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A MULTIPERIOD ASSET PRICING MODEL WITH
UNOBSERVABLE MARKET PORTFOLIO

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**A Multiperiod Asset Pricing Model With
Unobservable Market Portfolio**

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Abstract

This paper develops a general equilibrium theory of asset prices which is simple and yet does not require any estimation of the stochastic properties of the market portfolio or aggregate consumption. We show that risk-return equilibrium relationships can be presented in terms of conditional covariances of asset returns with the next-period riskless rate. If capital markets satisfy the minimum set of conditions that are necessary to produce the Intertemporal Sharpe-Lintner model, the pricing model presented here will hold. The present model, similar to the Sharpe-Lintner pricing equation, can be used to calculate the required rate of return on a risky project, or to evaluate the performance of a portfolio. Because the rate of return on the market portfolio does not have to be observed, empirical tests or applications of the model will not be subject to Roll's [13] criticism.

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*I have benefited from my conversation with H. Varian and the comments of two anonymous referees. I am, of course, responsible for any remaining errors.

A Multiperiod Asset Pricing Model With Unobservable Market Portfolio

This paper develops a multiperiod general equilibrium theory of asset prices which is simple and yet does not require any estimation of the stochastic properties of the market portfolio or aggregate consumption. We demonstrate that the systematic risk of asset returns can be measured through their conditional covariances with the next-period riskless rate of interest. The theory is developed using the minimum set of assumptions that produce the Intertemporal Sharpe-Lintner (ISL) model (see Constantinides [4]). Thus, there is a direct relationship between the ISL model and the present model--the present model holds if and only if the ISL model holds. However, our results do not require any estimation of the stochastic properties of the market portfolio, and it does not, therefore, give rise to the issue discussed in Roll [13].

Several research efforts have produced multiperiod asset pricing models (e.g., Fama [7], Merton [12], Long [10], Rubinstein [16], Stapleton and Subrahmanyam [18], Lucas [11], Breeden [2], Constantinides [4,5], Bhattacharya [1], and Cox, Ingersoll, and Ross [5]). Some of these studies have presented risk-return relationships that measure the riskiness of securities through covariances of their returns with the return on the market portfolio (e.g., Merton [12], Long [10], Stapleton and Subrahmanyam [18], and Constantinides [4,5]). On the other hand, other models use covariances of asset returns with changes in aggregate consumption to present risk-return relationships.

The market portfolio and aggregate consumption are, however, not readily observable, and thus, tests of asset pricing models have been rather inconclusive. The market portfolio includes assets that their returns cannot be observed, and hence the return on the true market portfolio has to be estimated using the return on only a subset of assets. Aggregate consumption, on the

other hand, must include aggregate consumption expenditures on nondurable goods as well as aggregate consumption flows derived from durable goods. Presently, no data on the latter are available and this component of aggregate consumption must, therefore, be estimated.

Before a formal discussion of the model is made, we present the main results of the paper along with the intuition behind it. Part I will then present the assumptions and the definitions of the paper. Part II presents the pricing equation, and Part III summarizes the paper.

This paper shows that the expected rates of return on risky securities satisfy the following equilibrium relationship:

$$E_t \tilde{R}_i(\phi_{t+1}) = R_f(\phi_t) + \gamma(\phi_t) \text{Cov}_t[\tilde{R}_i(\phi_{t+1}), \tilde{R}_f(\phi_{t+1})], \quad (1)$$

where ϕ_t is the state of the economy at t ; $\tilde{R}_i(\phi_{t+1})$ is one plus the one-period rate of return on risky asset i over $(t, t+1)$; $R_f(\phi_t)$ is one plus the one-period riskless rate of interest available at t ; E_t is the expectation operator conditional on information available at t ; $\gamma(\phi_t)$ is the market price of risk at t for risky assets that have normally distributed returns over $(t, t+1)$. Note, the next-period riskless rate, which is not known at t , is used in the covariance function that appears in equation (1).

According to equation (1), the expected rate of return on asset i is equal to the current riskless rate plus a risk premium, which is equal to the riskiness of asset i , $\text{Cov}_t[\tilde{R}_i(\phi_{t+1}), \tilde{R}_f(\phi_{t+1})]$, multiplied by the market price of risk, $\gamma(\phi_t)$. In this model, the "interest-rate-based" beta of asset i can be calculated as follows:

$$\beta_i(\phi_t) = \frac{\text{Cov}_t[\tilde{R}_i(\phi_{t+1}), \tilde{R}_f(\phi_{t+1})]}{\text{Var}_t[\tilde{R}_f(\phi_{t+1})]}$$

We can interpret equation (1) as follows. The fundamental result of the ISL model is the mean-variance efficiency of the market portfolio; this paper shows that when the market portfolio satisfies this condition, the equilibrium relationship of equation (1) will be satisfied. In the ISL asset pricing model, the systematic risk of securities is measured through covariances of asset returns with the return on the market portfolio. If we can find a random variable such that its intertemporal random behavior is totally caused by random changes in the value of aggregate wealth, then, under certain conditions, the market portfolio will no longer have to be observed; instead, this random variable can be used to measure the systematic risk of securities. This paper shows that the riskless rate is indeed such a random variable.

When capital markets satisfy the conditions that are necessary to produce the ISL model, all uncertainties about the next-period riskless rate are solely caused by the random changes in the value of aggregate wealth--i.e., one plus the riskless rate of interest is equal to the ratio of the current marginal utility of wealth and the expected value of the next-period marginal utility of wealth (see eq. (10)); this ratio is random only because the value of aggregate wealth is random.

We show that the conditional covariance of a risky asset with the next-period riskless rate is proportional to the conditional covariance of the same asset with the return on the market portfolio. Furthermore, the ratio of the two conditional covariances is independent of the risky asset.¹ The riskless rate of interest will, therefore, provide enough information about the systematic risk of securities so that risk-return equilibrium relationships that are in terms of covariances of asset returns with the return on the market portfolio can be presented in terms of covariances of asset returns with the next-period riskless rate.

The implications of the present model are rather significant. The model can be applied to the same type of problems that the ISL model has been traditionally applied to, while empirical estimation of the parameters of the model will not be subject to Roll's criticism. The model, for instance, can be used to calculate the present value of an uncertain income stream, or to measure the cost of capital of a risky project.

I. Assumptions and Definitions

We employ the following notation: t = time subscript; ϕ_t = state of the economy; $\tilde{R}_i(\phi_{t+1})$ = one plus the rate of return on asset $i=1, \dots, N_t$ over $(t, t+1)$; N_t = number of investment opportunities available at t ; $R_f(\phi_t)$ = one plus the riskless rate of return available at t ; $U(C_s, \dots, C_T)$ = utility of lifetime consumption; X_{it} = the percentage of wealth invested in asset i at t after ϕ_t is observed (note, $\sum_i X_{it} = 1$); $W(\phi_t)$ = investor's wealth at t prior to consumption at t .

We also make the following assumptions:

(A.1) Perfect Markets: Investment opportunities are traded in perfect markets, riskless borrowing and lending rates are equal, and short sales of all assets are allowed.

(A.2) Aggregation: The aggregation problem is solved, and hence, equilibrium prices are determined as if investors were identical.² Alternatively, we can construct a composite investor (see Constantinides[5]) and use his/her utility of lifetime consumption to derive our results. Although in this case we must assume that markets are complete.

(A.3) State independent utility: Each investor's utility of lifetime consumption does not explicitly depend upon the state of the economy. Because,

with heterogeneous investors, aggregation requires additively separable utility functions, we also assume that $U(C_s, \dots, C_T) = \sum_{t=s}^T U(C_t, t)$.

(A.4) **Rational expectations:** Investors have rational expectations; i.e., when making optimal investment-consumption decisions, investors take all available and relevant information (including the structure of the economy) into account. In the context of proposition 1, this assumption implies that optimal investments in financial securities are zero.

(A.5) **Investment opportunities:** At time t there are n_t competitive value maximizing firms that invest in real assets. The rate of return on equity of firm i over $(t, t+1)$, conditional on the state of the economy at t , ϕ_t , is normally distributed with mean $\bar{R}_i(W(\phi_t), t)$ and standard deviation $\sigma_i(W(\phi_t), t)$. In other words, returns on real assets can have nonstationary distribution provided that the nonstationarity is only a function of aggregate wealth and time. Hence, given $W(\phi_t)$, returns on real assets over $(t, t+1)$ cannot depend upon ϕ_t . This situation can arise if, for example, production technology of firm i is

$$\tilde{K}_i(\phi_{t+1}) = K_i(\phi_t) [\eta_i(W(\phi_t), t) \tilde{\epsilon}_i(t+1) + \mu_i(W(\phi_t), t)],$$

where, $K_i(\phi_t)$ is the amount of capital invested in production technology i (note, $K_i(\phi_t) > 0$ for all ϕ_t and i), $\mu_i(\cdot)$ and $\sigma_i(\cdot)$ are deterministic functions of $W(\phi_t)$, and $\tilde{\epsilon}_i(t+1)$ is a stationary normally distributed random variable.

There are also $N_t - n_t$ financial securities (e.g., bonds, options etc.) that have zero net supply (firms issue only equities), and their returns can have general nonstationary probability distributions.

Assumption A.5 of this paper is less restrictive than assumption A.4 of Constantinides [4] because we allow returns on real assets to follow nonstationary random processes.³

A representative investor will solve the following optimization problem:

$$J(W(\phi_s), \phi_s, s) = \max_{X_{it}, C_t} E_s \left\{ \sum_{t=s}^T U(C_t, t) \right\}$$

Subject to

$$\tilde{W}(\phi_t) = [W(\phi_{t-1}) - C_t] \left\{ \sum_{i=1}^{N_t} [X_{it} (\tilde{R}_i(\phi_t) - R_f(\phi_{t-1}))] + R_f(\phi_{t-1}) \right\}$$

The investor's wealth at the beginning of t , after the state of the economy, ϕ_t , is revealed, is $W(\phi_t)$; then, he/she chooses the optimal values of C_t and X_{it} . This optimization problem has been extensively discussed in the literature (e.g., see Fama [7], Long [9] and [10], and Constantinides [4]); thus, we have presented only the essential aspects of it.

II. An Interest-Rate-Based Asset Pricing Model

The first order conditions for the representative investor's expected utility maximization problem are:

$$U'(C_t, t) - J'(W(\phi_t), \phi_t, t) = 0, \quad (2a)$$

$$J'(W(\phi_t), \phi_t, t) - R_f(\phi_t) E_t [\tilde{J}'(W(\phi_{t+1}), \phi_{t+1}, t+1)] = 0, \quad (2b)$$

$$E_t \{ \tilde{J}'(W(\phi_{t+1}), \phi_{t+1}, t+1) [\tilde{R}_i(\phi_{t+1}) - R_f(\phi_t)] \} = 0, \quad (2c)$$

where $i=1, \dots, N_t$, and $U'(\cdot)$ and $J'(\cdot)$ are, respectively, partial derivatives of $U(\cdot)$ and $J(\cdot)$ with respect to C_t and $W(\phi_t)$. Throughout the paper partial derivatives will be denoted by primes.

Equation (2a) represents the familiar "envelop" condition; equation (2b) states that when optimal consumption-investment policies are followed, one plus the expected rate of decline in the marginal utility of wealth will be equal to one plus the riskless rate of interest; equation (2c) is the most basic equilibrium relationship among securities and it will play a crucial role in the

derivation of our results. In particular, using the definition of conditional covariance, equation (2c) can be written as follows (we use $J'(\phi_t)$ to represent $J'(W(\phi_t), \phi_t, t)$).

$$E_t \tilde{R}_i(\phi_{t+1}) = R_f(\phi_t) - [E_t J'(\phi_{t+1})]^{-1} \text{Cov}_t[\tilde{J}'(\phi_{t+1}), \tilde{R}_i(\phi_{t+1})], \quad (3)$$

and equation (2b) can be solved for the equilibrium riskless rate.

$$R_f(\phi_t) = \frac{J'(\phi_t)}{E_t[\tilde{J}'(\phi_{t+1})]}. \quad (4)$$

We now present a proposition that is central to the results of this paper.

Proposition 1. Assume that conditions specified in A.1 through A.5 hold.

Then, the representative investor's derived utility of lifetime consumption will be state independent.

Proof. See Fama [7] and Constantinides [4].

The basic result of Proposition 1 is that the representative investor's derived utility of lifetime consumption depends only upon his/her current wealth and time; i.e.,

$$J(W(\phi_s), s) = \max \left\{ E_s \sum_{t=s}^T U(C_t, s) \right\}.$$

Also note that following Fama [7] and Constantinides [4] the investor behaves as if he/she maximizes

$$J(W(\phi_t), t) = \max \{ U(C_t, t) + E_t \tilde{J}(W(\phi_{t+1}), t+1) \}$$

subject to the wealth constraint.

Two important implications of the proposition are:

(i) $J'(W(\phi_t), t)$, the current marginal utility of wealth is only a function of current aggregate wealth and time.

(ii) $E_t[\tilde{J}'(W(\phi_{t+1}), \phi_{t+1}, t+1) = g(W(\phi_t), t);$ i.e., the conditional expected value of the next-period marginal utility of wealth is a function of current wealth and time.

We now present two properties of normally distributed random variables that will prove to be important.

Assume that, conditional on y , random variables \tilde{x} and \tilde{z} have a bivariate normal probability distribution. Also assume that $h(\tilde{x})$ is a continuous real function with a continuous first derivative such that $E_y[|h'(\tilde{x})|] < \infty$.

Lemma 1.

$$\text{Cov}_y[h(\tilde{x}), \tilde{z}] = E_y[h'(\tilde{x})] \text{Cov}_y[\tilde{x}, \tilde{z}]. \quad (5)$$

Proof. Please see Stein[19] and Rubinstein[16].

Lemma 2.

$$\frac{\partial}{\partial y} E_y[h(\tilde{x})] = \mu' E_y[h'(\tilde{x})] + \frac{\sigma'}{\sigma} E_y[h'(\tilde{x})(\tilde{x} - \mu)], \quad (6)$$

where, conditional on y , $\tilde{x} \sim N(\mu, \sigma^2)$ and μ' and σ' are, respectively, partial derivatives of μ and σ with respect to y . Because y and x may not be bivariate normal random variables both μ and σ may be functions of y .

Proof. See the Appendix.

Because the aggregation problem is solved (see assumption (A.2)), equilibrium prices are set as if investors were identical; the representative investor's wealth, therefore, includes only real assets (i.e., no financial securities are held by identical investors). Since, conditional on the state of the economy at t , returns on real assets and thus the next-period level of aggregate wealth are normally distributed, we can apply lemma 1 to the conditional covariance between the marginal utility of wealth and the return on real asset i ; the result is:

$$\text{Cov}_t [\tilde{J}'(W(\phi_{t+1}), t+1), \tilde{R}_i(\phi_{t+1})] = E_t [\tilde{J}''] \text{Cov}_t [\tilde{W}(\phi_{t+1}), \tilde{R}_i(\phi_{t+1})].$$

Substituting the above expression into equation (3) we have:

$$E_t \tilde{R}_i(\phi_{t+1}) = R_f(\phi_t) + \lambda(\phi_t) \text{Cov}_t [\tilde{W}(\phi_{t+1}), \tilde{R}_i(\phi_{t+1})] \quad (7)$$

where, $\lambda(\phi_t) = -E_t [\tilde{J}''(W(\phi_{t+1}), t+1)] / E_t [\tilde{J}'(W(\phi_{t+1}), t+1)]$.

Equation (7) is, of course, the multiperiod version of the SL model. As was discussed before, the main difficulty in any empirical testing of equation (7) is that the market portfolio, $\tilde{W}(\phi_{t+1})$, is not observable. The objective of this paper is to represent the last term on the right hand side of equation (7) in terms of observable economic variables (e.g., the riskless rate of interest).

The equilibrium level of the one-period riskless rate was presented in equation (4). Here, we discuss that equation and the determinants of the riskless rate of interest in more details.

One plus the riskless rate of interest is

$$\begin{aligned} R_f(\phi_t) &= \frac{J'(W(\phi_t), t)}{E_t [J'(W(\phi_{t+1}), t+1)]} \\ &= R_f(W(\phi_t), t), \end{aligned} \quad (8)$$

where the results of proposition 1 are used to obtain the last line of the above expression. Hence, the current riskless rate of interest is a function of current wealth and time. The next-period riskless rate, which, given all available information at t , is a random variable can, therefore, be stated as follows:

$$\begin{aligned}\tilde{R}_f(\phi_{t+1}) &= \frac{\tilde{J}'(W(\phi_{t+1}), t+1)}{\tilde{E}_{t+1}[\tilde{J}'(W(\phi_{t+2}), t+2)]} \\ &= \tilde{R}_f(W(\phi_{t+1}), t+1).\end{aligned}\tag{9}$$

In other words, the next-period riskless rate is not known at t only because the next-period value of aggregate wealth is not known. In this economy, random variations in the value of aggregate wealth represent the aggregate risk. Furthermore, random behavior of other endogenous aggregate economic variables (e.g., the riskless rate, returns on long-term default-free bonds, and the market price of risk) are also driven by the random behavior of aggregate wealth.

If we can apply the results of lemma 1 to the conditional covariance of the next-period riskless rate with the return on real asset i , we will be able to show that this covariance is proportional to the conditional covariance of the same asset with aggregate wealth. That is to say,

$$\text{Cov}_t[\tilde{R}_f(W(\phi_{t+1})), \tilde{R}_i(\phi_{t+1})] = E_t[\tilde{R}_f'(W(\phi_{t+1}))\text{Cov}_t[\tilde{W}(\phi_{t+1}), \tilde{R}_i(\phi_{t+1})]].\tag{10}$$

We know from lemma 1 that the above expression is correct if

$E_t[|\tilde{R}_f'(W(\phi_{t+1}))|] < \infty$. Using the expression for $\tilde{R}_f(\cdot)$ and the results of lemma 2, we have (dependence on $W(\phi_{t+1})$ is suppressed):

$$\begin{aligned}\frac{\partial R_f(t+1)}{\partial W(t+1)} &= \\ R_f(t+1) &\left[\frac{\sigma(t+1)\mu'(t+1) - \sigma'(t+1)\mu(t+1)}{\sigma(t+1)} B(t+1) + \frac{\sigma'(t+1)}{\sigma} D(t+1) - A(t+1) \right],\end{aligned}\tag{11}$$

where

$$A(t+1) = -[J''(t+1)] [J'(t+1)]^{-1},$$

$$B(t+1) = -E_{t+1}[\tilde{J}''(t+2)] E_{t+1}[\tilde{J}'(t+2)]^{-1},$$

$$D(t+1) = -E_{t+1}[\tilde{J}''(t+2)W(t+2)] E_{t+1}[\tilde{J}'(t+2)]^{-1},$$

$$\mu(t+1) = E_{t+1}[\tilde{W}(\phi_{t+2})],$$

$$\sigma(t+1) = \text{Var}_{t+1}[\tilde{W}(\phi_{t+2})],$$

$\sigma'(t+1)$ and $\mu'(t+1)$ = partial derivatives of $\sigma(t+1)$ and $\mu(t+1)$ with respect to $W(\phi_{t+1})$.

If the expression on the right hand side of equation (11) is denoted by $\tilde{a}(\phi_{t+1})$, then we require that

$$E_t[|\tilde{a}(\phi_{t+1})|] < \infty. \quad (12)$$

This requirement does not appear to be very restrictive; for example, if $\tilde{W}(\phi_{t+1})$ and $\tilde{W}(\phi_{t+2})$ are independent, then $\tilde{a}(\phi_{t+1})$ will be equal to $-\tilde{R}_f(\phi_{t+1})\tilde{A}(t+1)$, or if $\tilde{W}(\phi_{t+1})$ and $\tilde{W}(\phi_{t+2})$ are bivariate normal random variables, then

$$\tilde{a}(\phi_{t+1}) = \tilde{R}(\phi_{t+1}) \left[\frac{\text{Cov}[\tilde{W}(\phi_{t+1}), \tilde{W}(\phi_{t+2})]}{\text{Var}[\tilde{W}(\phi_{t+1})]} \tilde{B}(t+1) - \tilde{A}(t+1) \right].$$

In both instances the requirement is satisfied if the representative investor's degree of risk aversion is bounded. We will assume that equation (12) is satisfied and denote this assumption by A.6.

Proposition 2. Assume that the conditions specified in A.1 through A.6 are satisfied. Then the following relationship will hold for all assets that, conditional on $W(\phi_t)$, have normally distributed returns.

$$E_t \tilde{R}_i(\phi_{t+1}) = R_f(\phi_t) + \gamma(\phi_t) \text{Cov}_t[\tilde{R}_i(\phi_{t+1}), \tilde{R}_f(\phi_{t+1})], \quad (13)$$

where $\gamma(\phi_t)$ is defined as $\lambda(\phi_t)\{E_t[a(\phi_{t+1})]\}^{-1}$.

Proof. From equation (10) we can solve for the conditional covariance of $\tilde{W}(\phi_{t+1})$ and $\tilde{R}_i(\phi_{t+1})$; substituting the results into equation (7), the above expression will be obtained.

In the above equilibrium relationship, the market price of risk is denoted by $\gamma(\phi_t)$; to test this relationship empirically we do not, however, have to know $\gamma(\phi_t)$. Similar to the approach often employed to present the ISL in a testable form, we can write equation (13) for another risky asset or a portfolio of real assets and thereby eliminate $\gamma(\phi_t)$. That is to say, we can write

$$E_t \tilde{R}_i(\phi_{t+1}) = R_f(\phi_t) - \frac{\text{Cov}_t[\tilde{R}_i(\phi_{t+1}), \tilde{R}_f(\phi_{t+1})]}{\text{Cov}_t[\tilde{R}_p(\phi_{t+1}), \tilde{R}_f(\phi_{t+1})]} \{E_t \tilde{R}_p(\phi_{t+1}) - R_f(\phi_t)\} \quad (14)$$

The reader should note that though $\tilde{R}_i(\phi_{t+1})$ is, by assumption, normally distributed, $\tilde{R}_f(\phi_{t+1})$ is certainly not. Because for a risk averse investor the marginal utility of wealth is not negative, and hence, from equation (9) we can see that $\tilde{R}_f(\phi_{t+1}) > 0$, which indicates that the riskless rate cannot be normally distributed.

If the one-period riskless rate is for some reason unobservable, then another equilibrium relationship in terms of covariances of asset returns with prices of long-term default-free bonds can be employed. For example, the current equilibrium price of a bond that will pay one dollar upon maturity, T , is

$$\begin{aligned} F(\phi_t, t, T) &= \frac{E_t[\tilde{J}'(W(\phi_T), T)]}{J'(W(\phi_t), t)}, \\ &= F(W(\phi_t), t, T). \end{aligned} \quad (15)$$

Note that the next-period price of the bond will depend upon the next-period value of aggregate wealth; hence, we can use conditional covariances of asset returns with prices of long-term bonds to present a risk-return equilibrium relationship similar to the one stated in equation (14). There is, however, one major difficulty in using the "bond-based" measure of risk to test or apply the model: theoretically we know that bond prices follow nonstationary random processes.⁵ Thus, empirical tests of the bond-based model are likely to be as complex as tests of the ISL model. The interest-rate-based model presented in section II, on the other hand, may avoid this problem for at least theoretically there is no reason to believe that the riskless rate follows a nonstationary random processes. However, whether the riskless rate of interest actually follows a stationary random process is an empirical question (one can, of course, build models that would generate a specific type of behavior by the riskless rate of interest (e.g., Cox, Ingersoll, and Ross[6] and Breeden[3])).

In contrast to the consumption-based and the arbitrage pricing models, the interest-rate-base model is a direct result of the mean-variance efficiency of the market portfolio. Hence, from a theoretical view, both the ISL model and the interest-rate-based model can be criticized on the same grounds (e.g., their rather strict common assumptions). The interest-based model should, therefore, be viewed as an attempt to introduce some new insights into the ISL model and to present an alternative equilibrium relationship that can be tested empirically.

III. Conclusion

This paper develops a multiperiod asset pricing model that does not require any estimation of the stochastic properties of the market portfolio or aggregate consumption. The results are developed using the framework set forth by Fama [7] and Constantinides [4], and consequently will hold in capital markets that

satisfy the minimum set conditions that produce the multiperiod SL asset pricing model.

We demonstrate that intertemporal random changes in the riskless rate can be used to measure the nondiversifiable risk of securities. In particular, because the "market-portfolio-based" beta of a risky asset is proportional to the "interest-rate-based" beta of the same asset, we can present risk-return equilibrium relationships that do not depend on covariances of asset returns with the return on the market portfolio.

As is the Sharpe-Lintner model, the pricing model of this paper is expressed in real terms. Observed data on asset returns are, on the other hand, expressed in nominal terms. Hence, an interesting extension of our results paper could be to relax the assumption that prices of consumption goods are nonrandom.⁶

Appendix

Proof of Lemma 2. First consider the following property of conditional normal random variables.

$$E_Y[h'(\tilde{x})] = E_Y\left[h(\tilde{x}) \frac{x - \mu(Y)}{\sigma(Y)^2}\right]. \quad (A1)$$

Equation (A1) is obtained by applying lemma 1 to $\text{Cov}_Y[h(\tilde{x}), \tilde{x}]$. Now we prove lemma 2. Let

$$f(x;Y) = [2\pi\sigma(Y)^2]^{-2} \exp\left\{-\frac{[x - \mu(Y)]^2}{2\sigma(Y)^2}\right\}.$$

Then,

$$\begin{aligned} \frac{\partial}{\partial y} E_Y[h(\tilde{x})] &= \int_{-\infty}^{+\infty} h(x) \frac{\partial f(x;Y)}{\partial y} dx \\ &= \int_{-\infty}^{+\infty} h(x) Q(x,Y) f(x;Y) dx, \end{aligned} \quad (A2)$$

where

$$Q(x,Y) = \frac{\sigma'(Y)}{\sigma(Y)} \left[\frac{[x - \mu(Y)]^2}{\sigma(Y)^2} - 1 \right] + \frac{\mu'(Y)[x - \mu(Y)]}{\sigma(Y)^2}.$$

Lemma 2 is proved by taking the expectation of the above expression and applying equation (A1) to the result.

Notes

¹Similar results are also discussed in Kazemi [8].

²Assume that investor i maximizes $U_i(C_0) + \rho_i E_0 V_i(C_1)$, where

$-U_i'(C_0)/U_i''(C_0) = A_i + B_i C_0$ and $-V_i'(C_1)/V_i''(C_1) = A_i + B_i C_1$. Rubinstein[15] presents the following conditions for aggregation: (i) All individuals are homogeneous; (ii) All individuals have the same beliefs, $\rho_i = \rho$, and $B_i = B \neq 0$; (iii) All individuals have the same beliefs and $B_i = B = 0$; (iv) All individuals have the same beliefs and resources, $A_i = A = 0$, and $B_i = B = 1$; (v) Markets are complete and $B_i = B = 0$; (vi) Markets are complete, all individuals have the same resources, $\rho_i = \rho$, $A_i = A = 0$, and $B_i = B = 1$. Because returns on real assets are normally distributed, the utility of the representative investor must be defined for normally distributed wealth. For further discussion of these conditions see Constantinides [4], footnote 8.

³[3] assumes that real asset returns have stationary joint probability distributions which belong to the family of two-fund separating distributions (e.g., see Ross [14]). [5] allows nonstationary returns on real assets.

⁴To apply lemma 1 to $\text{Cov}_t[\tilde{J}'(W(\phi_{t+1}), t+1), \hat{R}_i(\phi_{t+1})]$, we must restrict the utility function of the representative investor to the class where $E_t |J''(W(\phi_{t+1}), t+1)| < \infty$. Because the utility function is assumed to be concave (i.e., $J''(W(\phi_{t+1}), t+1) < 0$), the actual restriction is $E_t J''(W(\phi_{t+1}), t+1) > -\infty$.

⁵This is due to the fact that any stochastic process representing bond prices must satisfy the boundary condition $F(\phi_T, T, T) = 1$.

⁶As discussed in Fama[7] and Constantinides[4], once this assumption is relaxed, the indirect utility function of the representative investor becomes state dependent.

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