to

$$0 = \frac{\partial h_1}{\partial x_2} = \frac{\partial^2 \chi}{\partial x_2\, \partial x_3} \quad \text{at} \quad x_1 = \pm d$$

or

$$s(x) = 0 \quad \text{at} \quad x = \pm 1 .$$

Similarly, the condition $j_2 = 0$ reduces to

$$(D^2 - a^2)\ s = 0 \quad \text{at} \quad x = \pm 1 .$$

In view of Equation (14), only one of the last two conditions should be imposed.

7. Solution by Approximative Series

Equations (12) to (14) indicate that solutions are possible either for $f(x)$ even, $s(x)$ even, and $\theta(x)$ odd, or for $f(x)$ and $s(x)$ odd, and $\theta(x)$ even. This observation shall be used to obtain two classes of solutions by expanding $s(x)$ as an even series and $\theta(x)$ as an odd series, or by expanding $s(x)$ as an odd series and $\theta(x)$ as an even one. In both expansions, the expanded series shall satisfy the boundary conditions imposed upon the functions these series represent.

It can be proved (and will be done in detail) that when the fluid in neutrally stable with, $\gamma = 0$, Equations (12) to (14) become:

$$(D^2 - c^2)\,(R\theta) = -\,R\,D(P_r f) \quad , \tag{12}$$

$$(D^2 - c^2)\,(D^2 - a^2)\,(P_r f) = b(D^2 - a^2)\,[Q\,\tfrac{\eta}{\kappa}\,s(x)] - D(R\theta), \tag{13}$$

$$(D^2 - c^2)\,[Q\,\tfrac{\eta}{\kappa}\,s(x)] = -\,Qb(P_r f) \tag{14}$$

where

$$c^2 = a^2 + b^2 \; .$$

In view of the boundary conditions and the oddness or evenness of $\theta(x)$ and $s(x)$, the following series are chosen to express these functions:

$$(-R\theta) = \sum_{n=1,2,3\ldots}^{\infty} A_n'\cos(n\pi x)\;,\; [\tfrac{\eta}{\kappa}\;Q\;s(x)] = \sum_{n=1,2,3\ldots}^{\infty} B_n'\sin(r \tag{15'}$$

or

$$(-R\theta) = \sum_{1,3,5\ldots}^{\infty} A_n''\sin(\tfrac{n\pi x}{2}),\; [\tfrac{\eta}{\kappa}\,Q\;s(x)]$$

$$= \sum_{1,3,5\ldots}^{\infty} B_n''\cos(\tfrac{n\pi x}{2}) \quad . \tag{15''}$$

Substituting these values into Equation (13), one obtains:

$$(D^2 - c^2)\,(D^2 - a^2)\,(P_r f) = -\sum_{1,2,3}\Big(b[(n\pi)^2 + a^2]B_n' + (n\pi)\,A_n'\Big)\sin(n\pi x), \tag{16'}$$

$$(D^2 - c^2)\,(D^2 - a^2)\,(P_r f) = -\sum_{1,3,5}\Big(b[(\tfrac{n\pi}{2})^2 + a^2]B_n'' - (\tfrac{n\pi}{2})\,A_n'\Big)\cos(\tfrac{n\pi x}{2}) . \tag{16''}$$

One now defines, for convenience:

$$M_a^{n'} = [(n\pi)^2 + a^2], \quad M_a^{n''} = [(\tfrac{n\pi}{2})^2 + a^2],$$

$$M_c^{n'} = [(n\pi)^2 + c^2] \quad \text{and} \quad M_c^{n''} = [(\tfrac{n\pi}{2})^2 + c^2].$$

The general solution of Equations (16') and (16'') is:

$$P_r f(x) = \sum_{1,2,3} \frac{bB_n'(M_a^{n'}) + (n\pi)A_n'}{(M_a^{n'})\,(M_c^{n'})}\Big(-\sin(n\pi x) + K_n' \sinh(ax) + L_n' \sinh(cx) + M_n' \cosh(ax) + N_n' \cosh(cx)\Big) \tag{17'}$$

$$P_r f(x) = \sum_{1,3,5} \frac{b\,B_n''(M_a^{n''}) - (\tfrac{n\pi}{2})\,A_n''}{(M_a^{n''})\,(M_c^{n''})}\Big(-\cos(\tfrac{n\pi x}{2}) + K_n'' \sinh(ax) + L_n'' \sinh(cx) + M_n''\cosh(ax) + N_n'' \cosh(cx)\Big) \tag{17''}$$

Demanding that Equations (17') and (17'') satisfy

$$f = Df = 0 \quad \text{at} \quad x = \pm 1 \;,$$

one can evaluate $K_n' - N_n'$ and $K_n'' - N_n''$ in terms of A_n and B_n, and obtain:

$$M_n' = N_n' = 0, \quad K_n'' = L_n'' = 0,$$

$$L_n' = \frac{-(n\pi)\cos(n\pi)\sinh(a)}{a\cosh(a)\sinh(c) - c\cosh(c)\sinh(a)} ,$$

$$K_n' = \frac{(n\pi)\cos(n\pi)\sinh(c)}{a\cosh(a)\sinh(c) - c\cosh(c)\sinh(a)} ,$$

$$M_n'' = \frac{(\frac{n\pi}{2})\sin(\frac{n\pi}{2})\cosh(c)}{c\sinh(c)\cosh(a) - a\sinh(a)\cosh(c)} ,$$

$$N_n'' = \frac{-(\frac{n\pi}{2})\sin(\frac{n\pi}{2})\cosh(a)}{c\sinh(c)\cosh(a) - a\sinh(a)\cosh(c)} .$$

By substituting newly-acquired values for $(P_r f)$ as well as the expansions for $(R\theta)$ and $(Q_K^n s)$ in Equations (12) and (14), one obtains:

$$\sum_{n=1,2\ldots}^{\infty} (M_c^{n'}) B_n' \sin(n\pi x) = Q\, b \sum_{1,2,3}^{\infty} \frac{b\, B_n' (M_a^{n'}) + (n\pi) A_n'}{(M_a^{n'})(M_c^{n'})} \Big(- \sin(n\pi x) + K_n' \sinh(ax) + L_n' \sinh(cx)\Big) , \qquad (18')$$

$$\sum_{n=1,3,5..}^{\infty} (M_c^{n''}) B_n'' \cos(\frac{n\pi x}{2}) = Q\, b \sum_{1,3,5..} \frac{b\, B_n'' (M_a^{n''}) - (\frac{n\pi}{2}) A_n''}{(M_a^{n''})(M_c^{n''})} \Big(- \cos(\frac{n\pi x}{2}) + M_n'' \cosh(ax) + N_n'' \cosh(cx)\Big) , \qquad (18'')$$

$$\sum_{1,2,3} (M_c^{n'}) A_n' \cos(n\pi x) = - R \sum_{1,2,3} \frac{b(M_a^{n'}) B_n' + (n\pi) A_n'}{(M_a^{n'}) (M_c^{n'})} \Big(- (n\pi) \cos(n\pi x) + K_n' a \cosh (ax) + L_n' c \cosh (cx)\Big) , \qquad (19')$$

and

$$\sum_{1,3,5...} (M_c^{n''}) A_n'' \sin(\frac{n\pi x}{2}) = - R \sum_{1,3,5} \frac{b(M_a^{n''}) B_n'' - (\frac{n\pi}{2}) A_n''}{(M_c^{n''}) (M_a^{n''})} \Big((\frac{n\pi}{2}) \sin (\frac{n\pi x}{2}) + M_n'' a \sinh (ax) + N_n'' c \sinh (cx)\Big) \qquad (19'')$$

The terms sinh (ax), sinh (cx), cosh (ax), and cosh (cx) will now be expanded in terms of sines or cosines, and the coefficients of $\sin(n\pi x)$, $\cos(n\pi x)$, $\sin(\frac{n\pi x}{2})$, or $\cos(\frac{n\pi x}{2})$ on the two sides of Equations (18'), (19'), (18''), and (19'') will be equated. The totality of equations linear in (A_n')s - (B_n')s or (A_n'')s -(B_n'')s yields a secular equation stating that the determinant consisting of the coefficients of the A's and B's is zero. A good approximation is achieved by considering only the first harmonic, terms formed by sine or cosine of (πx) in Equations (18') and (19') and $(\frac{\pi x}{2})$ in Equations (18'') and (19''). This amounts to equating the first element of the determinant to zero. The results are:

$$R = \frac{(M_a^{1'}) (M_c^{1'})^2}{\pi^2 F'} \left(1 + \frac{Q\, b^2}{(M_c^{1'})^2} F' \right) , \qquad (20')$$

where

$$F' = \left(1 + \frac{2\pi^2}{a\ \coth(a) - c\ \coth(c)}\ \left[\frac{1}{(M_a^{1'})} - \frac{1}{(M_c^{1'})}\right]\right),$$

and

$$R = \frac{(M_a^{1''})\ (M_c^{1''})^2}{\pi^2\ F''}\left(\ 1 + \frac{Q\ b^2}{(M_c^{1''})^2} F''\right), \qquad (20'')$$

where

$$F'' = \left(1 + \frac{2\ (\pi/2)^2}{a\ \tanh\ (a)\ -\ c\ \tanh\ (c)}\ \left[\frac{1}{(M_a^{1''})} - \frac{1}{(M_c^{1''})}\right]\right)$$

The effect of the imposed magnetic field on marginal stability has the following features (see Figures 3-6): (a) For $b=0$, Q does not affect R, or convection is not influenced by electromagnetic effect if there are no x_2 variations; (b) For $b \neq 0$, the magnetic field raises R required for convection; (c) The modes for which the stream function ψ is odd with respect to x are more stable than the modes for which this stream function is even.

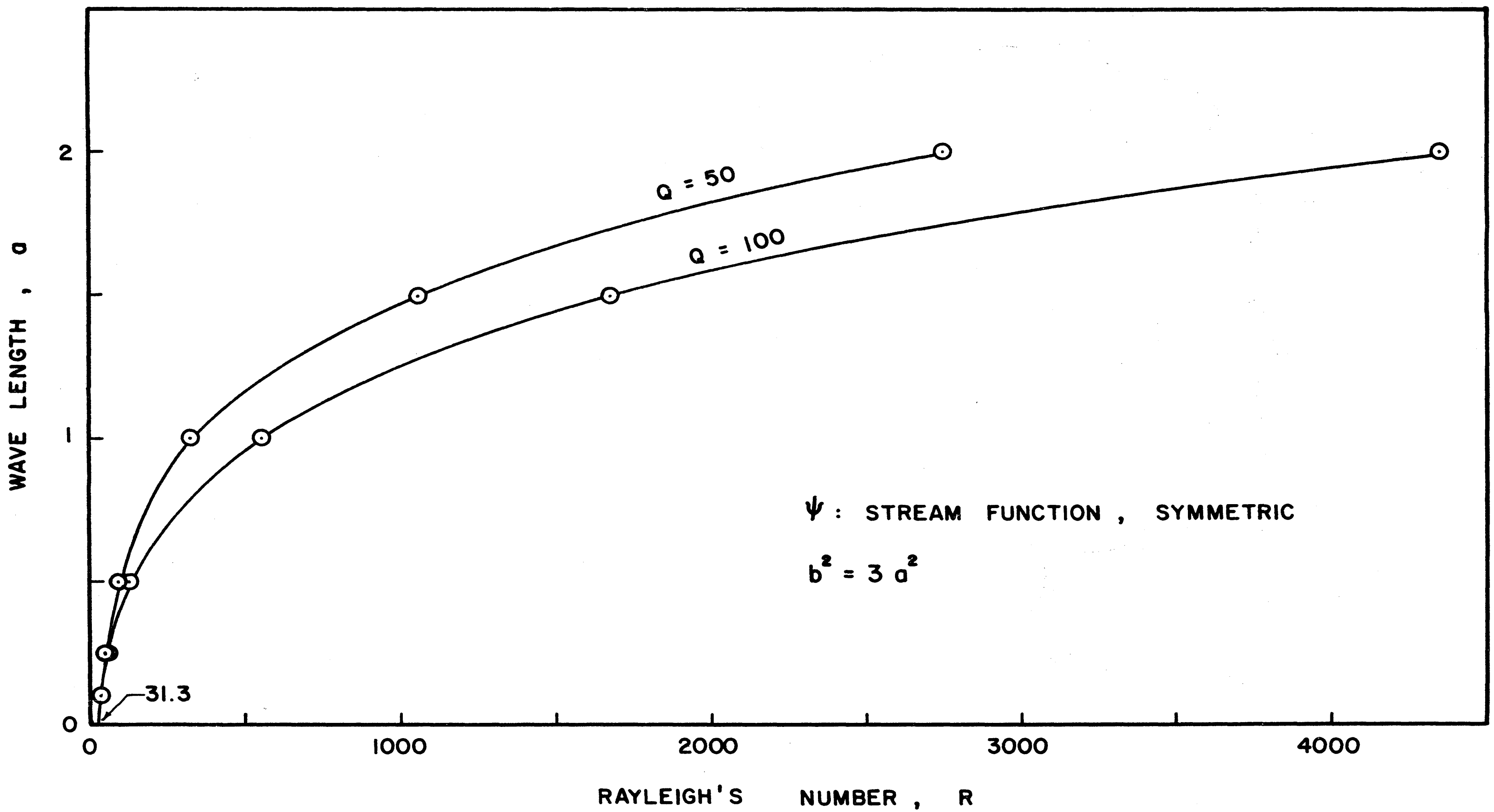

Figure 3. Q-R-a-b Relationship - H_2 Imposed.

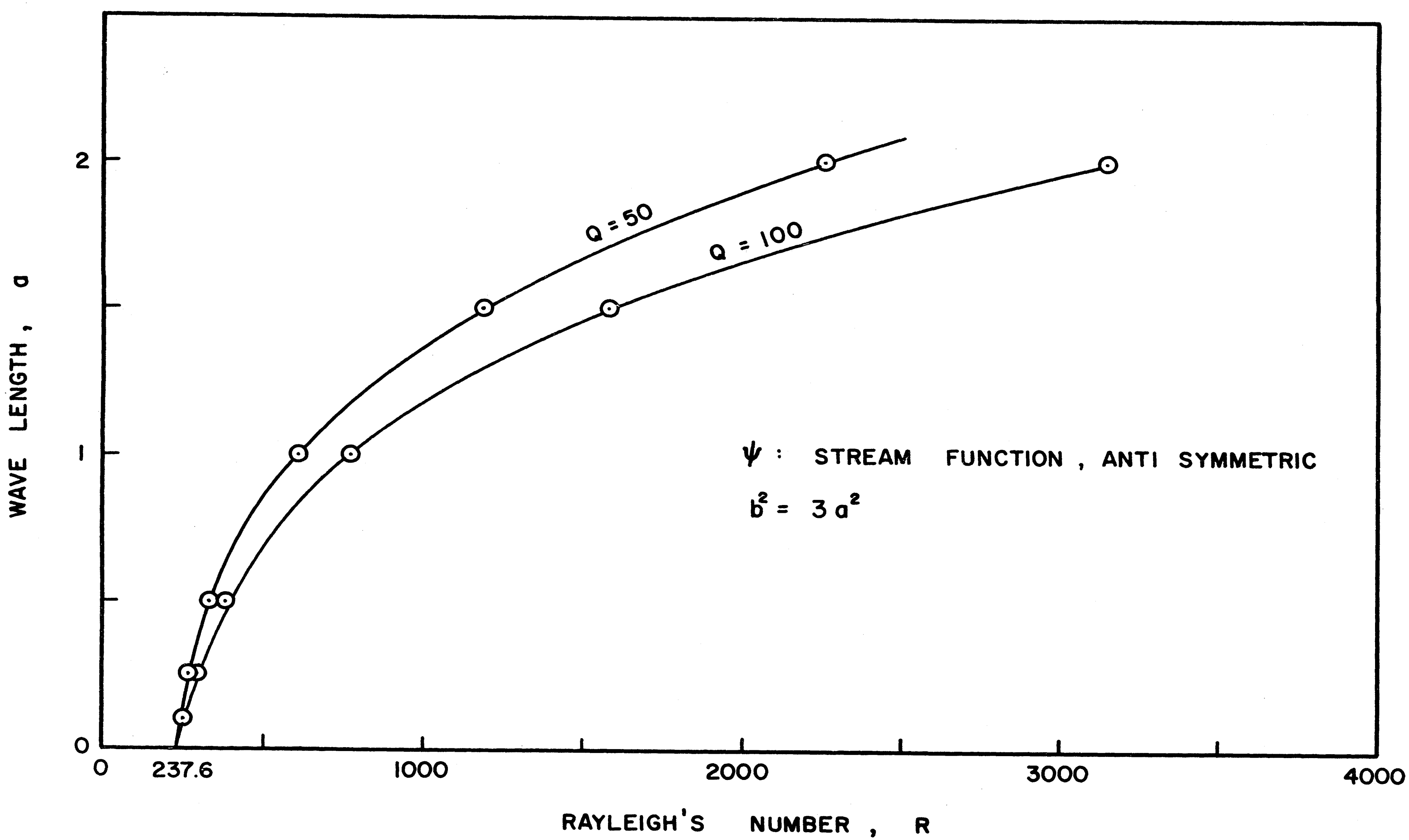

Figure 4. Q-R-a-b Relationship - H_2 Imposed.

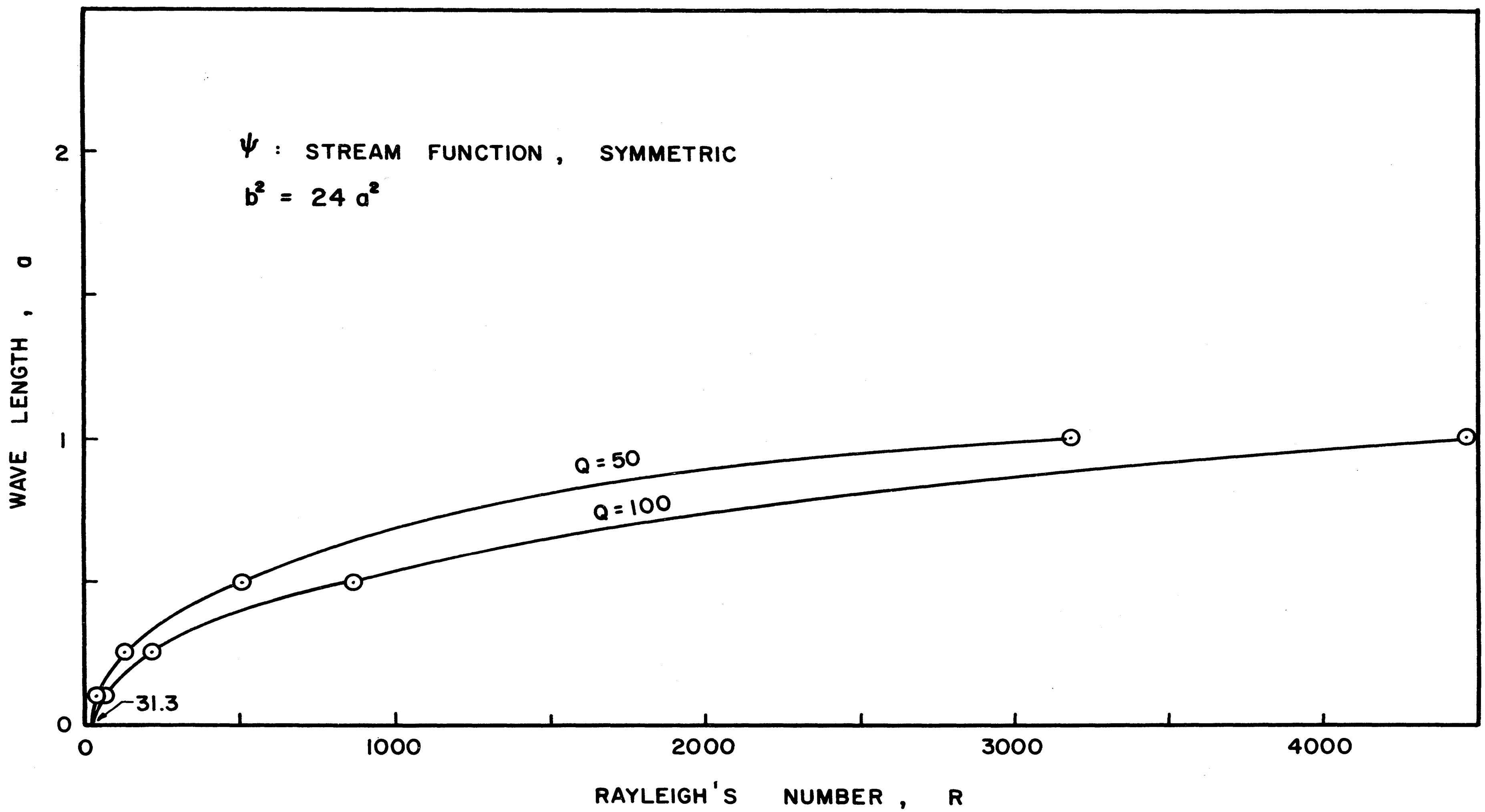

Figure 5. Q-R-a-b Relationship - H_2 Imposed.

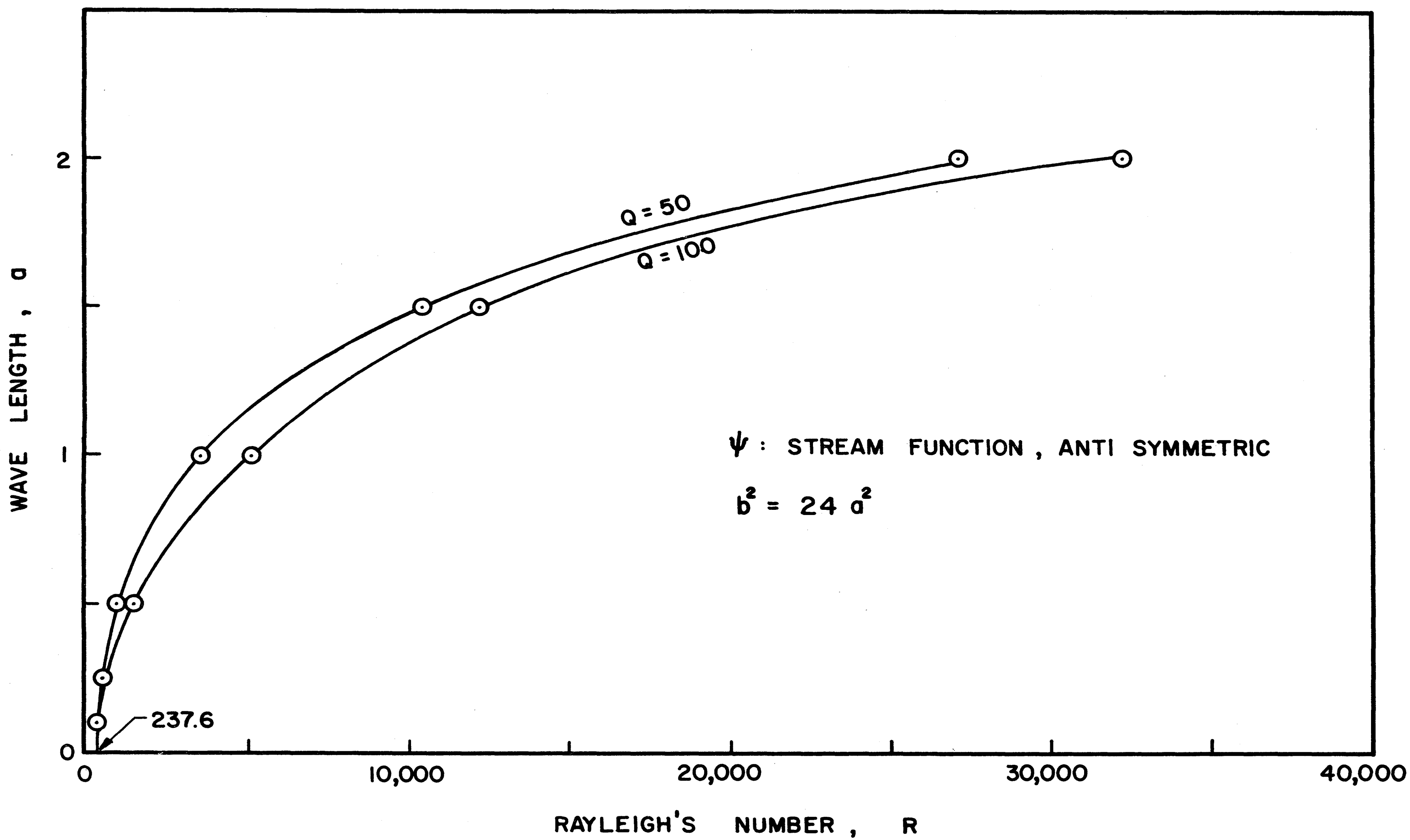

Figure 6. Q-R-a-b Relationship - H_2 Imposed.

CHAPTER IV

MAGNETIC FIELD NORMAL TO WALLS

In this chapter, the stabilizing effect of a magnetic field normal to the two parallel walls containing the fluid is examined. The fluid is again under a negative vertical temperature gradient. Since the modes of convection of any wave length have a vertical velocity component u_3, the term $(\vec{u} \times \vec{H})$ is non-zero for all modes. Consequently, interaction between velocity and magnetic fields will always be present. This interaction will be shown to have a stabilizing effect.

8. Reduction to Ordinary Differential Equations

Since the imposed field is uniform throughout the fluid, there is no imposed current.

The walls are assumed to be, relatively, very permeable magnetically and insulative thermally. This type of wall can in fact be achieved by a thin coating of insulative material on iron walls. If mass rather than heat diffusion is considered, no coating is needed. The boundary conditions would therefore be:

$$\frac{\partial T'}{\partial x_1} = 0 \quad \text{and } h_2 = h_3 = 0 \quad \text{at} \quad x_1 = \pm d \ .$$

The basic equations for this particular case are:

$$\rho_o \left(\frac{\partial}{\partial t} - \nu\nabla^2\right) u_i = - \frac{\partial}{\partial x_i}\left(p' + \frac{\mu H_1 h_1}{4\pi}\right) + (0,0,\rho g\alpha T') + \frac{\mu H_1}{4\pi} \frac{\partial h_i}{\partial x_1} \ , \qquad (1) - (3)$$

$$(\frac{\partial}{\partial t} - \eta\nabla^2)\, h_i = H_1 \frac{\partial u_i}{\partial x_1} \quad , \qquad (4) - (6)$$

$$\frac{\partial u_i}{\partial x_i} = 0 \quad , \qquad (7)$$

$$\frac{\partial h_i}{\partial x_i} = 0 \qquad (8)$$

and

$$(\frac{\partial}{\partial t} - \kappa\nabla^2)\, T' = -\beta u_3 \qquad (9)$$

One assumes here that $u_2 = 0$. Consequently, Equation (5) and boundary conditions govern a time decaying h_2 (see Section 16). h_2, not appearing elsewhere in the system, will be taken to be zero.

Cross differentiation of Equations (1) and (3) yields:

$$\rho_o\, (\frac{\partial}{\partial t} - \nu\nabla^2)\, (\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}) = \frac{\mu H_1}{4\pi} \frac{\partial}{\partial x_1} (\frac{\partial h_1}{\partial x_3} - \frac{\partial h_3}{\partial x_1}) - \rho_o g\alpha \frac{\partial T'}{\partial x_1} \cdot (10)$$

In view of the vanishing of u_2 and h_2, stream functions ψ and χ can again be used:

$$u_1 = \frac{\partial\psi}{\partial x_3},\ u_3 = -\frac{\partial\psi}{\partial x_1},\ \frac{h_1}{H_1} = \frac{\partial\chi}{\partial x_3} \text{ and } \frac{h_3}{H_3} = -\frac{\partial\chi}{\partial x_1}\ .$$

The thermal and magnetic diffusion are given respectively by

$$(\frac{\partial}{\partial t} - \kappa\nabla^2)\, T' = \beta \frac{\partial\psi}{\partial x_1} \qquad (9')$$

and

$$(\frac{\partial}{\partial t} - \eta\nabla^2)\, \chi = \frac{\partial\psi}{\partial x_1} \qquad (11)$$

Equations (9'), (10), and (11) together with the boundary conditions govern the phenomenon under consideration. Equation (5), being satisfied, and Equation (2) governing variation in p', can be omitted from the differential system.

With

$$\psi = \nu\, f(x) \cos(az) \sin(by) \exp(\gamma\tau),$$

$$T' = (\beta d)\, \theta(x) \cos(az) \sin(by) \exp(\gamma\tau),$$

$$\chi = d\, s(x) \cos(az) \sin(by) \exp(\gamma\tau),$$

and

$$x = \frac{x_1}{d} \;, \quad y = \frac{x_2}{d} \;, \quad z = \frac{x_3}{d} \;\text{ and } \tau = \frac{\kappa t}{d^2} \;.$$

equations (9'), (10), and (11) become:

$$[\gamma - (D^2 - a^2 - b^2)]\; \theta(x) = D(P_r f), \tag{12}$$

$$[\gamma - P_r(D^2 - a^2 - b^2)](D^2 - a^2) f(x) = Q\, D(D^2 - a^2)\frac{\eta}{\kappa}\, s(x) + RD\theta \tag{13}$$

and

$$[\gamma - \frac{\eta}{\kappa}(D^2 - a^2 - b^2)]\; s(x) = D(P_r f) \quad . \tag{14}$$

The boundary conditions are:

$$D\theta = 0 \quad \text{at} \quad x = \pm 1$$

for the temperature disturbance. For $u_3 = u_1 = 0$ or $\frac{\partial\psi}{\partial x_1} = \frac{\partial\psi}{\partial x_3} = 0$

at $x_1 = \pm d$, one has

$$f = Df = 0 \quad \text{at} \quad x = \pm 1 \;.$$

In order to satisfy $h_2 = h_3 = 0$ at $x_1 = \pm d$, it is sufficient to have $\frac{\partial\chi}{\partial x_1} = 0$ at $x_1 = \pm d$ as $h_2 = 0$ identically, or

$$Ds = 0 \quad \text{at} \quad x = \pm 1 \;.$$

9. Exchange of Stability

As stated before, when the complex growth rate $\gamma(= \gamma_r + i\gamma_i)$ has the following significance; $\gamma_r > 0$ implies that the mode amplitude grows with time, $\gamma_r < 0$ indicates a decaying disturbance. Further, $\gamma_i \neq 0$ implies that the disturbance oscillates. The differential system, in general, has a time-dependent solution, depending on a, b, Q, and R. It will be proved, in this section, that neutrally stable modes (having $\gamma_r = 0$) are time independent altogether (having $\gamma = 0$).

Differentiating Equation (14) with respect to x and substituting the results into Equation (13), one obtains:

$$[\gamma - P_r(D^2 - a^2 - b^2)](D^2 - a^2)\, f = Q[-D^2(P_r f) + (\gamma + \tfrac{\eta}{\kappa} b^2)Ds] + RD\theta \,. \qquad (13')$$

Multiplying Equations (12) by $\overline{\theta(x)}$, complex conjugate of $\theta(x)$, (13') by $\overline{f(x)}$, (14) by $\overline{s(x)}$, and integrating each product in the modified equations from x = -1 to x = + 1, one obtains from Equation (12):

$$(\gamma + a^2 + b^2) \int_{-1}^{1} \theta(x).\overline{\theta(x)}dx - \int_{-1}^{1} D^2\theta\, \overline{\theta}\, dx = \int_{-1}^{1} D(P_r f)\overline{\theta(x)}\, dx . \qquad (15)$$

Integrating by parts the second term on the l.h.s. of Equation (15), one obtains:

$$\int_{-1}^{1} D^2\theta.\overline{\theta}\, dx = D\theta.\overline{\theta(x)}\Big|_{-1}^{1} - \int_{-1}^{1} |D\theta|^2 dx = -\int_{-1}^{1} |D\theta|^2 dx,$$

since $D\theta.\overline{\theta(x)}\Big|_{-1}^{1}$ vanishes, because of the homogeneous boundary conditions. Consequently, Equation (15) becomes:

$$(\gamma + a^2 + b^2)\, \Theta_0 + \Theta_1 = \int_{-1}^{1} D(P_r f)\, \overline{\theta(x)}\, dx \quad . \qquad (15)$$

Similarly, Equations (13') and (14) yield:

$$-[\gamma + P_r(a^2 + b^2)]\, a^2 F_0 - [\gamma + P_r(2a^2 + b^2)]\, F_1 - P_r F_2$$

$$= QP_r F_1 + Q(\gamma + \tfrac{\eta}{\kappa} b^2) \int_{-1}^{1} Ds.\overline{f(x)}dx + R\int_{-1}^{1} D\theta.\overline{f(x)}dx, \qquad (16)$$

and $$[\gamma + (\tfrac{\eta}{\kappa})(a^2 + b^2)]\ S_o + (\tfrac{\eta}{\kappa})S_1 = \int_{-1}^{1} D(P_r f)\ \overline{s(x)}dx, \qquad (17)$$

respectively, where:

$$\Theta_o = \int_{-1}^{1}|\theta|^2 dx,\ \Theta_1 = \int_{-1}^{1}|D\theta|^2 dx,\ \Theta_2 = \int_{-1}^{1}|D^2\theta|^2 dx,$$

$$F_o = \int_{-1}^{1}|f|^2 dx,\ F_1 = \int_{-1}^{1}|Df|^2 dx,\ F_2 = \int_{-1}^{1}|D^2 f|^2 dx,$$

$$S_o = \int_{-1}^{1}|s|^2 dx,\ S_1 = \int_{-1}^{1}|Ds|^2 dx \text{ and } S_2 = \int_{-1}^{1}|D^2 s|^2 dx\ .$$

Integrating by parts the last term in Equation (16), one obtains:

$$\int_{-1}^{1} D\theta.\overline{f(x)}dx = + \theta\ \overline{f(x)}\Big|_{-1}^{1} - \int_{-1}^{1}\theta\ \overline{Df}\ dx = - \int_{-1}^{1}\theta\ \overline{Df}\ dx = \overline{(-\int_{-1}^{1}\overline{\theta}.Dfdx)}\,,$$

Similarly:

$$\int_{-1}^{1} Ds.\overline{f(x)}dx = -\overline{(\int_{-1}^{1}\overline{s}.Df.dx)}\ .$$

Utilizing the last two relations, and inserting Equations (15) and (17) into Equation (16), one obtains:

$$-[\gamma + P_r(a^2 + b^2)]a^2 F_o - [\gamma + P_r(2a^2 + b^2)]\ F_1 - P_r F_2 =$$

$$Q\left\{P_r F_1 - \frac{1}{P_r}\left([(\gamma + \tfrac{\eta}{\kappa} b^2) + \tfrac{\eta}{\kappa} a^2]\ S_o + (\tfrac{\eta}{\kappa})\ S_1\right)(\gamma + \tfrac{\eta}{\kappa} b^2)\right\}$$

$$- \frac{R}{P_r}[(\overline{\gamma} + a^2 + b^2)\Theta_o + \Theta_1] \qquad (18)$$

Equation (18) can be separated into its real and imaginary parts:

$$\gamma_i\left\{a^2 F_o + F_1 - \frac{Q}{P_r}(\tfrac{\eta}{\kappa})\ [a^2 S_o + S_1] + \frac{R}{P_r}\ \Theta_o\right\} = 0, \qquad (18)_i$$

and

$$\gamma_r \left\{ a^2 F_o + F_1 - \frac{Q}{P_r} (\frac{\eta}{\kappa}) [a^2 S_o + S_1] - \frac{R}{P_r} \Theta_o \right\} =$$

$$- a^2(a^2 + b^2) P_r F_o - (2a^2 + b^2) P_r F_1 - P_r F_2 - Q\Big\{P_r F_1$$

$$- \frac{1}{P_r}[|\gamma + \frac{\eta}{\kappa} b^2|^2 S_o + (\frac{\eta}{\kappa})^2 a^2 b^2 S_o + (\frac{\eta}{\kappa})^2 b^2 S_1]\Big\}$$

$$+ \frac{R}{P_r} [(a^2 + b^2) \Theta_o + \Theta_1] . \qquad (18)_r$$

To complete the proof of the principle of the "exchange of stabilities", one needs more relations between the quantities defined so far. Multiplying each side of Equation (14) by its complex conjugate and integrating from x = -1 to x = + 1, one gets:

$$|\gamma + \frac{\eta}{\kappa} b^2|^2 S_o + (\frac{\eta}{\kappa})^2 a^4 S_o + (\frac{\eta}{\kappa})^2 S_2 + 2(\gamma_r + (\frac{\eta}{\kappa}) b^2)(\frac{\eta}{\kappa}) S_1$$

$$+ 2a^2 (\frac{\eta}{\kappa})^2 S_1 + 2[\gamma_r + (\frac{\eta}{\kappa}) b^2] a^2 (\frac{\eta}{\kappa}) S_o = P_r^2 F_1 . \qquad (19)$$

From which two inequalities are obtained, so long as $\gamma_r \geqslant 0$:

$$|\gamma + (\frac{\eta}{\kappa}) b^2|^2 S_o + a^2 b^2 (\frac{\eta}{\kappa})^2 S_o + b^2 (\frac{\eta}{\kappa})^2 S_1 \leqslant P_r^2 F_1 , \qquad (20)$$

and

$$(\frac{\eta}{\kappa})^2 S_2 \leqslant P_r^2 F_1 . \qquad (21)$$

Integrating by parts, one obtains:

$$\int_{-1}^{1} D^2 s \, \overline{s(x)} \, dx = Ds.\overline{s(x)}\Big|_{-1}^{1} - \int_{-1}^{1} |Ds|^2 dx = - \int_{-1}^{1} |Ds|^2 dx,$$

since the first term on the right-hand side vanishes. By applying Schwarz's inequality to the last equation, one obtains:

$$S_2 . S_o \geqslant S_1^2 \qquad (22)$$

Finally, one can prove that if $\int_{-1}^{1} s(x)dx = 0$, then:

$$\int_{-1}^{1} |s|^2 dx \leqslant \int_{-1}^{1} |Ds|^2 \, dx$$

or

$$S_o \leqslant S_1 , \tag{23}$$

where the equality sign holds only when $s(x) = 0$ identically. In this case, $\int_{-1}^{1} s(x)dx$ is indeed zero, so long as $\gamma \neq 0$, as can readily be seen by integrating Equation (14) with respect to x from -1 to 1. Therefore, Equation (23) holds.

Combining Equations (21) and (22), one obtains:

$$\left(\frac{\eta}{\kappa}\right)^2 \frac{S_1^2}{S_o} \leqslant P_r^2 F_1 ,$$

and, in view of Equation (23),

$$\left(\frac{\eta}{\kappa}\right)^2 S_1 \leqslant P_r^2 F_1 \tag{24}$$

Returning now to Equation (17) and integrating its right-hand side by parts once, utilizing Schwarz's inequality, one gets:

$$\left(\frac{\eta}{\kappa}\right)^2 S_1^2 \leqslant P_r^2 F_o S_1 , \tag{25}$$

so long as $\gamma_r \geqslant 0$. From Equations (23) and (25), it follows that:

$$\left(\frac{\eta}{\kappa}\right)^2 S_o \leqslant P_r^2 F_o \tag{26}$$

Combining Equations (24) and (26), one obtains:

$$\left(\frac{\eta}{\kappa}\right)^2 [a^2 S_o + S_1] \leqslant P_r^2 [a^2 F_o + F_1] , \tag{27}$$

when the equality sign holds for the trivial case of no disturbance.

Equation $(18)_i$ can be satisfied by either $\gamma_i = 0$ or by the vanishing of the bracket coefficient of γ_i. If f and s are not zero identically, $\gamma_i \neq 0$ implies, because of Equation (27), that R is negative so long as $\frac{QP_r}{\eta/\kappa} \leqslant 1$ and $\gamma_r \geqslant 0$. However, if R is negative, the bracket coefficient of γ_r in $(18)_r$ is positive. Since the r. h. s. of $(18)_r$ is negative, in view of Equation (20), γ_r is negative, contrary to the original assumption. Thus under the assumption that $\frac{QP_r}{\eta/\kappa} \leqslant 1, \gamma_r \geqslant 0$ is possible only if $\gamma_i = 0$. Consequently, the mode corresponding to neutral stability is time-independent.

It should be mentioned that the limitation upon Q is independent of the wave length in the y or z direction. Physically, it means that one can, for any mode, impose an interfering magnetic field which would make the principle hold. This is not the case for the velocity fields considered in Chapters II and III. Nevertheless, one can prove the principle of "exchange of stability" subject to limitation upon Q involving wave lengths a or b.

10. Solution and Approximations

The relationship between a, b, Q, and R shall be obtained for the time-idependent mode by the approximation method carried out in Chapter III. The governing equations in Chapter IV are of the same structure as those in Chapter III. In this chapter, however, $\theta(x)$ and s(x) are either both odd or both even; therefore, the following expansions are chosen:

$$(-R\theta) = \sum_{1,2,3\ldots} A'_n \cos(n\pi x)\ , \quad [\frac{\eta}{\kappa} Q\ s(x)] = \sum_{1,2,3..} B'_n \cos(n\pi x);$$

or

$$(-R\theta) = \sum_{1,3,5\ldots} A''_n \sin(\frac{n\pi x}{2}), \quad [\frac{\eta}{\kappa} Q\ s(x)] = \sum_{1,3,5..} B''_n \sin(\frac{n\pi x}{2}).$$

In these expansions, boundary conditions for both θ and s are satisfied. By substituting $\theta(x)$ and $s(x)$ in Equation (13) and utilizing the remaining boundary conditions, $f(x)$ is obtained. Then approximating f, s and θ each by a single harmonic, one obtaines for $f(x)$ odd and $\theta(x)$ and $s(x)$ even:

$$R' = \frac{(M_c^{1'})^2 \, (M_a^{1'})}{F'} \left[1 + \frac{Q}{(M_c^{1'})^2} F' \right] , \tag{28}$$

where

$$F' = \pi^2 \left\{ 1 + \frac{2\pi^2}{a \coth(a) - c \coth(c)} \left[\frac{1}{M_a^{1'}} - \frac{1}{M_c^{1'}} \right] \right\}$$

and for $f(x)$ even and $\theta(x)$ and $s(x)$ odd:

$$R'' = \frac{(M_a^{1''}) \, (M_c^{1''})^2}{F''} \left[1 + \frac{QF''}{(M_c^{1''})^2} \right] , \tag{29}$$

where

$$F'' = \left(\frac{\pi}{2}\right)^2 \left\{ 1 + \frac{2(\pi/2)^2}{a \tanh(a) - c \tanh(c)} \left[\frac{1}{(M_a^{1''})} - \frac{1}{(M_c^{1''})} \right] \right\}$$

The quantities $(M_a^{1'})$, $(M_a^{1''})$, $(M_c^{1'})$, and $(M_c^{1''})$ have the same definitions as given in Section 7.

Equations (28) and (29) indeed show that the least stable mode is that of zero wave length, and that for given Q and a $R'' < R'$. Hence, modes in which f is even are less stable. These equations also show that the effect of the imposed magnetic field is present for any wave length, as the term including Q is not zero for any wave length (including a = 0). In this case, the magnetic field would raise the critical Rayleigh number required for convection. These results are ploted in Figures 7-10.

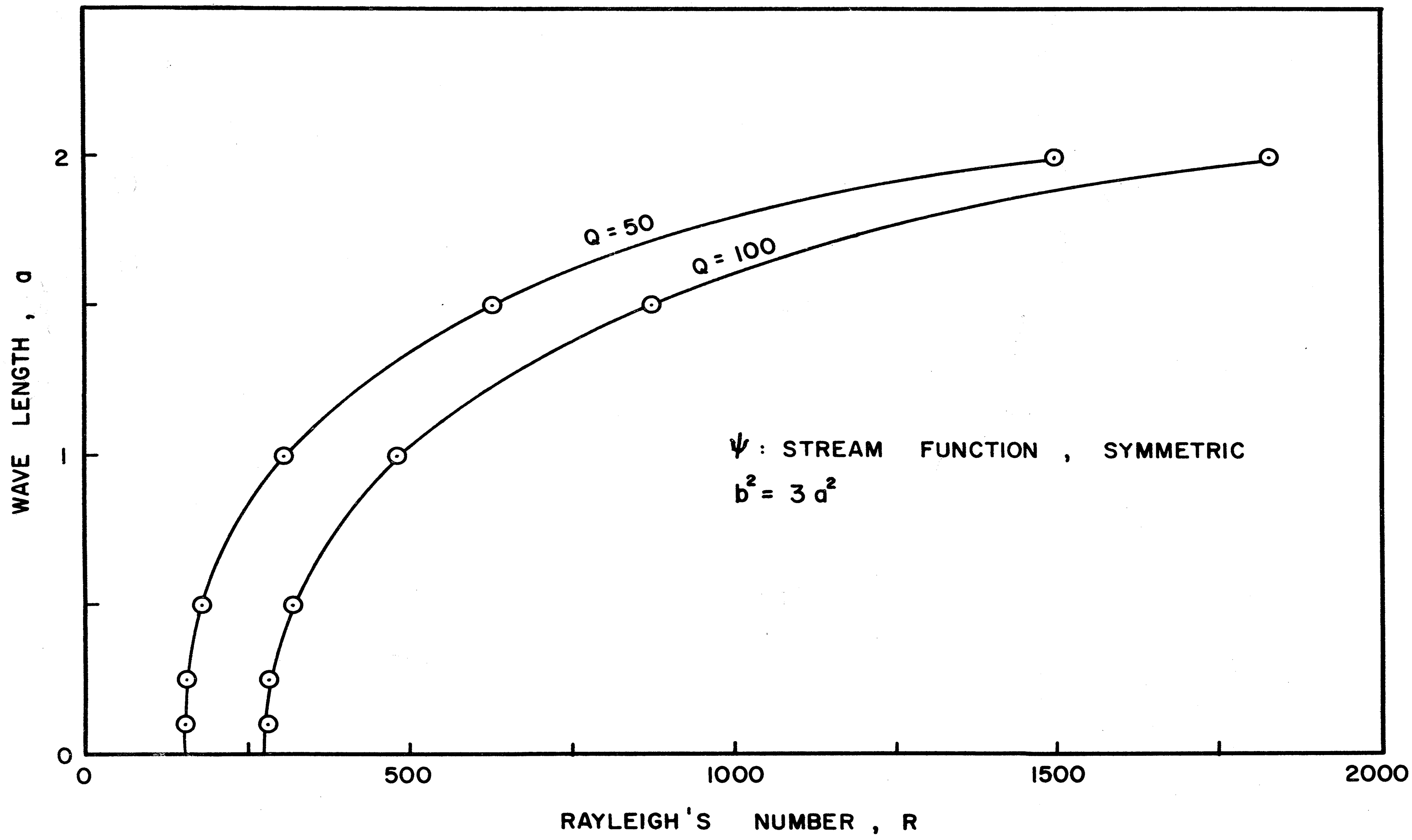

Figure 7. Q-R-a-b Relationship - H_1 Imposed.

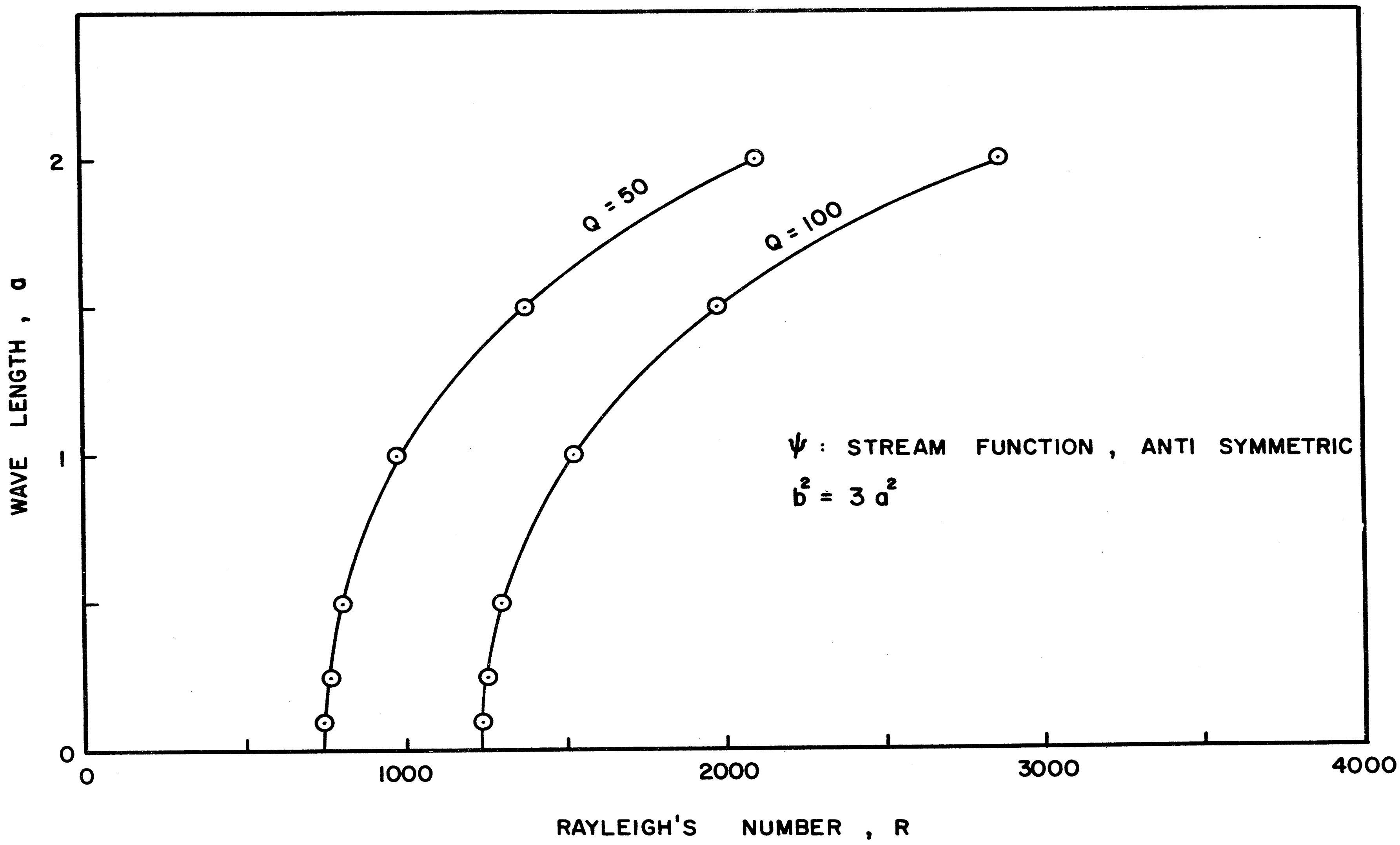

Figure 8. Q-R-a-b Relationship - H_1 Imposed.

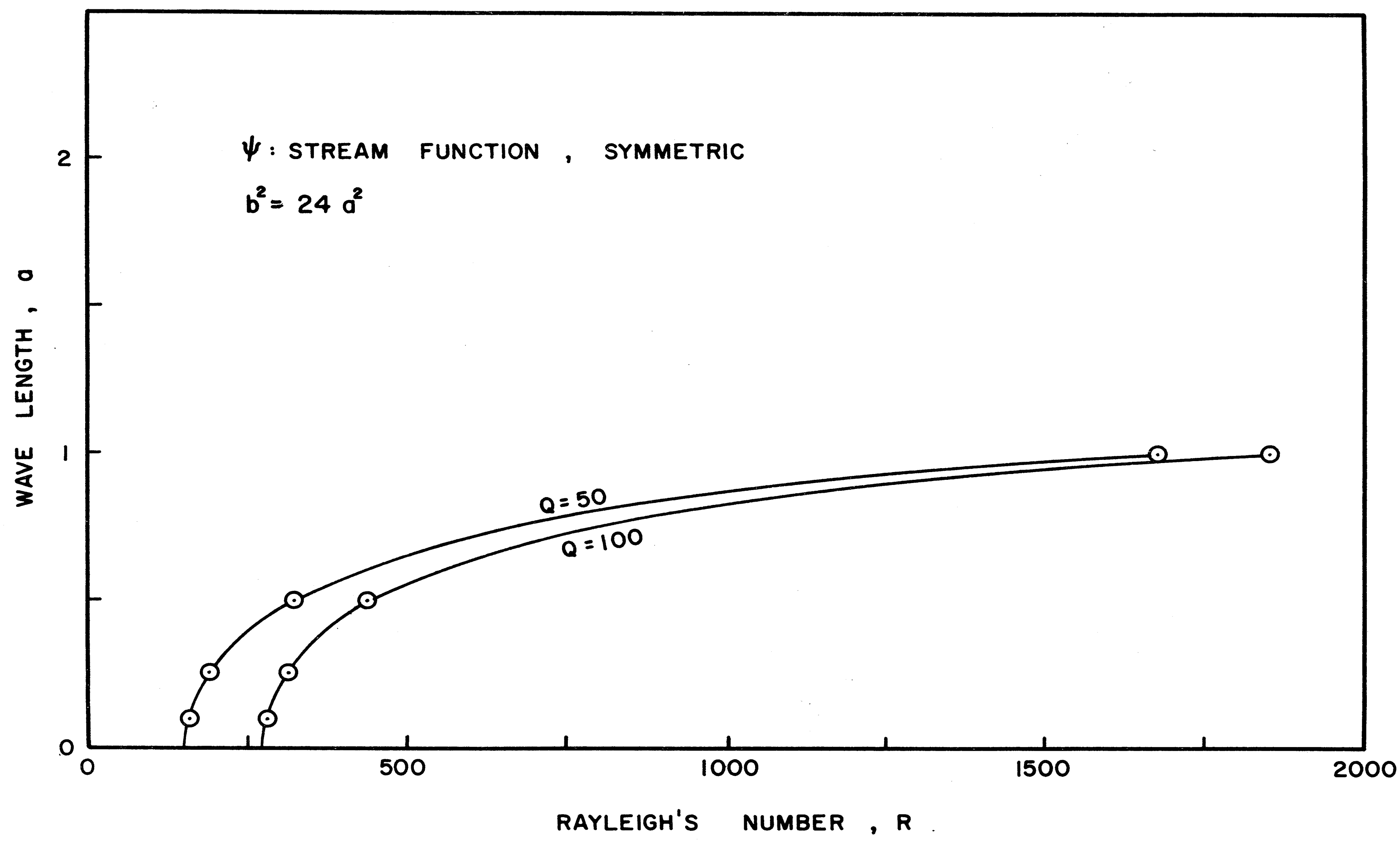

Figure 9. Q-R-a-b Relationship - H_1 Imposed.

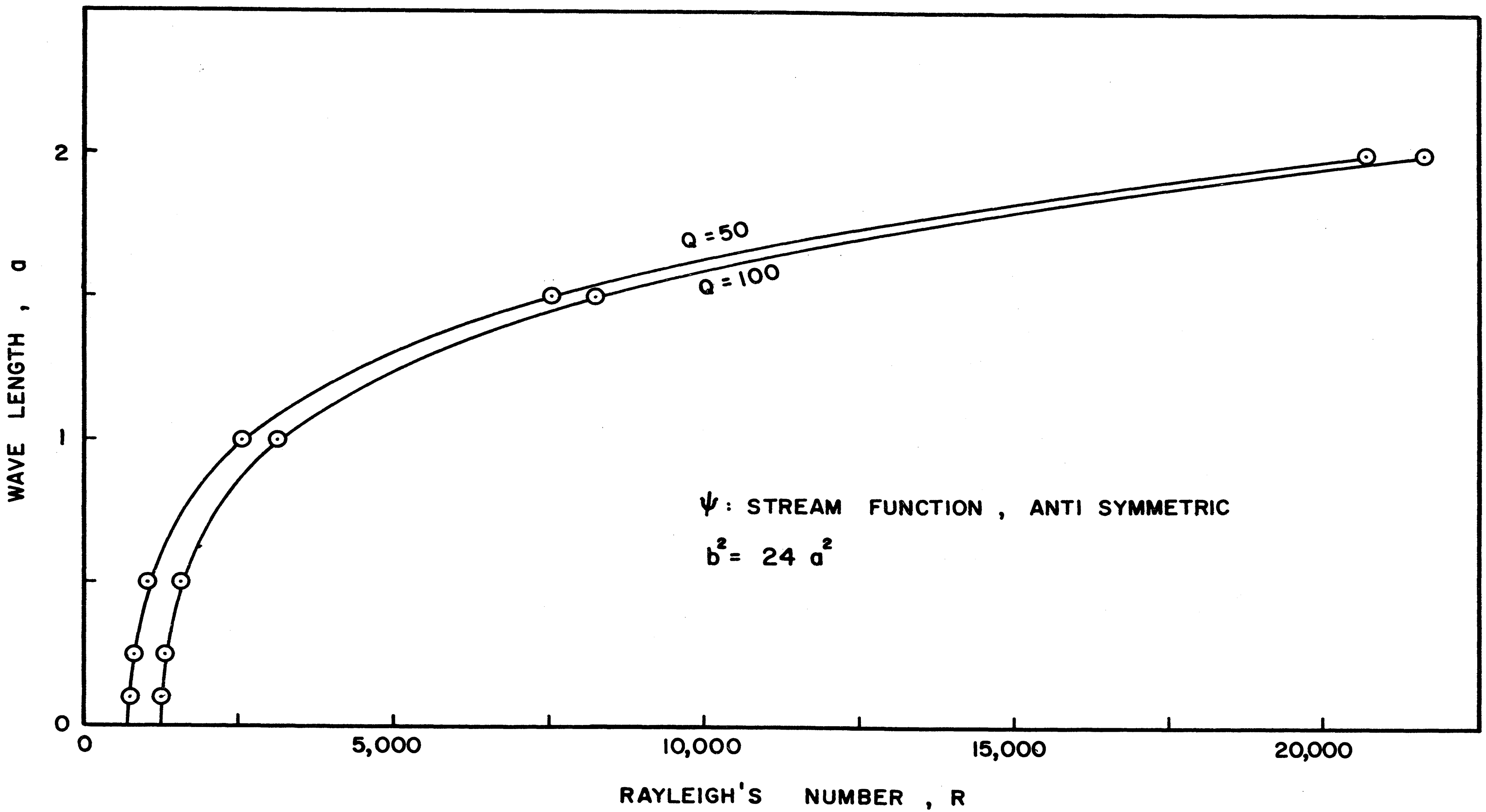

Figure 10. Q-R-a-b Relationship - H_1 Imposed.

CHAPTER V

EFFECT OF ELECTROMAGNETIC FIELDS ON STABILITY
OF A FLUID IN A CIRCULAR TUBE

In this chapter, the stability of an infinitely high circular column of fluid, with an adverse temperature gradient and a vertical uniform magnetic field will be considered. Apart from the geometrical configuration of the container, this case is the same as that discussed in Chapter II. The purpose of this chapter is, however, mainly to demonstrate an analysis which is not used in Chapter II and is more general.

11. Splitting of the General Problem into Two Parts

The governing equations, Equations (II.1) to (II.9), transformed to cylindrical coordinates, with (x_1, x_2, x_3) indicating (r, θ, z),

$$\rho_o \left[\frac{\partial u_1}{\partial t} - \nu\left(\nabla^2 u_1 - \frac{2}{x_1}\frac{\partial u_2}{\partial x_2} - \frac{u_1}{x_1{}^2}\right)\right] = \frac{\mu H_3}{4\pi}\frac{\partial h_i}{\partial x_3} - \frac{\partial}{\partial x_1}\left(p' + \frac{\mu H_3 h_3}{4\pi}\right), \quad (1)$$

$$\rho_o \left[\frac{\partial u_2}{\partial t} - \nu\left(\nabla^2 u_2 + \frac{2}{x_1{}^2}\frac{\partial u_1}{\partial x_2} - \frac{u_2}{x_1{}^2}\right)\right] = \frac{\mu H_3}{4\pi}\frac{\partial h_2}{\partial x_3} - \frac{1}{x_1}\frac{\partial}{\partial x_2}\left(p' + \frac{\mu H_3 h_3}{4\pi}\right), (2)$$

$$\rho_o\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)u_3 = \frac{\mu H_3}{4\pi}\frac{\partial h_3}{\partial x_3} - \frac{\partial}{\partial x_3}\left(p' + \frac{\mu H_3 h_3}{4\pi}\right) + \rho_o g\alpha T', \quad (3)$$

$$\frac{\partial h_1}{\partial t} - \eta\left(\nabla^2 h_1 - \frac{2}{x_1{}^2}\frac{\partial h_2}{\partial x_2} - \frac{h_1}{x_1{}^2}\right) = H_3 \frac{\partial u_1}{\partial x_3} , \quad (4)$$

$$\frac{\partial h_2}{\partial t} - \eta(\nabla^2 h_2 + \frac{2}{x_1^2}\frac{\partial h_1}{\partial x_2} - \frac{h_2}{x_1^2}) = H_3 \frac{\partial u_2}{\partial x_3} \; , \tag{5}$$

$$(\frac{\partial}{\partial t} - \eta\nabla^2)h_3 = H_3 \frac{\partial u_3}{\partial x_3} \; , \tag{6}$$

$$\frac{1}{x_1}\frac{\partial(x_1 u_1)}{\partial x_1} + \frac{1}{x_1}\frac{\partial u_2}{\partial x_1} + \frac{\partial u_3}{\partial x_3} = 0 \; , \tag{7}$$

$$\frac{1}{x_1}\frac{\partial(x_1 u_1)}{\partial x_1} + \frac{1}{x_1}\frac{\partial h_2}{\partial x_2} + \frac{\partial h_3}{\partial x_3} = 0 \; , \tag{8}$$

and

$$(\frac{\partial}{\partial t} - \kappa\nabla^2)T' = -\beta u_3 \; . \tag{9}$$

The boundary conditions, as in Chapter II, are:

$$\frac{\partial T'}{\partial x_1} = 0 \quad \text{and} \quad j_2 = j_3 = 0 \quad \text{at} \quad x_1 = d/2 \; .$$

However, here, unlike Chapter II, no velocity component or variation with respect to any coordinate will be assumed to vanish a priori.

By cross-differentiation of Equations (1) and (2), (2) and (3), and (3) and (1) as well as (4) and (5), one eliminates the p' terms and obtains:

$$\rho_o \left[\frac{\partial \zeta_1}{\partial t} - \nu(\nabla^2\zeta_1 - \frac{2}{x_1^2}\frac{\partial \zeta_2}{\partial x_2} - \frac{\zeta_1}{x_1^2})\right] = \mu H_3 \frac{\partial j_1}{\partial x_3} + \rho_o g\alpha \frac{1}{x_1}\frac{\partial T'}{\partial x_2} \; , \tag{10}$$

$$\rho_o \left[\frac{\partial \zeta_2}{\partial t} - \nu(\nabla^2 \zeta_2 + \frac{2}{x_1^2} \frac{\partial \zeta_1}{\partial x_2} - \frac{\zeta_2}{x_1^2})\right] = \mu H_3 \frac{\partial j_2}{\partial x_3}$$

$$- \rho_o g\alpha \frac{\partial T'}{\partial x_1} , \qquad (11)$$

$$\rho_o \left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\zeta_3 = \mu H_3 \frac{\partial j_3}{\partial x_3} , \qquad (12)$$

and

$$\left(\frac{\partial}{\partial t} - \eta\nabla^2\right) j_3 = \frac{H_3}{4\pi} \frac{\partial \zeta_3}{\partial x_3} \qquad (13)$$

where ζ_1, ζ_2 and ζ_3 are the three vorticity components.

The system under consideration, consisting of nine equations and accompanying boundary conditions, and governing eight dependent variables, shall be split into two systems. Each such system shall consist of fewer governing equations, fewer boundary conditions and governing fewer quantities.

It can be shown that any velocity field u_i satisfying Equation (7) can be split into fields $u_i^{(1)}$ and $u_i^{(2)}$ such that:

$$u_1^{(1)} + u_1^{(2)} = u_1 ,$$

$$u_2^{(1)} + u_2^{(2)} = u_2 ,$$

$$u_3^{(1)} = u_3 ,$$

$$\frac{\partial u_1^{(2)}}{\partial x_1} + \frac{u_1^{(2)}}{x_1} + \frac{1}{x_1} \frac{\partial u_2^{(2)}}{\partial x_2} = 0$$

and

$$\frac{\partial(x_1 u_2^{(1)})}{\partial x_1} - \frac{\partial u_1^{(1)}}{\partial x_2} = 0 \ .$$

This split is possible, as the five differential relations govern five newly-defined quantities. Since u_i satisfies Equation (7), so does $u_i^{(2)}$. Consequently, any incompressible flow can be split into two, flow I for which $\zeta_3 = 0$ and flow II for which $u_3 = 0$, where flows I and II are both incompressible. This conclusion can in fact be shown to be a particular case of Helmholtz's Theory. The separation procedure will be preformed for all pertinent variables. From Equations (9) and (6), T' and h_3 can be shown to depend upon u_3 or $u_3^{(1)}$, so that $T' = T'^{(1)}$ and $h_3 = h_3^{(1)}$. However, from Equations (12) and (13), $j_3 = j_3^{(2)}$, as it depends on ζ_3, and in flow I ζ_3 is zero. All other quantities will be associated with both flows:

$$j_2 = j_2^{(1)} + j_2^{(2)}, \qquad j_1 = j_1^{(1)} + j_1^{(2)},$$

$$h_2 = h_2^{(1)} + h_2^{(2)} \text{ and } h_1 = h_1^{(1)} + h_1^{(2)} \ .$$

Since the governing equations are linear, it is sufficient to have each of the set of quantities, $(\quad)^{(1)}$ and $(\quad)^{(2)}$, satisfy these equations separately, so that the sum $[(\quad)^{(1)} + (\quad)^{(2)}]$ would satisfy them, too.

In the spirit of former chapters, one seeks a time-independent solution assuming:

$$u_1 = (\frac{2\nu}{d})\ U(r)\ \sin(n\theta)\ \cos(az) \ ,$$

$$u_2 = (\frac{2\nu}{d})\ V(r)\ \cos(n\theta)\ \cos(az) \ ,$$

$$u_3 = (\frac{2\nu}{d})\ W(r)\ \sin(n\theta)\ \sin(az) \ ,$$

$$h_1 = H_3 \, X(r) \sin(n\theta) \sin(az) \; ,$$

$$h_2 = H_3 \, Y(r) \cos(n\theta) \sin(az) \; ,$$

$$h_3 = H_3 \, Z(r) \sin(n\theta) \cos(az) \; ,$$

and

$$T' = \frac{\beta d/2}{a} \, \theta(r) \sin(n\theta) \sin(az) \; ,$$

where

$$r = \frac{x_1}{d/2} \, , \quad x_2 = \theta \quad \text{and} \quad z = \frac{x_3}{d/2} \; .$$

12. Development for Sub-System I

The continuity of the flow and the magnetic field requires:

$$DU^{(1)} + \frac{U^{(1)}}{r} - n \frac{V^{(1)}}{r} + a \, W^{(1)} = 0 \tag{14}$$

and

$$DX^{(1)} + \frac{X^{(1)}}{r} - n \frac{Y^{(1)}}{r} - a \, Z^{(1)} = 0 \tag{15}$$

where

$$D = \frac{d}{dr} \; .$$

As $j_3 = \zeta_3 = 0$ in this flow, the corresponding differential relations are:

$$D(rV^{(1)}) = nU^{(1)} \; , \tag{16}$$

and

$$D(rY^{(1)}) = nX^{(1)} \; . \tag{17}$$

Since each of the three components' fields is governed by two differential relations, fields can be uniquely defined by scalar quantities.

With:

$$(rV^{(1)}) = n\,\psi(r) \quad , \quad (rY^{(1)}) = n\,\chi(r) \; ,$$

$$U^{(1)} = D\,\psi(r) \quad , \quad X^{(1)} = D\,\chi(r)$$

and

$$W^{(1)} = -\frac{1}{a}\,(L_n + a^2)\psi(r) \; , \quad Z^{(1)} = \frac{1}{a}\,(L_n + a^2)\chi ,$$

where

$$L_n = (\frac{1}{r}\,DrD - \frac{n^2}{r^2} - a^2) \quad \text{and} \quad (L_n + a^2) = (\frac{1}{r}\,DrD - \frac{n^2}{r^2}) \; .$$

Equations (16) and (17) are automatically satisfied. Currents as well as vorticity components will be expressed in terms of the newly-defined functions as follows:

$$\zeta_1^{(1)} = -\frac{1}{a}\,[\frac{\nu}{(d/2)^2}]\,\frac{n}{r}\,(L_n\psi)\cos(n\theta)\,\sin(az) \; , \tag{18}$$

$$j_1^{(1)} = \frac{1}{a}\,(\frac{H_3}{4\pi d/2})\,\frac{n}{r}\,(L_n\,\chi)\cos(n\theta)\,\cos(az) \; , \tag{19}$$

$$\zeta_2^{(1)} = \frac{1}{a}\,[\frac{\nu}{(d/2)^2}]\,D(L_n\psi)\,\sin(n\theta)\,\sin(az) \; , \tag{20}$$

and

$$j_2^{(1)} = -\frac{1}{a}\,(\frac{H_3}{4\pi d/2})\,D(L_n\chi)\,\sin(n\theta)\,\cos(az) \quad . \tag{21}$$

Inserting Equations (18)-(21) into Equation (12), and setting to zero for time-independency, one obtains:

$$-\rho_o\,\frac{\nu}{(d/2)^2}\,(L_n - \frac{1}{r^2})\,(-\frac{1}{a}\,\frac{\nu}{(d/2)^2})\,\frac{n}{r}\,(L_n\psi)$$
$$+\,\rho_o\,\frac{2n}{a}\,(\frac{\nu}{(d/2)^2})^2\,D(L_n\psi) = -\,\mu H_3\,(\frac{H_3}{4\pi d/2})\,\frac{1}{d/2}\,\frac{n}{r}\,(L_n\chi)$$
$$+\,\frac{\rho_o g\alpha}{a}\,n\,\frac{\theta}{r} \quad . \tag{22}$$

Since for any function of r

$$(L_n - \frac{1}{r^2})\,(\frac{f(r)}{r}) = -\frac{2}{r^2}\,D\,f(r) + \frac{1}{r}\,L_n f\ , \tag{23}$$

Equation (22) yields.

$$L_n L_n\,(P_r\psi) = a\,Q\,\frac{\eta}{\kappa}\,L_n\chi + R\theta\ , \tag{24}$$

where

$$Q = \frac{\mu^2\sigma(H_3)^2(d/2)^2}{\rho_o\nu}\ ,\quad P_r = \frac{\nu}{\kappa}\ \text{and}\ R = -\frac{g\alpha\beta(d/2)^4}{\nu\kappa}\ .$$

It should be noted that Equation (24), though derived from Equation (10), implies satisfaction of Equation (11) too, since ψ and χ uniquely define velocity and magnetic field respectively. The dependency of $h_i^{(1)}$ on $u_i^{(1)}$ can also be uniquely determined by inserting the values of $h_2^{(1)}$, $h_1^{(1)}$, and $u_2^{(1)}$, expressed in terms of ψ and χ, into Equation (5), thus yielding:

$$-(\frac{\eta}{\kappa})\,\frac{1}{(d/2)^2}\,[L_n - \frac{1}{r^2}]\,(n\,\frac{\chi}{r}\,H_3) - 2(\frac{\eta}{\kappa})\,\frac{n}{(d/2)^2}\,\frac{D\chi}{r}\,H_3 =$$

$$-\,a\,\frac{H_3}{d/2}\,\frac{\nu}{(d/2)}\,n\,\frac{\psi}{r}\ , \tag{25'}$$

which, with the aid of Equation (23), becomes:

$$(\frac{\eta}{\kappa})\,L_n\chi = a(P_r\psi)\ . \tag{25}$$

Finally, Equation (9) yields:

$$-L_n\,\theta(r) = (L_n + a^2)\,(P_r\psi)\ . \tag{26}$$

13. Development for Sub-System II

Since this flow and the associated magnetic field have two components each, conventional stream function will be used to define these fields:

$$(rU^{(2)}) = n\varphi(r) \quad , \quad V^{(2)} = D\varphi \ ,$$

$$rX^{(2)} = n\Omega(r) \quad \text{and} \quad Y^{(2)} = D\Omega \ ,$$

so that continuity is automatically satisfied. The vorticity and current components are accordingly:

$$\zeta_3 = \zeta_3^{(2)} = \frac{\nu}{(d/2)^2} (L_n + a^2)\, \varphi(r) \cos(n\theta) \cos(az) \quad ,$$

$$j_3 = j_3^{(2)} = \frac{H_3}{4\pi d/2} (L_n + a^2)\, \Omega(r) \cos(n\theta) \sin(az),$$

$$\zeta_2^{(2)} = - \frac{\nu}{(d/2)^2} (\frac{an\varphi}{r}) \sin(n\theta) \sin(az) \quad ,$$

$$j_2^{(2)} = (\frac{H_3}{4\pi d/2}) (\frac{an\Omega}{r}) \sin(n\theta) \cos(az) \quad ,$$

$$\zeta_1^{(2)} = \frac{\nu}{(d/2)^2} (aD\varphi) \cos(n\theta) \sin(az) \quad ,$$

and

$$j_1^{(2)} = - (\frac{H_3}{4\pi d/2}) (a\ D\Omega) \cos(n\theta) \cos(az) \ .$$

Inserting the appropriate values in Equation (12), one gets:

$$0 = L_n (L_n + a^2) (P_r\varphi) + a\ Q(L_n + a^2) (\frac{\eta}{\kappa} \Omega) \ . \qquad (27)$$

Inserting the appropriate values of h_i and u_i in either Equation (4) or Equation (5), utilizing Equation (23), one obtains:

$$0 = L_n(\frac{\eta}{\kappa} \Omega) \ - a(P_r\varphi) \ . \qquad (28)$$

In this flow, temperature effects are not being dealt with, and Equations (27) and (28) govern only two components of velocity (both horizontal) and two of the magnetic field. Physical considerations may lead to the conjecture that $\varphi = \Omega = 0$ identically, as buoyancy force does not appear to play its part. This indeed would be the case if the boundary conditions governing $(\quad)^{(2)}$ quantities were homogeneous. However, the boundary conditions contain quantities of the form $[(\quad)^{(1)} + (\quad)^{(2)}]$. Therefore $(\quad)^{(2)}$ quantities, being linked to convection terms via the boundary conditions, do not necessarily decay with time.

14. Boundary Conditions and Solution

Combining Equations (24) to (26) as well as Equations (27) and (28), one obtains:

$$[L_n^3 - (R - a^2Q)\, L_n - Ra^2]\, (P_r\psi) = 0 \tag{29}$$

and

$$[L_n \cdot L_n + a^2Q]\, (L_n + a^2)\, (P_r\varphi) = 0 \tag{30}$$

Equations (29) and (30) can be rewritten:

$$(L_n + \alpha_1^2)\, (L_n + \alpha_3^2)\, (L_n + \alpha_3^2)\, (P_r\psi) = 0 \quad , \tag{29'}$$

$$(L_n + p_1^2)\, (L_n + p_2^2)\, (L_n + a^2)\, (P_r\varphi) = 0 \tag{30'}$$

where α_i^2 and p_i^2 are functions of R, Q, and a. The general solution for ψ and φ are of the forms:

$$P_r\psi = \sum_{i=1}^{3} A_i\, J_n\, (\alpha_i r) \;+\; \sum_{i=1}^{3} B_i Y_n (\alpha_i r)$$

and

$$P_r\varphi = \sum_{i=1}^{2} C_i\, J_n(p_i r) + \sum_{i=1}^{2} D_i Y_n (p_i r) + E_n\, r^n + E_{-n}\, r^{-n} \, .$$

Wishing to avoid singularities at $r = 0$, one sets $B_i = D_i = E_{-n} = 0$ so that only six, out of the original 12, constants are left to be determined.

The condition of no slip at the boundary requires:

$$u_1 = u_2 = u_3 = 0 \quad \text{at} \quad x_1 = d/2$$

or

$$D\psi + \frac{n\varphi}{r} = 0 \quad \text{at} \quad r=1 , \tag{31}$$

$$\frac{n\psi}{r} + D\varphi = 0 \quad \text{at} \quad r=1 , \tag{32}$$

and

$$(L_n + a^2)\,\psi = 0 \quad \text{at} \quad r=1 \quad . \tag{33}$$

An electrically very conductive wall requires:

$$j_2 = j_3 = 0 \quad \text{at} \quad x_1 = d/2 \quad .$$

Therefore

$$j_3 = j_3^{(2)} = 0$$

or

$$(L_n + a^2)\,\Omega = 0 \quad \text{at} \quad r=1 \quad .$$

From Equation (27), this boundary condition can be written:

$$L_n(L_n + a^2)\,\psi = 0 \quad \text{at} \quad r=1 \tag{34}$$

Also:

$$j_2 = j_2^{(1)} + j_2^{(2)} \quad \text{at} \quad x_1 = d/2$$

implies

$$-\frac{1}{a} D(L_n\chi) + \frac{an\Omega}{r} = 0 \quad \text{at} \quad r=1 \quad .$$

Equations (25) and (28) will be used to express the last boundary condition in terms of ψ and φ. Differentiating (25), one gets:

$$D(L_n\chi) = \frac{\kappa}{\eta} a\, D(P_r\psi) \tag{35}$$

Since $(L_n + a^2)\,\Omega = 0$ at $r=1$ Equation (28) yields

$$a\,\frac{\eta}{\kappa}\,\Omega - (P_r\varphi) = 0 \quad \text{at} \quad r=1 \;.$$

Therefore, the condition $j_2 = 0$ yields:

$$D\psi + \frac{n\varphi}{r} = 0 \quad \text{at} \quad r=1 \;.$$

This condition has been required for $u_1 = 0$ (Equation (31)). The duplication is due to the physical requirement that $j_3 = 0$ on the boundary, implies the vanishing of j_2 there, which is explainable in the following fashion. If one lets the vertical potential drop vanish on r=1 <u>for any angular position θ or height z</u>, as one does by considering the mode of the form $f(r)\cos(n\theta)\cos(az)$, one makes the container an equipotential surface.

The physical condition imposed upon the temperature disturbance is:

$$\frac{\partial T'}{\partial x_1} = 0 \quad \text{at} \quad x_1 = d/2$$

or

$$D\theta = 0 \quad \text{at} \quad r=1 \;.$$

Differentiating Equation (24) and utilizing Equation (35), one obtains for this condition

$$D(L_nL_n + a^2Q)\,(P_r\psi) = 0 \quad \text{at } r=1 \;. \tag{36}$$

Equation (30) when expanded is:

$$\left[\left(\frac{1}{r}\,DrD - \frac{n^2}{r} - a^2\right)\left(\frac{1}{r}\,DrD - \frac{n^2}{r^2} - a^2\right) + a^2Q\right]\,(L_n + a^2)(P_r\varphi) =$$

$$[(L_n + a^2) - (a\sqrt{1 + i\sqrt{Q}})^2][(L_n + a^2) - (a\sqrt{1-i\sqrt{Q}})^2]$$
$$(L_n + a^2)(P_r\varphi) = 0 \tag{30'}$$

Hence, in (30'), $p_1 = \overline{p_2}$, both being complex for non-zero a and Q. Therefore, $(p_1 r) = (\overline{p_2 r})$ and $J_n(p_1 r) = \overline{J_n(p_2 r)}$. If one requires φ to be real, one should have:

$$C_2 + C_1 = 0 \qquad (37)$$

Thus, one has six homogeneous boundary conditions and six undetermined constants.

Two more conditions are required at the origin. One requires $\theta(0) = 0$ because T' is expressed in the form $\theta(r)\sin(n\theta)\cos(az)$ and is single-valued at the origin. Also demanding u_2 to be finite at the origin:

$$ru_2 = 0 \quad \text{at} \quad r=0$$

Therefore:

$$r(V^{(1)} + V^{(2)}) = r(\frac{n\psi}{r} + D\varphi) = 0 \quad \text{at} \quad r=0 \qquad (38')$$

and

$$\theta(0) = (L_n L_n + a^2 Q)\psi = 0 \quad \text{at} \quad r=0 \qquad (39')$$

by virtue of Equations (24) and (25). These conditions are automatically satisfied if $n \geqslant 1$ as $J_n(w)$ vanishes at w=0 for $n \geqslant 1$.

A solution for n=1 can also be constructed, however, by setting $\varphi = \Omega = 0$. In this case, Equation (34) would not be imposed, there being no j_3 in this case. Equation (37) is automatically satisfied, as $\varphi = 0$ identically. However, Equations (31), (32), (33), and (36) will remain (in a somewhat simplified form due to the vanishing of φ), as well as (38') and (39'). With six boundary conditions, the six coefficients of the $Y_1(\alpha_i r)$ and the $J_1(\alpha_i r)$ can be determined, not assuming $B_i = 0$.

In general:

$$Y_1(w) = J_1(w)\ln(w) - \frac{1}{2}\left(\frac{w}{2}\right)^{-1} - \frac{1}{2}\sum_{k=0}^{\infty}(-1)^k \frac{H_k + H(k+1)}{k!\,(k+1)!}\left(\frac{w}{2}\right)^{(2k+1)}.$$

Thus at the vicinity of the origin $Y_1(w)$ behaves like $-(w)^{-1}$, as the first r.h.s. term as well as the power series vanish there. $J_1(w)$ vanishes for w=0 too. Therefore, Equation (38') when $\varphi=0$ reads:

$$\psi = 0 \qquad \text{at} \qquad r = 0 \tag{38}$$

or

$$0 = \lim_{r \to 0} \left(B_1 \frac{1}{\alpha_1 r} + B_2 \frac{1}{\alpha_2 r} + B_3 \frac{1}{\alpha_3 r}\right) .$$

Consequently:

$$\sum_{i=1}^{3} \frac{B_i}{\alpha_i} = 0 .$$

Equation (39'), by virtue of Equation (38), reduces to:

$$L_n L_n \psi = 0 \qquad \text{at} \qquad r=0 . \tag{39}$$

In similar fashion, this yields:

$$\sum_{i=1}^{3} (\alpha_i^2 - a^2)^2 \frac{B_i}{\alpha_i} = 0 .$$

The other four boundary conditions at r=1 yield four homogeneous algebraic equations linear in the A's and B's, whose coefficients are the Bessel functions $J_1(\alpha_i)$s and $Y_1(\alpha_i)$s.

For the axisymmetric case, n = 0, solution can be obtained for $\varphi = \Omega = 0$ again. Further, if n = 0, definitions of V and Y yield:

$$V = V^{(1)} = n\psi = 0$$

and

$$Y = Y^{(1)} = n\chi = 0 .$$

This is understandable, since this flow is axisymmetric with ζ_3 and j_3 equal to zero.

The governing equation for $(P_r\psi)$ is the same as before, namely (29) with n = 0. The boundard conditions are:

$$D\psi = 0 \quad \text{at} \quad r = 1 \tag{31}$$

and

$$(L_n + a^2)\,\psi = 0 \quad \text{at} \quad r = 1 \ . \tag{33}$$

The thermal and electromagnetic boundary conditions are both implied by:

$$D(L_nL_n + a^2Q)\,\psi = 0 \quad \text{at} \quad r = 1 \tag{36}$$

All the boundary conditions are expressed in terms of first or higher degree derivatives of ψ, but the function ψ itself has no physical meaning when n = 0. Differentiating (29) by r and setting n = 0, one has:

$$D(L_o + \alpha_1^2)\,(L_o + \alpha_2^2)(L_o + \alpha_3^2)\,(P_r\psi) =$$

$$(L_1 + \alpha_1^2)\,(L_1 + \alpha_2^2)\,(L_1 + \alpha_3^2)\,[D(P_r\psi)] = 0$$

as

$$L_1 = D\,\frac{1}{r}\,Dr - a^2 \ .$$

The solution for $D(P_r\psi)$ would be, in general, in terms of six Bessel functions of order 1. The linear combination of the three J_1's is the solution sought. Again in this case there are just three boundary conditions.

According to the results in [6] for an infinitely high circular cylindrical column, in the absence of an electromagnetic field, the governing equations and boundary conditions for the eigenfunctions are:

$$L^3\Psi = R(L + a^2)\,\Psi$$

$$\Psi = D\Psi = L^2\Psi = 0 \quad \text{at } r = 1 \ ,$$

where L in [6] is equal to L_1 here, and where axisymmetry is assumed.

The governing differential system here is:

$$L_1^3\,[D(P_r\psi)] = [(R-a^2Q)L_1 + R\,a^2]\,[D(P_r\psi)]$$

$$D(P_r\psi) = (L_o + a^2)\,(P_r\psi) = D(L_oL_o + a^2Q)(P_r\psi) = 0 \text{ at } r=1$$

The second boundary condition can be simplified to:

$$D[D(P_r\psi)] = 0 \quad \text{at} \quad r=1 \quad .$$

Similarly, the third boundary condition, utilizing the first one, reduces to:

$$L_1^2\,[D(P_r\psi)] = 0 \quad \text{at} \quad r=1 \quad .$$

Finally, Ψ and $D(P_r\psi)$ have essentially the same physical significance, both being proportional to u_2. Thus, both systems are, in general, similar and identical for $Q = 0$.

The solutions of the differential systems have now been reduced to homogeneous algebraic equations linear in the constant multipliers of the Bessel functions or the powers r^n. From these, one can find the secular equation in the usual form. Thus the general three-dimensional solution for the pertinent differential equations satisfying the boundary conditions can be found for any integral value n.

Results for axisymmetry were found to be compatible with other work done in the field. Also, if $a = 0$, the effect of the magnetic field is nil. Consequently the least stable mode, being independent of x_3, is unreflected by the presence of the magnetic field. For $n \neq 0$, the solution requires the handling of a 6x6 determinant, and has not been carried out numerically.

CHAPTER VI

EFFECT OF VERTICAL ELECTRIC CURRENT ON THE STABILITY
OF FLUID CONTAINED BETWEEN TWO WALLS

In this chapter, the effect of a vertical current on the stability of a conducting fluid contained between two walls, with a negative vertical temperature gradient, will be investigated. What happens inside the walls is also considered in this chapter. Numerical results, however, have been obtained for simpler circumstances.

15. Splitting of the General Problem Into Two Parts

The fluid and the containing walls shall be assumed to be under a constant uniform potential gradient in the x_3 direction, so that the imposed current density is in the fluid $(-J_3)$ and inside the walls $(-J_3 \frac{\sigma^*}{\sigma})$. Accompanying magnetic field will be x_3 and time independent too, in the fluid

$$4\pi(-J_3) = \frac{\partial H_2}{\partial x_1} - \frac{\partial H_1}{\partial x_2}$$

will hold. And in the walls, one requires:

$$4\pi(-J_3 \frac{\sigma^*}{\sigma}) = \frac{\partial H_2^*}{\partial x_1} - \frac{\partial H_1^*}{\partial x_2} .$$

A solution of the form:

$$H_1 = (4\pi J_3)x_2 \quad , \quad H_1^* = (4\pi J_3 \frac{\sigma^*}{\sigma})x_2 ,$$

and

$$H_2 = H_2^* = H_3 = H_3^* = 0$$

is possible, provided the boundary conditions are satisfied.

Assuming no surface current in the fluid-wall boundary, one has:

$$H_2 = H_2^* \quad \text{and} \quad H_3 = H_3^* \quad \text{at} \quad x_1 = \pm d \ .$$

Preservation of magnetic flux requires:

$$\mu H_1 = \mu^* H_1^* \quad \text{at} \quad x_1 = \pm d \ .$$

Equality of electric potential gradients along the boundary requires:

$$\frac{1}{\sigma}\left(\frac{\partial H_3}{\partial x_1} - \frac{\partial H_1}{\partial x_3}\right) = \frac{1}{\sigma^*}\left(\frac{\partial H_3^*}{\partial x_1} - \frac{\partial H_1^*}{\partial x_3}\right)$$

and

$$\frac{1}{\sigma}\left(\frac{\partial H_2}{\partial x_1} - \frac{\partial H_1}{\partial x_2}\right) = \frac{1}{\sigma^*}\left(\frac{\partial H_2}{\partial x_1} - \frac{\partial H_1^*}{\partial x_2}\right)$$

at

$$x_1 = \pm d \ .$$

As only H_1 and H_1^* do not vanish, both being x_3 independent, boundary conditions to be satisfied are:

$$\mu H_1 = \mu^* H_1^* \quad \text{at} \quad x_1 = \pm d \ ,$$

and

$$\frac{1}{\sigma}\frac{\partial H_1}{\partial x_2} = \frac{1}{\sigma^*}\frac{\partial H_1^*}{\partial x_2} \quad \text{at} \quad x_1 = \pm d \ .$$

This is possible if:

$$\mu\sigma = \mu^*\sigma^* \quad \text{or} \quad \eta = \eta^* \ .$$

The type of field under consideration ($H_1 \prec x_2$) can, however, be produced by other means without assuming $\eta = \eta^*$.

Seeking the least stable mode, which, according to former results is likely to be independent of x_3, variation with x_3 will be neglected. Consequently, Equation (I.8) now assumes the following forms:

$$\rho_o \left(\frac{\partial}{\partial t} - \nu\nabla^2\right) u_1 = -\frac{\partial p'}{\partial x_1} - \mu h_2 J_3 \quad , \tag{1}$$

$$\rho_o \left(\frac{\partial}{\partial t} - \nu\nabla^2\right) u_2 = -\frac{\partial p'}{\partial x_2} + \mu h_1 J_3 + \mu H_1 j_3 \ , \tag{2}$$

and

$$\rho_o \left(\frac{\partial}{\partial t} - \nu\nabla^2\right) u_3 = -\mu H_1 j_2 + \rho_o g\alpha T' \ . \tag{3}$$

Unlike former cases, due to the non-uniformity of H_1, Equation (I.9) reads:

$$\left(\frac{\partial}{\partial t} - \eta\nabla^2\right)h_1 = \frac{-\partial}{\partial x_2}(u_2 H_1) \quad , \tag{4}$$

$$\left(\frac{\partial}{\partial t} - \eta\nabla^2\right)h_2 = \frac{\partial}{\partial x_1}(u_2 H_1) \ , \tag{5}$$

and

$$\left(\frac{\partial}{\partial t} - \eta\nabla^2\right)h_3 = H_1 \frac{\partial u_3}{\partial x_1} \quad . \tag{6}$$

Also, continuity requires:

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0 \tag{7}$$

and

$$\frac{\partial h_1}{\partial x_1} + \frac{\partial h_2}{\partial x_2} = 0 \tag{8}$$

Temperature disturbance distribution is governed by:

$$\left(\frac{\partial}{\partial t} - \kappa\nabla^2\right)T' = -\beta u_3 \tag{9}$$

Inside the wall, disturbance quantities T'^*, h_1^*, h_2^*, and h_3^* are governed by:

$$(\frac{\partial}{\partial t} - \eta^*\nabla^2)\, h_i^* = 0 \quad , \qquad (4)^*\text{-}(6)^*$$

$$(\frac{\partial}{\partial t} - \kappa^*\nabla^2)\, T'^* = 0 \; , \qquad (9)^*$$

and

$$\frac{\partial h_1^*}{\partial x_1} + \frac{\partial h_2^*}{\partial x_2} = 0 \quad , \qquad (8)^*$$

since velocity terms vanish inside the wall.

Boundary conditions are:

$$u_i = 0 \qquad \text{at} \qquad x_1 = \pm\, d \; ,$$

assuming no slip at the boundary. Temperature disturbance as well as heat flow continuity requires:

$$T' = T'^* \quad \text{and} \quad k\,\frac{\partial T'}{\partial x_1} = k^*\,\frac{\partial T'^*}{\partial x_1} \quad \text{at} \quad x_1 = \pm\, d \; .$$

Continuity of flux or magnetic field components require:

$$h_2 = h_2^* \quad \text{and} \quad h_3 = h_3^* \quad \text{at} \quad x_1 = \pm\, d \; .$$

and

$$\mu h_1 = \mu^* h_1^* \quad \text{at} \quad x_1 = \pm\, d \; .$$

Finally, by equating electrical potential on both sides of the boundary, one gets:

$$\frac{1}{\sigma}\,(\frac{\partial h_1}{\partial x_2} - \frac{\partial h_2}{\partial x_1}) = \frac{1}{\sigma^*}\,(\frac{\partial h_1}{\partial x_2} - \frac{\partial h_2^*}{\partial x_1})$$

and

$$\frac{1}{\sigma}\,(\frac{\partial h_3}{\partial x_1}) = \frac{1}{\sigma^*}\,(\frac{\partial h_3^*}{\partial x_1}) \quad \text{at} \quad x_1 = \pm\, d \; .$$

It is possible to split the large differential system governing many variables into two differential systems each consisting of fewer equations, fewer boundary conditions, governing fewer variables and therefore easier to deal with. System I consists of Equations (1), (2), (4), (4)*, (5), (5)*, (7), (8), and (8)* with the relevant boundary conditions. System II consists of Equations (3), (6), (6)*, (9), and (9)*, and appropriate boundary conditions governing u_3, h_3, h_3^*, T', and T'*.

16. Decay of Disturbances Represented by System I

In dealing with cases of stability, one encounters a set of homogeneous governing differential equations accompanied by homogeneous boundary conditions. It is physically understandable that if such differential system lacks a term representing a motivating force but does include terms representing energy dissipation, it would govern solutions representing decaying disturbances. For example, the system governing h_2 in Chapters II and IV, as magnetic diffusivity is dissipative in its nature and there is no motivating term in (II.5) or (IV.5). Another example is System I here. Decay of a disturbance governed by this differential system will be proved, for once, rigorously.

Cross differentiating Equations (1) and (2), and utilizing Equation (8), one obtains:

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) = -\frac{\mu H_1}{\rho_o}\frac{\partial j_3}{\partial x_1} \quad . \tag{10}$$

In view of Equations (7), (8), and (8)*, one can make use of the stream functions $\tilde{\psi}$, $\tilde{\chi}$, and $\tilde{\chi}*$:

$$u_1 = \frac{\partial\tilde{\psi}}{\partial x_2}, \quad u_2 = -\frac{\partial\tilde{\psi}}{\partial x_1}, \quad h_1 = \frac{\partial\tilde{\chi}}{\partial x_2}, \quad h_2 = -\frac{\partial\tilde{\chi}*}{\partial x_1},$$

$$h_1^* = \frac{\partial\tilde{\chi}*}{\partial x_2} \quad \text{and} \quad h_2^* = -\frac{\partial\tilde{\chi}*}{\partial x_1} \quad .$$

Combining Equations (4) and (5) as well as (4)* and (5)*, one obtains:

$$\left(\frac{\partial}{\partial t} - \eta\nabla^2\right)\tilde{\chi} = -(H_1 u_2) = H_1 \frac{\partial\tilde{\psi}}{\partial x_1}, \qquad (11)^*$$

and

$$\left(\frac{\partial}{\partial t} - \eta\nabla^2\right)\tilde{\chi}* = 0 \; .$$

Rewriting Equation (10) in terms of the stream functions, one gets:

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\nabla^2\tilde{\psi} = \frac{\mu H_1}{\rho_o 4\pi}\frac{\partial}{\partial x_1}(\nabla^2\tilde{\chi}), \qquad (10')$$

with

$$\tilde{\psi} = \nu\psi(x,y)\exp(\gamma\tau), \quad \tilde{\chi} = (4\pi J_3 d^2)\,\chi(x,y)\exp(\gamma\tau),$$

$$\tilde{\chi}* = (4\pi J_3 d^2)\,\chi*(x,y)\exp(\gamma\tau)$$

$$x = \frac{x_1}{d}, \quad y = \frac{x_2}{d} \quad \text{and} \quad \tau = \frac{\kappa t}{d^2}$$

one obtains:

$$[\gamma - (\tfrac{\eta}{\nu})\nabla_1^2]\,\chi = y\frac{\partial\psi}{\partial x}, \qquad (11'')$$

$$[\gamma - (\tfrac{\eta^*}{\nu})\nabla_1^2]\,\chi* = 0 \;, \qquad (11'')*$$

$$(\gamma - \nabla_1^2)\,\nabla_1^2\psi = Q(\tfrac{\eta}{\nu})\,y\frac{\partial}{\partial x}(\nabla_1^2\chi)\;, \qquad (10'')$$

where

$$Q = \frac{\mu^2\sigma(4\pi J_3)^2 d^4}{\rho_o \nu} \quad \text{and} \quad \nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad .$$

The boundary conditions in terms of the stream functions are:

$$\psi = \frac{\partial\psi}{\partial x} = 0 \quad \text{at} \quad x = \pm 1 \quad ,$$

$$\psi \rightarrow 0 \quad \text{as} \quad |y| \rightarrow \infty$$

for velocity components. Electromagnetic quantities are governed by:

$$\frac{1}{\sigma}\nabla_1^2\chi = \frac{1}{\sigma^*}\nabla_1^2\chi^* \quad , \quad \mu\frac{\partial\chi}{\partial y} = \mu^* \frac{\partial\chi^*}{\partial y}$$

and

$$\frac{\partial\chi}{\partial x} = \frac{\partial\chi^*}{\partial x} \quad \text{at} \quad x = \pm 1 \ ,$$

by

$$\chi^* \rightarrow 0 \quad \text{as} \quad |x| \rightarrow \infty$$

and by

$$\chi^*, \chi \rightarrow 0 \quad \text{as} \quad |y| \rightarrow \infty \ .$$

At the boundary ($x = \pm 1$) the r.h.s. of Equation (11') vanishes; thus, subtracting (11") from (11")* $(\frac{\mu^*}{\mu})$, one gets:

$$\gamma[\chi - (\frac{\mu^*}{\mu})\chi^*] - [(\frac{\eta}{\nu})\nabla_1^2\chi - (\frac{\eta^*}{\nu})\frac{\mu^*}{\mu}\nabla_1^2\chi^*] = 0 \ .$$

Since the vanishing of one bracket term in the last equation implies the vanishing of the other, and since $\sigma\mu = \sigma^*\mu^*$, either

$$\mu^*\chi^* = \mu\chi \quad \text{at} \quad x = \pm 1$$

or

$$\frac{1}{\sigma}\nabla_1^2\chi = \frac{1}{\sigma^*}\nabla_1^2\chi^* \quad \text{at} \quad x = \pm 1 \quad .$$

Also

$$\frac{\partial \chi}{\partial x} = \frac{\partial \chi^*}{\partial x} \quad \text{at} \quad x = \pm 1 \ .$$

Multiplying Equation (11'') by $\bar{\chi}$, Equation (11'')* by $\bar{\chi}^*$ and Equation (10'') by $\bar{\psi}$, integrating throughout the volume within which these functions are defined, and utilizing Green's theorem, one obtains:

$$\gamma X_o + \left(\frac{\eta}{\nu}\right) X_1 - \iint_{\Sigma} \bar{\chi} \frac{\partial \chi}{\partial n} d\Sigma = \iiint_v y \frac{\partial \psi}{\partial x} \bar{\chi} \, dv \ , \tag{12}$$

$$\gamma X_o^* + \left(\frac{\eta^*}{\nu}\right) X_1^* - \iint_{\Sigma^*} \bar{\chi}^* \left(\frac{\partial \chi^*}{\partial n}\right) d\Sigma^* = 0 \ , \tag{12)*}$$

and

$$-\gamma \Psi_1 - \Psi_2 = Q \left(\frac{\eta}{\nu}\right) \iiint_v y\bar{\psi} \frac{\partial}{\partial x} (\nabla_1^2 \chi) \, dv \ . \tag{13'}$$

Substituting Equation (11'') into the r.h.s. of Equation (13'), one obtains:

$$-\gamma \Psi_1 - \Psi_2 = Q \iiint_v y\bar{\psi} \frac{\partial}{\partial x} \left[\gamma\chi - y \frac{\partial \psi}{\partial x}\right] dv \ , \tag{13}$$

where the direction of n is defined to be perpendicular to the surface of integration, $\iiint_v \ldots dv$ indicates integration of the volume including fluid, $\iiint_{v^*} \ldots dv^*$ indicates integration over volume of walls, $\iint_{\Sigma} \ldots d\Sigma$ and $\iint_{\Sigma^*} \ldots d\Sigma^*$ indicate corresponding respective surface integrations. In Equations (12), (12)*, and (13):

$$X_o = \iiint_v |\chi|^2 dv \ , \quad X_1 = \iiint_v |\nabla\chi|^2 dv \ ,$$

$$X_o^* = \iiint_{v^*} |\chi^*|^2 dv^* \ , \quad X_1^* = \iiint_{v^*} |\nabla\chi^*|^2 dv \ ,$$

$$\Psi_o = \iiint_v |\psi|^2 dv \ , \quad \Psi_1 = \iiint_v |\nabla\psi|^2 dv \ , \quad \Psi_2 = \iiint_v |\nabla^2\psi|^2 dv$$

and

$$\Psi_3 = \iiint_V y^2 \left|\frac{\partial\psi}{\partial x}\right|^2 dv \quad .$$

Combining Equations (12) and (12)* by utilizing the boundary conditions, one obtains:

$$\gamma[X_o + \frac{\mu^*}{\mu} X_o^*] + \frac{\eta}{\nu} [X_1 + \frac{\mu^*}{\mu} X_1^*] = \iiint_V y \frac{\partial\psi}{\partial x} \bar{X} \, dv \quad . \qquad (14)$$

Rewriting Equation (13), integrating $y^2 \frac{\partial^2\psi}{\partial x^2} \bar{\psi}$ with respect to x once, and using the boundary conditions for ψ and $\frac{\partial\psi}{\partial x}$, one gets:

$$-\gamma\Psi_1 - \Psi_2 - Q\Psi_3 = Q\gamma \iiint_V y \bar{\psi} \frac{\partial X}{\partial x} dv = Q\gamma\overline{(\iiint_V y \psi \frac{\partial \bar{X}}{\partial x} dv)} =$$

$$-Q\gamma\overline{(\iiint_V y \frac{\partial\psi}{\partial x} \bar{X} \, dv)} \quad . \qquad (13')$$

Combination of Equations (13') and (14) yields:

$$-\gamma\Psi_1 - \Psi_2 = Q[\Psi_3 - |\gamma|^2(X_o + \frac{\mu^*}{\mu} X_o^*) - \gamma(\frac{\eta}{\nu}) (X_1 + \frac{\mu^*}{\mu} X_1^*)] \quad .(15)$$

Multiplying each side of Equations (11'') and (11'')* by their complex conjugates, and integrating over their respective volume, one gets:

$$|\gamma|^2 X_o + 2\gamma_r(\frac{\eta}{\nu}) X_1 + (\frac{\eta}{\nu})^2 X_2 - \gamma(\frac{\eta}{\nu}) \iint_\Sigma X\frac{\partial\bar{X}}{\partial n} \, d\Sigma$$

$$- \bar{\gamma} (\frac{\eta}{\nu}) \iint_\Sigma \bar{X} \frac{\partial X}{\partial n} d\Sigma = \Psi_3 \qquad (16)$$

and

$$|\gamma|^2 X_o^* + 2\gamma_r(\frac{\eta^*}{\nu}) X_1^* + (\frac{\eta^*}{\nu})^2 X_2^* - \gamma(\frac{\eta^*}{\nu}) \iint_{\Sigma^*} X^* \frac{\partial\bar{X}^*}{\partial n} d\Sigma^*$$

$$- \bar{\gamma}(\frac{\eta^*}{\nu}) \iint_{\Sigma^*} \bar{X}^* \frac{\partial X^*}{\partial n} d\Sigma^* = 0 \qquad (17)$$

When the last two equations are combined by using the boundary conditions, it follows that:

$$\Psi_3 = |\gamma|^2(X_o + \frac{\mu^*}{\mu} X_o^*) + 2\gamma_r\ (\frac{\eta}{\nu})\ (X_1 + \frac{\mu^*}{\mu} X_1^*)$$

$$+ (\frac{\eta}{\nu})^2\ (X_2 + \frac{\mu^*}{\mu} X_2^*) \qquad . \qquad (19)$$

Equation (15) can be split into its real and imaginary parts:

$$-\gamma_r\ \Psi_1 = \Psi_2 + Q\ [\Psi_3 - |\gamma|^2(X_o + \frac{\mu^*}{\mu} X_o^*) - \gamma_r \frac{\eta}{\nu}\ (X_1 + \frac{\mu^*}{\mu} X_1^*)] \quad (15)_r$$

and

$$0 = \gamma_i\ [\Psi_1 - Q(\frac{\eta}{\nu})\ (X_1 + \frac{\mu^*}{\mu} X_1^*)]\ . \qquad (15)_i$$

Substituting from Equation (19) to Equation $(15)_r$, one gets:

$$-\gamma_r\ \Psi_1 = \Psi_2 + Q(\frac{\eta}{\nu})^2\ (X_2 + \frac{\mu^*}{\mu} X_2^*) + \gamma_r\ Q(\frac{\eta}{\nu})\ (X_1 + \frac{\mu^*}{\mu} X_1^*)$$

Since Ψ_1, Ψ_2, etc. are positive definite, $\gamma_r \leq 0$ and $\gamma_r = 0$ only when $\tilde{\Psi} = \tilde{X} = \tilde{X}^* = 0$ identically. Hence, if there is a two-dimensional, this disturbance will decay with time.

17. Solution for System II

In the spirit of former chapters, the non-trivial time-independent solution of system II shall be sought.

With

$$\frac{x_1}{d} = x \quad , \quad \frac{x_2}{d} = y \quad ,$$

$$T' = \frac{\nu}{\kappa}\ (\beta d)\ \theta(x,y)\ , \quad h_3 = \frac{\nu}{\eta}\ (4\pi J_3 d)\ \varphi(x,y)$$

and

$$u_3 = \frac{\nu}{d}\ f(x,y)$$

one gets:

$$-\nabla_1^2 f = Q\, y \frac{\partial \varphi}{\partial x} - R\theta \ , \tag{20}$$

$$-\nabla_1^2 \varphi = y \frac{\partial f}{\partial x} \ , \tag{21}$$

and

$$\nabla_1^2 \theta = f \ . \tag{22}$$

So far, what happens inside the fluid and walls has been considered, and the fluid has been considered to extend to infinity in the direction of x and y. Numerical solutions will be obtained for the case in which the fluid is bounded at $x = \pm 1$ and $y = \pm b$; under some circumstances, one may let b become relatively larger than 1. What happens inside the wall will not be considered. Consideration of a column (or slab) extending to infinity is somewhat artificial. Furthermore, in practice, disturbances are likely to be concentrated over a finite part of the fluid domain.

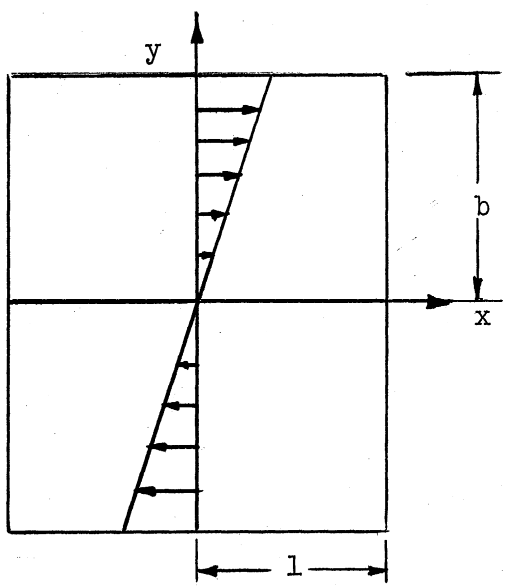

Figure 11. Rectangular Cross-Section of the Column.

A magnetic field varying linearly with y can be formed in several ways. Such a field can be formed by having $\eta = \eta^*$ (Section 15) at the boundary walls at $x = \pm 1$ extending to infinity in the y direction, or by having iron masses behind thin glass walls at $x = \pm 1$. In this case, walls at $y = \pm b$ shall be considered thermally as well as electrically very conductive, and walls at $x = \pm 1$ shall be considered thermally and electrically conductive. This choice is artificial, but the main purpose of the calculation is to show the difference in the effectiveness of electric current in inhibiting different modes of convection. The boundary conditions to be imposed upon disturbance quantities are therefore:

$$\frac{\partial T'}{\partial x_1} = 0 \quad \text{or} \quad \frac{\partial \theta}{\partial x} = 0 \quad \text{at} \quad x = \pm 1$$

$$T' = 0 \quad \text{or} \quad \theta = 0 \quad \text{at} \quad y = \pm b$$

$$h_3 = 0 \quad \text{or} \quad \varphi = 0 \quad \text{at} \quad x = \pm 1$$

$$\frac{\partial h_3}{\partial x_2} = 0 \quad \text{or} \quad \frac{\partial \varphi}{\partial y} = 0 \quad \text{at} \quad y = \pm b$$

and

$$u_3 = 0 \quad \text{or} \quad f = 0 \quad \text{at} \quad x = \pm 1 \quad \text{and} \quad y = \pm b.$$

The function f can be expanded as an infinite series:

[a] $$f = \sum_m \sum_n A_{mn} \sin(m\pi x) \cos\left(\frac{2n-1}{2} \frac{\pi y}{b}\right) ,$$

or

[b] $$f = \sum_m \sum_n A_{mn} \cos\left(\frac{2m-1}{2} \pi x\right) \sin\left(\frac{n\pi y}{b}\right) .$$

These are not the only types of series expansion of f that satisfy the boundary conditions at $x = \pm 1$ and $y = \pm b$. However, one need not consider the case in which f is even both x and y-wise, since on physical grounds (continuity) one must require:

$$\int_{-1}^{1}\int_{-b}^{b} u_3 \, dxdy = 0 \quad , \qquad \text{or} \qquad \int_{-1}^{1}\int_{-b}^{b} f \, dxdy = 0,$$

so long as one deals with the stability of a column contained within a certain volume. Also, from previous knowledge, it can be expected that the velocity field described by f odd in x and y is likely to be a relatively stable one. Therefore, only the situation where f is in the form [a] and [b] will be investigated.

For case [a], with $\alpha_m = m\pi$ and $\beta_n = \frac{2n-1}{2}\frac{\pi}{b}$, Equation (22) becomes:

$$\nabla_1^2\theta = \sum_{n=1,2,3..}\ \sum_{m=1,2,3} Amn \sin(\alpha_m x) \cos(\beta_n y) \ .$$

the solution of which is:

$$\theta = \sum_{n=1,2}\ \sum_{m=1,2} Amn \left[- \frac{\sin(\alpha_m x) \cos(\beta_n y)}{\alpha_m^2 + \beta_n^2} \right]$$

$$+ \sum_{n=1,2..} L_n \sinh(\beta_n x) \cos(\beta_n y) \qquad . \qquad (23)$$

In this solution:

$$\theta = 0 \qquad \text{at} \qquad y = \pm b$$

is automatically satisfied. By adjusting the complementary solution to satisfy the boundary conditions, L_n's can be evaluated. Since:

$$D\theta\Big|_{x=\pm 1} = 0 = \sum_n \sum_m Amn \left[- \frac{\alpha_m(-1)^m \cos(\beta_n y)}{\alpha_m^2 + \beta_n^2} \right]$$

$$+ \sum_n L_n\beta_n \cosh(\beta n) \cos(\beta_n y) \ ,$$

one has:

$$L_n = \frac{1}{\beta_n \cosh(\beta_n)} \sum_m \frac{Amn\ (-1)^m \alpha_m}{\alpha_m^2 + \beta_n^2} \qquad (24)$$

Substituting assumed f in Equation (21), one gets:

$$-\nabla_1^2 \varphi = \sum_m \sum_n Amn(\alpha_m)\ y \cos(\alpha_m x) \cos(\beta_n y)$$

thus

$$\varphi = \sum_m \sum_n \frac{Amn(\alpha_m)}{\alpha_m^2 + \beta_n^2} [-\cos(\alpha_m x)\ y \cos(\beta_n y)$$

$$+ \frac{2\beta_n}{\alpha_m^2 + \beta_n^2} \cos(\alpha_m x) \sinh(\beta_n y)] + \sum_m M_m \cos(\alpha_m x) \sinh(a_m y)$$

$$+ \sum_n K_n \cosh(\beta_n x) \sin(\beta_n y) \qquad (25)$$

Again with the aid of the given boundary conditions, M_m and K_n can be evaluated. Since:

$$\frac{\partial \varphi}{\partial y} = 0 \quad \text{at} \quad y = \pm b .$$

$$0 = \sum_m \sum_n \frac{Amn\ (\alpha_m)}{\alpha_m^2 + \beta_n^2} [\beta_n b (-1)^{n+1}] \cos(\alpha_m x)$$

$$+ \sum_m M_m\ (\alpha_m) \cosh(\alpha_m b) \cos(\alpha_m x)$$

thus

$$M_m = - \frac{1}{\cosh(\alpha_m b)} \sum_n \frac{Amn(\beta_n b)}{(\alpha_m^2 + \beta_n^2)} (-1)^{n+1} \qquad (26)$$

Letting $x = \pm 1$ in Equation (25), multiplying by $\sin(\beta_n y)$ and integrating with respect to y from y = b to y = ± b, one has:

$$0 = K_n \cosh(\beta_n) + \sum_m M_m \cos(\alpha_m) a_n^m$$

$$+ \sum_m \sum_k \frac{Amk\ (\alpha_m)}{\alpha_m^2 + \beta_k^2} \cos(\alpha_m) \left[-b_n^k + \frac{2\beta_k}{\alpha_m^2 + \beta_k^2} \delta_n^k \right] , \qquad (27)$$

in which

$$a^m_i = \int_{-b}^{b} \sinh(\alpha_m y) \sin(\beta_i y)\, dy ,$$

$$b^n_i = \int_{-b}^{b} y \cos(\beta_n y) \sin(\beta_i y) dy ,$$

and

$$\delta^i_j = \text{Kronecker delta}$$

Finally, the newly-evaluated φ and θ will be inserted in Equation (20). By expressing all quantities on both sides of the equations in terms of $\sin(\alpha_m x)\cos(\beta_n y)$, and equating all $\sin(\alpha_m x) \cos(\beta_n y)$ terms on both sides of Equation (20), for every pair (m,n) one obtains:

$$Amn\ [\alpha_m^2 + \beta_n^2] = Q \left\{ - \sum_k \frac{\alpha_m^2}{\alpha_m^2 + \beta_k^2} Amk\ C_m^k \right.$$

$$+ 2 \sum_k \frac{\alpha_m^2}{(\alpha_m^2 + \beta_k^2)^2} \beta_k\ Amk\ d_m^k$$

$$\left. - \sum_k K_k \beta_k\ e_m^k\ d_m^k + \sum_m M_m (\alpha_m)\ f_n^m \right\}$$

$$+ R \left\{ \frac{Amn}{\alpha_m^2 + \beta_n^2} - L_n\ e_n^n \right\} , \qquad (28)$$

in which

$$\int_{-b}^{b} y \sinh(\alpha_m y) \cos(\beta_i y)\, dy = f_i^m , \int_{-b}^{b} \sinh(\beta_n x) \sin(\alpha_i x) dx = e_i^n$$

$$\int_{-b}^{b} y \sin(\beta_n y) \cos(\beta_i y) dy = d_i^n \text{ and } \int_{-b}^{b} y^2 \cos(\beta_n y) \cos(\beta_i y) dy = c_i^n$$

Equations (28), (24), (26), and (27) yield an infinite set of homogeneous algebraic equations governing the A's. By setting the determinant of the coefficients of A's equal to zero, one gets the desired secular equation for Q and R for non-trivial solution of the system. By letting m run from 1 to m', and n from 1 to n', a (m'x n') determinant, an approximation to the desired relationship between Q and R, is obtained. Letting m' = 1 and n' = 2, one gets:

$$0 = \overline{A_{11}} \Big[R[1 + 2 \frac{\alpha_1^{\;1}}{\alpha_1^2 + \beta_1^2} \frac{\tanh(\beta_1)}{\beta_1}] - (\alpha_1^2 + \beta_1^2)^2$$

$$- Q \Big\{ \alpha_1^2 [\frac{b^2}{3} - \frac{2}{(2\beta_1)^2}] - \frac{\alpha_1^2}{\alpha_1^2 + \beta_1^2} (1 + \frac{1}{2} \frac{\tanh \beta_1}{\beta_1})$$

$$+ \frac{\tanh(\beta_2)}{\beta_2} (\frac{2\beta_1 \beta_2}{\beta_2^2 - \beta_1^2}) \frac{2\alpha_1^2}{\alpha_1^2 + \beta_2^2} - \frac{\tanh \beta_2}{\beta_2} (\frac{2\alpha_1 \beta_2}{\alpha_1^2 + \beta_2^2})^2 \frac{2\beta_1^2}{\beta_2^2 - \beta_1^2}$$

$$+ \tanh(\alpha_1 b)\, (\alpha_1 b) \frac{2\beta_1^2}{\alpha_1^2 + \beta_1^2} - \frac{4\alpha_1^2 \beta_1^2}{(\alpha_1^2 + \beta_1^2)^2} \Big\} \Big]$$

$$+ \overline{A_{12}}\, Q \Big\{ \frac{8\alpha_1\beta_1\beta_2}{(\beta_2^2 - \beta_1^2)^2} - \frac{2\alpha_1^2}{(\alpha_1^2 + \beta_2^2)} \frac{2\beta_1\beta_2}{\beta_2^2 - \beta_1^2}$$

$$- \frac{\tanh(\beta_1)}{\beta_1} \frac{2\alpha_1^2}{\alpha_1^2 + \beta_1^2} \left(\frac{\beta_1\beta_2}{\beta_2^2 - \beta_1^2}\right) - \frac{\tanh(\beta_1)}{\beta_1} \frac{2\alpha_1\beta_1\beta_2}{(\alpha_1^2 + \beta_1^2)^2}$$

$$- \frac{\tanh(\beta_2)}{\beta_2} \frac{2\alpha_1^2}{\alpha_1^2 + \beta_2^2} \frac{\beta_1\beta_2}{\beta_2^2 - \beta_1^2} + (\alpha_1 b) \tanh(\alpha_1 b) \frac{2\beta_1\beta_2}{\alpha_1^2 + \beta_1^2}$$

$$\left. - \frac{4\alpha_1^2\beta_1\beta_2}{(\alpha_1^2 + \beta_1^2)^2} \right\} .$$

$$0 = \overline{A_{11}} \left\{ Q \frac{8\alpha_1\beta_1\beta_2}{(\beta_2^2 - \beta_1^2)^2} + \frac{4\alpha_1^2\beta_1\beta_2}{(\alpha_1^2 + \beta_1^2)(\beta_2^2 - \beta_1^2)} \right.$$

$$+ \frac{\tanh(\beta_1)}{\beta_1} \frac{\alpha_1^2}{\alpha_1^2 + \beta_1^2} \frac{2\beta_1\beta_2}{\beta_2^2 - \beta_1^2} - \frac{\tanh(\beta_2)}{\beta_2} \frac{2\alpha_1^2\beta_1\beta_2}{(\alpha_1^2 + \beta_2^2)^2}$$

$$+ \frac{\tanh(\beta_2)}{\beta_2} \frac{\alpha_1^2}{\alpha_1^2 + \beta_2^2} \frac{2\beta_1\beta_2}{\beta_2^2 - \beta_1^2} + (\alpha_1 b) \tanh(\alpha_1 b) \frac{2\beta_1\beta_2}{\alpha_1^2 + \beta_2^2}$$

$$\left. - \frac{4\alpha_1^2\beta_1\beta_2}{(\alpha_1^2 + \beta_2^2)^2} \right\} + \overline{A_{12}} \left[R\left[1 + \frac{2\alpha_1^2}{\alpha_1^2 + \beta_2^2} \frac{\tanh\beta_2}{\beta_2}\right] - (\alpha_1^2 + \beta_2^2)^2 \right.$$

$$- Q \left\{ \alpha_1^2 \left[\frac{b^2}{3} - \frac{2}{(2\beta_2)^2}\right] - \frac{\alpha_1^2}{\alpha_1^2 + \beta_2^2} \left(1 + \frac{1}{2} \frac{\tanh\beta_2}{\beta_2}\right) \right.$$

$$- \frac{\tanh\beta_1}{\beta_1} \left(\frac{2\alpha_1\beta_1}{\alpha_1^2 + \beta_1^2}\right)^2 \frac{2\beta_2^2}{\beta_1^2 - \beta_2^2} + \frac{\tanh\beta_1}{\beta_1} \left(\frac{2\beta_1\beta_2}{\beta_2^2 - \beta_1^2}\right)^2 \frac{2\alpha_1^2}{\alpha_1^2 + \beta_1^2}$$

$$+ (\alpha_1 b) \tanh(\alpha_1 b) \frac{2\beta_2^2}{\alpha_1^2 + \beta_2^2} - \left(\frac{2\alpha_1\beta_2}{\alpha_1^2 + \beta_2^2}\right)^2 \Bigg\}\Bigg] \quad (29)_a$$

in which

$$\overline{A_{11}} = A_{11}/(\alpha_1^2 + \beta_1^2) \text{ and } \overline{A_{12}} = A_{12}/(\alpha_1^2 + \beta_2^2)$$

Special attention should be paid to the first two terms in the bracket coefficient of the diagonal elements.

$$R\left(1 + 2 \frac{1}{1 + (\frac{2n-1}{2b})^2} \frac{\tanh(\beta_n)}{\beta_n}\right) - \pi^4 \left[1 + (\frac{2n-1}{2b})^2\right]^2 \quad (30)_a$$

If Q = 0, the determinant reduces to one having only diagonal terms of the form of $(30)_a$. Such a determinant would vanish by having one of its terms of the form $(30)_a$ equal to zero. Vanishing of the nth diagonal term yields an approximation for R_n, the Rayleigh's number corresponding to the convection mode having u_3 varying as $\cos(\frac{2n-1}{2b} \pi y)$, when electromagnetic interaction is neglected. By setting term $(30)_a$ equal to zero, having n = 1 and b very large one gets

$$R_1 = \pi^4/3 = 32.5$$

which is in good agreement with 31.3 obtained in [6].

18. Other Solution and Comparison

If f is now assumed to be of expansion form [b]:

$$f = \sum_m \sum_n Amn \cos(\frac{2m-1}{2} \pi x) \sin(\frac{n\pi y}{b}),$$

in which

$$\lambda_m = \left(\frac{2m-1}{2}\pi\right) \quad \text{and } \omega_n = \left(\frac{n\pi}{b}\right) .$$

Resulting algebraic equations are:

$$0 = \overline{A_{11}} \Bigg[R[1 + 2 \frac{\coth\omega_1}{\omega_1} \frac{\lambda_1^2}{\lambda_1^2 + \omega_1^2}] - (\lambda_1^2 + \omega_1^2)^2$$

$$- Q \Bigg\{ \lambda_1^2 b^2 [\frac{1}{3} - \frac{1}{2\pi^2}] - \frac{\lambda_1^2}{\lambda_1^2 + \omega_1^2} (1 + \frac{1}{2} \frac{\coth\omega_1}{\omega_1})$$

$$- \frac{\coth\omega_2}{\omega_2} \frac{(2\lambda_1 \omega_2)^2}{(\lambda_1^2 + \omega_2^2)^2} \frac{2\omega_1 \omega_2}{\omega_2^2 - \omega_1^2}$$

$$+ \frac{\coth\omega_2}{\omega_2} \left(\frac{2\omega_2}{\omega_2^2 - \omega_1^2}\right)^2 \frac{2\lambda_1^2 \omega_1 \omega_2}{\lambda_1^2 + \omega_2^2} - \left(\frac{2\lambda_1 \omega_1}{\lambda_1^2 + \omega_1^2}\right)^2$$

$$+ (\lambda_1 b) \coth(\lambda_1 b) \frac{2\omega_1^2}{\lambda_1^2 + \omega_1^2} \Bigg\} + \overline{A_{12}}\, Q \Bigg\{ \frac{8\lambda_1^2 \omega_1 \omega_2}{(\omega_2^2 - \omega_1^2)^2}$$

$$- \frac{4\lambda_1^2}{\lambda_1^2 + \omega_2^2} \frac{\omega_2^2}{\omega_2^2 - \omega_1^2} - \frac{\coth\omega_1}{\omega_1} \frac{2\lambda_1 \omega_1 \omega_2}{(\lambda_1^2 + \omega_1^2)^2}$$

$$- \frac{\coth\omega_1}{\omega_1} \frac{2\lambda_1^2}{\lambda_1^2 + \omega_1^2} \frac{\omega_1 \omega_2}{\omega_2^2 - \omega_1^2} - \frac{\coth\omega_2}{\omega_2} \frac{2\lambda_1^2}{\lambda_1^2 + \omega_2^2} \frac{\omega_2^2}{\omega_2^2 - \omega_1^2}$$

$$- \frac{4\lambda_1^2 \omega_1\omega_2}{(\lambda_1^2 + \omega_1^2)^2} + (\lambda_1 b) \coth(\lambda_1 b) \frac{2\omega_1 \omega_2}{\lambda_1^2 + \omega_1^2} \Bigg\}$$

$$0 = Q \overline{A_{11}} \Bigg\{ \frac{8\lambda_1^2 \omega_1 \omega_2}{(\omega_1^2 - \omega_2^2)^2} - \frac{4\lambda_1^2}{\lambda_1^2 + \omega_1^2} \frac{\omega_1^2}{\omega_1^2 - \omega_2^2}$$

$$- \frac{\coth\omega_2}{\omega_2} \frac{2\lambda_1^2 \omega_1 \omega_2}{(\lambda_1^2 + \omega_2^2)^2} - \frac{\coth\omega_2}{\omega_2} \frac{2\lambda_1^2}{\lambda_1^2 + \omega_2^2} \frac{\omega_1 \omega_2}{\omega_1^2 - \omega_2^2}$$

$$- \frac{\coth\omega_1}{\omega_1} \frac{2\lambda_1^2}{\lambda_1^2 + \omega_1^2} \frac{\omega_1^2}{\omega_1^2 - \omega_2^2} - \frac{4\lambda_1^2 \omega_2 \omega_1}{(\lambda_1^2 + \omega_2^2)^2}$$

$$+ (\lambda_1 b) \coth(\lambda_1 b) \frac{2\omega_1 \omega_2}{\lambda_1^2 + \omega_2^2} \Bigg\} + \overline{A_{12}} \Bigg[R[1 + 2 \frac{\coth\omega_2}{\omega_2} \frac{\lambda_1^2}{\lambda^2 + \omega_2^2}]$$

$$- (\lambda_1^2 + \omega_2^2)^2 - Q \Bigg\{ \lambda_1^2 b^2 (\frac{1}{3} - \frac{1}{8\pi^2}) - \frac{\lambda_1^2}{\lambda_1^2 + \omega_2^2} (1 + \frac{1}{2} \frac{\coth\omega_2}{\omega_2})$$

$$- \frac{\coth\omega_1}{\omega_1} (\frac{2\lambda_1 \omega_1}{\lambda_1^2 + \omega_1^2})^2 \frac{2\omega_1 \omega_2}{\omega_1^2 - \omega_2^2}$$

$$+ \frac{\coth\omega_1}{\omega_1} (\frac{2\lambda_1 \omega_1}{\omega_2^2 - \omega_1^2})^2 \frac{2\omega_1 \omega_2}{\lambda_1^2 + \omega_1^2} - (\frac{2\lambda_1 \omega_2}{\lambda_1^2 + \omega_2^2})^2$$

$$+ (\lambda_1 b) \coth(\lambda_1 b) \frac{2\omega_2^2}{\lambda_1^2 + \omega_2^2} \Big\}\Big]. \qquad (29)_b$$

Again diagonal elements have terms of the type

$$R[1+ \frac{2\lambda_1^2}{\lambda_1^2 + \omega_n^2} \frac{\coth\omega_n}{\omega_n}] - [\lambda_1^2 + \omega_n^2]^2 \qquad (30)_b$$

which yield the critical R_n for convection modes having u_3 vary as $\sin(\frac{n\pi y}{b})$ by setting the (nth) $(30)_b$ term to zero.

Since Equations $(29)_a$ and $(29)_b$ have been arrived at by approximation, the limitation upon utilizing these to derive numerical results should be carefully observed. Equation (23) shows that insofar as θ is condemned, no approximation with respect to y is involved in the calculations. By setting $(30)_a$ to zero, an approximation of R, is obtained because of the representation of $\sinh(\beta_n x)$ by $\sin(\alpha_1 x)$ term. However, Equation (25) shows that $y \frac{\partial\varphi}{\partial x}$ yields the terms $y^2 \cos(\beta_n y)$, $y \sinh(\alpha_m y)$ and $y \sin(\beta_n y)$ in Equation (20). Hence, there is also an error in these calculation arising from approximation in the y direction, when one considers φ as represented by only a few harmonics. This error is compounded with errors arising from approximation in the x direction. However, the y-error is of a relatively higher significance than the x-error, especially if b is large. Thus, in order to solve the system for large b, many y harmonics should be taken into consideration. The use of only two y harmonics is chiefly to demonstrate the method, and can be utilized with confidence only for small b.

Equating the determinants of the coefficients in $(29)_a$ and $(29)_b$ to zero, one gets the Q-R quadratic relationship for b = 2:

$$33.6Q^2 + Q[2932-R(46.2)] + [R(1.53)-236]\ [R(2-562)-119.5] = 0 \qquad (31)_a$$

and

$$3.6Q^2 + Q[874-R(18.9)] + [R(1.695)-24.4]\ [R(1.128)-152] = 0 \qquad (31)_b$$

In Figure (12), the relationship between Q and R has been plotted for both flows. The solid curves represent valid approximations, the dotted lines represent the expected contributions if more x harmonics are used in calculations, and the dashed dotted lines do not represent valid approximations. The Q corresponding to any R_n of each of the flows is zero.

Figure (12) also indicates that the imposed electric current does affect convection inasmuch as there is interaction between velocity and magnetic fields. In both flows, the term $(\vec{u} \times \vec{H})$ is not zero, for y = 0 and s in fact $\vec{j}(u_3 H_1)$, a vector in direction y. The term $\nabla x(\vec{u} \times \vec{H})$ contains therefore a term $\frac{\partial}{\partial x_1}(u_3\ H_1)$ which represents the amount of interaction of velocity and magnetic fields, and is clearly of bigger magnitude in flow [a] than in flow [b], for b = 2. Consequently, flow [b] is less stable with as well as without the imposed current. In view of the reasonable results obtained for b = 2, it is hoped that the method used can be applied to any b.

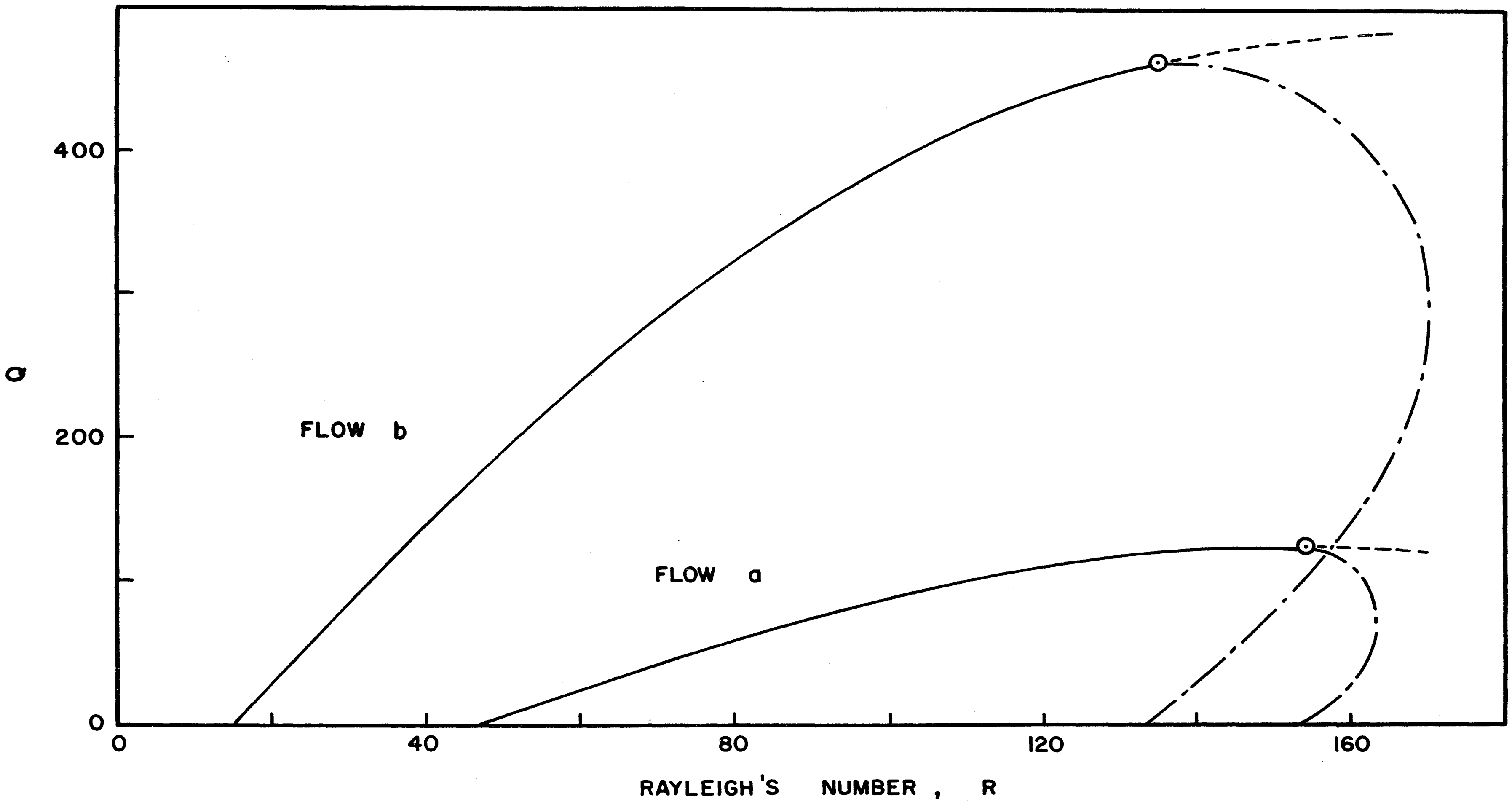

Figure 12. Q-R Relationship - J_3 Imposed.

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