Research Support
University of Michigan Business School

July 1998

A THEORY OF COSTLY SEQUENTIAL BIDDING

WORKING PAPER #98028

BY

KENT D. DANIEL
NORTHWESTERN UNIVERSITY

AND

DAVID HIRSHLEIFER
UNIVERSITY OF MICHIGAN BUSINESS SCHOOL

July 30, 1998

A Theory of Costly Sequential Bidding

KENT D. DANIEL AND DAVID HIRSHLEIFER*

We propose a model of sequential bidding for a valuable object, such as a takeover target, when it is costly submit or revise a bid. An implication of the model is that bidding occurs in repeated jumps, a pattern that is consistent with certain types of natural auctions such as takeover contests. The jumps in bid communicate bidders' information rapidly, leading to contests that are completed with a small numbers of bids. The model provides several new results concerning revenue and efficiency relationships between different auctions, and provides an information-based interpretation of delays in bidding.

*Daniel: Kellogg Graduate School of Management, Northwestern University, 2001 Sheridan Road, Evanston, IL 60208 and NBER; Hirshleifer: School of Business Administration, University of Michigan, 701 Tappan Street, Ann Arbor, MI 48109-1234. We thank Michael Barclay, Dan Bernhardt, Sushil Bikhchandani, Henry Cao, Peter DeMarzo, Michael Fishman, David Levine, Tracy Lewis, Canice Prendergast, Raghu Rajan, John Riley, Lars Stole, Robert Wilson, Jeffrey Zwiebel and seminar participants at UC Berkeley, UCLA, the University of Chicago, Columbia University, Duke University, the University of Illinois at Urbana-Champaign, Indiana University, the London School of Economics, the University of Michigan, New York University, Northwestern University, Ohio State University, Stanford University, and participants of the Western Finance Association meetings at Santa Fe and the American Economic Association meetings at New Orleans for valuable comments and suggestions, and Ralph Bachmann and David Heike for very helpful research assistance.

A Theory of Costly Sequential Bidding

Several markets for unique and valuable objects proceed as variations of English or common open ascending bid auctions. Examples include both formally organized auctions such as Sotheby's, spontaneous auctions such as takeover contests, and (with modification) some privatizations such as the U.S. government auctions for rights to personal communication services spectrum.[1]

In the standard theory of second price auctions (see William Vickrey (1961)), the bidder with the highest valuation for the object wins the auction at a price equal to the valuation of the second-highest-valuation bidder. The conclusion conventionally drawn is that this is also the outcome in sequential English auctions. After an initial minimum bid, each bidder, in turn, submits a bid equal to the previous bid plus the minimum bid increment unless the resulting bid would be higher than his valuation, in which case he passes. The fairly compelling idea is that until his valuation is reached, a bidder who is behind may as well make another try, since he has nothing to lose by doing so and potentially may gain.

We refer to this outcome as the *ratchet solution*. The ratchet solution has two implications that are often at variance with actual bidding behavior. First, bidding is predicted to increase by small amounts, with bidders dropping out after many bids. In practice, bidding for corporate acquisitions typically moves in large jumps, and ends after a few bids.[2] Peter Cramton (1997) provides evidence of frequent large jumps in the bidding for personal communication spectrum (PCS) rights auctioned by the U.S. government. Ralph Cassady (1967, p. 75) observes about private auctions that "... [the would-be buyer] may offer a high price at an early stage in the proceedings in the hope of scaring off competitors." Similarly,

[1] See John McMillan (1994) for a good summary of the PCS spectrum auction design.

[2] The *Wall Street Journal* reports large initial bid premia in several takeovers. For example, Mattel offered a 73 percent premium ($2.2 billion) in a $5 billion unsolicited bid for Hasbro (*WSJ* 1/25/96), and Johnson & Johnson's August 1994 agreement to buy Neutrogena at $35.25 per share was a 70 percent premium over the price two weeks before the bid. Sandoz's 1994 bid for Gerber was for $53, compared to a preceding day price of $35; several similar transactions were reported. Jumps in bidding are common after the initial bid as well. For example, the takeover bidding for Conoco in 1981 saw an initial bid of $70 per share by Seagrams over a market stock price of $58.875. This was followed several weeks later by a competing bid of $87.50 from a Conoco management/DuPont group, and 11 days later by a $90 per share bid from Mobil (see Richard S. Ruback (1982)). Bidding in the 1982 takeover contest for Cities Service went from a stock price of $35.50 to an initial bid of $45, to a competing bid of $63 (see Richard S. Ruback (1983)). And in 1984-85, bidding for Unocal jumped from a stock price of $48 to an initial offer of $54, to a competing bid of $78 (J. Fred Weston, Kwang S. Chung and Susan E. Hoag 1990, p. 616-19,522).

Cassady suggests (p. 81) that, in practice, making a high "keep-out" can be a better strategy than making a low initial bid which would then have to be increased if competition arose.

Second, the conventional analysis does not examine incentives to wait to see what competitors do before making an offer. However, in formal auctions, if there are no bids at the required opening bid, the auctioneer will often lower the required opening level until he hears a bid. After this, the bid sometimes then progresses to a level higher than the initial required opening bid (Cassady (1967)). Similarly, in the market for corporations, takeover rumors about possible bidders for a target firm often circulate for significant periods of time before an offer appears, followed in some cases by competition between multiple bidders for the target.[3]

A reason for these discrepancies between theoretical predictions and empirical observations may be that, in some spontaneous auctions, submitting or revising a bid is costly. David Hirshleifer and Ivan Png (1990) suggest that the costs of takeover bidding include "... fees to counsel, investment bankers, and other outside advisors, the opportunity cost of executive time, [and] the cost of obtaining financing for the bid." Also, in the U.S. some mandated S.E.C. information filings have to be repeated with each bid revision.[4] Consistent with this, H. Nejat Seyhun (1997) notes that unsuccessful takeover bidders experience stock returns of -0.7 percent, in contrast with positive 0.7 percent for successful bidders, a significant difference. He argues that "If the bidder firms do not succeed, they are stuck with the costs while they do not enjoy the synergistic benefits of the takeover."

Hirshleifer and Png further observe that the ratchet solution is highly sensitive to the introduction of costs of bidding, however small. This sensitivity is most evident if valuations are common knowledge. In this case, the effect of a small positive bidding cost is extreme: the highest valuation bidder wins the object at the reservation price, and the other bidders quit immediately rather than waste the cost of bidding.

It may be unfair to hold the symmetric information case against the ratchet solution, since a standard rationale for bidding up to one's valuation is that a bidder cannot be sure how much its opponents value the object. However, consider an equilibrium in which

[3]For example, Paramount was the subject of takeover rumors for several months before Viacom first announced its bid in September 1993; one week later, QVC announced a higher bid.

[4]In addition, if the takeover will be associated with restructuring of the bidder and target, then real investment and operating decisions of the bidder may be hampered by continuing uncertainty over whether merger will occur.

bidders make jumps in their bids in order to *signal* their valuations. Just as in the symmetric information case, once the bidding process has revealed enough information to persuade a bidder that his cause is lost, he should avoid bid costs by quitting immediately.[5]

This paper offers an equilibrium solution to a private valuation, sequential auction model with a costly bidding process. We show that no matter how small the bidding cost, in equilibrium bidding occurs in a series of informative jumps, and that there exist equilibria in which the auction terminates after only a few bids. Furthermore, bidders with low valuations will sometimes delay their bids in order to assess the strength of their competition. We therefore analyze bidding as a *learning process.* The conventional ratchet solution describes an outcome that minimizes the rate of learning subject to the constraint that at least some learning occurs in each round. We focus on an equilibrium that *maximizes* the rate of relevant learning. This allows the auction to end rapidly, which economizes on costs of bidding.

In this model, a bidder makes a high initial bid or raises the bidding by a discrete amount in order to signal a high valuation. Since the first bid perfectly reveals that bidder's valuation, even a small bid cost deters the other bidder, if his valuation is low, from submitting a bid, just as in the symmetric information case mentioned above. If the other bidder responds with a higher bid, he need not fully reveal his valuation, only to signal a sufficiently high valuation that the first bidder will be intimidated into quitting.

There are two benefits of signalling (bidding high) related to forcing out competitors. First, deterring a competitor whose valuation is above the bid of the signalling bidder allows that bidder to buy at a lower price. Second, driving out a competitor early reduces the expected bid costs to be incurred. The cost of signalling is that a bidder may pay more than was necessary to drive out a competitor with valuation is below the signalling bid.

If costs of bidding are literally zero, there are multiple equilibria. The two extreme equilibria are the ratchet solution, and the signalling equilibrium we explore here. However, with zero costs the signalling equilibrium is weak: even after a competitor's fully revealing bid has persuaded a bidder that he is destined to lose, with zero bid costs, bidding up to one's valuation is a weakly dominant strategy.[6] However, for any positive bid cost, however,

[5] Another justification for ratcheting behavior is that your opponent may inadvertently make incorrect moves (*i.e.*, "tremble"). If the likelihood of trembles is small (or infinitesimal, as in the Trembling Hand Perfection equilibrium concept of Selten), the potential expected gain from winning as a result of an opponent's tremble is small, and so is outweighed by even a modest bid cost.

[6] Suppose your opponent's valuation exceeds yours. You don't save any bidding costs by dropping out,

small, the signalling equilibrium is strong. Once a bidder is persuaded that his opponent is going to win (because of his higher valuation), he strictly prefers to quit.

Our model is related to three papers on bidding in takeover contests. Michael J. Fishman (1988) examines a setting in which bidding is costless, but the second bidder must pay an initial investigation cost to learn its valuation of the acquisition target. If the first bidder learns of a high valuation it makes a high preemptive bid which deters the second bidder from investigating and entering the contest. Fishman assumes, however, that if the first bid does accommodate investigation by the second bidder, the ratchet solution ensues.

Hirshleifer and Png (1990) examine a model of sequential bidding with both investigation costs and costs of bidding in a three-type example to examine the desirability of facilitating competition in takeover contests. The current paper achieves greater tractability by focusing on costs of bidding when investigation is costless.

Sugato Bhattacharyya (1990) provides models based on either an initial investigation cost or a one-time uninformative entry fee. If the second bidder enters then, like Fishman, Bhattacharyya assumes that the target firm is sold according to the ratchet solution. He also shows that the bid level can be used as a signal of a bidder's valuation.

Because the Fishman and Bhattacharyya papers examine one-time costs of bidding, they find a single jump occurs in which the first bidder signals his valuation. Their models also exclude delay. In our paper it is costly to bid at every round, so every bid is at a significant premium relative to the preceding one. At every turn, the bidder either passes, or jumps higher in order to signal non-negligible information about his valuation. We focus on equilibria in which a jump in bid either signals the bidder's exact valuation, or signals that his valuation is so high that further competition by the other bidder would, in equilibrium, be unprofitable. This strong informativeness of bidding completes the auction rapidly, which economizes on bid costs.

A rather different approach to jumps in bidding is provided by Christopher Avery (1998) in a setting with common rather than private valuations. In his model, the initial bidding serves as a means of selecting and coordinating upon the asymmetric equilibrium to be

and if your opponent quits by mistake you are glad you bid again. So you may as well 'give it a try.' So when bid costs are precisely zero, the signalling equilibrium is not trembling hand perfect. More generally, if there is a finite nontrivial probability of "trembles," then the willingness of bidders to quit will depend on a comparison of the tremble probability (which encourages bidding) with the bid cost.

employed in a later round of bidding.[7]

Several recent papers examine alternative forms of strategic interaction in takeover auction models. Bhagwan Chowdhry and Vikram Nanda (1993) examine the role of debt as a commitment to bid aggressively. Mike Burkart (1995) investigates a private value auction/takeover setting, and shows that a toehold can induce a potential bidder to bid more aggressively, and can lead to inefficient allocation of the object. In a common value setting, Jeremy Bulow, Ming Huang and Paul Klemperer (1996) show that the initial shareholder bids more aggressively, and as a result competitors bid less aggressively.

We apply the model to derive a number of implications for costly sequential auctions relating to the choice to delay, and what a bidder learns by observing the other delay; the relations between bidding schedules, bidder profits, and seller revenues in the signalling equilibrium of the costly sequential bidding (CSB) auction with other more familiar auctions (with entry fees or minimum bids); the asymptotic optimality of the CSB auction when bid costs are small; bidder preferences regarding order of moves; and the relationship between initial bid jumps and subsequent ones.

The remainder of the paper is structured as follows. Section I outlines the economic setting. Section II reviews the ratchet solution. Section III examines the equilibrium when bidding is costless, and Section IV the equilibrium with costs. Section V compares the sequential auction with other auctions, and Section VI concludes.

I. The Economic Setting

A single good is to be auctioned to two potential bidders. The i'th bidder's valuation for the good, θ_i, is independent of the other bidder's valuation, and its distribution is given by the strictly increasing and twice differentiable probability distribution function $F_i(\theta)$ on $[\underline{\theta}, \overline{\theta}]$, where $\underline{\theta} \geq 0$.[8] Bidders are assumed to maximize their expected profit, which is defined as his valuation of the object θ_i less the amount he pays b_i (if he succeeds in buying it), and

[7]Other authors have examined bids as signals in settings with repeated sales. Sushil Bikhchandani (1988) examines reputation effects in repeated auctions and Sushil Bikhchandani and Chi-fu Huang (1989) provides a signalling model for auctions of objects with resale markets.

[8]Assuming that that all individuals' distributions have the same upper and lower bounds is not essential, but considerably simplifies the notation.

less any bidding costs incurred.[9]

The order of moves is predetermined.[10] The first bidder (*FB*) moves first, and may either pass or submit a bid of b_1 greater than or equal to the minimum bid $\underline{b}$. *FB*'s action is revealed to everyone. The second bidder (*SB*) then can either make a bid $b_2 \geq b_1$ (or $\underline{b}$, if *FB* passed) or pass. Any number of passes may occur before bidding begins. The auction ends when a bid is followed by the other bidder passing.

A bidder incurs a cost of γ each time he bids, regardless of whether he ultimately succeeds in buying the object. If he passes, he pays nothing. The quantity γ is a pure transaction cost of bidding, and paying it does not yield any information to the bidder about either his own valuation or those of his competitors.

II. A Discussion of the Ratchet Solution

As discussed earlier, the conclusion conventionally drawn is that, in sequential English auctions, each bidder in turn should submit a bid equal to the previous bid plus the minimum bid increment as long as the resulting bid is less than his valuation. We refer to this outcome as the *ratchet solution.*

The ratchet solution is a Perfect Bayesian Equilibrium in a setting with minimum bid increments and zero cost of bidding. It is always a weakly dominant strategy to continue to bid as long as the bid level is less than your valuation. (The formal details are unclear in a setting with no minimum bid increment.) Thus, the object will be sold to the highest valuation bidder at a price close to the valuation of the second highest bidder.

Additionally, the ratchet solution is the only trembling-hand perfect equilibrium. Even if bidder A has somehow credibly signalled that his valuation exceeds bidder B's, it is still in B's interest to bid until his valuation is reached, because A possibly may have trembled to an incorrect signal, or may mistakenly pass if B bids again. Since B will continue bidding until just before his valuation is reached, there is no reason for A to try to signal (by raising the bid by more than the minimum bid increment), which entails a risk of paying more than B's valuation.

[9]We model this as a pure private value auction. However, as long as there is at least some private component, there is an incentive to bid high to signal and intimidate the competing bidder, so we would expect effects similar to those modelled here to apply in setting with correlated valuations.

[10]We will show that the expected profit for the bidder is independent of the order of moves. Thus, the equilibrium is consistent with an endogenous order of moves.

In contrast, if bid cost $\gamma > 0$, ratchet behavior is not weakly dominant. If *SB* is certain that *FB* has higher valuation, *SB* should quit rather than waste his bidding cost. Furthermore, for a given bid cost and with a sufficiently small minimum bid increment, ratchet behavior by all bidders ensures negative expected profits for all bidders because the bid cost is incurred many times, and so does not constitute an equilibrium.

Although not the focus of our analysis, when bidding is costly there may exist equilibria that are analogous to the ratchet solution, in which the equilibrium jump in the bid is by a "small" amount that conveys little information about the bidder. High valuation bidders are therefore pooled with low valuation bidders, and successive bids only gradually peel the lowest valuation bidders off of the pool. Such pooling equilibria are wasteful. Learning is slow, so many rounds of bidding cost will be incurred, whereas in a signalling equilibrium the bidding ends quickly. Moreover, in a pooling equilibrium there is likely to be an incentive for a high valuation bidder to defect by jumping to a higher bid to signal his type. If this incentive to drive out competitors is stronger for a high valuation bidder than for a low valuation bidder, this defection may credibly reveal high valuation, breaking the pooling equilibrium.

III. A Signalling Equilibrium with Costless Bidding

We now present an equilibrium with costless bidding in which the first bidder (*FB*) makes a high bid which perfectly reveals his valuation. The costless case is useful as a tractable form of the model that lends itself to comparison with the ratchet solution and other costless-bidding auction mechanisms. With zero bid costs, the equilibrium given here is weak: *FB* is indifferent between making the truth revealing bid and any lower bid. Also, as discussed in Section II, this equilibrium is not trembling hand perfect. However, these weaknesses of the equilibrium obtain only for a bidding cost of exactly zero. In Section IV we show that when bidding is costly the signalling equilibrium is strong, and that *FB* strictly prefers to make the truth revealing bid. Thus, the behavior described here is best viewed as the limiting case of the equilibrium as positive bidding costs *approach* zero.

For simplicity, in this section we rule out defections that involve passing with the intent of bidding later. This is dealt with in a later section, which shows that such defections do not increase expected profits, and with positive bid costs, strictly reduce expected profits.

We maintain the assumptions outlined in Section I. We will focus on an equilibrium in which *FB* makes a bid that fully reveals his type, and *SB* responds with either: (1) a bid at $\hat{\theta}_1$, the signalled valuation of *FB*, in which case *FB* will then pass and *SB* will win; or (2) by passing himself, in which case *FB* will win. This signalling equilibrium is weak in the sense that each bidder is indifferent between making his equilibrium bid and any lower bid. In the remainder of this section we state and verify the equilibrium.

A. The Proposed Equilibrium

As a solving method we examine *FB*'s decision assuming that he plans to bid once (given equilibrium behavior on the part of *SB*). The next subsection verifies that defections involving multiple bids do not increase expected profits. Under the equilibrium conjecture that *FB* plans on making a single bid, *FB* maximizes his expected profit:

$$(\theta_1 - b_1)\, F_2\left(\hat{\theta}_1\left(b_1\right)\right). \tag{1}$$

FB's gain if he does win the auction is $\theta_1 - b_1$. Since *FB's* bid of b_1 signals his valuation to be $\hat{\theta}_1(b_1)$. Based on the equilibrium conjecture that *SB* quits if and only if his valuation is below *FB*'s signalled valuation, *FB*'s probability of winning with this first bid is $F_2\big(\hat{\theta}_1\left(b\right)\big)$.

In equilibrium *SB* passes if $\theta_2 < \hat{\theta}_1$,[11] and bids $b_2 = \hat{\theta}_1$ and wins if $\theta_2 > \hat{\theta}_1$. *SB*'s equilibrium bid is a direct result of the assumption that, in response to an out-of-equilibrium move by *SB* of bidding less than $\hat{\theta}_1$, *FB* would infer *skeptically* that *SB*'s bid is virtually as high as his valuation, *i.e.*, that $\hat{\theta}_2(b_2) = b_2$.[12]

The out-of-equilibrium beliefs that support the proposed equilibrium are consistent with the intuition underlying the Intuitive Criterion of In-Koo Cho and David M. Kreps (1987).[13] This leads to an equilibrium in which the object is efficiently allocated to the highest-valuation bidder. However, there exist other equilibria involving more 'credulous' beliefs

[11] As discussed in Section II, with zero bidding costs passing is a weakly dominated strategy. However, if there is a positive bidding cost, however small, then after θ_1 is revealed, *SB* with $\theta_2 < \hat{\theta}_1$ strongly prefers to pass. Thus, the equilibrium described here is a limiting case of the strong equilibria when there are positive bidding costs (as in Section IV).

[12] This inference is based on the conjecture that *SB* bid (very close to) his full valuation, which is no better than passing. In other words, *FB* is highly skeptical about *SB*'s valuation. Less extreme skepticism yields essentially the same results. An inference that θ_2 is slightly higher valuation than b_2 still supports the equilibrium. When the inference is $b_2 + \epsilon(b_2)$, where $0 < \epsilon(b_2) < \theta_1 - b_2$, *FB* believes his valuation is higher and hence, with $\epsilon(\cdot)$ sufficiently small, can win with a bid very close to b_2. This would deter *SB*'s defection.

[13] The Cho and Kreps (1987) equilibrium is formally defined only in settings with a discrete set of possible types.

which share some important general properties demonstrated here, such as jumps in bids as signals of value and asymptotic optimality as bid costs become small. These credulous equilibria involve partial pooling, so that a higher valuation bidder will sometimes be bluffed out by a lower valuation bidder. Kent D. Daniel and David Hirshleifer (1998) analyze the equilibria in which *FB* makes a credulous inference on seeing an out-of-equilibrium bid by *SB*; see also Bhattacharyya (1990).

Given his beliefs, *FB* with valuation $\theta_1 > b_2$ would respond to a bid $b_2 < \hat{\theta}_1$ by bidding (infinitesimally higher than) b_2, expecting that *SB* would be forced to pass. Similarly, *FB* would continue to employ the strategy of upping *SB*'s bid slightly until *SB* passes or until *SB*'s bid equals or exceeds θ_1. Thus, *SB* cannot gain from a bid below $\hat{\theta}_1(b_1)$, and always follows the equilibrium strategy.

In deriving the signalling schedule, it is convenient to let $b_1(\hat{\theta}_1)$ denote the inverse function $\hat{\theta}_1^{-1}(\cdot)$, *i.e.*, the amount that must be bid to signal a valuation of $\hat{\theta}_1$. The inverse is single-valued under the conjecture, to be verified, that the equilibrium bid is a strictly increasing function of *FB*'s type. *FB* can thus be viewed as maximizing his profits over his signalled valuation,

$$\pi_1(\theta_1) = \max_{\hat{\theta}_1} F_2(\hat{\theta}_1)\left[\theta_1 - b_1(\hat{\theta}_1)\right] - \gamma. \tag{2}$$

This problem is essentially identical to that solved by a bidder in a static, symmetric first price sealed bid auction (see, *e.g.* Paul Milgrom and Robert Weber (1982) and John Riley (1989)).[14] However, in a conventional first price sealed bid auction this problem is symmetric among the bidders, whereas in our setting the bidders are positioned differently. Differentiating with respect to $\hat{\theta}_1$ and equating to zero gives the first order condition for the optimal signalled type $\hat{\theta}_1$; we will later verify that this is the global optimum. Since we seek an equilibrium in which bids are fully and truthfully revealing, we then set $\hat{\theta}_1 = \theta_1$. Suppressing subscripts, this results in the linear first order differential equation

$$F(\theta)b'(\theta) + b(\theta)f(\theta) - \theta f(\theta) = 0,$$

which simplifies to

$$\frac{d}{d\theta}\left(F(\theta)b(\theta)\right) = \theta f(\theta). \tag{3}$$

[14]In both settings, $\hat{\theta}_1$ is the critical value for the opponent's valuation below which the bidder wins the auction, and $b_1(\hat{\theta}_1)$ is the amount a bidder needs to offer to achieve that critical value.

Imposing the initial condition that some type θ_1^* submits the minimum bid of $\underline{b}$, we obtain the unique solution:

$$b_1(\theta_1) = \frac{1}{F_2(\theta_1)}\left(\int_{\theta_1^*}^{\theta_1} s f_2(s)ds + \underline{b}F_2(\theta_1^*)\right). \tag{4}$$

The following lemma provides the relevant initial condition.

Lemma 1 *If:*

1. *If $\underline{b} > \underline{\theta}$ then $\theta_1^* = \underline{b}$, and $b_1(\underline{b}) = \underline{b}$.*

2. *$\underline{b} \leq \underline{\theta}$, then $\theta_1^* = \underline{\theta}$ and $\lim_{\theta \to \underline{\theta}^+} b_1(\theta_1) = \underline{\theta}$.*

Proof: See the appendix.

The intuition for part 1 is that, first $\theta_1^* > \underline{b}$ cannot be an equilibrium: a bidder with valuation $\theta_1 > \underline{b}$ (but less than θ_1^*) could obtain a positive expected profit from bidding $\underline{b}$, versus zero from passing, and would therefore bid. Also, θ_1^* cannot be less than $\underline{b}$, as a bidder with valuation less than $\underline{b}$ would lose money by bidding.

The intuition for part 2 is that as θ approaches $\underline{\theta}$, a limiting bid greater than $\underline{\theta}$ cannot be an equilibrium because the low-type bidders would lose money. Also, a limiting bid less than $\underline{\theta}$ cannot be an equilibrium: a bidder with valuation close to $\underline{\theta}$, who is almost sure to lose, would have an incentive to bid higher to mimic a higher type. He would thereby obtain a non-negligible expected profit. Thus, the only equilibrium is that the limit is $\underline{\theta}$.[15]

Part 2 does not specify $b(\underline{\theta})$. Since $F_2(\underline{\theta}) = 0$, the bid schedule (4) is not defined at $\theta_1 = \underline{\theta}$. The lowest type $\underline{\theta}$ has a zero probability of winning in a revealing equilibrium, so he would be equally well off with a bid of $\underline{b}$, $\underline{\theta}$, or any other bid.[16] In contrast, if $\underline{b} > \underline{\theta}$, then $\theta_1^* = \underline{b}$, and the bid schedule (4) is defined at $\theta = \underline{b}$.

With these alternative initial conditions, by a standard theorem for linear first order differential equations, the solution to (3) exists and is unique so long as $f(\theta)$ is continuous on $(\underline{\theta}, \overline{\theta})$. If we define $\underline{x} \equiv \max\{\underline{\theta}, \underline{b}\}$ as the minimum bid that will ever be made, then *FB*

[15]This result is dependent on our assumption that $F_i(\theta)$ is twice differentiable, implying that there is no mass-point at $\theta = \bar{\theta}$.

[16]Surprisingly, the most relevant equilibrium has a discontinuous bid schedule in which the lowest type bids $\underline{b} < \underline{\theta}$. We will see in Section IV that when bidding costs are positive (however small), the lowest type *FB* to make an offer bids $\underline{b}$ (*not* $\underline{\theta}$), and the bidding schedule is continuous. In the limit as the bidding costs go to zero, the bidding schedule converges pointwise to the schedule just described (with a discontinuity at $\underline{\theta}$). (Since the limit function is discontinuous, the convergence is not uniform.).

will pass if $\theta_1 < \underline{b}$, and will otherwise bid:

$$
\begin{aligned}
b_1(\theta_1) &= \begin{cases} \frac{1}{F_2(\theta_1)}\left[\int_{\underline{x}}^{\theta_1} s f_2(s)ds + \underline{x}F_2(\underline{x})\right] & \text{if } \theta_1 > \underline{x} \\ \underline{x} & \text{if } \theta_1 \le \underline{x} \end{cases} \\
&= E\left[\tilde{x} | \tilde{\theta}_2 \le \theta_1\right], \qquad (5)
\end{aligned}
$$

where random variable $\tilde{x}$ is defined by

$$
\tilde{x} \equiv \begin{cases} \theta_2 & \text{if } \theta_2 \ge \underline{b} \\ \underline{b} & \text{if } \theta_2 < \underline{b}. \end{cases}
$$

If $\underline{b} < \underline{\theta}$, meaning that the minimum bid is not a constraint, then $E\left[\tilde{x}|\tilde{\theta}_2 \le \theta_1\right] = E\left[\tilde{\theta}_2|\tilde{\theta}_2 \le \theta_1\right]$.[17] In either case, the bidding schedule in equation (5) requires the bid to be equal to what the bidder would on average pay if he wins, according to the ratchet solution in an English auction with a minimum bid of $\underline{b}$.

To verify that the first order condition from the optimization problem (2) yields a global maximum, first define *FB's* expected profit as a function of his signalled valuation as

$$
\begin{aligned}
\Pi_1(\hat{\theta}_1, \theta_1) &= \left[\theta_1 - b(\hat{\theta}_1)\right] F_2(\hat{\theta}_1) \\
&= \theta_1 F_2(\hat{\theta}_1) - \int_{\underline{\theta}}^{\hat{\theta}_1} s f(s) ds. \qquad (6)
\end{aligned}
$$

Then the second order condition becomes

$$
\left.\frac{\partial^2 \Pi}{\partial^2 \hat{\theta}_1}\right|_{\hat{\theta}_1 = \theta_1} = -f_2(\theta) < 0.
$$

In a fully revealing equilibrium, $\hat{\theta}_1 = \theta_1$. Substituting this into (6) and integrating by parts gives

$$
\pi(\theta) = [\theta - b(\theta)]F_2(\theta) = \int_{\underline{\theta}}^{\theta} F_2(s)ds. \qquad (7)
$$

This is the same expected profit as in the ratchet solution, and the same as *SB*'s expected profit. Thus both players are indifferent between this equilibrium and the ratchet solution, are just willing to follow this bid level, and are indifferent between bidding first and second. Thus, the standard bidder-profit-equivalence expected profit function for optimal static auctions obtains for both *FB* and *SB*.[18]

[17] *SB*'s bid schedule can also be viewed as taking the form of equation (5) with 1's and 2's reversed, and with a minimum bid of θ_1. That is, *SB* will bid only if $\theta_2 > \theta_1$, and the bid will be $b_2(\theta_2) = E[\theta_1|\theta_1 < \theta_2] = \theta_1$ since *FB's* valuation is known at the time of *SB*'s decision.

[18] Bidders' profits and bid schedules can also be derived using an envelope condition argument (see, *e.g.*, Milgrom and Weber (1982)).

Thus, with zero bid costs the truth-revealing equilibrium in the sequential bidding auction provides identical bidder expected profits and seller expected revenue to those of efficient auctions such as sealed bid first and second price auctions and the ratchet solution. The easiest way to see this is to observe that *SB*'s payoffs are identical to those obtained in the ratchet solution. Since *FB* and *SB* both make the same profits as in other efficient auctions, and since the object is sold to the highest valuation bidder, the seller's profits are also identical to other efficient auctions. These revenue and profit equivalences will continue to hold in an equilibrium with positive bid cost, taking the limit as the bid cost approaches zero.

B. Weak Optimality of the Proposed Equilibrium

The previous section showed that *FB* does at least as well bidding according to the schedule given in (5) as making a defection in which only a single bid is planned (given equilibrium behavior by *SB*) . To verify that this solution is an equilibrium, we also need to show that it is unprofitable for *FB* to defect by (1) making an initial low bid b_1' (resuming subscripts) such that $\hat{\theta}_1(b_1') < \theta_1$, and then (2) bidding a second time if *SB* does not then pass. With such a 'low-bid' strategy *FB* could pay a lower price for the item if *SB* passes, but could still potentially win the auction by rebidding, if *SB* bids.[19]

Suppose that *FB* defects to a low-bid strategy. If *SB* responds with a pass, *FB* wins and pays less than he would have in the proposed equilibrium. However, if *SB* responds with a bid, we show in this section that *FB* pays more if he wins on a second (or later) bid. These effects offset, so his expected profit is equal with the low-bid strategy or with the proposed equilibrium strategy. This confirms the proposed behavior as a weak perfect Bayesian equilibrium. In Section IV we show that when bidding is costly, his profit with the low-bid defection is strictly lower, so the proposed equilibrium is strong.

FB's expected profit from the low-bid strategy will depend on *SB*'s inference on seeing the off-equilibrium action of a second bid by *FB*. We make the assumption (to be verified as part of an equilibrium) that *SB* interprets such an action as a new and (this time) truthful separating bid.[20] Therefore, to determine *FB's* profit from the low-bid strategy, we examine

[19] *FB* would never bid above the proposed equilibrium bid level. If *FB* were to do so, the only equilibrium bid *SB* can make in response is $b_2 = \hat{\theta}_1(b_1) > \theta_1$. *FB* would never respond to this bid since it exceeds valuation. But if *FB* plans only a single bid, then we have shown that the bid level given in (5) is optimal.

[20] We make this assumption in order to be tough on our candidate for equilibrium. This type of belief

a defection in which, after an initial low bid and a responding equilibrium bid by *SB*, *FB* bids so as to signal his true valuation as θ_1.[21]

Given *FB*'s first bid of b_1 such that $\hat{\theta}_1(b_1) < \theta_1$, and given that *SB* responds with an equilibrium bid of $b_2 = \hat{\theta}_1(b_1)$, *FB* now knows that $\theta_2 \in \left[b_2, \overline{\theta}\right]$ with distribution $F_2(\theta_2|\theta_2 > b_2)$. *FB*'s problem in signalling his signalled valuation at this stage is isomorphic with the problem on his first bid in the setting of the previous section in which he had only one chance to bid. Thus, the form of the solution for the bid is the same as in (5), with two differences. First, the unconditional distribution and density functions for θ_2 must be replaced with the functions conditional on $\theta_2 \in [b_2, \overline{\theta}]$,

$$(8) \qquad F_2(\theta_2|\theta_2 > \hat{\theta}_1) = \frac{F_2(\theta_2) - F_2(\hat{\theta}_1)}{1 - F_2(\hat{\theta}_1)} \quad \text{and} \quad f_2(\theta_2|\theta_2 > \hat{\theta}_1) = \frac{f_2(\theta_2)}{1 - F_2(\hat{\theta}_1)}.$$

Second, we now have the initial condition that the minimum valuation *FB* who would wish to bid again must at least match *SB*'s bid. When bid costs are zero, the lowest type *FB* that would at least match a bid of b_2 has valuation $t = b_2 = \hat{\theta}_1$.

Replacing the distribution and density functions in (5) with those given in (8), and using the above initial condition gives the schedule for *FB's* second bid as a function of his true valuation θ_1 and the valuation signalled by his first round bid:

$$_2b_1\left(\theta_1, \hat{\theta}_1\right) = \frac{\int_{\hat{\theta}_1}^{\theta_1} t f_2(t|t > \hat{\theta}_1)dt}{F_2(\theta_1|\theta_2 > \hat{\theta}_1)} = E[\theta_2|\hat{\theta}_1 < \theta_2 < \theta_1].$$

FB's expected profit from the two rounds of bidding is therefore

$$\begin{aligned} \pi_2(\hat{\theta}_1) &= Pr(\theta_2 < \hat{\theta}_1)\left[\theta_1 - b_1(\hat{\theta}_1)\right] + Pr(\hat{\theta}_1 < \theta_2 < \theta_1)\left[\theta_1 - {_2b_1}(\theta_1)\right] \\ &= Pr(\theta_2 < \hat{\theta}_1)\left(\theta_1 - E[\theta_2|\theta_2 < \hat{\theta}_1]\right) \\ &\quad + Pr(\hat{\theta}_1 < \theta_2 < \theta_1)\left(\theta_1 - E[\theta_2|\hat{\theta}_1 < \theta_2 < \theta_1]\right) \end{aligned}$$

revision on *SB*'s part encourages defection because in effect *FB* is able to say "I deceived you before, but this time I am signalling truthfully" and *SB* will believe him. If instead, once convinced by the first signal that $\theta_1 = \hat{\theta}_1$, *SB* would never increase his assessment of θ_1 (or at most would update it only to γ plus the new bid), *SB* would be less likely to quit, and a defection to a low bid would become even less profitable. The proposed belief is consistent with the concept of perfect Bayesian equilibrium.

[21]Suppose that there is no net gain to bidding low on the first bid with the plan of countering any bid by *SB* with a *second* and revealing bid. Then an iterative argument can be used to show that there will be no gain underbidding on the second bid in the expectation of making a third revealing bid. This is based on the idea that the second round bidding problem (foreseeing a third round of bidding) is similar to the first round bidding problem (foreseeing a second round of bidding). The appendix (Proof of Proposition 1) shows that there is no net gain to any finite or infinite arbitrary sequence of underbidding in many rounds.

$$
\begin{aligned}
&= Pr(\theta_2 < \theta_1)\,(\theta_1 - E[\theta_2|\theta_2 < \theta_1]) \\
(9) \qquad &= Pr(\theta_2 < \theta_1)\,[\theta_1 - b_1(\theta_1)]\,.
\end{aligned}
$$

The expected profit is therefore equal to the expected profit when *FB* contemplates only a single bid (see equation (6)). Thus, there is no gain to the low-bid defection. The appendix demonstrates that for any dynamic defection strategy by *FB* that ultimately signals value, the total expected profits are equal to that obtained with only a single bid. Any such defection that does not ultimately signal value accurately generates strictly lower expected profits. Thus, no defection generates expected total profits greater than (6). We therefore have:

Proposition 1 *If there are two risk neutral bidders who can bid costlessly, then there exists a weak perfect Bayesian equilibrium such that:*

1. *FB's bid, as shown in equation (5), is equal to what FB would on average pay, given that he wins, in a ratchet solution.*
2. *SB, if he wins, pays* $\max\{\underline{b}, \theta_1\}$, *exactly what he pays in a ratchet solution.*
3. *Based on 1. and 2., both bidders and the seller are indifferent between the signalling equilibrium and the ratchet solution, given risk neutrality on the part of all three. If the seller is risk-averse, he prefers the signalling equilibrium.*
4. *Since the expected profit conditional on valuation* θ_i *is equal for a first and second bidder, bidders are indifferent between moving first and second.*
5. *The probability that SB makes a bid is decreasing with the level of the first bid.*

Proof: See appendix.

The evidence of Robert H. Jennings and Michael A. Mazzeo (1993) that a high takeover bid premium is associated with a lower probability of competing offers, is consistent with implication 5. The model has a further empirical implication:

Proposition 2 *The jump between the first and the second bid,* $\theta_1 - b_1$ *is increasing in the initial bid* b_1 *for all identically distributed valuation densities* f *such that* $b'(\theta) < 1$.[22] *satisfying either:*

[22]From equation (4), $\lim_{\theta \to \theta^{*},+} b(\theta) = \theta^*$; and since the highest valuation bidder makes positive expected profit, $b(\overline{\theta}) < \theta$. So averaging b' over the range $[\theta^*, \overline{\theta}]$ shows that,

$$\frac{1}{\overline{\theta} - \theta^*} \int_{\theta^*}^{\overline{\theta}} b'(s)ds = \frac{b(\overline{\theta}) - b(\theta^*)}{(\overline{\theta} - \theta^*)} < 1.$$

To prove that the bid jump size $\theta - b(\theta)$ is monotonically increasing in $b(\theta)$ we derive special cases under which $b'(\theta) < 1$ is true for all θ, not just on average.

1. $f'(\theta) \leq 0$ *for all* θ.

2. $f'(\theta) = k$ *for all* θ, *where* $k > 0$,

3. $f''(\theta) < 0$ *and* $b(\theta) = \frac{1}{F(\theta)} \int_{\underline{\theta}}^{\theta} tf(t)dt \geq \frac{1}{2}(\theta + \underline{\theta})$ *for all* θ.

Proof: See appendix

This implication has not, to our knowledge been tested.

IV. The Equilibrium with Positive Bidding Costs

The previous section demonstrated that, with zero bidding costs, the first bidder (*FB*) is indifferent between the equilibrium bid and any lower bid. With positive bidding costs, we now show that the equilibrium behavior becomes strongly optimal. Our procedure to derive an equilibrium will be similar to that of Section III. First, we derive the equilibrium bidding schedules assuming single-bid strategies, *i.e.*, that *FB* plans to bid only once (if the other bidder follows his equilibrium strategy). Then, we examine *FB's* general maximization problem, contemplating multiple-bid defections, to verify that this single-bid strategy is strictly optimal even when multiple bids are allowed.

When there are delays in bidding, the first player (*FB*) may not be the first to submit an offer. *FB* with a valuation close to $\underline{\theta}$ will not bid initially, because the potential gain from bidding is lower than the bidding cost γ that must be paid. If *SB* also passes, he reveals a low valuation, which raises *FB*'s assessment of his probability of winning, then *FB* may indeed enter the bidding later.

To illustrate the equilibrium, consider a setting where the minimum bid $\underline{b} = 0$, where both bidder's valuations are drawn from a uniform distribution on $[0, 1]$, and where the bidding cost $\gamma = 0.01$. In the first round of bidding, a *FB* with a valuation of $\theta_1 = 0.1$ will clearly choose to pass rather than bid: the reason is that his unconditional probability of having the highest valuation is only 0.1, and his profit if he wins with a bid of $\underline{b} = 0$, is 0.1. Thus, his expected profit, net of bid cost is $(0.1 \cdot 0.1) - 0.01 = 0$. On the other hand, if he passes now, he may have a chance to bid and earn positive profits in the future, providing *SB*'s valuation turns out to be low.[23]

[23]Delay is encouraged by the assumption that there is no cost of passing with the intent of bidding later. Realistically, such delay is probably costly. However, the evidence and examples in the introduction suggest that the cost of bidding probably exceeds the cost of delaying one move. In the model, delay would still

FB will only want to bid if his valuation is greater than a critical value denoted θ_1^*. The equilibrium derived in Subsection A implies that $\theta_1^* = 0.14$ in this example. If $\theta_1 > \theta_1^*$, *FB* will bid according to the schedule shown in Figure 1,[24] and this bid will reveal his valuation. Then, analogous with the costless analysis, if *SB*'s valuation is below *FB*'s signalled valuation $(\theta_2 \leq \hat{\theta}_1)$, *SB* will pass and *FB* will win. If $\theta_2 > \hat{\theta}_1$, then *SB* will bid $\hat{\theta}_1 - \gamma$, and *FB* will then pass and *SB* will win.

In the case $\theta_1 < 0.14$, *FB* will pass. Now *SB* infers correctly that *FB*'s valuation is below 0.14. *SB* then knows he faces a weak opponent and is therefore willing to bid even if θ_2 is considerably lower than 0.14. We will show in Subsection A that *SB* will bid if $\theta_2 > \theta_2^* = 0.05$ according to the schedule in the Figure labeled "*SB*, 1st Round." *SB* will never bid to signal a valuation higher than θ_1^*. Since *SB* knows at this point that $\theta_1 < \theta_1^*$, he can win with certainty by bidding high enough to signal a valuation of θ_1^*. Any higher bid would be wasteful.

Finally, suppose that $\theta_1 < 0.14$ and $\theta_2 < \theta_2^*$. Both *FB* and *SB* will pass in the first round, and *FB* will have a second chance to bid. *SB*'s pass has revealed to *FB* that $\theta_2 < \theta_2^* = 0.05$. Knowing that *SB* is very weak, *FB* is now willing to bid with any valuation $\theta_1 > \theta_3^* = 0.03$. If *FB*'s valuation is above this cutoff, he will bid according to the schedule in the Figure labeled 'FB, 2nd Round.' Passing can continue indefinitely. We will show later that, as long as one of the bidders valuations is greater than $\underline{b}+\gamma$ (in this case 0.01), a bid will eventually occur, and the highest valuation bidder will win the auction.

In the general analysis, we call the first player who in equilibrium submits a bid *FTB* (*First-To-Bid*), and his opponent *STB* (*Second-To-Bid*). Recall that the auction only ends after a bid is followed by a pass. Let F and S subscripts designate *FTB* and *STB* respectively. Let an n subscript refer to the n'th move after a sequence of $n-1$ passes, $n \geq 1$ (so an odd n refers to *FB* and an even n to *SB*). Let $b_n(\theta_F)$ be the bid of *FTB* as a function of his valuation when the first bid occurs in the n'th move, and let $\hat{\theta}_n(b_n)$ be the inference by *STB* about *FTB*'s valuation if the first bid is made on the n'th move.

Proposition 3 *If bidding costs are $\gamma > 0$, then there exists a perfect Bayesian equilibrium such that:*

occur so long as bidding is more costly than passing. A possible further model extension would be to allow for costless quitting, as distinct from costly delay.

[24]This comes from equation (4), with the boundary condition that θ_1^* is equal to *FB*'s bidding cutoff.

1. *Following a sequence of $n-1$ passes ($n \geq 1$), at move n FTB with a valuation of θ_F will submit a bid of $b_n(\theta_F)$ if $\theta_F \geq \theta_n^*$, where θ_n^* is a constant, and pass otherwise.*

2. *Suppose that after a sequence of $n-1$ passes FTB submits a bid of $b_n(\theta_F)$. Then FTB's valuation is correctly inferred by STB to be θ_F, and STB with valuation θ_S will bid $b_S = \theta_F - \gamma$ if $\theta_S > \hat{\theta}_n(b_n)$ and pass otherwise.*

3. *FTB's bidding schedule is given by:*

$$(10) \quad b_n(\theta_F) = \begin{cases} \frac{1}{F_S(\theta_F)}\left[\underline{b}F_S(\theta_n^*) + \int_{\theta_n^*}^{\theta_F} t f_S(t)dt\right] & \text{if } \theta_F \leq \theta_{n-1}^* \\ \frac{1}{F_S(\theta_{n-1}^*)}\left[\underline{b}F_S(\theta_n^*) + \int_{\theta_n^*}^{\theta_{n-1}^*} t f_S(t)dt\right] & \text{if } \theta_F > \theta_{n-1}^*, \end{cases}$$

   where $F_S(\cdot)$ and $f_S(\cdot)$ are the prior distribution and density functions of STB's valuation.

4. *The inference schedule $\hat{\theta}_n(b_n)$ is given by the inverse of the bidding function above for $b_n \in [\underline{b}, b_n(\theta_{n-1}^*))$.*

5. *The sequence of critical valuations θ_n^*, $n \geq 0$ has the properties that*

   (a) *$\theta_{n+1}^* < \theta_n^*$,*

   (b) *$\theta^* \equiv \lim_{n\to\infty} \theta_n^* = \max\{\underline{\theta}, \underline{b} + \gamma\}$.*

   The sequence is defined by $\theta_0^* \equiv \overline{\theta}$, and for $n > 0$ the iterative relation:

$$(11) \qquad \gamma\left[F_S(\theta_{n-1}^*) - F_S(\theta^*)\right] = \int_{\theta^*}^{\theta_n^*} (t - \underline{b}) f_S(t)dt.$$

   where $\theta^* \equiv \max\{\underline{\theta}, \underline{b} + \gamma\}$.

6. *If $\theta_1, \theta_2 < \underline{b} + \gamma$, the object is not sold. Otherwise, it is sold to the highest valuation bidder.*

This perfect Bayesian equilibrium is supported by the out-of-equilibrium belief that, if at step 2, STB submits a bid $b_S < \theta_F - \gamma$, FTB believes that STB's valuation is (close to) $b_S + \gamma$, and consequently revises his bid to just above b_S.

To prove this proposition we need to show three things: (i) given the proposed equilibrium, if $\theta_1 > \gamma + \underline{b}$ or $\theta_2 > \gamma + \underline{b}$, then the object is sold to the highest valuation bidder, and otherwise is not sold (Part 6.); (ii) bidding versus passing is optimal behavior as specified, and (iii) bidding according to the bidding schedule is optimal behavior.

Point (i) is simple. Based on the rules as specified earlier, at any given point in the game, in this equilibrium a bidder quits only if the other bidder has made an offer sufficient to signal a higher valuation. Therefore, if the object is sold at all, it goes to the high valuation bidder.

If neither valuation can generate positive profits from a bid at the minimum bid, the object will not be sold.

Points (ii) and (iii) will be established in Subsections A, B and C. *FTB*'s maximization problem when he bids in a given round is to choose his bid b_F so as to maximize his expected profit given equilibrium response by *STB*,

$$\max_{b_F} \; (\theta_F - b_F)\, F_S^*\left(\hat{\theta}(b_F)\right) - \gamma, \tag{12}$$

where $F_S^*(\cdot)$ is the distribution function of *STB*'s valuation conditional on *FTB*'s current information, which may include the information that *STB* has passed in one or more prior bidding rounds. Comparing with the problem in (2), it is clear that precisely the same first order condition applies, leading to the same differential equation, (3), with the same solution (4). When bidding is costly, a lower range of types will not bid because their expected gross profit would be less than the cost of submitting a bid. Thus, on the first move in which a bid occurs, some *FTB* type $\theta_F^* > \underline{\theta}$ can credibly separate from all lower types by submitting a bid of $\underline{b}$, no matter how low $\underline{b}$ is set. Imposing the boundary condition that θ_n^* be the lowest valuation type to bid and that such a bidder makes the minimum bid $\underline{b}$ gives the upper branch of equation (10). *FTB* knows that $\theta_S \leq \theta_{n-1}^*$. So when θ_F is so high that $b_n(\theta_F) \geq \theta_{n-1}^*$, FB has signalled as strongly as necessary to drive out *STB* with certainty. This confirms the 'topping out' of the bid schedule given in the lower branch of of equation (10). This confirms (iii) under the restriction that FTB plans to bid once (given equilibrium behavior by *STB*) . Subsection C examines general defection strategies involving bidding below the equilibrium bid with the plan of bidding again later.

The lowest valuation *FTB* to bid on a move must be indifferent between bidding and not; if bidding were strictly preferable, a slightly lower type would have incentive to mimic, as in the proof of Lemma 1. To derive this boundary condition, we must determine the entire sequence of critical values, which we do next.

A. Derivation of the Critical Value Sequence

This section derives the equilibrium bid versus pass sequence of critical values. In this equilibrium, the first bidder will bid only if his valuation is above a certain critical value θ_1^*, and will pass otherwise, giving *SB* the opportunity to make a first bid if $\theta_2 > \theta_2^*$. We calculate the decreasing sequence θ_n^* by solving for the valuation of the bidder who is indifferent between bidding zero and passing in the n'th move, assuming there have been $n-1$ prior passes.

If n is odd, the bidder under consideration is FB, and if n is even, this is SB. This bidder will submit a bid only if his expected profit from doing so is at least as high as his expected profit from passing. Given continuity, a bidder who submits the minimum bid of $\underline{b}$ must be indifferent between bidding and passing. Therefore, to calculate θ_n^*, the type who will submit a bid of $\underline{b}$, we equate the expected profit from bidding $\underline{b}$ to the expected profit from passing.

If the n'th move bidder has valuation θ_n^*, he becomes first to bid by submitting a minimum bid of $\underline{b}$. His expected profit from bidding is his payoff if he wins $(\theta_n^*-\underline{b})$, times the probability that he wins, minus the bid cost γ:

$$\pi_n^B = (\theta_n^* - \underline{b})Pr(\theta_S < \theta_n^* | \theta_S < \theta_{n-1}^*) - \gamma = (\theta_n^* - \underline{b}) \left[\frac{F_S(\theta_n^*)}{F_S(\theta_{n-1}^*)} \right] - \gamma. \tag{13}$$

His profit from passing depends on whether his opponent, in the next round, (a) passes, (b) makes a non-top-out bid, or (c) makes a top-out bid. In the appendix, we show that equating the profits from bidding and passing for type θ_n^* yields the critical value sequence for passing, as given in (11) in Part 5 of Proposition 3.

B. Strong Optimality of Bid-versus-Pass Rule

We now demonstrate another part of the conjectured equilibrium (see item (ii) after Proposition 3): that it is optimal for an n'th move bidder whose valuation is above the bidding cutoff θ_n^* to bid, while a bidder with lower valuation passes. The terminology of FTB (first to bid) applies to the bidder who *in equilibrium* bids first, even when we discuss defections in which he does otherwise. We show that bidding is strongly optimal if $\theta_F > \theta_F^*$. Specifically, the expected payoff for FTB with $\theta_F > \theta_F^*$ is strictly higher if he follows an equilibrium strategy of bidding in round n than if he defects by passing, with the plan of bidding again in round $n+1$.

Consider the decision of FTB with $\theta_F > \theta_n^*$ at move n given that $n-1$ prior passes have occurred. FTB believes that the other bidder's valuation is below θ_n^*, since STB did not bid previously. The conditional distribution for STB's valuation is therefore

$$G_S(\theta) \equiv Pr(t < \theta | t < \theta_{n-1}^*) = \frac{F_S(\theta)}{F_S(\theta_n^*)}.$$

Bidders' profits and bid schedules can be written using an envelope condition argument (see, *e.g.*, Milgrom and Weber (1982)). *FTB*'s move-n optimized expected profits, analogous to equation (2), are

$$\pi_F^n(\theta_F) = \max_{\hat{\theta}_F} G_S(\hat{\theta}_F)\left[\theta_F - b_F(\hat{\theta}_F)\right] - \gamma. \tag{14}$$

Using the envelope theorem and assuming that the player with the higher valuation always wins the auction, the derivative of (14) evaluated at $\hat{\theta}_F = \theta_F$ is

$$\frac{d\pi(\theta_F)}{d\theta_F} = G_S(\theta_F).$$

Integrating, the equilibrium expected profit for *FTB* with $\theta_F > \theta_F^*$ who follows the *E*quilibrium strategy of bidding at move n is:

$$\pi_F^E(\theta_F) = \pi_F^E(\theta_F^*) + \int_{\theta_F^*}^{\theta_F} G_S(t)dt, \tag{15}$$

where the first term on the RHS is the expected profit of the lowest type to bid.

Now, define

$$H_S(\theta_F) \equiv Pr(t < \theta_F | t > \theta_F^*) = \frac{G_S(\theta_F) - G_S(\theta_F^*)}{1 - G_S(\theta_F^*)}.$$

Using the envelope condition, *FTB*'s expected profit from *D*efecting by passing at move n, and then bidding to maximize expected profits in the next round is:

$$\begin{aligned}\pi_F^D(\theta_F) &= \pi_F^D(\theta_F^*) + G_S(\theta_F^*)(\theta_F - \theta_F^*) + [1 - G_S(\theta_F^*)]\int_{\theta_F^*}^{\theta_F} H_S(t)dt \\ &= \pi_F^E(\theta_F^*) + G_S(\theta_F^*)(\theta_F - \theta_F^*) + \int_{\theta_F^*}^{\theta_F} \left[G_S(t) - G_S(\theta_F^*)\right] dt. \end{aligned} \tag{16}$$

The first term on the RHS, $\pi_F^D(\theta_F^*)$, is the expected profit of a bidder with $\theta_F = \theta_F^*$ who passes in the n'th round. Since type θ_F^* is indifferent to passing and bidding, this is the same as $\pi_F^E(\theta_F^*)$ in the RHS of (15). The next two terms come from analyzing the profit of *FTB* with valuation $\theta_F > \theta_F^*$ relative to *FTB* with valuation $\theta_F = \theta_F^*$.

After *FTB* makes an out-of-equilibrium pass, there two possible *STB* responses to consider. First, if $\theta_S < \theta_F^*$, which happens with probability $G_S(\theta_F^*)$, then *STB* will either pass or signal his true valuation. In either case *FTB* will win the auction and will obtain an extra profit $\theta_F - \theta_F^*$ above what *FTB* with $\theta_F = \theta_F^*$ would earn, which gives the second term.[25]

[25]Given the out-of-equilibrium beliefs, *FTB* with $\theta_F > \theta_F^*$ optimally makes the same response to a *STB* that signalled $\theta_S < \theta_F^*$ as would a *FTB* with $\theta_F = \theta_F^*$, by bidding $\hat{\theta}_S - \gamma$.

Second, if $\theta_S \geq \theta_F^*$, which happens with probability $[1 - G_S(\theta_F^*)]$, then *STB* will make a 'top-out' bid, signalling a valuation of $\theta_S \geq \theta_F^*$. Then *FTB* may want to bid again. However, if *FTB*'s valuation is only slightly greater than θ_F^*, then he is almost sure to lose to a *STB* who has signalled valuation $\theta_S \geq \theta_F^*$. The net extra payoff for *FTB* of this type from making a second bid is negative, so he will pass. Thus it is only a *FTB* with valuation $\theta_F \geq \theta_P^* > \theta_F^*$ who will want to bid. Moreover, for $\theta_F = \theta_P^*$, the net incremental payoff from making a second bid must be zero. The envelope condition gives *FTB*'s bidder's profit as an integral of the conditional distribution function for *STB*'s valuation (H_S). This gives the third term of (16).

Taking the difference between the profit from the bidding and passing strategies gives

$$\pi_F^E(\theta) - \pi_F^D(\theta) = \int_{\theta_F^*}^{\theta_P^*} [G_S(s) - G_S(\theta_F^*)]\, ds.$$

Since $\theta_P^* > \theta_F^*$, this quantity is positive, so a bidder with $\theta_F > \theta_F^*$ would never defect by passing in the n'th round with the plan of bidding in the next round.

C. Strong Optimality of Adhering to the Bidding Schedule

FTB can only defect from the equilibrium strategy in three ways: he can pass, and he can bid either above or below the equilibrium bid. The previous subsection showed that *FTB* would never wish to defect by passing. Also, it is clear that *FTB*, with valuation θ_F, would never make a high bid: were he to do so, signalling his valuation as $\hat{\theta} > \theta_F$, STB's only equilibrium responses would be to bid $\hat{\theta} - \gamma$ or pass. In either case, *FTB* would not bid again, so this defection would necessarily only involve a single bid. However, we have already shown that, if FTB make only one bid, he prefers to make a truth revealing bid.

This leaves us with one possible remaining defection to consider: a 'low-bid' strategy in which *FTB* makes an initial low bid, and then potentially makes other bids. We first examine a defection where, if *FTB* does make a second bid it truthfully reveals his valuation. We then show that the argument that rules out this defection also rules out all multiple bid defection.

Let the valuation of *FTB* be θ (F subscript omitted). Suppose that *FTB* contemplates a low-bid defection, signalling $\hat{\theta} < \theta$ on his first bid, with the plan to bid truthfully next round if *STB* responds with a counterbid. Let θ_n^* denote the lowest valuation *FTB* who, in equilibrium, bids at move n, $G_S(\cdot)$ the probability distribution function for STB's valuation conditional on *FTB* making his first bid at move n (*i.e.*, conditional on STB's valuation lying in $[\underline{\theta}, \theta^*_{n-1})$). In the rest of this subsection, We suppress the n subscript on θ^* and the S subscript on probability distributions.

We now calculate the profit from the alternative strategy of defecting to a lower bid and signalling a valuation of $\theta^* \leq \hat{\theta} < \theta$.[26] If *STB* bids in response, we assume that *FTB* then either passes or counters with a further, and this time truthful signalling bid. Since bidding is costly, if FTB's valuation θ were only slightly above $\hat{\theta}$, he would not find it profitable to bid again. There is a new critical value, θ', the minimum value of θ such that *FTB* would be just willing to bid a second time in response to the equilibrium counterbid by *STB* of $\hat{\theta} - \gamma$. Let $\pi^D(\theta; \hat{\theta})$ denote *FB*'s expected payoff from defecting to a low bid which signals a valuation of $\hat{\theta} < \theta$. Arguments almost identical to those used in Subsection B in developing equation (16) show that

$$\pi^D(\theta; \hat{\theta}) = \pi^D(\hat{\theta}; \hat{\theta}) + G(\hat{\theta})(\theta - \hat{\theta}) + [1 - G(\hat{\theta})] \int_{\theta'}^{\theta} H(s)ds$$

where

$$H(s) = \frac{G(s) - G(\hat{\theta})}{1 - G(\hat{\theta})}$$

is the distribution of *STB* conditional on his valuation being greater than $\hat{\theta}$. Also, since $\pi^D(\hat{\theta}; \hat{\theta})$ is *not* a defection, this profit is equal to the equilibrium profit for *FB* of type $\hat{\theta}$.

If, however, *FTB* follows the equilibrium strategy, then by the envelope theorem, his expected profit is:

$$\begin{aligned} \pi^E(\theta) &= \pi^E(\theta^*) + \int_{\theta^*}^{\theta} G(s)ds \\ &= \pi^E(\theta^*) + G(\hat{\theta})(\theta - \hat{\theta}) + [1 - G(\hat{\theta})] \int_{\hat{\theta}}^{\theta} H(s)ds, \end{aligned}$$

where the E superscript denotes *E*quilibrium profit, and the second equality follows by simple algebra. The difference is

$$\pi^E(\theta) - \pi^D(\theta; \hat{\theta}) = [1 - G(\hat{\theta})] \int_{\theta'}^{\hat{\theta}} H(s)ds.$$

We have demonstrated that $\theta' > \hat{\theta}$, meaning this quantity must be positive, meaning the defection to the low bid strategy is not profitable.

[26]Recall that a *FTB* with valuation θ^* makes an equilibrium bid of $\underline{b}$. Therefore it is impossible to signal a valuation lower than θ^* by bidding.

To verify the equilibrium, we must also show that a strategy involving *any* number of planned bids is less profitable than the equilibrium strategy. To see this, note that the second bid problem of *FTB* in the two-bid strategy is isomorphic to the bidding problem in the single bid strategy. So, the expected profit from the second bid will be reduced if on the second bid *FTB* does not reveal his true type, but instead underbids and than bids a third time if *STB* bids in response. This reasoning extends indefinitely to show that any strategy involving more than a single equilibrium bid will yield a lower expected profit. This completes the proof of Proposition 3.

D. Delay, Profits, and Asymptotic Efficiency

To further illustrate the properties of the equilibrium, consider the example used at the beginning of this section where $\underline{b} = 0$, and $\theta, \theta \sim U[0,1]$. In this case, the relation in equation (11) simplifies to

$$\theta_1^* = \sqrt{\gamma(2-\gamma)}.$$

So, for $\gamma = 0.2$, $\theta_1^* = 0.6$. Also, by (11),

$$\theta_n^* = \sqrt{\gamma(2\theta_{n-1}^* - \gamma)}.$$

Using this relation, the sequence of bidding critical values is plotted in Figure 2 for $n = 1, 26$. Also, Figure 1 shows the first four bidding schedules for the uniform $[0,1]$ case for $\gamma = 0.01$. These examples suggests that reasonably modest values of γ lead to relatively large effects on bidding schedules and on delay. The introduction suggested that sequential bidding frequently arises as a spontaneous auction mechanism in the absence of formal organization of the market. The following corollary describes the asymptotic optimality of the signalling equilibrium in the costly sequential bidding auction.

Corollary 1 1. *Taking the limit as $\gamma \to 0$, the equilibrium of the costly sequential bidding auction described in Proposition 3 approaches the zero cost equilibrium of Proposition 1. Thus, for small γ, delays in bidding vanish, the equilibrium is asymptotically efficient, and for a risk averse seller is preferable to a FPSB auction and the Ratchet Solution.*

2. *The probabilities of delays in bidding increase monotonically with the cost of bidding.*

Proof: Part 1 follows directly from Proposition 3, and by setting $n = 1$ in Part 5(b) and $\theta_1^* = \underline{b}$. Part 2 can be verified by parametrically differentiating (11) in Proposition 3 to show that $d\theta^*/d\gamma > 0$.

Envelope condition techniques can be extended to calculate bidder expected profits. A very tractable expression results.

Proposition 4 *FB's and SB's net expected profits in the skeptical equilibrium of a two-bidder Costly Sequential Bidding auction with bid cost γ and minimum bid given $\underline{b}$ are*

$$\pi_1(\theta) = \int_{\underline{b}+\gamma}^{\theta} F_2(s)ds \qquad \textit{and} \qquad \pi_2(\theta) = \int_{\underline{b}+\gamma}^{\theta} F_1(s)ds. \tag{17}$$

Proof: See appendix.

This is very similar in form to past 'revenue equivalence' auction results. It is surprising to find such a similar form in an asymmetric auction involving dynamic stochastic learning and delay, stochastic incurrence of deadweight bid costs, use by a given bidder of very different bid schedules under different contingencies, and 'topping out' of bids.

V. Comparisons with Alternative Auction Mechanisms

We compare here profits and bidding strategies in a sequential auction with those of alternative auctions. In the next subsection we compare the profits of bidders and the seller in the Costly Sequential Bidding (CSB) auction to a standard static auction, the First Price Sealed Bid (FPSB) auction, and show that the CSB auction is asymptotically optimal as bidding costs become small. In Subsection B, we compare bid schedules in these auctions.

A. Profit Comparisons

In this subsection we show that there is bidder profit equivalence between the CSB auction and the FPSB auction. This conclusion is surprising given the seeming complexity of behavior in the CSB auction equilibrium, and such structural differences as sequentiality and the resulting possibilities of signalling and of delay, and stochastically incurred deadweight bid costs.

Corollary 2 *FB's and SB's net expected profits in the skeptical equilibrium of a two-bidder CSB auction with bid cost γ and minimum bid given $\underline{b}$ are equal to to the expected bidder profits in a two-bidder FPSB auction with a minimum bid $\underline{b}+\gamma$ and zero bid cost. The sellers'* *expected revenues are smaller in the CSB auction by the expected bid costs to be incurred. The expected bid costs incurred are less than 2γ.*

This follows directly from Proposition 4. Expected bid costs incurred are less than 2γ, because the bidding ends in at most 2 bids, and sometimes does not start at all (or ends in one bid).

Since the object is allocated to the same bidder in both the CSB and the FPSB auctions, the total social value gross of bid costs generated by the two auctions is the same. The difference in expected social value is the difference in expected bid costs incurred. The expected bidder profits are also the same (by Proposition 4). Therefore the seller's expected revenue is lower by the expected bid costs incurred in the CSB auction.

As the bid cost γ approaches zero, the minimum type that will eventually make a bid approaches $\underline{b}$. Thus, *FB*'s bid schedule and the seller's expected revenue approach those of the first price sealed bid auction.

It is interesting to compare the profits in (17) with those of a FPSB auction with a one-shot *entry fee*:

Proposition 5 *The expected bidder profits in a CSB auction with bid cost γ and minimum bid $\underline{b}$ are equal to expected bidder profits in a FPSB auction with minimum bid $\underline{b}$, and where the only cost of bidding is an entry fee $\gamma' < \gamma$ set so that a potential bidder will bid if and only if his valuation exceeds $\underline{b} + \gamma$. The seller's expected revenues differ in the two auctions by the expected bid costs to be incurred.*

Bidders do equally well in the CSB auction as in this FPSB auction because the bidder-profit-equivalence integral starts with $\underline{b} + \gamma$ and runs up to the bidder's valuation in both auctions. In both auctions the object is sold if and only if the maximum valuation exceeds $\underline{b} + \gamma$. Thus, the same profits are realized in the same states of the world in the two auctions. In other words, gross of bid costs incurred, combined bidder-seller expected profits are identical in the two auctions. However, bid costs incurred differ. Since bidder profits (net of bid costs) are the same, seller expected profits must differ by the difference in the expected bid costs incurred. As a result, if bid costs are positive but small, the CSB auction is close to optimal.

We would expect sellers to incur the costs of designing and organizing formal auction schemes, such as sealed bid auctions, to optimize revenues when more valuable objects are involved. Nevertheless, corporate takeovers are usually allowed to proceed spontaneously as natural sequential bidding auctions. Propositions 2 and 5 offer a possible explanation: if bid costs are low, the "spontaneous" sequential bidding auction is approximately optimal.[27]

B. Bidding Schedule Comparisons

This section will show that the bidding strategy of *FB* but not *SB* is identical to that of a sealed bid first price auction with an entry fee $e > \gamma$, where each bidder gets to privately observe his valuation before deciding whether to incur the entry fee. As discussed in earlier sections, the differential equation for *FB's* bid in the CSB auction is the same as in the FPSB auction. Therefore, the bidding strategy for the *FB* in the CSB auction will be the same as for the bidders in FPSB if the initial conditions are matched.

The lowest type *FB* in the CSB auction to bid in the first round (θ_1^*) nets a strictly positive expected profit. He does so because he is determined as the individual who is indifferent between bidding and passing. His opportunity cost of bidding is his positive expected profit from passing with the option to bid later. In contrast, the lowest type in the FPSB auction earns 0 if he doesn't bid. Thus, given equal minimum bids in the two auctions, the bidding cutoff is lower in the FPSB auction than for *FB* in the CSB auction. It follows that in order to have the same critical cutoff in the two auctions, the entry fee in the FPSB auction must be higher than the bid cost in the CSB auction. This discussion implies the following result.

Proposition 6 *For any two-bidder Costly Sequential Bidding auction with bid cost* γ*, there exists a two-bidder First Price Sealed Bid auction with the same minimum bid* $\underline{b}$ *and with an entry fee* $e > \gamma$ *such that FB's bid schedule in the CSB auction is identical to each of the bidders' schedules in the FPSB auction.*

For several reasons, bidder expected profits and seller expected revenues are not equivalent in these two auctions. *SB*'s response in the CSB auction is to match *FB*'s revealed type; this differs from the bidders' behavior in the FPSB auction. Furthermore, the bid costs differ from the entry fee, and different numbers of individuals may incur these expenses. Also, *FB* obtains profits from later rounds of the CSB auction while the FPSB auction is one-shot.

[27] A related argument on approximate auction optimality is made by Jeremy Bulow and Paul Klemperer (1996). They show that an auction with one more bidder is generally superior to a negotiation, suggesting that "it is more worthwhile for a seller to devote resources to expanding the market than to collecting information and making the calculations required to figure out the best mechanism."

VI. Conclusion

The traditional ratchet solution for sequential ascending auctions predicts that bidding will always proceed by minimal increments. This minimizes the rate of relevant learning, and maximizes the number of rounds of bidding that takes place before the auction ends. This paper considers the effects of bid revision costs which must be paid every time a bid is submitted or revised. We propose an alternative equilibrium in which each bidder bids a substantial increment above the minimum or previous bid, and in which the amount of the bid is a signal of valuation. The equilibrium we analyze maximizes the rate of relevant learning, so that the each bidder needs to bid at most once.

A high valuation bidder makes a high bid so as to signal that he is willing to bid considerably more for an object. This will intimidate a competitor with lower valuation into quitting. A competitor with higher valuation makes a further jump, which signals a valuation so high that it forces the initial bidder to quit. Even with zero bid costs, this is a weak equilibrium in the sense that the payoff to a bidder who bids off the equilibrium path is equal to the payoff from making an equilibrium bid. The equilibrium is strong for any positive level of bid revision costs, however small. The structure of the equilibria relies on the out-of-equilibrium beliefs of the bidders. We focus on a set of 'skeptical beliefs' which lead to an equilibrium with the property that the higher valuation bidder always wins the auction. The model also implies empirically that high initial jumps in the bid will tend to result in higher subsequent jumps.

The rapid learning of our analysis is at the opposite extreme to the ratchet solution. In reality, bidders often bid repeatedly. We conjecture that if a bidder receives progressively more precise signals of his valuation (either exogenously, or based on a choice to investigate more precisely before submitting higher bids), he will sometimes revise his bid upwards to signal the receipt of new favorable information.[28] In this scenario, each bid still reveals the bidder's best estimate of his valuation. This information process implies that bidders will sometimes make many bids, each at a significant premium over their opponent's previous

[28]For example, it is likely that further information sometimes arrives during takeover contests. A takeover target may make private or public disclosures during the contest that are relevant for valuation. Also, a target may take defensive actions such as "scorched earth" defensive strategies which affect the valuation of the bidder. Furthermore, a contest can stimulate analysts to generate valuation-relevant information about the target and potential synergies.

bid.[29]

When bidding is costly, the model implies that bidders may delay before starting to bid, and these delays may be arbitrarily long before the first bid is made. Delay reveals that a bidder believes he is unlikely to win, and therefore does not find it worthwhile to incur the bid cost. A bidder who is initially too pessimistic to bid may gain confidence about his chance of winning after seeing his competitor pass. So long as any bidder has a valuation exceeding the minimum bid plus the bid cost, bidding must eventually commence. The fact that a competitor waits unavoidably communicates that his valuation is low, which increases the gains to other bidders of making a bid. The bidding schedule followed by the first bidder is the same as in a static first-price sealed bid auction in which the bidder must pay an entry fee in order to submit an offer.

Bidding schedules, profits, and seller revenues in the costly sequential bidding auction are related to those of traditional static auctions. When bidding is costly, the first bidder's bidding schedule as a function of his valuation is identical to that in a *first price* sealed bid auction with identical minimum bid and with an entry fee set above the bid cost. After a first bid, the amount actually paid by a victorious second bidder is identical to that in a *second-price* sealed bid auction.

Somewhat surprisingly, traditional revenue equivalence results can be extended, with modification, to this costly dynamic auction. When bidding is costless, the expected revenue to each of the bidders is the same as in the ratchet solution. Bidders' net expected profits in the costly sequential bidding auction have the same form as profits in a First Price Sealed Bid Auction with an entry fee lower than the bid cost; they are therefore indifferent as to the order of moves. In addition the costly sequential auction generates the same expected bidder profits as a costless first price sealed bid auction with a minimum bid set equal to the bid cost. Sellers' expected profits are lower in the sequential auction by the expected bid costs incurred.

Furthermore, the costly sequential bidding auction is asymptotically optimal as the bid cost approaches zero. This offers a possible explanation for the persistence of such 'sponta-

[29] A related extension could potentially explain why a bidder may sometimes jump above his own previous bid in the absence of an intervening bid (consistent with auction evidence of Cramton (1997)). Self-jumps of this type are by assumption impossible in our model since the auction would end before such an opportunity arose. However, in a setting that matched the PCS auction more closely, the arrival of further favorable information could create a further incentive to signal, and hence a self-jump.

neous auctions' even in markets for high-value objects such as firms.

Several models of takeover auctions have applied the ratchet solution (perhaps with an initial entry or investigation cost) to address such issues as debt as a commitment device in takeovers, the effects of value reducing defensive strategies, and optimal regulation of takeover contests (see, e.g., Chowdhry and Nanda (1993) and Elazar Berkovitch and Naveen Khanna (1990)). It will be interesting to see what new implications can be derived applying an auction models where bidding occurs in a series of jumps.

To sum up, we have provided a model in which, owing to bidding costs, at each opportunity a bidder either passes or jumps to increase the bid by a substantial margin over the previous one. This communicates bidders' information rapidly, leading to contests that are completed with small numbers of bids. This general pattern is consistent with certain types of natural sequential auctions, such as takeover contests. The model provides several new results concerning revenue and efficiency relationships between different auctions, and provides an information-based interpretation of delays in bidding.

A Appendix - Proof of Propositions

Proof of Lemma 1:

Part 1: No player will ever bid above his valuation, so $b(\theta) \leq \theta$. So $\lim_{\theta \to \underline{\theta}^+} b(\theta) \leq \underline{\theta}$. We next show that $\lim_{\theta \to \underline{\theta}^+} b(\theta) \geq \underline{\theta}$. Assume to the contrary that $\lim_{\theta \to \underline{\theta}^+} b(\theta) < \underline{\theta}$. Then by continuity for any $\epsilon > 0$, however small, $\exists\ \theta' > \underline{\theta}$ such that $b(\theta') = \underline{\theta} - \epsilon$. Now, the expected profit of a bidder with valuation $\underline{\theta} + \delta$ ($\delta > 0$) who makes an equilibrium bid is $F(\underline{\theta} + \delta)[\underline{\theta} + \delta - b(\underline{\theta} + \delta)]$, which goes to zero as $\delta \to 0$. However, a mimicking bid of $b(\theta')$ yields expected profit $F(\theta')(\epsilon + \delta)$, which remains bounded away from 0 as $\delta \to 0$. Therefore, for sufficiently small δ, a mimicking bid yields higher profit than a bid at the proposed level. This contradicts the assumption, so $\lim_{\theta \to \underline{\theta}^+} b(\theta) \geq \underline{\theta}$ Taken together, these two restrictions imply that $\lim_{\theta \to \underline{\theta}^+} b(\theta) = \underline{\theta}$.

Part 2: No bidder with valuation below the minimum bid can profit from making an offer. So the lowest type that can break even by bidding is $\theta_1 = \underline{b}$, and this type can break even only if $b_1(\underline{b}) = \underline{b}$.

Proof of Proposition 1:

We showed that the optimal bid of a bidder who plans to bid once is

$$b(\theta_1) = E[\theta_2 | \underline{\theta} < \theta_2 < \theta_1].$$

Now suppose that *FB* defects to a low-bid strategy. First, *FB* signals a valuation of θ', then a valuation of θ'', and so on until *SB* drops out or until he finally signals a valuation of θ_1, his true valuation. (It is not optimal to quit before signalling one's true valuation.) With such a strategy, *FB's* probability of winning is the same, $F(\theta_1)$. His expected payment is

$$\begin{array}{ll} E[\theta_2 | \underline{\theta} < \theta_2 < \theta'] \cdot F(\theta') & + E[\theta_2 | \theta' < \theta_2 < \theta''] \cdot [F(\theta'') - F(\theta')] \\ + E[\theta_2 | \theta'' < \theta_2 < \theta'''] \cdot [F(\theta''') - F(\theta'')] & + E[\theta_2 | \theta''' < \theta_2 < \theta_1] \cdot [F(\theta_1) - F(\theta''')] \end{array}$$

This sums to

$$E[\theta_2 | \underline{\theta} < \theta_2 < \theta_1] \cdot F(\theta_1).$$

Thus, the expected payment is independent of the number of intermediate bids *FB* makes.

The rest of the proposition is proven in the text except for the part 3 claim that the seller prefers the signalling equilibrium. In the event that $\theta_2 < \theta_1$, in the signalling equilibrium the seller receives $E[\theta_2 | \theta_2 < \theta_1]$, and in the Ratchet Solution receives θ_2. Thus, if $\theta_2 < \theta_1$, then the Ratchet Solution expected payment is $R = E[\theta_2 | \theta_2 < \theta_1] + \theta_2 - E[\theta_2 | \theta_2 < \theta_1] = S + \theta_2 - E[\theta_2 | \theta_2 < \theta_1]$. If $\theta_2 \geq \theta_1$, $S = R$. Let

$$\epsilon \equiv \begin{cases} 0 & \text{if } \theta_2 \geq \theta_1, \\ \theta_2 - E[\theta_2 | \theta_2 < \theta_1] & \text{if } \theta_2 < \theta_1. \end{cases}$$

$$\begin{aligned} E[\epsilon | \theta_1] &= E[\epsilon | \theta_2 < \theta_1] Pr(\theta_2 < \theta_1) + E[\epsilon | \theta_2 \geq \theta_1] Pr(\theta_2 \geq \theta_1) \\ &= (E[\theta_2 | \theta_2 < \theta_1] - E[\theta_2 | \theta_2 < \theta_1]) \, Pr(\theta_2 < \theta_1) \\ &= 0. \end{aligned}$$

So ϵ is white noise. $\|$

Proof of Proposition 2: The second bid is equal to *FB*'s valuation. Therefore the jump between the first and the second bid is the difference between *FB*'s valuation and *FB*'s bid. Suppressing 1 and 2 subscripts, this is increasing with θ_1 if

$$\frac{d\left[\theta_1 - b_1(\theta_1)\right]}{d\theta_1} > 0, \text{ i.e., } \frac{db_1(\theta_1)}{d\theta_1} < 1.$$

Differentiating the bidding schedule in (4) with respect to θ gives

$$\frac{db_1(\theta_1)}{d\theta_1} = \frac{f(\theta_1)}{F(\theta_1)}\left[\theta_1 - b_1(\theta_1)\right].$$

Substituting in equation (7) gives:

$$\frac{db_1(\theta_1)}{d\theta_1} = \theta_1 F(\theta_1) - \int_{\theta^*}^{\theta_1} F(t)dt.$$

Part 1: For f such that $f'(\theta) \leq 0 \; \forall\theta$,

$$\begin{aligned} F(\theta_1) &= \int_{\theta^*}^{\theta_1} f(t)dt \\ &\geq f(\theta_1)(\theta_1 - \theta^*), \quad \text{so} \end{aligned}$$

$$\begin{aligned} \frac{db_1(\theta_1)}{d\theta_1} &= \frac{f(\theta_1)}{F(\theta_1)}(\theta_1 - b_1(\theta_1)) \\ &< \frac{f(\theta_1)}{F(\theta_1)}(\theta_1 - \theta^*) \\ &\leq \frac{f(\theta_1)}{f(\theta_1)(\theta_1 - \theta^*)}(\theta_1 - \theta^*) \\ &= 1. \end{aligned}$$

Part 2: If $f'(\theta) = k$, then f and F are:

$$f(\theta) = \frac{2(\theta_1 - \theta^*)}{(\overline{\theta} - \theta^*)^2}, \quad F(\theta) = \frac{(\theta_1 - \theta^*)^2}{(\overline{\theta} - \theta^*)^2}.$$

Direct substitution gives

$$\frac{db_1(\theta_1)}{d\theta_1} = \frac{f(\theta_1)}{[F(\theta_1)]^2}\int_{\theta^*}^{\theta_1} F(t)dt = \frac{2}{3}.$$

Part 3: For any density f such that $f''(\theta) < 0 \; \forall\theta$,

$$F(\theta_1) > \frac{1}{2}f(\theta_1)(\theta_1 - \theta^*),$$

because the RHS is the area of the right triangle inscribed under the density $f(\theta)$. Therefore,

$$\begin{aligned}
\frac{db_1(\theta_1)}{d\theta_1} &= \frac{f(\theta_1)}{F(\theta_1)}\left[\theta_1 - b_1(\theta_1)\right] \\
&< \frac{f(\theta_1)}{\frac{1}{2}f(\theta_1)(\theta_1-\theta^*)}\left[\theta_1 - b_1(\theta_1)\right] \\
&< \frac{f(\theta_1)}{\frac{1}{2}f(\theta_1)(\theta_1-\theta^*)}\left[\theta_1 - \frac{1}{2}(\theta_1+\theta^*)\right] \\
&< \frac{2}{(\theta_1-\theta^*)}\left(\frac{\theta_1-\theta^*}{2}\right) \\
&= 1.
\end{aligned}$$

Derivation of the critical value sequence for passing, (11) in Proposition 3

We refer to a bidder as the n'th move bidder or *FTB* if this is his *equilibrium* behavior. The n'th move bidder's profit if he passes is calculated by considering the three possibilities for the other bidder's valuation: (1) $\theta_S \geq \theta_n^*$, (2) $\theta_n^* > \theta_S \geq \theta_{n+1}^*$ and (3) $\theta_S < \theta_{n+1}^*$. In Case 1, the n'th move bidder will lose the auction and pay no bidding cost. In Case 2, the other bidder will submit an equilibrium bid in the next move, signalling his valuation, to which the n'th move bidder, in order to win the auction, will respond with a bid of $\theta_S - \gamma$ and win. In Case 2, the n'th move bidder will also have to pay the bidding cost of γ. In Case 3, the other bidder will pass in the $(n+1)$'th move. Therefore the n'th move 'equilibrium strategy' given his initial defection is to submit a bid signalling that his valuation is greater than or equal to θ_{n+1}^*, the other bidder's maximum possible valuation at this stage. We will omit S subscripts which belong to all density and distribution functions in the proof. Then the n'th move bidder's expected profit from passing in the n'th move is

$$\begin{aligned}
\pi_P &= \theta_n^*\frac{F(\theta_n^*)}{F(\theta_{n-1}^*)} - \int_{\theta_{n+1}^*}^{\theta_n^*}(t-\gamma)\frac{f(t)}{F(\theta_{n-1}^*)}dt \\
&\quad -\frac{1}{F(\theta_{n+1}^*)}\left[\underline{b}F(\theta_{n+2}^*) + \int_{\theta_{n+2}^*}^{\theta_{n+1}^*} t\frac{f(t)}{F(\theta_{n+1}^*)}dt\right]\frac{F(\theta_{n+1}^*)}{F(\theta_{n-1}^*)} - \gamma\frac{F(\theta_n^*)}{F(\theta_{n-1}^*)}. \qquad (18)
\end{aligned}$$

The first term (which comes from Cases 2 and 3) is just *FTB*'s valuation of the object times the probability of winning (which occurs when $\theta_S < \theta_n^*$) conditional on $\theta_S < \theta_{n-1}^*$. The second term is the n'th move bidder's expected payment for the object in Case 2. The third term is the n'th move bidder's bid (in brackets, equation (10) with n replaced by $n+2$) given Case 3, times the probability that this will occur, $Pr(\theta_S < \theta_{n+1}^* | \theta_S < \theta_{n-1}^*)$. In Case 3, the n'th move bidder will signal that his valuation is θ_{n+1}^*, which is the $n+1$'th move bidder's maximum valuation given that he passes a second time. Finally, the fourth term gives the expected bidding costs incurred in the future by the n'th move bidder given that he passes. (He incurs future bid costs if the $n+1$'th move bidder has $\theta_S < \theta_n^*$), the probability of which is conditioned on knowing that $\theta_S < \theta_{n-1}^*$).) Simplifying this expression gives

$$\pi_P = \frac{1}{F(\theta_{n-1}^*)}\left[\theta_n^*F(\theta_n^*) - \int_{\theta_{n+2}^*}^{\theta_n^*} tf(t)dt - \gamma F(\theta_{n+1}^*) - \underline{b}F(\theta_{n+2}^*)\right]. \qquad (19)$$

Equating π_B and π_P gives an iterative relation for the critical value levels.

$$\gamma\left[F(\theta_{n-1}^*) - F(\theta_{n+1}^*)\right] = \int_{\theta_{n+2}^*}^{\theta_n^*}(t-\underline{b})f(t)dt \qquad (20)$$

To derive the general equation, we substitute in $n = i, i+2, \ldots, i+2k, \ldots$ into this equation, recalling that, by definition, $\theta_0^* = \overline{\theta}$, and taking the telescoping sum of the resulting equations:

$$\begin{aligned}
\gamma\left[F(\theta_{i-1}^*) - F(\theta_{i+1}^*)\right] &= \int_{\theta_{i+2}^*}^{\theta_i^*} (t-\underline{b})f(t)dt \\
\gamma\left[F(\theta_{i+1}^*) - F(\theta_{i+3}^*)\right] &= \int_{\theta_{i+4}^*}^{\theta_{i+2}^*} (t-\underline{b})f(t)dt \\
&\vdots \\
\gamma\left[F(\theta_{i+2k-3}^*) - F(\theta_{i+2k-1}^*)\right] &= \int_{\theta_{i+2k}^*}^{\theta_{i+2k-2}^*} (t-\underline{b})f(t)dt,
\end{aligned}$$

which sums to

$$\gamma\left[F(\theta_{i-1}^*) - F(\theta_{i+2k-1}^*)\right] = \int_{\theta_{i+2k}^*}^{\theta_i^*} (t-\underline{b})f(t)dt. \tag{21}$$

The θ_n^*'s are monotonically decreasing, because the more passes there have been, the lower a bidder's valuation has to be to make him indifferent between bidding and not. To simplify, we define $\theta^* \equiv \max\{\underline{\theta}, \underline{b}+\gamma\}$ and show that $\lim_{n\to\infty} \theta_n^* = \theta^*$. To show this, we consider two cases. In Case I, $\underline{\theta} < \underline{b}+\gamma$. In case II, $\underline{\theta} \geq \underline{b}+\gamma$.

In Case I, we need to show that $\theta^* \equiv \lim_{n\to\infty} \theta_n^* = \underline{b}+\gamma$. Suppose instead that, in equilibrium, $\theta^* > \underline{b}+\gamma$. Let $\delta \equiv (1/2)[\theta^* - (\underline{b}+\gamma)]$, which is positive in Case I. A bidder with valuation $\theta = \theta^* - \delta$ who makes a first bid of $\underline{b}$ at some move will make a positive profit of $\theta^* - \delta - \underline{b} - \gamma$ with positive probability, as opposed to zero profits according to the equilibrium strategy (always passing). Thus, a bidder with valuation $\theta^* - \delta < \underline{b}+\gamma$ will eventually bid. This contradicts the premise that the limit of the critical values $\theta^* > \underline{b}+\gamma$. Therefore, in Case I, $\theta^* = \underline{b}+\gamma$.

Next, suppose that $\theta^* < \underline{b}+\gamma$, so that $\delta < 0$. The profit of a bidder with $\theta = \theta^* - \delta$ who at any round makes the minimum bid of $\underline{b}$ and wins will be $(\theta^* - \delta) - \underline{b} - \gamma < 0$, as opposed to zero profits if he always passes. Thus, a bidder with valuation $\theta^* - \delta > \theta^*$ will never bid. This contradicts the premise that $\theta^* < \underline{b}+\gamma$. Therefore, in Case I, $\theta^* = \underline{b}+\gamma$.

Similar arguments for Case II shows $\theta^* = \underline{\theta}$ is the only critical value consistent with equilibrium.

Thus, taking the limit of (21) as $k \to \infty$ gives the iterative definition of the θ_i^* sequence given in (11) in Proposition 3. (This applies to θ_1^* as well since we define $\theta_0^* \equiv \overline{\theta}$.)

Proof of Proposition 4: Consider a bidder (*FB* or *SB*) with valuation θ at an arbitrary point in the game tree along the equilibrium path. In the absence of any prior bid by the other bidder, he will become *FTB* at some move n. For conditioning expectations, we will sometimes use an integer variable (e.g., 'n') to denote the event that the first bid occurs in the n'th move. Let $\pi(\theta|n)$ be defined as the expected profit of the given bidder if the first bid is in the n'th move, *i.e.*, he is *STB*. The given bidder must have parity matching n, *i.e.*, he must be *FB iff* n is odd, and *SB* iff n is even. For the remainder of the proof, without risk of ambiguity we suppress all subscripts on distribution and density functions. Let s denote the valuation of the other bidder. We define the conditional distribution function

$$\begin{aligned}
G(\theta) &= Pr(s < \theta|n) \\
&= Pr(s < \theta|s < \theta_{n-1}^*) \\
&= \frac{F(\theta)}{F(\theta_{n-1}^*)}.
\end{aligned} \tag{22}$$

Similarly, the density is

$$g(\theta) = \frac{f(\theta)}{F(\theta^*_{n-1})}.$$

We distinguish two cases.

Case 1: $\theta^*_n < \theta < \theta^*_{n-1}$.

In this case, the bidder makes zero profits unless he is *FTB*, because if the other bidder is *FTB*, his valuation is at least $s > \theta^*_{n-1}$. Also, in this case, if the first bid is made in the n'th move, it is never so high that the flat part of the bid schedule is reached where *FTB* is sure of winning. The conditional expected profit of *FTB* given that the first bid occurs on the n'th turn is

$$\begin{aligned} \pi(\theta|n) &= G(\theta)\left[\theta - \left(\frac{\int_{\theta^*_n}^{\theta} sg(s)ds}{G(\theta)}\right)\right] \\ &= \frac{1}{F(\theta^*_n)}\left[F(\theta)\theta - \int_{\theta^*_n}^{\theta} sf(s)ds\right]. \end{aligned} \tag{23}$$

The unconditional expected profit is the probability-weighted sum of conditional expected profits. However, we have seen that the conditional expected profit when the bidder is not *FTB*is zero. Therefore, the expected profit is the probability of becoming *FTB*, $F(\theta^*_{n-1})$, multiplied by (23). Thus,

$$\begin{aligned} \frac{d\pi(\theta)}{d\theta} &= F(\theta^*_{n-1})\frac{d\pi(\theta|n)}{d\theta} \\ &= F(\theta) + f(\theta)\theta - f(\theta)\theta \\ &= F(\theta). \end{aligned} \tag{24}$$

Case 2: $\theta^*_{n-1} < \theta < \theta^*_{n-2}$.

The fact that $\theta^*_{n-1} < \theta$ leads to two key differences. First, when the first bid is made on the n'th move, the bidder is sure to win and bids on the flat part of his bid schedule. Second, the bidder may win even if the other bidder bids on move $n-1$.

Consider first the case where the other bidder has valuation $s < \theta^*_{n-1}$, which occurs with probability $F(\theta^*_{n-1})$, so that the first bid is in move n. Then *FTB*'s conditional expected profits can be written relative to the profits of a bidder with valuation θ^*_{n-1} as

$$\pi(\theta|n) = \pi(\theta^*_n|n) + \int_{\theta^*_n}^{\theta^*_{n-1}} G(s)ds + (\theta - \theta^*_{n-1}). \tag{25}$$

Here the first term is the conditional expected profit of a bidder with valuation θ^*_n. By a standard envelope condition argument, since we are examining a revealing continuous bid schedule, a bidder with valuation θ^*_{n-1} will have profits equal to this first profit plus the second term above (the integral). This integrates just up to the point where the conditional distribution function hits 1. Finally, since the bidder actually has valuation $\theta > \theta^*_{n-1}$, he makes an additional profit given by the last term. He makes this additional profit with certainty since he is sure to win.[30]

Consider next the case where the other bidder has valuation $\theta^*_{n-1} < s < \theta^*_{n-3}$, which occurs with probability $F(\theta^*_{n-3}) - F(\theta^*_{n-1})$, so that the first bid is in move $n-1$. (If s is

[30]The last quantity could be folded into the integral by making the upper limit of integration be θ.

above this range, the other bidder will win for sure, and the profits we are calculating are zero.)

Now define the conditional distribution function $H(\theta)$ as the probability the bidder is high given that the other bidder bid first in move $n-1$, *i.e.*,

$$\begin{aligned} H(\theta) &= Pr(s < \theta | n-1) \\ &= Pr(s < \theta | \theta^*_{n-1} < s < \theta^*_{n-3}) \\ &= \frac{F(\theta)}{F(\theta^*_{n-3}) - F(\theta^*_{n-1})}. \end{aligned}$$

Similarly, the density

$$h(\theta) = \frac{f(\theta)}{F(\theta^*_{n-3}) - F(\theta^*_{n-1})}.$$

Now the bidder's conditional expected profit can be calculated by integrating over possible values of FTB at $n-1$,

$$\begin{aligned} \pi(\theta|n-1) &= \int_{\theta^*_{n-1}}^{\theta} (\theta - s)h(s)ds \\ &= \theta[H(\theta) - H(\theta^*_{n-1})] - \int_{\theta^*_{n-1}}^{\theta} sh(s)ds \\ &= \frac{1}{F(\theta^*_{n-3}) - F(\theta^*_{n-1})}\left(\theta[F(\theta) - F(\theta^*_{n-1})] - \int_{\theta^*_{n-1}}^{\theta} sf(s)ds\right). \end{aligned} \tag{26}$$

Since unconditional expected profit is a probability weighted average of conditional expected profits, we multiplying (25) by $F(\theta^*_{n-1})$, multiply (26) by $F(\theta^*_{n-3}) - F(\theta^*_{n-1})$, sum, and integrate by parts the last integral in (26). This gives

$$\begin{aligned} \pi(\theta) &= \pi(\theta^*_n|n)F(\theta^*_{n-1}) + \int_{\theta^*_n}^{\theta^*_{n-1}} F(s)ds + \theta F(\theta^*_{n-1}) - \theta^*_{n-1}F(\theta^*_{n-1}) + \theta F(\theta) - \theta F(\theta^*_{n-1}) \\ &\quad - \int_{\theta^*_{n-1}}^{\theta} sf(s)ds \\ &= \pi(\theta^*_n|n)F(\theta^*_{n-1}) + \int_{\theta^*_n}^{\theta} F(s)ds \\ &= \pi(\theta^*_n) + \int_{\theta^*_n}^{\theta} F(s)ds, \end{aligned}$$

where the last equality holds because

$$\pi(\theta^*_n) = \pi(\theta^*_n|n)F(\theta^*_{n-1}) + \pi(\theta^*_n|s > \theta^*_{n-1})[1 - F(\theta^*_{n-1})], \tag{27}$$

and the last term on the RHS is zero, because if $s > \theta^*_{n-1}$, then the bidder with $\theta = \theta^*_n$ always loses so his conditional profit in the above equation is zero.

Differentiating with respect to θ shows that just as in Case 1, equation (24),

$$\frac{d\pi(\theta)}{d\theta} = F(\theta). \tag{28}$$

Since equation (28) obtains for values of θ satisfying either Case 1 or Case 2, and n is arbitrary, and since $\lim_{n\to\infty} \theta_n^* = \max\{\underline{\theta}, \underline{b} + \gamma\}$, (28) holds for all $\theta > \underline{b} + \gamma$. Thus,

$$\pi(\theta) = \int_{\underline{b}+\gamma}^{\theta} F(\theta) d\theta.$$

This last quantity is the profit to a bidder in a FPSB auction with minimum bid of $\underline{b} + \gamma$.

Thus, the auctions are bidder-profit equivalent. Since stochastic deadweight costs are incurred, the CSB auction yields lower exp. revenues for the seller by the expected bid costs incurred. ||

References

Avery, Christopher. "Strategic Jump Bidding in English Auctions." *The Review of Economic Studies*, April 1998, *65* (2), pp. 185–210.

Berkovitch, Elazar and Khanna, Naveen. "How Target Shareholders Benefit from Value-Reducing Defensive Strategies." *Journal of Finance*, 1990, *45*, pp. 137–56.

Bhattacharyya, Sugato. "The Analytics of Takeover Bidding: Initial Bids and Their Premia." Working Paper, Carnegie Mellon University, 1990.

Bikhchandani, Sushil. "Reputation in Repeated Second Price Auctions." *Journal of Economic Theory*, 1988, *46*, pp. 97–119.

___ **and Huang, Chi-fu.** "Auctions with Resale Markets: An Exploratory Model of Treasury Bill Markets." *Review of Financial Studies*, 1989, *2* (3), pp. 311–339.

Bulow, Jeremy and Klemperer, Paul. "Auctions versus Negotiations." *American Economic Review*, March 1996, *86* (1), pp. 180–194.

___ ; **Huang, Ming, and Klemperer, Paul.** "Toeholds and Takeovers." Stanford University Working Paper, November 1996.

Burkart, Mike. "Initial Shareholdings and Overbidding in Takeover Contests." *Journal of Finance*, December 1995, *50* (5), pp. 1491–1515.

Cassady, Ralph. *Auctions and Auctioneering*, California: University of California Press, 1967.

Cho, In-Koo and Kreps, David M. "Signaling Games and Stable Equilibria." *Quarterly Journal of Economics*, 1987, *102*, pp. 179–221.

Chowdhry, Bhagwan and Nanda, Vikram. "The Strategic Role of Debt in Takeover Contests." *Journal of Finance*, June 1993, *48* (2), pp. 731–46.

Cramton, Peter. "The FCC Spectrum Auctions: An Early Assessment." *Journal of Economics and Management Strategy*, 1997, *6* (3), pp. 431–496.

Daniel, Kent D. and Hirshleifer, David. "Competitive Bidding and Bluffing." Northwestern University Working Paper, 1998.

Fishman, Michael J. "A Theory of Pre-emptive Takeover Bidding." *Rand Journal of Economics*, 1988, *19* (1), pp. 88–101.

Hirshleifer, David and Png, Ivan. "Facilitation of Competing Bids and the Price of a Takeover Target." *Review of Financial Studies*, 1990, *2* (4), pp. 587–606.

Jennings, Robert H. and Mazzeo, Michael A. "Competing Bids, Target Management Resistance, and the Structure of Takeover Bids." *Review of Financial Studies*, Winter 1993, *6* (4), pp. 883–910.

McMillan, John. "Selling Spectrum Rights." *Journal of Economic Perspectives*, Summer 1994, *8* (3), pp. 145–162.

Milgrom, Paul and Weber, Robert. "A Theory of Auctions and Competitive Bidding." *Econometrica*, 1982, *50*, pp. 1089–1122.

Riley, John. "Expected Revenues from Open and Sealed Bid Auctions." *Journal of Economic Perspectives*, Summer 1989, *3* (3), pp. 41–50.

Ruback, Richard S. "The Conoco Takeover and Stockholder Returns." *Sloan Management Review*, Winter 1982, pp. 12–13.

____ . "The Cities Service Takeover: A Case Study." *Journal of Finance*, May 1983, *38*, pp. 319–30.

Seyhun, H. Nejat. *Investment Intelligence: Tips from Insider Trading*, Cambridge: MIT Press, 1997.

Vickrey, William. "Counterspeculation, Auctions and Competitive Sealed Tenders." *Journal of Finance*, March 1961, *16* (1), pp. 8–37.

Weston, J. Fred; Chung, Kwang S., and Hoag, Susan E. *Mergers, Restructuring and Corporate Control*, Engelwood Cliffs, NJ: Prentice Hall, 1990.

Figure 1: First Four Bidding Schedules, Uniform $[0,1]$ Example, $\gamma = 0.01$

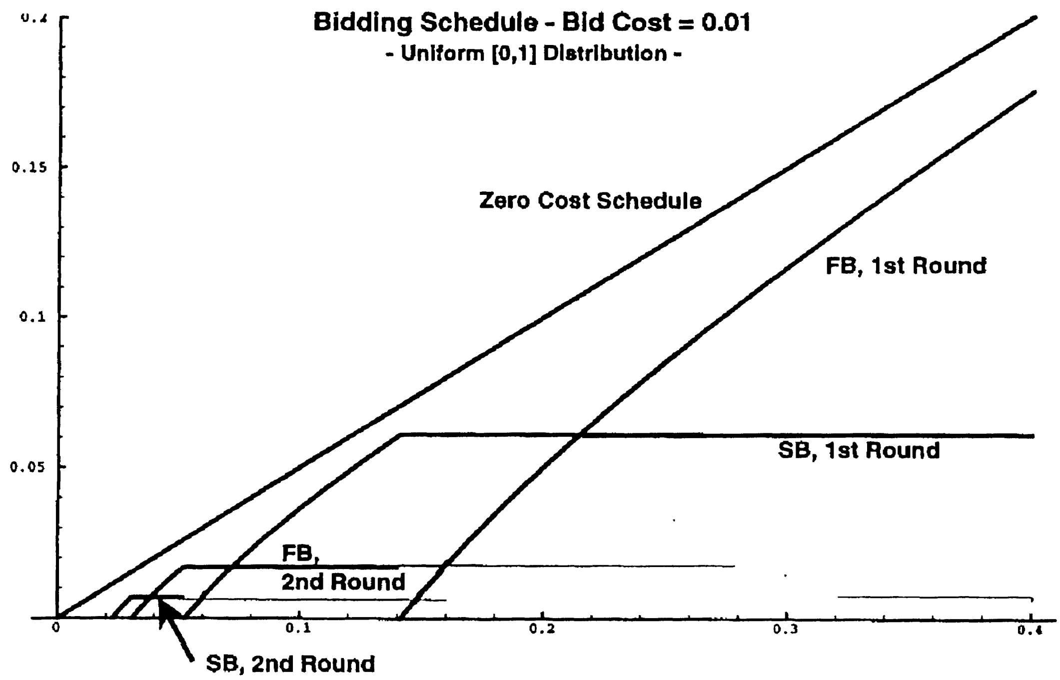

Figure 2: Bidding Critical Value Sequence, $n = 1, 26$.