SOME PROPERTIES OF WEIGHTED
TARDINESS PROBLEMS

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Some Properties of Weighted Tardiness Problems

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Abstract

We consider scheduling of n jobs available simultaneously to be scheduled

on a single machine. The criterion is to minimize total weighted tardiness.

We identify a necessary condition for adjacent jobs in optimal sequences.

Further it is shown that the well known Modified Due Date Rule is a special

case of the necessary condition identified by us. Lastly, we identify circum-

stances under which first job in an optimal sequence can be determined without

fully solving the problem.

Consider the following single machine weighted tardiness problem--we have

a set of n jobs to be scheduled simultaneously on a single machine. Each job

has associated with it a triple--(w_i, p_i, d_i) representing the weight, process-

ing time and due date of J_i (job i). Let c_i represent the completion time of

job i for a particular sequence. We wish to schedule the jobs such that the

sum of weighted tardiness is minimized--i.e.,

$$\sum_{i=1}^{i=n} w_i \max (o, c_i - d_i)$$

is minimized.

It has been shown by Lenstra [1977] that the above problem is NP-Complete.

In the past, various researchers developed enumerative methods for solving

this problem. Noteworthy among these methods are the approaches developed by

Rinnooy Kan et al [1975], Picard and Queyranne [1978] and Schrage and Baker

[1978]. These procedures make use of the dominance tests developed by Emmons

[1969] for average tardiness problems and extended for the weighted tardiness

problem by Rinnooy Kan et al [1975] in structuring the problem as a global pre-

cedence network. The purpose of this note is to develop an additional local

precedence relationship among adjacent jobs in optimal sequences.

Proposition 1: Consider any two adjacent jobs in an optimal sequence for the

single machine weighted tardiness problem. Either the following con-

dition holds good or an alternate optimal sequence can be constructed

by interchanging the adjacent jobs which satisfies the following

condition.

$$\frac{w_{[i]}}{p_{[i]}} \left(1 - \frac{(d_{[i]}-t-p_{[i]})^+}{p_{[i+1]}} \right) \geq \frac{w_{[i+1]}}{p_{[i+1]}} \left(1 - \frac{(d_{[i+1]}-t-p_{[i+1]})^+}{p_{[i]}} \right)$$

where [i] denotes the index of the job in ith position, x^+ denotes

max (0,x) and t is the start time for $J_{[i]}$.

<u>Proof</u>: In the optimal sequence under consideration, jobs are either early (being on time is considered as early case) or tardy. We show through pairwise interchange arguments that either our condition is satisfied or an alternate optimal sequence is constructed in which that condition is satisfied.

We have 16 cases to consider (E is early and T is tardy). Without loss of generality, assume that i is the index of job occupying ith position and j be the index of jobs occupying i+1th position.

Case #	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Before interchange																
Job i	E	E	E	E	E	E	E	E	T	T	T	T	T	T	T	T
Job j	E	E	E	E	T	T	T	T	E	E	E	E	T	T	T	T
After interchange																
Job i	E	E	T	T	E	E	T	T	E	E	T	T	E	E	T	T
Job j	E	T	E	T	E	T	E	T	E	T	E	T	E	T	E	E

Note that cases 2, 4, 9, 10, 12, 13 and 14 are logically inconsistent. In case 1, if the condition is not satisfied, we merely interchange jobs i and j without losing optimality. In all other cases, the condition can be shown to be necessary through pairwise interchange arguments. Details are omitted here for the sake of brevity. ∎

Further, it is interesting to note that the Corollary 2 developed by Picard and Queyranne [3] is a special case of Proposition 1. Also, we further show that a very effective dispatch rule developed by Baker and Bertrand [6] for the average tardiness problem ($w_i=1, i=1,....n$) is a special case of Proposition 1. The details are given below.

Baker and Bertrand [1982] had developed a very effective dispatch type heuristic rule for the average tardiness problem—i.e., $w_i = 1$ for all jobs. The rule is as follows—for all jobs yet to be scheduled, determine modified due date (d'_i) as follows:

$$d'_i = \max(d_i, t+p_i).$$

Schedule next the jobs that has earliest modified due date.

Proposition 2: MDD is a necessary condition for any optimal sequence for the static average tardiness problems.

Proof: We establish the claim by showing that the Modified Due Date Rule can be derived from Proposition 1. Since $w_i = 1$ for all jobs in the unweighted case, Proposition 1 can be rewritten as

$$\frac{1}{p_i}\left\{1 - \frac{(d_i-t-p_i)^+}{p_j}\right\} \geq \frac{1}{p_j}\left\{1 - \frac{(d_j-t-p_j)^+}{p_i}\right\}$$

$$p_j\left\{1 - \frac{(d_i-t-p_i)^+}{p_j}\right\} \geq p_i\left\{1 - \frac{(d_j-t-p_j)^+}{p_i}\right\}$$

$$p_j - (d_i-t-p_i)^+ \geq p_i - (d_j-t-p_j)^+$$

$$-p_i - (d_i-t-p_i)^+ \geq -p_j - (d_j-t-p_j)^+$$

$$p_i + (d_i-t-p_i)^+ \leq p_j - (d_j-t-p_j)^+$$

$$p_i + \max(0, d_i-t-p_i) \leq p_j + \max(0, d_j-t-p_j)$$

$$-(t+p_i)+p_i+\max(t+p_i,d_i) \leq -(t+p_j)+p_j+\max(t+p_j,d_j)$$

$$\max(d_i,t+p_i) \leq \max(d_j,t+p_j)$$

∎

Proposition 1 is easy to implement in any search procedure for weighted tardiness problem. It can be applied at any point in an enumerative procedure to ensure local optimality, thus reducing the search for optimal solution.

Further, it may be noted that Proposition 1 can easily be used irrespective of

whether the enumerative method generates the schedule in forward direction or

backward direction.

Further, Proposition 1 can be used as a stopping rule in some situations

in identifying optimal first job. In most practical situations, we are con-

cerned with identifying next job to be scheduled on the machine and not neces-

sarily develop the full schedule.

Proposition 3: If the job with the highest $\frac{w}{p}$ ratio is tardy even if scheduled

first, then there exists at least one optimal sequence in which it

should be scheduled first.

Proof: The proof is by contradiction. Let k be the index of the job with the

highest $\frac{w}{p}$ ratio. Suppose that there exists an optimal sequence

such that J_i immediately precedes J_k and let t be the start time for

J_i. By Proposition 1,

$$\frac{w_i}{p_i}\left(1 - \frac{(d_i - t - p_i)^+}{p_j}\right) \geq \frac{w_k}{p_k}\left(1 - \frac{(d_k - t - p_k)^+}{p_j}\right)$$

Note that since $d_k < p_k$, right hand side is always $\frac{w_k}{p_k}$ and LHS is

never more than $\frac{w_i}{p_i}$. Since $\frac{w_k}{p_k} \geq \frac{w_i}{p_i}$ (by definition), this condradicts

the supposition that the original sequence was optimal. Thus there

can be no optimal sequence in which J_k is preceded by another job.

This implies that J_k should be scheduled first irrespective of the

rest of the schedule.

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of Proposition 1 which is actually a stronger version of the result than

claimed by Morton and Rachamadugu [1982] and also for pointing out the vali-

dity of their result ~~not~~ for a case ⟨*not*⟩ analyzed in their earlier work.

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