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ERROR BOUNDS FOR EOQ

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Abstract

In this paper we explore the properties of the discounted total cost function for the economic order quantity. We show that it is convex. Further, it is shown that the Classical Economic Order Quantity (based on Wilson's formula) is not less than the true optimum value based on discounting. Bounds for the discounted reorder interval (or order quantity) based on average cost analysis are also provided. Further, we analytically show that larger the ratio of non-capital related holding charges to the total holding charges, more adverse is the effect on the accuracy of the average cost analysis.

ERROR BOUNDS FOR EOQ

Introduction

It is well known that the classical Economic Order Quantity, which is based on average cost analysis, is an approximation for the conceptually more rigorous discounted cost version. It is clear that the objective of maximizing the net present value of shareholder wealth leads to the use of net present value as the criterion for decision making. For a more detailed exposition, see the arguments presented by Grubbstrom [1]. In prior analyses (Hadley and Whitin [2]), it was shown that the total cost function based on average cost analysis is convex, and using that property, the optimal order quantity was determined. However, properties of the discounted total cost function have remained unexplored. Many researchers recognize that the classical economic order quantity is an excellent first order approximation to the discounted cost version. Hadley and Whitin [3] discussed this in a problem form in their classical book on analysis of inventory systems. This was further verified by Hadley [2] in an extensive computational study. Trippi and Lewin [5] also addressed the problem of determining optimal lot sizes using discounting. However, they implicitly assumed that there are no noncapital related holding charges such as material handling, insurance and warehousing etc. In this paper, we analyze the general case. We show that the discounted total cost function for the general case is convex. We further show that, for the general case, the reorder interval (or the order quantity) derived using average cost analysis is never less than the true optimum based on discounting. Relative error bounds for the optimum reorder interval (or order quantity) are provided based on average cost analysis. These bounds are useful in search methods used to determine the optimum reorder interval (or order quantity) for the discounted model. Further, we derive an expression for the relative total cost error if the classical reorder interval (or order quantity) based on the average cost analysis is used. It is shown that the accuracy of the average cost analysis is dependent, among other factors, on the ratio of noncapital related holding charges such as material handling and warehousing to the total holding charges. Thus using the analysis provided in this paper, the decision maker can verify if the average cost analysis has provided acceptable approximate solution. If not, search methods can be used to determine more accurate order quantities.

Notation

T : Reorder interval

D : Demand rate

r : Discount rate

p : Price per unit

S : Set up cost or Order cost

h : Inventory carrying cost, exclusive of capital charges (in dollars per unit per period)

$NPV(T)$: Net present value if the reorder interval is T

$ANN(T)$: Annualized cost per period if the reorder interval is T

$AVC(T)$ = Cost per period based on the average cost analysis if the reorder interval is T

T_E = Best reorder interval using the classical average cost analysis

$= \sqrt{2S/D(h + pr)}$

T_o = Reorder interval based on the discounted cost approach (true optimum).

Analysis

Since the assumptions made in determining the classical economic order quantity (or reorder interval) are well known, they are not repeated here. The reader is refered to Hax and Candea [4].

If we discount all future costs at a discount rate r,

$$NPV(T) = S + \int_0^T hD(T - \ell)e^{-r\ell}\, d\ell + DTp + e^{-rT}\, NPV(T) \qquad (1)$$

$$= \frac{S}{1 - e^{-rT}} + \frac{1}{1 - e^{-rT}} \int_0^T hD(T - \ell)e^{-r\ell}\, d\ell + \frac{DTp}{1 - e^{-rT}} \qquad (2)$$

Equation (2) indicates the net present value of all cash outflows if the reorder interval is T. In order to compare this with conventional average cost analysis, we use ANN(T), an alternate, but equivalent measure. ANN(T) represents the equivalent uniform cash flow stream that generates the same NPV(T). Since we are considering an infinite horizon (from the basics of discounting),

$$ANN(T) = r * NPV(T)$$

$$= \frac{Sr}{1 - e^{-rT}} + \frac{r}{1 - e^{-rT}} \int_0^T hD(T - \ell)e^{-r\ell}\, d\ell + \frac{DTpr}{1 - e^{-rT}} \qquad (3)$$

$$= \frac{Sr}{1 - e^{-rT}} + \frac{hD}{r}\,\frac{rT - 1 + e^{-rT}}{1 - e^{-rT}} + \frac{DprT}{1 - e^{-rT}} \qquad (4)$$

For small values of rT, the above expression can be approximated as

$$ANN(T) \approx \frac{S}{T} + \frac{hDT}{2} + Dp + \frac{DprT}{2}$$

$$\approx \frac{S}{T} + \frac{1}{2}(h + rp)DT + Dp \qquad (5)$$

However, based on average cost analysis,

$$AVC(T) = \frac{S}{T} + \frac{1}{2}(h + rp)DT + Dp \qquad (6)$$

Dp is a constant and independent of the decision variable T. In most discussions of average cost analysis, it is not included in the cost expression that is to be minimized. It is included here in (6) to show that the total cost per period based on the average cost analysis is approximately the same as ANN(T). It is well known that (6) is convex. The classical Economic Order Quantity or reorder interval (T_E) are determined using the convexity properties of (6).

However, properties of ANN(T) have remained unexplored. We show that ANN(T) is also convex.

<u>Remark 1</u>: ANN(T) is convex.

<u>Proof</u>: Rewriting (4),

$$\text{ANN}(T) = \frac{Sr}{1 - e^{-rT}} + (Dpr + hD) \frac{T}{1 - e^{-rT}} - \frac{Dh}{r} \tag{7}$$

$$\text{ANN}'(T) = - \frac{Sr^2 e^{-rT}}{(1 - e^{-rT})^2} + (Dpr + hD) \frac{1 - e^{-rT} \div rTe^{-rT}}{(1 - e^{-rT})^2} \tag{8}$$

$$\text{ANN}''(T) = \frac{Sr^3 e^{-rT} (1 + e^{-rT})}{(1 - e^{-rT})^3}$$

$$+ \frac{(Dpr + hD)re^{-rT}}{(1 - e^{-rT})^3} \{(2 + rT)e^{-rT} - (2 - rT)\} \tag{9}$$

ANN(T) is convex if and only if

$$\text{ANN}''(T) \geq 0$$

It can easily be seen that all terms in (9) except the term in curled parenthesis are positive. Consider the term in curled parenthesis.

$$(2 + rT)e^{-rT} - (2 - rT) = \frac{1}{e^{rT}} \{(2 + rT) - (2 - rT)e^{rT}\}$$

$$= \frac{1}{e^{rT}} \{rT(1 + e^{rT}) + 2(1 - e^{rT})\}$$

$$= \frac{1}{e^{rT}} \left\{ \sum_{n=3}^{\infty} \frac{r^n T^n}{n!} (n - 2) \right\}.$$

$$\geq 0$$

Hence ANN"(T) $\geq$ 0 and ANN(T) is convex. ■

Since ANN(T) is proportionate to NPV(T), NPV(T) is also convex. We next explore the optimality of the classical reorder interval which is based on the average cost analysis.

Remark 2: The reorder interval (or order quantity) based on average cost analysis is never less than the optimum value derived using discounting.

Proof: It is well known that

$$T_E = \sqrt{\frac{2S}{D(h + pr)}} \tag{10}$$

where T_E is reorder interval based on average cost analysis. Let T_o be the true optimum based on discounted cost analysis. Since ANN(T) is convex, T_o is determined by setting ANN'(T) zero. Reconsider (8).

$$\text{ANN}'(T) = 0 \Rightarrow \frac{Sr^2}{D(h + pr)} = e^{rT_o} - 1 - rT_o$$

$$= \frac{r^2 T_o^2}{2} + \frac{r^3 T_o^3}{6} + \frac{r^4 T_o^4}{24} \cdots\cdot$$

$$\frac{2S}{D(h + pr)} = T_o^2 + \frac{rT_o^3}{3} + \frac{r^2 T_o^4}{12} \cdots\cdot \tag{11}$$

Using (10) in (11),

$$T_E^2 = T_o^{\,2} + \frac{rT_o^{\,3}}{3} + \frac{r^2T_o^{\,4}}{12} \quad \ldots .$$

Since $T_o \geq 0$, the above expression can be rewritten as,

$$T_E^{\,2} - T_o^{\,2} \geq 0$$

$$T_E \geq T_o \qquad \blacksquare$$

It may be noted that Remarks 1 and 2 are generalizations of the results derived by Trippi and Lewin [5]. As noted earlier, their analysis is a special case where it is assumed that the noncapital inventory related cash outflows such as material handling, insurance, taxes and warehouse leasing do not exist.

Error Bounds

From equation (11) we can derive an error bound for the classical reorder interval. Equation (11) can be rewritten as

$$\frac{T_E - T_o}{T_o} = \sqrt{1 + \frac{rT_o}{3} + \frac{r^2T_o^{\,2}}{12} + \frac{r^3T_o^{\,3}}{60} \quad \ldots .} - 1 \qquad (12a)$$

Since $T_E \geq T_o$, the above expression can be rewritten as

$$\frac{T_E - T_o}{T_o} \leq \sqrt{1 + \frac{rT_E}{3} + \frac{r^2T_E^{\,2}}{12} + \frac{r^3T_E^{\,3}}{60} \quad \ldots .} - 1$$

$$\leq \sqrt{\frac{2(e^{rT_E} - rT_E - 1)}{rT_E}} - 1 \qquad (12b)$$

Hence, equation (12) provides ex-post relative error bound on the optimal reorder interval (T_o). Using Remark 2 and expression (12), we bound T_o as given below.

$$rT_E \left\{ \frac{rT_E}{\sqrt{2(e^{rT_E} - rT_e - 1)}} \right\} \leq rT_o \leq rT_E \qquad (13)$$

Table 1 shows the ex-post relative error for the reorder interval $((T_E - T_o)/T_o)$ based on the value of T_E and compares it with the exact value of the relative error. It can be seen that for all practical purposes, the ex-post relative error is a good approximation of the exact relative error.

TABLE 1

RELATIVE ERROR BOUNDS FOR rT_E

rT_E	Ex-post relative error for T_E in percent = expression (12b) * 100	Exact value of relative error in percent $= \frac{T_E - T_o}{T_o} * 100$
0.05	0.85	0.84
0.10	1.72	1.69
0.15	2.63	2.56
0.21	3.57	3.45
0.26	4.54	4.35
0.32	5.55	5.26
0.37	6.60	6.19
0.43	7.68	7.14
0.49	8.81	8.10
0.55	9.98	9.08
0.61	11.20	10.07
0.67	12.46	11.09
0.73	13.78	12.12
0.79	15.15	13.16
0.86	16.57	14.23
0.92	18.06	15.32
0.99	19.61	16.42
1.06	21.23	17.55
1.13	22.92	18.69
1.20	24.69	19.86

Note: Computations are shown upto a value of $rT_E = 1.2$. For example, if rT_E equals 1.2 and r is 20% per year, T_E is 6 years!

Table 2 compares T_o with the ex-post lower bound and upper bounds for it based on T_E. These bounds are from the expression (13). It can be seen that the ex-post lower bound on T_o is much tighter than the upper bound.

TABLE 2

BOUNDS FOR rT_o BASED ON THE AVERAGE COST ANALYSIS

rT_E	Ex-post lower bound on rT_o from (13)	Exact value of rT_o	Upper bound on rT_o $(=rT_E)$
0.05042	0.05000	0.05000	0.05042
0.10169	0.09997	0.10000	0.10169
0.15385	0.14990	0.15000	0.15385
0.20689	0.19976	0.20000	0.20689
0.26087	0.24953	0.25000	0.26087
0.31578	0.29917	0.30000	0.31578
0.37167	0.34866	0.35000	0.37167
0.42854	0.39797	0.40000	0.42854
0.48644	0.44706	0.45000	0.48644
0.54538	0.49589	0.50000	0.54538
0.60540	0.54444	0.55000	0.60540
0.66651	0.59266	0.60000	0.66651
0.72875	0.64052	0.65000	0.72875
0.79215	0.68796	0.70000	0.79215
0.85674	0.73494	0.75000	0.85674
0.92254	0.78141	0.80000	0.92254
0.98959	0.82733	0.85000	0.98959
1.05793	0.87264	0.90000	1.05793
1.12757	0.91729	0.95000	1.12757
1.19857	0.96122	1.00000	1.19857

Note: Computations are shown upto a value of $rT_E = 1.2$. For $rT_E = 1.2$ and and $r = 20\%$ per year, T_E is 6 years!

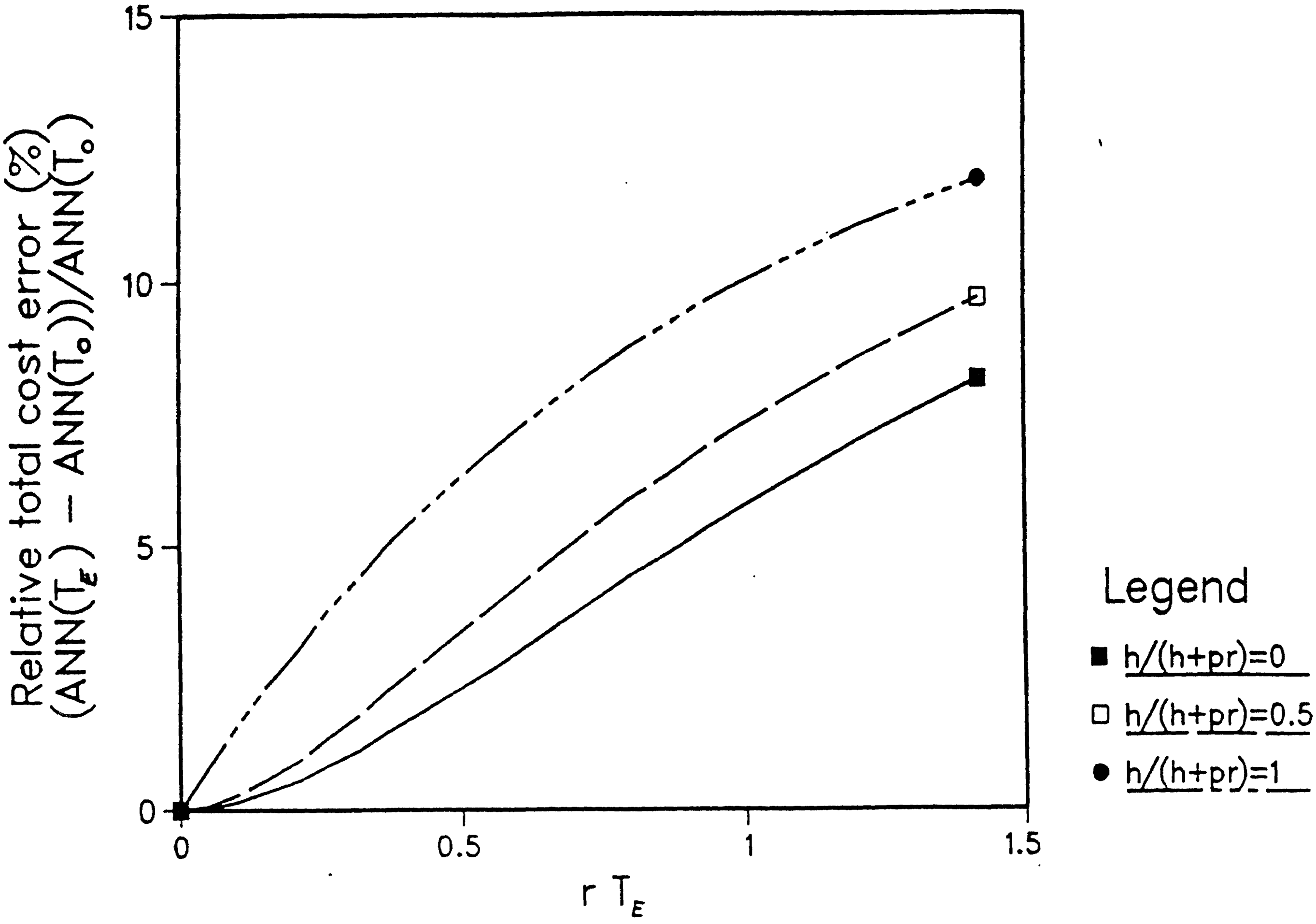
Similarly, we derive the relative error value for the discounted cost function. Exact value of the relative error for the discounted total cost function are obtained from:

$$\frac{ANN(T_E) - ANN(T_o)}{ANN(T_o)} = \frac{r(T_E - T_o) - (e^{-rT_o} - e^{-rT_E})}{(1 - e^{-rT_E}) \left\{ e^{rT_o} - \frac{h}{h + pr} \right\}} \tag{14}$$

Details of the derivation are shown in the appendix. Note that T_E and T_o are related by expression (12a). Thus how close an approximation T_E is to T_o depends on r. h/(h + pr) is the ratio of non-capital related holding charges (such as warehousing and material handling) to the total handling charges. The relative error for the discounted total cost function depends on this ratio. This is shown in Figure 1.

FIGURE 1

EFFECT OF h/(h + pr) ON
RELATIVE DISCOUNTED TOTAL COST ERROR



It is clear from Figure 1 that the ratio h/(h + pr) influences the relative error for the discounted total cost function. As the ratio of non-capital related holding charges (such as material handling and warehousing) to total hold-charges increases, the relative error for the discounted total cost function increases.

Conclusion

We have shown in this paper that the discounted total cost function is also convex. Further, we have established that the reorder interval (or order quantity) based on average cost analysis is not less than the true optimum based on the discounted cost version. We also provide ex-post lower and upper bounds for the reorder interval (or order quantity). Computational results show that the lower bound is tight. Relative error for the discounted total cost is influenced by the proportion of non-capital related holding charges such as material handling and warehousing in the total holding costs. When this ratio is high, the reorder interval (or order quantity) based on the average cost analysis may be inadequate. However, the bounds for T_o based on T_E provided by us will be useful in search procedures for determining T_o.

References

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APPENDIX

Derivation for the relative discounted total cost error. Using expression (7),

$$(ANN(T_o) - ANN(T_E))/ANN(T_o)$$

$$= \frac{Sr^2(e^{-rT_E} - e^{-rT_o}) + (Dpr + hD)r\,\{T_E(1 - e^{-rT_o}) - T_o(1 - e^{-rT_E})\}}{(1 - e^{-rT_E})\{Sr^2 + (Dpr^2 + hDr)T_o - hD(1 - e^{-rT_o})\}} \qquad (I)$$

At T_o, derivative of ANN(T) equals zero. This implies (using expression (11)),

$$Sr^2 = (Dpr + hD)(e^{rT_o} - 1 - rT_o) \qquad (II)$$

Substituting II in I, relative discounted total cost error can be rewritten as

$$= \frac{(e^{-rT_E} - e^{-rT_o})(e^{rT_o} - 1 - rT_o) + r\{T_E(1 - e^{-rT_o}) - T_o(1 - e^{-rT_E})\}}{(1 - e^{-rT_E})\{e^{rT_o} - 1 - (\frac{h}{h + pr})(1 - e^{-rT_o})\}},$$

With a little algebraic manipulation, above expression can be rewritten as

$$= \frac{r(T_E - T_o) - (e^{-rT_o} - e^{-rT_E})}{(1 - e^{-rT_E})\{e^{rT_o} - (\frac{h}{h + pr})\}}$$

■