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THE DETERMINATION OF OPTIMAL CUSTOMER SELECTION AND ALLOCATION POLICIES FOR FINITE QUEUES IN PARALLEL

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BACKGROUND OF THIS PAPER

This paper is based on research in multiple channel queueing systems. The intent is to develop a methodology which can be utilized to derive optimal customer selection and allocation policies for this class of waiting line problems.

The Determination of Optimal Customer Selection and Allocation Policies for Finite Queues in Parallel

1. Introduction

The determination of procedures for routing customers to servers is a problem of substantial importance in queueing system design and control. The "classical" results for a single channel queue assume that the nth customer is accepted for service unless the system is "full" at the epoch of his arrival [see, for instance, Cox and Smith (3)]. Recently, Cinlar (1,2) has developed models for situations where the acceptance of the nth customer depends probabilistically upon the acceptance (or non-acceptance) of the (n-1)st customer. An "every-other" customer selection rule and a random customer selection rule are two special cases of this selection mechanism. Cinlar (1,2) has also developed results for a single channel queue where the selection of the nth customer depends both upon the customer type and upon the state of the system at the epoch of his arrival. The case of queueing with selective rejection considered by Scott (12), in which low priority customers are rejected when the queue reaches some specified subcapacity length in order to leave room for potential high priority customers, is a special case of this second type of customer selection rule.*

For multiple service channels in parallel, the system operator is faced with the additional problem of allocating accepted customers to a service channel. When the allocation rule depends upon the allocation of the previous customer, the customer type, or the state of a single channel, Cinlar's previously referenced results can be used to decompose the arrival stream so that

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*The mathematics utilized in deriving this special case of queueing with selective rejection from the results of Cinlar is developed in Appendix A.

the arrival processes to each channel can be considered independently. There are, however, many situations where this decomposition of the arrival process is not sufficient. For instance, under a shortest queue-length customer allocation rule, which has been considered for two service channels by Haight (4) and Kingman (9), the customer allocation depends upon the state of all queues at the epoch of arrival. To analyze these more complex situations Hall and Disney (5) have developed results for a system of queues in parallel where the customer selection and allocation policy depends upon the customer type and the state of all queues in the system at the epoch of his arrival.

Even after the performance of a particular system has been characterized, the question remains as to which customer selection and allocation rules are best for the system under consideration. For instance, in a single channel queue with selective rejection, what is the optimal rejection level for low priority customers? In a multiple channel queue, what is the optimal allocation pattern for arriving customers?

A special case of this latter problem has been considered by Miller (10). In the situation considered by Miller, n customer classes arrive independently at a reserver system with no storage. A fixed reward is accumulated when customers of each class are accepted for service, and the objective is to select or reject customers to maximize the expected reward received over an infinite planning horizon. Another special case is considered by Ireland (7), who develops control policies for r servers when infinite queues are allowed to form and when the objective is to sequentially minimize the expected retention cost for the next K customers.

In this paper, we shall consider the general problem of optimal customer selection and allocation for systems of finite queues in parallel. The intent is to show how these systems can be modeled so that Markov renewal programming can be utilized to derive optimal customer selection and allocation policies. Within this modeling framework, the models of Scott, Miller and Ireland become special cases. Moreover, an exceptionally wide range of operating policies can be studied for a large class of queueing systems. The basic model will be developed in Section 2; various applications of this model will then be considered in Section 3.

2. Development of the General Allocation Model

Assume that m customer types arrive at the queueing system according to a semi-Markov process with transition probabilities given by $A_{ij}(x) =$

$$\Pr\{X_n \leq x,\ Z_n = j \mid Z_{n-1} = i\};\ i,j=1,\ldots,m \qquad [1]$$

where X_n is the time between the arrival of the (n-1)st and nth customer and Z_n is the type of the nth customer.

Denote the state of the system of queues at the epoch of the nth arrival by $\{S_n^1,\ldots,S_n^r\} = \{\underline{S}_n\}$, where S_n^k is the number of customers in the sub-system formed by server k, k=1,...,r, and $S_n^k = s_k$; $s_k=0,1,\ldots,q_k$, where q_k is the capacity of sub-system k. Assume that the service times in the k service channels are negative exponential with channel-dependent rate u_k, k=1,...,r.

Let Y_n be equal to k if customer n is assigned to queue k and let $Y_n=0$ if customer n is rejected (not accepted) by the system. Under a given customer selection and allocation policy g, we assume that the probabilistic properties of the switch Y_n are completely specified when the customer type and system state are known. That is, we assume that the following conditional probabilities can be determined:

$$\Pr^g\{Y_n=\ell \mid Z_o,\ldots,Z_n;\ \underline{S}_o,\ldots,\underline{S}_n;\ Y_o,\ldots,Y_{n-1}\}$$

$$= \Pr^g\{Y_n=\ell \mid Z_n=i,\ \underline{S}_n=\underline{s}\} = b_\ell^g\ (i,\underline{s}) \qquad [2]$$

for $\ell = 0,1,\ldots,r$; $i=1,\ldots,m$; $s_i^k=0,1,\ldots,q_k$; $k=1,\ldots,r$ and for $g \in G$, the set of all feasible selection and allocation policies for the queueing system under consideration.

For this basic model, Hall and Disney (5) have proven that:

(i) The r+2 dimensional process $\{Z_n,S_n^1,\ldots,S_n^r,X_n\}$ is equivalent to a finite state semi-Markov process.

(ii) The r+1 dimensional process $\{Z_n,S_n^1,\ldots,S_n^r\}$ is a finite Markov chain.

(iii) The r dimension process $\{S_n^1,\ldots,S_n^r\}$ is Markov if and only if the arrival process is a renewal process.

Further, for a given selection and allocation policy g, transition distributions in the $\{Z_n,\underline{S}_n,X_n\}$ process are given by:

$$\Pr^g\{\underline{S}_n=\underline{t},\ Z_n=j,\ X_n\le x \mid \underline{S}_{n-1}=\underline{s},\ Z_{n-1}=i\} = \Pr^g\{i,\underline{s};\ j,\underline{t},x\}$$

$$= \sum_{k=0}^{r} b_k^g(i,\underline{s}) \int_o^x \prod_{\ell=1}^{r} \phi_k(t_\ell,s_\ell,x)\,dA_{ij}(x) \qquad [3]$$

for $i,j=1,\ldots,m,$

$s_k,t_k=0,1,\ldots,q_k,$

$k=1,\ldots,r$

where $b_k^g(i,\underline{s})$ is given by [2], $dA_{ij}(x)$ is derived from [1] and $\phi_k(t_\ell,s_\ell,x) = \Pr\{S_n^\ell=t_\ell \mid S_{n-1}^\ell=s_\ell,Y_{n-1}=k,X_n=x\}$, so that the $\phi_k(t_\ell,s_\ell,x)$ are given by the following relations:

Case i: $k = \ell$ [the nth arrival is assigned to the ℓth queue in parallel]

$$\phi_k(t_\ell, s_\ell, x) = \begin{cases} a_{s_\ell+1-t_\ell}(\ell,x) & 0<t_\ell\leq s_\ell+1 \quad s_\ell = 0,\ldots, q_\ell-1 \\ 1 - \sum_{k=0}^{s_\ell} a_k(\ell,x) & t_\ell = 0 \quad s_\ell = 0,\ldots, q_\ell-1 \\ 0 & \text{otherwise} \end{cases}$$

Case ii: $k \neq \ell$ [the nth arrival is not assigned to the ℓth queue in parallel]

$$\phi_k(t_\ell, s_\ell, x) = \begin{cases} a_{s_\ell-t_\ell}(\ell,x) & 0<t_\ell\leq s \quad s_\ell = 1,\ldots, q_\ell \\ 1 - \sum_{k=0}^{s_\ell-1} a_k(\ell,x) & t_\ell = 0 \quad s_\ell = 1,\ldots, q_\ell \\ 0 & \text{otherwise} \end{cases}$$

where

$$a_k(\ell,x) = \frac{e^{-u_\ell x}(u_\ell x)^k}{k!} \qquad \begin{array}{l} k = 0,1,\ldots \\ \ell = 1,\ldots,r \end{array}$$

The one-step transitions in the Markov chain $\{Z_n, \underline{S}_n\}$ are obtained from (3) as $Pr^g\{i,\underline{s};\ j,\underline{t},\infty\}$.

To determine an optimal customer selection and allocation policy from the feasible policy set G, it is necessary to develop a reward structure for the system under consideration. In general, we shall allow this reward to depend on the nth customer type and the system state at the epoch of his arrival, the (n-1)st customer type and the system state at the epoch of his arrival, the time between the (n-1)st and nth arrival, and the decision made at the epoch of the (n-1)st arrival. Then, corresponding to the transition distribution $Pr^g\{i,\underline{s};\ j,\underline{t},x\}$, a fixed reward $R^g(i,\underline{s};j,\underline{t},x)$ is accumulated if admission and allocation policy $g\epsilon G$ is followed. Assuming that the set of feasible decisions G is finite, since the underlying stochastic process is semi-Markov it is possible to find the selection and allocation policy which maximizes the total expected return for both finite and infinite horizon problems by utilizing the well-defined technique of Markov Renewal programming [see for instance Jewell (8)].

In actuality, it is likely that the transition probabilities for the process $\{Z_n, \underline{S}_n, X_n\}$ and the reward structure for the process are less general than those considered in this section. In these special cases, it is possible to develop more explicit characterizations, and we shall do this for some special classes of problems in the next section.

3. Applications to some specific Queueing Systems

(i) Single Channel Queue with Selective Rejection.

Denoting the state of the single service at the epoch of customer arrival channel by S_n, under the assumptions of the previous sections the 3 dimensional process $\{Z_n, S_n, X_n\}$ is equivalent to a semi-Markov process. We assume that at the epoch of arrival of a customer, we have two potential decisions--accept the customer or reject him. Letting $Y_n = 1$ if the nth customer is accepted and zero otherwise, from (2) we can write

$$b_1^1(i,k) = \begin{cases} 1 & k<N \\ 0 & k=N \end{cases} \quad \& \quad b_1^2(i,k) = 0$$

where k is the system state at the epoch of arrival, N is the system capacity, g=1 if we decide to accept the customer, and g=2 otherwise.

Letting u denote the channel service rate, we have

$$p^g(i,k;\, j,\ell,x) = \int_0^x \phi^g(\ell,k,x)\,dA_{ij}(x) \qquad g=1,2$$

where

$$\phi^1(\ell,k,x) = \begin{cases} a_{k+1-\ell}(x) & 0<\ell\leq k+1 \\ 1 - \sum_{g=o}^{k} a_g(x) & \ell=0 \\ 0 & \text{otherwise} \end{cases}$$

$$\phi^2(\ell,k,x) = \begin{cases} a_{k-\ell}(x) & 0<\ell\leq k \\ 1 - \sum_{g=o}^{k-1} a_g(x) & \ell=0 \\ 0 & \text{otherwise} \end{cases}$$

and $a_p(x) = \dfrac{e^{-ux}(ux)^p}{p!}$

Observe that the decision to select or reject a customer affects only $b^g(i,k)$ in this situation. One possible generalization is to let dAij(x), the transition distribution for the arrival process, depend upon the decision. For instance, such a dependence prevails when the decision to accept (or reject) customers is made known to future customers, thereby affecting their arrival patterns.

To develop the reward structure, we shall assume that the rewards accumulated are independent of the time between arrivals. Let r_j denote the fixed reward which is accumulated when a customer of type i is accepted for service, let α be the system start-up cost, and let β be the system shut-down cost. Further let P_i be a penalty cost (loss of good will) accumulated when a customer of type i is rejected by the system. Then

$$R^1(i,k;\ j,\ell) = \begin{cases} r_i & 0<\ell\leq N \quad 0<k<N \\ r_i-\alpha & k=0 \\ r_i-\beta & \ell=0 \end{cases}$$

$$R^2(i,k;\ j,\ell) = \begin{cases} -p_i & 0<\ell\leq N \quad 0\leq k\leq N \\ -p_i-\beta & \ell=0 \end{cases}$$

In this special case the reward does not depend upon the type of the next customer. Furthermore, the dependence of the rewards upon the state $S_{n-1}=k$ arises only through the system set-up cost when $k=0$. A more sophisticated reward structure might assume that the gain from selecting a customer of type i goes down as more customers are in the system, reflecting both the additional burden placed on waiting facilities and the potential cost incurred by the customer because of delays in receiving service.

Since neither the reward structure nor the decision structure influence the times between transitions in the process $\{Z_n, S_n, X_n\}$, the optimal decision policy can be found by applying the techniques of Markov programming to the embedded chain $\{Z_n, S_n\}$. Solution procedures are simplified in this case, as indicated by Howard (6).

(ii) Multiple Channel Queues with no Storage.

In the situation considered by Miller, let $S_n^j = 0$ if server j is idle at the epoch of customer arrival and let $S_n^j = 1$ if server j is serving a customer at this time, $j=1,\dots,r$. Let $Y_n=k$ if the n^{th} customer is routed to the k^{th} server and $Y_n=0$ if the n^{th} customer is rejected, and let the admissible policies $g\in G$ correspond to these values of Y_n so that from (2):

$$b_k^g(i,\underline{s}) = \begin{cases} 1 & \text{if } k=g>0 \text{ and } s_k=0 \\ 0 & \text{otherwise} \end{cases}$$

for values of $k=0,\ldots, r$

To develop a reward structure, assume again that the rewards accumulated are independent of the time between customer arrivals. Let r_{ik} denote the fixed reward when a customer of type i is serviced by server k, let α_k be the start-up cost for channel k, and let β_k be the shut-down cost of channel k.* Then

$$R^g(i,\underline{s};\ j,\underline{t},x) = \begin{cases} r_{ig}+\alpha_g & \text{for } s_g = 0,\ t_g\neq 0 \\ r_{ig} + \alpha_g - \beta_g & \text{for } s_g=t_g=0 \\ 0 & \text{otherwise} \end{cases}$$

As in the previous example, since the reward structure is independent of x, Markov programming techniques can be utilized to find the optimal policy for the embedded chain $\{Z_n, S_n^1, \ldots, S_n^r\}$.

(iii) Extensions to more Complex Networks

The results of this paper apply directly to networks of parallel service channels. Moreover, extensions to certain types of parallel-series networks can be made by observing that the departure process from a GI/G/1 queue is itself a semi-Markov process. When this departure process serves as the arrival process to a system of negative exponential parallel servers, the results of this paper are immediately applicable when one is faced with the problem of optimal customer selection and allocation.

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*The specific situation considered by Miller assumes a simplified reward structure in which $r_{ik}=r_i$, $i=1,\ldots, m$, and $\alpha_k=\beta_k=0$.

Appendix A

Queueing with Selective Rejection

1. Introduction

Consider the following general class of queueing problems: customers of several types arrive randomly at a queueing system where priority (type-one) customers are admitted to the queue unless the waiting line is full; however, customers of the remaining types are denied admittance if the queue has reached some pre-specified sub-capacity length. The motivation for designing queueing systems with this selective rejection mechanism is to maintain waiting room for potential priority (type-one) customers, and such an operating policy is apt to prevail in many applications.

A specific version of this problem was formulated by Scott (12), who developed analytical results for the case of two-customer types under the assumption of independent, negative-exponential arrival and service processes. An application considered by Scott arises in automotive repair. Here a dealer accepts customers who have purchased a car from him for service unless the garage is full. Other customers are rejected when the queue reaches some specified sub-capacity length in order to leave room for the arrival of potential type-one customers.

The objective of this paper is to generalize Scott's model to include m customer types ($o<m<\infty$) whose arrival at the system can be described by a semi-Markov process. Denoting the capacity of the system by N ($o<N<\infty$), customers of type j are refused service if there are R_j or more customers in the system, $j=1,\ldots,m$. [Without loss of generality we assume that $R_j \leq N$ and $R_1 = N$, the latter assumption implying that type-one customers are always accepted unless the system is full.] Analytical results describing the operation of this queueing system are developed by utilizing results on semi-Markovian queues in parallel. These systems have been extensively considered by Cinlar (1) and more recently by Hall & Disney (5). The application of these results to a queueing system with selective rejection yields the system parameters considered by Scott--the expected queue-length, probability of rejecting a certain type of customer, and the average rate of change of the arrival intensity. Moreover, many other system characteristics of interest can be derived, including characteristics of the time between rejections of customers of the various types, the time between overflows, and other related processes.

2. Basic Results

We begin the discussion by briefly summarizing the applicable results of Cinlar (1). These are then applied to describe the behavior of a queueing system with selective rejection.

Let T_n denote the epoch of arrival of the nth customer, $n = 0,1,2,\ldots$, and let $X_n = T_n - T_{n-1}$ be the time between arrivals ($n \geq 1$). Further let Z_n denote the type of the nth customer, where Z_n takes on the values $1,\ldots,m$. Let $A_{ij}(x) = \Pr\{X_n \leq x,\ Z_n = j \mid Z_{n-1} = i\}$, $i,j = 1,\ldots,m$, characterize the distribution of times between arrivals. Under these conditions it is known that the arrival process $\{X_n, Z_n;\ n = 0,1,2\ldots\}$ is equivalent to a semi-Markov process. [See, for instance, Pyke (11)].

Denote the state of the system at the instant of the nth arrival by S_n. Then S_n takes on the values $0,1,\ldots,N$, where N is the system capacity. Assume that the server operates according to a negative-expotential distribution with parameter μ.

Let Y_n be an indicator variable taking on the value 1 if the nth customer is accepted by the system and the value 0 if the customer is rejected by the system. Assume that the stochastic switch Y_n is governed by the conditional distribution $\Pr\{Y_n = \delta | Z_n, S_n\}$, where $\delta = 0$ or 1. That is, we assume that the acceptance of a customer depends both upon his type and the system state at the epoch of his arrival.

Under these conditions Cinlar (1) has proven that:

(i) the process $\{Z_n, S_n, X_n\}$ is semi-Markov,

(ii) the process $\{Z_n, S_n\}$ is a Markov chain,

(iii) the process $\{S_n\}$ is Markov if and only if $m = 1$.

Furthermore letting $\phi_h(k,1,x) = \Pr\{S_n = 1 | S_{n-1} = k, Y_{n-1} = h, X_n = x\}$ and letting $b_h(1,j) = \Pr\{Y_n = h | S_n = 1, Z_n = j\}$, the transitions in the process $\{Z_n, S_n, X_n\}$ obey:

$$\Pr\{S_n = 1, Z_n = j, X_n \le x | S_{n-1} = k, Z_{n-1} = i\} = P(i,k;\, j,1,x)$$

$$= \int_o^x [b_o(k,i)\phi_o(k,1,x) + b_1(k,i)\phi_1(k,1,x)]dA_{ij}(x) \qquad [1]$$

where

$$b_o(k,i) = 1 - b_1(k,i)$$

and

$$\phi_h(k,1,x) = \begin{cases} \alpha_{k-1+h}(x) & o<1\le k+h\le N+k \\ \sum_{n=i+h}^{\infty} \alpha_n(x) & 1 = o,\ i\le N \\ 0 & \text{otherwise} \end{cases}$$

with

$$\alpha_n(x) = \frac{e^{-\mu x}(\mu x)^n}{n!} \qquad n = 0,1,2,\ldots$$

The one-step transition probabilities in the underlying Markov chain $\{Z_n, S_n\}$ are given by

$$\Pr\{S_n = 1,\ Z_n = j \mid S_{n-1} = k,\ Z_{n-1} = i\} = P(i,\ k;\ j,\ 1,\infty)$$

$$= \int_0^\infty [b_o(k,i)\phi_o(k,1,x) + b_1(k,i)\phi_1(k,1,x)]dA_{ij}(x)$$

Given an initial distribution $P_o\ (i,k) = \Pr\{Z_o = i,\ S_o = k\}$ and the one step transition probablities, the imbedded process $\{Z_n,\ S_n\}$ is completely characterized. The theory of finite Markov chains can then be employed to examine both the transient and steady state behavior of the process.

3. Application to the Selective Rejection Problem

Results for a queueing system with selective rejection can then be derived by considering the following model for the stochastic switch Y_n.

Again letting $Y_n = 1$ when the nth customer is accepted and $Y_n = 0$ otherwise, we note that customers of type i are accepted for service whenever $S_n<R_i$, $i = 1,\ldots,N$ [$R_1 = N$ by our previous convention]. Then

$$b_1(i,j) = \Pr\{Y_n = 1 | Z_n = i,\ S_n = j\} = \begin{cases} 1 & j<R_i \quad i = 1,\ldots,m \\ 0 & j\geq R_i \end{cases}$$

From equation (1) the one-step transition probabilities for the semi-Markov process $\{Z_n,\ S_n,\ X_n\}$ become:

$$\Pr\{S_n = 1,\ Z_n = j,\ X_n\leq x | S_{n-1} = k,\ Z_{n-1} = i\} = P(i,\ k;\ j,1,x)$$

$$= \begin{cases} \int_0^x \phi_1(k,1,x)dA_{ij}(x) & \text{if } k<R_i \\ \int_0^x \phi_o(k,1,x)dA_{ij}(x) & \text{if } k\geq R_i \end{cases} \qquad [2]$$

and the one-step transition probabilities for the finite Markov chain $\{Z_n,\ S_n\}$ become:

$$P(i,\ k;\ j,1,\infty) = \begin{cases} \int_o^\infty \phi_1(k,1,x)dA_{ij}(x) & k<R_i \\ \int_o^\infty \phi_o(k,1,x)dA_{ij}(x) & k\geq R_i \end{cases} \qquad [3]$$

Hence, the imbedded Markov chain for the selective rejection queueing system has an exceptionally simple form. Letting $P_n(j,1) = \Pr\{Z_n = j,\ S_n = 1\}$ denote the probability the system is in state (j,1) at time n, we have the recursive relationship from the theory of Markov chains:

$$P_n(j,1) = \sum_{i=1}^{m} \sum_{k=o}^{N} P(i,\ k;\ j,1,\infty)\ P_{n-1}(i,k)$$

$$= \sum_{i\varepsilon S} \sum_{k=o}^{N} \int_o^\infty \phi_1(k,1,x)dA_{ij}(x)\ P_{n-1}(i,k)$$

$$+ \sum_{i\varepsilon \overline{S}} \sum_{k=o}^{N} \int_o^\infty \phi_o(k,1,x)dA_{ij}(x)\ P_{n-1}(i,k) \qquad n = 1,2,\ldots$$

where S is that subset of the integers $(1,\ldots,m)$ such that for $i\varepsilon S$, $k<R_i$ and $\overline{S}$ is the complement of S.

The distribution of the state of the system at the arrival of the nth customer is given by $P_n(1) = \sum_{j=1}^{m} P_n(j,1)$, and the expected number in the system at this epoch by $\sum_{1=o}^{N} \sum_{j=1}^{m} 1P_n(j,1)$.

The probability of rejecting a customer of type i at epoch n is

$$\Pr\{Y_n = 0|Z_n = i\} = \sum_{1=o}^{N} \Pr\{Y_n = 0|Z_n = i,\ S_n = 1\}\ \Pr\{S_n = 1\}$$

$$= \sum_{1=1}^{N} b_o(i,1)\ P_n(1) = \sum_{1=R_i}^{N} P_n(1)$$

When i=1 this is the probability of an over-flow from the queueing system [i.e.--the probability that the system is full at the epoch of the nth arrival].

The average rate of change of the arrival intensity at the epoch of the nth arrival is equivalent to the probability of rejection of the nth customer. Hence this parameter is given by the relationship.

$$\Pr\{Y_n = o\} = \sum_{i=1}^{m} \sum_{l=o}^{N} \Pr\{Y_n = o | Z_n = i,\ S_n = l\} \Pr\{Z_n = i,\ S_n = l\}$$

$$= \sum_{i=1}^{m} \sum_{l=o}^{N} b_o(i,l)\ P_n(i,l)$$

$$= \sum_{i=1}^{m} \sum_{l=R_i}^{N} P_n(i,l)$$

Analogous results for the system operation in steady state can be obtained by computing the limiting state probabilities $\lim_{n\to\infty} P_n(i,l)$ using conventional techniques from the theory of finite Markov chains. Steady state analogues for the system parameter developed above follow immediately.

We note that the one-step transition matrices $P(i, k; j,l,\infty)$ have a very simple form which facilitates numerical computation for this class of problems. For instance, consider the case of two customer types (m=2), where type 1 customers are admitted when $S_n<N$ and type 2 customers are admitted when $S_n<R$ $[R\leq N]$.

Letting the two-triple (z_n, s_n) denote the state of the system and

$$a_{ij}^{k} = \int_{o}^{\infty} \frac{e^{-\mu x}(\mu x)^k}{k!}\, dA_{ij}(x)$$

the one-step transition matrix for the process $\{Z_n, S_n\}$ is given by:

	(1,0)	...	(1,R)	...	(1,N)	(2,0)	...	(2,R)	...	(2,N)
(1,0)	$\sum_{1}^{\infty} a_{11}^{k}$	a_{11}^{0} 0	...		0	$\sum_{1}^{\infty} a_{12}^{k}$	a_{12}^{0} 0	...		0
(1,1)	$\sum_{2}^{\infty} a_{11}^{k}$	a_{11}^{1} a_{11}^{0} 0 ...			0	$\sum_{2}^{\infty} a_{12}^{k}$	a_{12}^{1} a_{12}^{0} 0	...		0
⋮		⋮			⋮		⋮			⋮
					0					
(1,N-1)	$\sum_{n}^{\infty} a_{11}^{k}$	a_{11}^{n-1} ...	a_{11}^{n-r}		a_{11}^{0}	$\sum_{n}^{\infty} a_{12}^{k}$	a_{12}^{n-1} ...	a_{12}^{n-r}	...	a_{12}^{0}
(1,N)	$\sum_{n}^{\infty} a_{11}^{k}$	a_{11}^{n-1} ...	a_{11}^{n-r}		a_{11}^{0}	$\sum_{n}^{\infty} a_{12}^{k}$	a_{12}^{n-1} ...	a_{12}^{n-r}	...	a_{12}^{0}
(2,0)	$\sum_{1}^{\infty} a_{21}^{k}$	a_{21}^{0} 0 ...			0	$\sum_{1}^{\infty} a_{22}^{k}$	a_{22}^{0}	0 ...		0
⋮	⋮				⋮					⋮
(2,R-1)	$\sum_{r}^{\infty} a_{21}^{k}$	a_{21}^{r-1} ...	a_{21}^{0} 0	...	0	$\sum_{r}^{\infty} a_{22}^{k}$	a_{22}^{r-1} ...	a_{22}^{0} 0	...	0
(2,R)	$\sum_{r}^{\infty} a_{21}^{k}$	a_{21}^{r-1} ...	a_{21}^{0} 0	...	0	$\sum_{r}^{\infty} a_{22}^{k}$	a_{22}^{r-1}	a_{22}^{0} 0	...	0
(2,R+1)	$\sum_{r+1}^{\infty} a_{21}^{k}$	a_{21}^{r} ...	a_{21}^{1} a_{21}^{0}	...	0	$\sum_{r+1}^{\infty} a_{22}^{k}$	a_{22}^{r} ...	a_{22}^{1} a_{22}^{0}	0 ...	0
⋮		⋮					⋮			⋮
(2,N)	$\sum_{n}^{\infty} a_{21}^{k}$	a_{21}^{n-1} ...			a_{21}^{0}	$\sum_{n}^{\infty} a_{22}^{k}$	a_{22}^{n-1} ...			a_{22}^{0}

To find the distribution of the time between rejections of customers of any type i (i=1,...,m), it is sufficient to note that this time interval is the recurrence time of the state $\{S_n = R_i, Z_n = i\}$ in the semi-Markov process $\{S_n, Z_n, X_n\}$. Thus, the times between rejections of each customer type form a renewal process, and the distribution of these times can be found from the transition probabilities P(i, k; j,l,x) using conventional techniques discussed by Pyke (11) and Cinlar (2).

When i = 1, $R_i = N$, and the recurrence time of the state $\{S_n = N, Z_n = 1\}$ is the time between overflows from the system.

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