ENGINEERING RESEARCH INSTITUTE
UNIVERSITY OF MICHIGAN
ANN ARBOR

FINAL REPORT

NOTES ON THE "WATER-ENTRY" PROBLEM

By

J. G. BERRY

R. M. COOPER

P. M. NAGHDI

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PREFACE

This is the final report on Project 2041-3.

Previously a technical report entitled "On The Equations of Motion of Cylindrical Shells" was issued, a copy of a paper based on this report is enclosed.

The "water-entry" problem (subject one of the contract) was carried out to the point where it became clear that for practical purposes the methods of analyses available at present are not suitable. Before a problem as complicated as this can be successfully solved, a great deal of fundamental research on dynamical problems of shells must be conducted.

FINAL REPORT

NOTES ON THE
"WATER-ENTRY" PROBLEM

1. Introduction

This report contains a partial analysis of the problem of determining the stresses which arise in a torpedo-like structure when it is fired or dropped into water. This is usually referred to as the "water-entry" problem.

For purposes of analysis, the torpedo is assumed to be composed of a thin circular cylindrical shell (the body), joined at one end by a hemisphere or an ogive (the nose) and at the other by a cone (the tail). The mass of any mechanism inside the body is assumed to be distributed in the nose, body, and tail in such a way that the density of the body is constant and the center of gravity remains unchanged.

It has been observed that, at least for relatively long torpedos, the failure of the structure occurs in the cylindrical portion. Therefore the "water-entry" problem may be idealized to that of the determination of the stresses in a thin circular cylindrical shell, loaded by inertia forces and subjected to appropriate time-dependent boundary conditions.

The authors appreciate that the idealization of the "water-entry" problem stated above may not correspond too closely to the truth. However, if any progress is to be made on this very complicated problem, some such simplification is essential.

2. Equations of Motion of Cylindrical Shells

The basic equations governing the displacements and stresses for an elastic cylindrical shell were discussed in detail in a previous report(1)*. Therefore only those equations which are essential to the present discussion will be recorded here.

*A superscript in brackets ( ) refers to the corresponding number in the bibliography.

The coupled equations of motion for a cylindrical shell are:

$$\frac{\partial^2 u}{\partial x^2} + \frac{(1-\nu)}{2}\frac{\partial^2 u}{\partial s^2} + \frac{(1+\nu)}{2}\frac{\partial^2 v}{\partial x \partial s} - \frac{\nu}{a}\frac{\partial w}{\partial x} + \frac{(1-\nu^2)}{Eh} P_x = 0 \qquad (1.a)$$

$$\left.\begin{aligned} &\frac{(1+\nu)}{2}\frac{\partial^2 u}{\partial x \partial s} + \frac{(1-\nu)}{2}(1+k)\frac{\partial^2 v}{\partial x^2} + (1+k)\frac{\partial^2 v}{\partial s^2} - \frac{1}{a}\frac{\partial w}{\partial s} \\ &+ ka\frac{\partial^3 w}{\partial x^2 \partial s} + ka\frac{\partial^3 w}{\partial s^3} + \frac{(1-\nu^2)}{Eh} P_s = 0 \end{aligned}\right\} \qquad (1.b)$$

$$\left.\begin{aligned} &\frac{\nu}{a}\frac{\partial u}{\partial x} - ka\frac{\partial^3 v}{\partial x^2 \partial s} - ka\frac{\partial^3 v}{\partial s^3} + \frac{1}{a}\frac{\partial v}{\partial s} - ka^2\frac{\partial^4 w}{\partial x^4} \\ &-2ka^2\frac{\partial^4 w}{\partial x^2 \partial s^2} - ka^2\frac{\partial^4 w}{\partial s^4} - \frac{w}{a^2} + \frac{(1-\nu^2)}{Eh} P_z = 0 \end{aligned}\right\} \qquad (1.c)$$

where $k = \frac{h^2}{12a^2}$; h = the thickness of the shell; a = the radius of the shell;

and u, v, and w are the displacements in the x, s, and z direction respectively (see figure 1). The quantities P_x, P_s, and P_z represent the effective external and body forces per unit area of the middle surface. They are given by

$$\left.\begin{aligned} P_x &= -\rho h\frac{\partial^2 u}{\partial t^2} + \bar{P}_x \\ P_s &= -\rho h\frac{\partial^2 s}{\partial t^2} + \bar{P}_s \\ P_z &= -\rho h\frac{\partial^2 w}{\partial t^2} + \bar{P}_z \end{aligned}\right\} \qquad (2)$$

where ρ is the density of the shell. $\bar{P}_x$, $\bar{P}_s$, and $\bar{P}_z$ are those components of the load which do not contain the displacements.

3. Calculation of $\bar{P}_x$, $\bar{P}_s$, and $\bar{P}_z$

The reactive forces which exist between the water and the torpedo (in general the nose) during the time of entry cause the torpedo to undergo rigid body motions. Therefore, the quantities $\bar{P}_x$, $\bar{P}_s$, and $\bar{P}_z$ will contain, in addition to gravity forces, inertia forces due to this motion. These inertia forces may or may not be large compared to the reactive forces at the time of entry; in any event, they must be calculated for purposes of comparison. Although the rigid-body motion of the torpedo is truly three-dimensional, motions other than those in the vertical plane are, in general, small and the additional refinement achieved by their consideration is not justifiable in the present analysis.

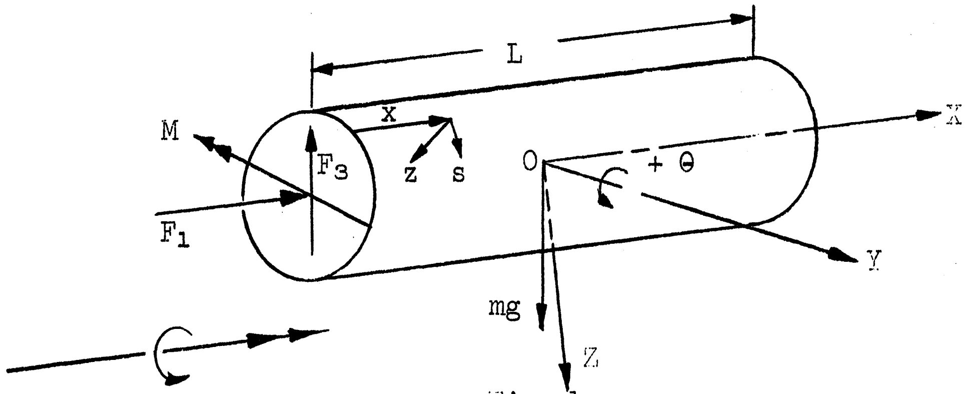

Fig. 1

Figure 1 shows the cylindrical shell together with the forces which act on it and the two coordinate systems being used. The forces F_1 and F_3 and the moment M are the gross reactions between the body and the nose and are assumed to be known. The displacements u, v, and w are measured with respect to the coordinate system x, s, and z where x is along a generator; s is in the circumferential direction, and z is directed along the inward normal to the cylinder. The coordinate system OX, OY, and OZ is fixed in the body with point O being the center of gravity of the complete structure. The axis OY remains horizontal and any rotation that may take place is about this axis through the angle Θ shown

The equations of motion of a rigid body referred to a set of axes fixed in that body are[1]

$$\left.\begin{aligned} m\left[\frac{d\vec{v}}{dt} + (\vec{\omega} \times \vec{v})\right] &= \vec{F} \\ \frac{d\vec{h}}{dt} + (\vec{\omega} \times \vec{h}) &= \vec{M} \end{aligned}\right\} \qquad (3)$$

where m = the mass of the torpedo,

$\vec{v}$ = the velocity vector,

$\vec{\omega}$ = the instantaneous angular velocity vector,

$\vec{F}$ = the applied force,

$\vec{h}$ = the angular momentum vector, and

$\vec{M}$ = the applied moment.

If i, j, and k are the unit vectors along the X, Y, and Z axis respectively, then for the case of plane motion and for the loadings shown in Fig. 1, it follows that

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1 See for example, reference (2), P. 132

$$\left.\begin{array}{c} \vec{v} = v_1\vec{i} + v_3\vec{k} \; ; \; \vec{\omega} = \omega_2\vec{j} \\ \vec{h} = h_2\vec{j} \; : \; h_2 = I_2\omega_2 \\ I_2 = \text{the mass moment of inertia about the OY axis.} \\ \vec{F} = (F_1 - mg \sin\Theta)\,\vec{i} + (-F_3 + mg\cos\Theta)\,\vec{k} \\ \vec{M} = -\,(M + F_3\bar{x})\,\vec{j}\;, \end{array}\right\} \quad (4)$$

$\bar{x}$ = the distance from the nose end of the cylinder to the center of gravity. The subscripts 1, 2, and 3 refer to the components in the X, Y, and Z directions respectively. Introducing (4) into (3), there results

$$\left.\begin{array}{c} m\left(\frac{dv_1}{dt} + v_3\,\omega_2\right) = F_1 - mg\sin\Theta \\ m\left(\frac{dv_3}{dt} - v_1\,\omega_2\right) = -F_3 + mg\cos\Theta \\ I_2\,\frac{d\omega_2}{dt} = -M - F_3\,\bar{x}\;. \end{array}\right\} \quad (5)$$

If $\vec{r}$ denotes the radius vector from the point 0 to any point in the middle surface of the body, then the velocity of that point, referred to the body fixed coordinate system (X, Y, Z), is

$$\vec{v}_r = \vec{\omega} \times \vec{r} \quad (6)$$

and the acceleration is[2]

$$\left.\begin{array}{rl} \vec{A}_r & = \frac{d}{dt}\vec{v}_r + \vec{\omega} \times \vec{v}_r \\ & = \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \vec{\omega} \times \vec{r}\;. \end{array}\right\} \quad (7)$$

where $\dot{\vec{\omega}} \times \vec{r}$ = the tangential acceleration and

$\vec{\omega} \times \vec{\omega} \times \vec{r}$ = the normal acceleration .

From equation (3), (4), and (7), the total acceleration of a point is found to be

$$\begin{array}{rl} \vec{A}_T = & \left\{\mathring{\omega}_2\,r_3 - \omega_2^2\,r_1 + \mathring{v}_1 + v_3\,\omega_2\right\}\vec{i} \\ & + \left\{-\mathring{\omega}_2\,r_1 - \omega_2^2\,r_3 + \mathring{v}_3 - v_1\,\omega_2\right\}\vec{k}\;. \end{array} \quad (8)$$

2 In both equations (6) and (7), the magnitude of $\vec{r}$ has been assumed to be constant. This implies the neglect of Coriolis acceleration ($2\vec{\omega} \times \mathring{\vec{r}}$). The term $\ddot{\vec{r}}$ has already been accounted for by equation (2).

It now remains to relate the coordinate system (X, Y, Z) to the system (x, s, z). Let the unit vectors in the direction of x, s, and z be i', j', and k'; then from figures 1 and 2 it can be seen that

$$\left.\begin{aligned} \vec{i}\,' &= \vec{i} \\ \vec{j}\,' &= \vec{j}\cos\beta + \vec{k}\sin\beta \\ \vec{k}\,' &= -\vec{j}\sin\beta + \vec{k}\cos\beta \end{aligned}\right\} \quad (9)$$

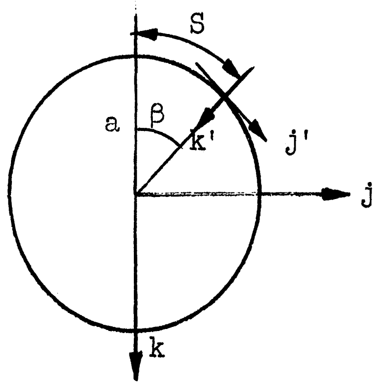

Fig. 2

and

$$\left.\begin{aligned} r_1 &= (x - \bar{x}) \\ r_2 &= a \sin\beta \\ r_3 &= -a\cos\beta \,. \end{aligned}\right\} \quad (10)$$

Using equations (8) and (9) and taking into account the weight of an element of the body, the expressions for $\bar{P}_x$, $\bar{P}_s$, and $\bar{P}_z$ are

$$\left.\begin{aligned} \bar{P}_x &= -\rho h\left\{\mathring{\omega}_2 r_3 - \omega_2^2 r_1 + \mathring{v}_1 + v_3\omega_2 + g\sin\Theta\right\} \\ \bar{P}_s &= -\rho h\left\{[-\mathring{\omega}_2 r_1 - \omega_2^2 r_3 + \mathring{v}_3 - v_1\omega_2]\sin\beta - g\cos\Theta\sin\beta\right\} \\ \bar{P}_z &= -\rho h\left\{[-\mathring{\omega}_2 r_1 - \omega_2^2 r_3 + \mathring{v}_3 - v_1\omega_2]\cos\beta - g\cos\Theta\cos\beta\right\} \,. \end{aligned}\right\} \quad (11)$$

In view of equations (5), (11) may be rewritten as

$$\left.\begin{aligned} \bar{P}_x &= -\rho h\left\{\mathring{\omega}_2 r_3 - \omega_2^2 r_1 + \frac{F_1}{m}\right\} \\ \bar{P}_s &= -\rho h\left\{-\mathring{\omega}_2 r_1 - \omega_2^2 r_3 - \frac{F_3}{m}\right\}\sin\beta \\ \bar{P}_z &= -\rho h\left\{-\mathring{\omega}_2 r_1 - \omega_2^2 r_3 - \frac{F_3}{m}\right\}\cos\beta \,. \end{aligned}\right\} \quad (12)$$

The angular velocity ω_2 is found from the third of equations (5).

4. Boundary Conditions

The true boundary conditions for the cylindrical shell are in reality continuity conditions, a requirement that the state of stress and deformation across the junctures between the nose, body, and tail must be continuous. In order to employ these conditions, three separate problems must be solved and then the stresses and displacements at the boundaries matched - truly a difficult and tedious task. Therefore, the continuity conditions must be replaced by "appropriate" boundary conditions which are as realistic as possible.

Whatever the conditions selected at the end $x = 0$, they are subject to the restriction that their integrated effect must give rise to the proper reactions (resultant force and moment) between the nose and the body (see Fig. L). At the other end ($x = L$) it may be assumed, justifiably, that any resultant force and moment which arises due to the mass of the tail can be neglected as compared to the large reactive forces between the water and the nose. Therefore, the boundary tractions at $x = L$ may be chosen to be self equilibrating.

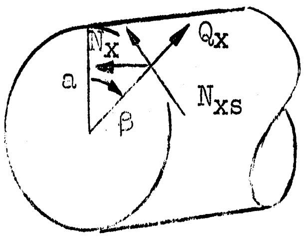

Fig. 3

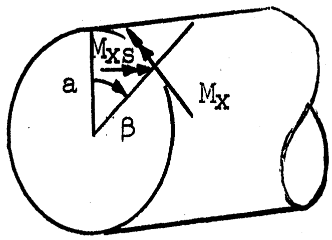

Fig. 4

Edge x=0 Showing Stress Resultants and Couples.

Referring to Figs. 1, 3, and 4 it can be seen that the boundary conditions at $x = 0$ must satisfy the conditions

$$\left.\begin{aligned}
&\int_0^{2\pi a} N_x\big|_{x=0}\, ds = -F_1 \\
&\int_0^{2\pi a} S_x\big|_{x=0} \cos\beta\, ds + \int_0^{2\pi a} T_x\big|_{x=0} \sin\beta\, ds = F_3 \\
&\int_0^{2\pi a} S_x\big|_{x=0} \sin\beta\, ds - \int_0^{2\pi a} T_x\big|_{x=0} \cos\beta\, ds = 0 \\
&a\int_0^{2\pi a} T_x\big|_{x=0}\, ds = 0 \\
&-a\int_0^{2\pi a} N_x\big|_{x=0} \cos\beta\, ds + \int_0^{2\pi a} M_x\big|_{x=0} \cos\beta\, ds = M \\
&a\int_0^{2\pi a} N_x\big|_{x=0} \sin\beta\, ds - \int_0^{2\pi a} M_x\big|_{x=0} \sin\beta\, ds = 0
\end{aligned}\right\} \quad (13)$$

where
$$S_x = Q_x + \frac{\partial M_{xs}}{\partial s} ;$$

$$\beta = s/a ;$$

and
$$T_x = N_{xs} - \frac{M_{xs}}{a} .$$

The definitions of the stress resultants and couples will be found in reference (1). The first three of (13) are force equilibrium and the last three are moment equilibrium equations. It is obvious from equations (13) that the stress resultants and couples, and consequently the displacements and their derivatives on the boundaries may be specified in many ways. If the problem at hand were statical, then Saint-Venants' principle would give assurance that any choice consistant with (13) will yield the same stresses except locally. However, in dynamical problems of shells, the boundary conditions affect the natural frequencies, and consequently the resonance points and amplification factors, in a manner which is not too well known.[3] Therefore some care should be taken in selection of the boundary conditions. Aside from physical considerations, the choice of boundary conditions has a profound effect on the complexity of the mathematical analysis, as will be shown in the next section.

5. Method of Solution

There are three standard techniques available for the solution of dynamical problems of elasticity when the boundary conditions are time-dependent. These three methods will be discussed in this section.

a. Whenever the boundary or edge conditions are of the force or moment type (specification of a force or moment at the edge), then Lagrange's equation can often be used to advantage. The calculation of the generalized work corresponding to the edge loadings is usually not too difficult. The difficulty arises in the determination of the generalized coordinates. In classical beam problems, for example, the generalized coordinates are the normal modes and their determination is a rather simple task. The situation in the case of shells, however, is far different. Here again, the normal modes are the generalized coordinates, but their determination is in general difficult.

If the displacements are assumed to have the form

[3] The one set of experiments which is available on the natural frequencies of cylindrical shells (3) is inconclusive. Most of the cylinders tested were very short and it appears conceivable that the actual boundary conditions were vastly different from those the investigators postulated.

$$\left.\begin{aligned} u &= \sum_{n=0}^{\infty} A_n\, e^{\frac{\lambda x}{a}} \cos\frac{ns}{a}\, e^{ipt} \\ v &= \sum_{n=0}^{\infty} B_n\, e^{\frac{\lambda x}{a}} \sin\frac{ns}{a}\, e^{ipt} \\ w &= \sum_{n=0}^{\infty} C_n\, e^{\frac{\lambda x}{a}} \cos\frac{ns}{a}\, e^{ipt} \end{aligned}\right\} \qquad (14)$$

and these expressions are substituted into equations (1), when $\bar{P}_x$, $\bar{P}_s$, $\bar{P}_z$ have been set equal to zero; there results the following characteristic equation for the determination of λ :

$$\begin{aligned} &\lambda^8 - [4n^2 - \frac{(3-\nu)}{(1-\nu)}\gamma p^2]\lambda^6 + [6n^4 - 2(2+\nu)n^2 + \frac{1-\nu^2}{k} - \gamma p^2 \\ &\cdot (3\frac{(3-\nu)}{1-\nu}n^2 + \frac{1}{k}) + \frac{2}{1-\nu}\gamma^2 p^4]\lambda^4 - [4n^6 - 2(3+\nu)n^4 + \frac{(5+3\nu)}{2}n^2 \\ &- \gamma p^2 (3\frac{(3-\nu)}{1-\nu}n^4 + \frac{2n^2}{k} + \frac{(3+2\nu)}{k}) + \gamma^2 p^4 (\frac{3-\nu}{1-\nu}\frac{1}{k} + \frac{4}{1-\nu}n^2)]\lambda^2 \qquad (15) \\ &+ [n^8 - 2n^6 + n^4 - \gamma p^2 (\frac{3-\nu}{1-\nu}n^6 + \frac{n^4}{k} + \frac{n^2}{k}) + \gamma^2 p^4 (\frac{3-\nu}{1-\nu}\frac{n^2}{k} + \frac{2}{1-\nu}n^4 \\ &+ \frac{2}{1-\nu}\frac{1}{k}) - \frac{2}{1-\nu}\frac{1}{k}\gamma^3 p^6] = 0 , \end{aligned}$$

where

$$\gamma = \frac{(1-\nu^2)\rho a^2}{E} .$$

Once the natural frequencies p are known, then (15) can be solved numerically. However, the natural frequencies are found by substituting (14) into the boundary conditions. Thus the seemingly paradoxical situation arises in which the p's cannot be determined until (15) is solved, but (15) cannot be solved until the p's are known. One way to circumvent this difficulty is to guess the form of λ; however, this has been done only for the case of pinned ends. For any other boundary conditions, exact values for λ have not been obtained and therefore the normal modes are unknown.

b. The second method, known as the Mindlin-Goodman[3] method, is applicable for stress as well as displacement boundary conditions. This method consists, essentially, of defining each displacement as the sum of two functions, one of which is chosen to remove the time-dependent edge conditions. The problem is thus transformed to a forced-vibration problem with homogeneous boundary conditions. For practical reasons, it is again essential that the normal modes be known, thus the same difficulty as outlined in Part a is again encountered.

c. The third method is to use the Laplace transform[4]. This method has the advantage that the time-dependent boundary conditions are handled in a straightforward manner. However, the task of obtaining the inverse transform can be very difficult.

The Laplace-transform approach was used on the problem of a cylindrical shell loaded at the end x = 0 (see Fig. 1) by the forces

$$F_1 = \bar{F}_1 \sin\frac{\pi t}{\tau} \qquad 0 \leqq t \leqq \tau$$
$$= 0 \qquad t > \tau$$
$$F_3 = \bar{F}_3 \sin\frac{\pi t}{\tau} \qquad 0 \leqq t \leqq \tau$$
$$= 0 \qquad t > \tau$$

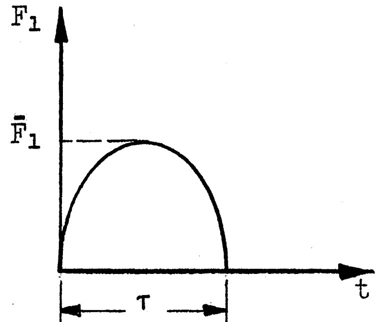

Fig. 5

where $\bar{F}_1$ and $\bar{F}_3$ are known constants.
The edge x = L is taken to be free and the shell is assumed to be unstrained at t = 0.

In view of the loading, the boundary conditions at x = 0 are chosen to be

$$\left.\begin{aligned} N_x\big|_{x=0} &= -\frac{F_1}{2\pi a}, \qquad & M_x\big|_{x=0} &= 0 \\ T_x\big|_{x=0} &= 0, & S_x\big|_{x=0} &= \frac{F_3}{\pi a}\cos\beta \end{aligned}\right\} \tag{16}$$

Equations (16) appear to be the simplest conditions which will satisfy equation (13). The boundary conditions at x = L are:

$$\left.\begin{aligned} N_x\big|_{x=L} &= 0 \qquad & M_x\big|_{x=L} &= 0 \\ T_x\big|_{x=L} &= 0 & S_x\big|_{x=L} &= 0 \end{aligned}\right\} \tag{17}$$

Before transforming the above problem to the Laplace plane, it is necessary to solve the third of equation (5) and then calculate $\bar{P}_x$, $\bar{P}_s$, and $\bar{P}_z$ from equation (12).

After transformation, the problem consists of the solution of three partial differential equations in the two independent variables x and s (equations (1), transformed), subject to eight boundary conditions (equations (16) and (17), transformed). The determination of the solutions of the system of three equations can be reduced to the determination of the complete solution of a single eighth-order nonhomogeneous equation[4] plus the particular integrals of two fourth-order equations.[5]

The single eighth-order equation has the transformed w(i.e.$\bar{w}$) as the dependent variable, and the homogeneous solution may be found by assuming

$$\bar{w} = \sum_{n=0}^{\infty} A_n \, e^{\frac{\lambda x}{a}} \cos \frac{ns}{a} \quad .$$

The resulting characteristic equation for the determination of λ is the same as equation (15) when p^2 is replaced by $-p^2$. Some preliminary computation has indicated that the roots of the transformed characteristic equation, at least for small values of n, have the form

$$\left.\begin{aligned} \lambda_{1,2,3,4} &= \pm c \pm i d \\ \lambda_{5,6} &= \pm \bar{c} \\ \lambda_{7,8} &= \pm i \bar{d} \end{aligned}\right\} \qquad (19)$$

where c, $\bar{c}$, d, and $\bar{d}$ are, as yet, unknown functions of the Laplace parameter p.

Using equations (19) and equations (15) of reference (1), expressions for the transformed displacements $\bar{u}$, $\bar{v}$, and $\bar{w}$ may now be found. These expressions involve eight arbitrary constants (one for each λ) as well as the unknown functions c, $\bar{c}$, d, and $\bar{d}$. Substitution of the expressions for $\bar{u}$, $\bar{v}$, and $\bar{w}$ into the transformed boundary conditions yields eight simultaneous equations for the eight arbitrary constants. The determinant of the coefficients of these constants is directly related to the natural-frequency equation, and the roots of this equation must be found before the inverse transform can be obtained.

The transformed frequency equation[6] cannot be solved immediately, since it involves the unknown functions c, $\bar{c}$, d, and $\bar{d}$. However, if equations (19) are substituted into the transformed characteristic equation, then a system of nonlinear algebraic equations is obtained. These equations must

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[4] Equation (15c) of reference (1), transformed.

[5] Equations (15a) and (15b) of reference (1), transformed.

[6] The transformed frequency equation, i.e. the expansion of the determinant mentioned above, has not been written down here because it would occupy at least 20 pages of this report.

be solved simultaneously with the transformed frequency equation to obtain the desired roots. This is clearly a task which exceeds all bounds of practicality.

The solution of the transformed frequency equation for the case $n = 0$ (symmetrical vibrations) can be carried out with considerably less difficulty, since the transformed characteristic equation can be factored and explicit expressions found for the λ's. After considerable computation, it was found that the symmetrical frequencies, for the dimensions used, were very close to the frequencies obtained from the membrane theory; intuitively this is exactly what one expects to find. Unfortunately, the symmetrical problem is of little interest as far as stress analysis of a torpedo is concerned and therefore the problem was not carried to completion.

6. Conclusions

Before a problem as complicated as the "water-entry" problem can be solved, a great deal of fundamental research must be done on dynamical problems of shells. The authors firmly believe that unless this fundamental research is undertaken there is no hope of obtaining any useful design formulas.

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