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A DEDUCTIVE APPROACH TO LINEAR OPTIMIZATION

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by

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BACKGROUND OF THIS PAPER

This is a draft of two chapters of a monograph on linear and nonlinear optimization. It is intended for students in optimization courses having interests in the mathematics of optimization.

Chapter 1. Mathematical Background

1.1. Introduction. Linear optimization theory can be developed either constructively or deductively. In the former approach, mathematical concepts are introduced which do doubly duty: they are used to prove the existence of optimal solutions and to calculate (and characterize) such solutions. Thus the constructive approach is an economical one and largely for this reason has become standard (Dantzig [4], Spivey and Thrall [12], and others).

Deductive approaches, used first in the development of the theory and still earlier in the development of the closely related theory of games, have disadvantages. They utilize mathematical concepts (such as convexity and separating hyperplanes) that are not normally found in a linear algebra course. Moreover, these concepts are needed to prove the important existence theorems, but once this task is completed, an entirely different set of mathematical concepts is needed to develop an algorithm for calculating the solutions whose existence has been assured. Thus these approaches, despite their mathematical elegance, are regarded as uneconomical.

A deductive approach, however, has superior features if one wishes to generalize to nonlinear optimization, since the tools of convexity, separating hyperplanes, etc., are then essential. To put it another way, the mathematical tools used to prove existence theorems in the linear case (and which are not used again) move to the center of the stage when one wishes to consider nonlinear optimization.

This monograph is concerned with a deductive approach which is largely algebraic and which minimizes the use of concepts from analysis. It is based upon the work of Tucker, some of which is summarized in a paper by Good [5], and assumes as a prerequisite a course in linear algebra.* Since such courses vary considerably in content, the discussion begins with a brief review of relevant concepts from the theory of bilinear forms and from orthogonality, and introduces the necessary concepts from analysis. Concepts that are found in the typical linear algebra course are either briefly mentioned or stated formally without proof (references to standard linear algebra texts are frequently indicated). Other concepts which are less likely to be covered in such a course are stated formally and sometimes proved, with the proofs of the remainder indicated as exercises for the reader.

1.2. Bilinear Forms. We begin in the spirit of linear algebra and introduce concepts for vectors that are coordinate free. Vectors will be denoted initially by Greek letters and their coordinates, relative to indicated bases, will be denoted by lower case Latin letters. Later, when only one basis is used for the discussion, we will identify a vector in terms of

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*A deductive approach that is exclusively algebraic can be found in paper 4 of Kuhn and Tucker [7]; a deductive approach that uses analysis can be found in Ben-Israel [1].

its coordinates, since it has only one such representation in this case. All vector spaces will be understood to be finite dimensional.

Definition. Let U and V be vector spaces over a field F and let f be a function which maps a pair of vectors α and β, where $\alpha \varepsilon U$ and $\beta \varepsilon V$, into F such that f is a linear function of α and β separately.

Now suppose $f(\alpha,\beta)$ is a bilinear form; then for α_1, $\alpha_2 \varepsilon U$, β_1, $\beta_2 \varepsilon V$ and a_1, $a_2 \varepsilon F$ we consider

$$f(a_1\alpha_1 + a_2\alpha_2, b_1\beta_1 + b_2\beta_2).$$

For the moment let $\delta = b_1\beta_1 + b_2\beta_2$; since f is linear in its first argument we have

$$\begin{aligned} f(a_1\alpha_1+a_2\alpha_2,\delta) &= a_1 f(\alpha_1,\delta) + a_2 f(\alpha_2,\delta) \\ &= a_1 f(\alpha_1,b_1\beta_1+b_2\beta_2) + a_2 f(\alpha_2,b_1\beta_1+b_2\beta_2). \end{aligned}$$

Now consider $f(\alpha_1, b_1\beta_1+b_2\beta_2)$; this is linear in the second argument, so

$$f(\alpha_1,b_1\beta_1 + b_2\beta_2) = b_1 f(\alpha_1,\beta_1) + b_2 f(\alpha_1,\beta_2)$$

and an analogous result holds for $f(\alpha_2,b_1\beta_1+b_2\beta_2)$. Hence

$$\begin{aligned} f(a_1\alpha_1+a_2\alpha_2,b_1\beta_1+b_2\beta_2) &= a_1b_1 f(\alpha_1,\beta_1) + a_1b_2 f(\alpha_1,\beta_2) \\ &\quad + a_2b_1 f(\alpha_2,\beta_1) + a_2b_2 f(\alpha_2,\beta_2). \end{aligned}$$

As a slight generalization, let U and V be vector spaces over a field F having, respectively, bases $\{\alpha_1, \cdots, \alpha_p\}$ and $\{\beta_1, \cdots, \beta_n\}$ and let

$$\alpha = \sum_{i=1}^{p} x_i \alpha_i \quad \text{and} \quad \beta = \sum_{j=1}^{n} y_j \beta_j \ .$$

Then for a bilinear form we can write

$$f(\alpha, \beta) = f\left[\sum_{i=1}^{p} x_i \alpha_i , \beta\right] = \sum_{i=1}^{p} x_i f(\alpha_i, \beta)$$

and for each i we have

$$f(\alpha_i, \beta) = f\left[\alpha_i, \sum_{j=1}^{n} y_j \beta_j\right] = \sum_{j=1}^{n} y_j f(\alpha_i, \beta_j).$$

Hence

$$f(\alpha, \beta) = \sum_{i=1}^{p} x_i \left[\sum_{j=1}^{n} y_j f(\alpha_i, \beta_j)\right] = \sum_{i=1}^{p} \sum_{j=1}^{n} x_i y_j f(\alpha_i, \beta_j).$$

Therefore, for any $\alpha \varepsilon U$ and $\beta \varepsilon V$, the values of the bilinear form $f(\alpha, \beta)$ are known when the pn values $f(\alpha_i, \beta_j)$ are known -- given that we have bases of U and V.

Now denote $f(\alpha_i, \beta_j) = b_{ij} \varepsilon F$ and define $B = [b_{ij}]$ to be the matrix representing the bilinear form with respect to the bases above. Relative to the same bases we can use the coordinate vectors

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

to represent the vectors α and β respectively; then

$$\begin{aligned} f(\alpha,\beta) &= \sum_{i=1}^{p} \sum_{j=1}^{n} x_i b_{ij} y_j \\ &= [x_1 \cdots x_p] \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{p1} & \cdots & b_{pn} \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \\ &= X^T BY. \end{aligned}$$

Since any p by n matrix can be regarded as the matrix of a bilinear form in p+n variables, many theorems about matrices have counterparts in the theory of bilinear forms and conversely. We will see some examples of this below.

In all the following we confine ourselves to the real field R and all bilinear forms will be understood to be defined for vector spaces over R.

Definition. Let f be a bilinear form; if $f(\alpha,\beta) = f(\beta,\alpha)$ for every $\alpha,\beta \in V(R)$, we say that f is symmetric.*

Theorem 1. A bilinear form $f(\alpha,\beta)$ is symmetric if and only if any matrix B representing $f(\alpha,\beta)$ is symmetric (i.e.,

*For this definition to have meaning the bilinear form f must be defined on pairs of vectors from the same vector space.

has the property that $B = B^T$).

Proof. The matrix $B = [b_{ij}]$ is determined by $f(\alpha_i, \alpha_j)$ and $B^T = [b_{ji}]$ is determined by $f(\alpha_j, \alpha_i)$, where the vectors α_i are basis vectors for the vector space. Now

$$b_{ji} = f(\alpha_j, \alpha_i) = f(\alpha_i, \alpha_j) = b_{ij},$$

so $B^T = B$.

Among other things, this theorem assures us that if a matrix of a bilinear form, expressed in terms of a given basis, equals its own transpose, then the matrix of the form in terms of any other basis will also be equal to its transpose.

Definition. If $f(\alpha,\alpha) = 0$ for every $\alpha \varepsilon V(R)$, then the bilinear form f is said to be skew-symmetric.

Theorem 2. A bilinear form $f(\alpha,\beta)$ is skew-symmetric if and only if any matrix representing f is skew-symmetric (i.e., has the property that $B = -B^T$).

Proof. For any $\alpha,\beta \varepsilon V(R)$ consider $\alpha+\beta$; we have from skew-symmetry $f(\alpha+\beta,\alpha+\beta) = 0$. Also

$$0 = f(\alpha+\beta,\alpha+\beta) = f(\alpha,\alpha) + f(\alpha,\beta) + f(\beta,\alpha) + f(\beta,\beta)$$
$$= f(\alpha,\beta) + f(\beta,\alpha).$$

Hence $f(\alpha,\beta) = -f(\beta,\alpha)$ and $B = -B^T$.

On the other hand, if $B = -B^T$, we have $f(\alpha,\beta) = -f(\beta,\alpha)$ for all $\alpha,\beta \varepsilon V(R)$. Then $f(\alpha,\alpha) = -f(\alpha,\alpha)$ so that

$$f(\alpha,\alpha) + f(\alpha,\alpha) = 0$$

or

$$(1+1)f(\alpha,\alpha) = 0.$$

Since $1+1 \neq 0$ in R, this means that $f(\alpha,\alpha) = 0$ so that f is skew-symmetric.

Both Theorems 1 and 2 can be generalized to vector spaces over any field F of characteristic $\neq 2$. We define this concept as follows. For any positive integer n, denote by n1 the sum 1+...+1 (n terms). If $n1 \neq 0$ for any positive integer n, then F is said to have characteristic 0. If n1 = 0 for some positive integer, let p be the smallest positive integer such that p1 = 0. Then p is called the characteristic of F. The field {0, 1} has characteristic 2 and the field {0, 1, 2} has characteristic 3.

Returning to the concepts of symmetry and skew-symmetry, if x is any element of a field with characteristic 2, then $x + x = 0$ and $x = -x$. Thus the properties of symmetry and skew-symmetry in this case coincide.

The characteristic p of a field, incidentally, is a prime number. A characteristic can be defined for other algebraic systems as well (for example, rings and integral domains). A discussion of the characteristic of a ring and domain appears in Birkhoff and MacLane [2], Chap. XIII; another good reference is Nomizu [10], Chap. 5.

Theorem 3. Any bilinear form defined on V(R) can be represented uniquely as the sum of a symmetric bilinear form and a skew-symmetric bilinear form.

Proof. Exercise; hint: write the symmetric form as $f_s(\alpha,\beta) = 1/2[f(\alpha,\beta) + f(\beta,\alpha)]$ and the skew-symmetric form as

$$f_{ss}(\alpha,\beta) = 1/2[f(\alpha,\beta) - f(\beta,\alpha)].$$

Theorem 3 can also be stated as a theorem on matrices: if B is any square matrix, it can be uniquely represented as the sum of a symmetric and a skew-symmetric matrix.

Definition. Let $f(\alpha,\beta)$ be a bilinear form defined on $V(R)$ such that

(i) $f(\alpha,\beta)$ is symmetric,

(ii) $f(\alpha,\beta)$ is positive definite (i.e., $f(\alpha,\alpha) \geqq 0$ for every $\alpha \varepsilon V(R)$ and $f(\alpha,\alpha) = 0$ if and only if $\alpha = 0$); then $f(\alpha,\beta)$ is called an inner product function.

Definition. A vector space over R with an inner product function is called an Euclidean space. Integer subscripts on a symbol indicating a vector space will denote the dimension of the space; thus $V_4(R)$ denotes a four-dimensional vector space over the real s and E_n denotes an Euclidean space of dimension n.

Examples.

(1) Let $f(\alpha,\beta) = x_1y_2 - x_2y_1$ in E_2 where $\alpha = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\beta = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, then

$$f(\alpha,\beta) = [x_1 \quad x_2] \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

and f is skew-symmetric.

(2) Let $\alpha, \beta \epsilon E_n$, let E_n have the unit vectors U_i as basis vectors (i.e., U_i has 1 for its i^{th} coordinate and zero elsewhere), let α and β have, respectively, the coordinate vectors

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ and } \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} .$$

Then

$$f(\alpha,\beta) = f(\Sigma x_i U_i, \Sigma y_j U_j) = X^T I Y = \Sigma x_i y_i,$$

the conventional inner product of two vectors. Note that the matrix of the form in this case is the identity matrix.

(3) Given a linear programming problem and its dual,

$$\min\ z = C^T X \qquad\qquad \max\ g = U^T B$$

subject to $\qquad\qquad$ subject to

$$AX \geqq B \qquad\qquad U^T A \leqq C^T$$

$$X \geqq 0 \qquad\qquad U \geqq 0$$

we get the inequality (for all feasible vectors X and U)

$$U^T B \leqq U^T AX \leqq C^T X.$$

The function $f(U,V) = U^T AX$ is a bilinear form.

<u>Exercises</u>

1. Write the following bilinear forms as a matrix product:

(a) $g(\alpha,\beta) = 2x_1y_1 + 3x_1y_2 - 3x_2y_1 + x_2y_2$,

(b) $f(\alpha,\beta) = x_1y_1 + 2x_1y_2 - x_1y_3 - 5x_2y_1 + 7x_2y_2 + 4x_2y_3$,

(c) $h(\alpha,\beta) = x_1y_1 - x_1y_2 - 3x_1y_3 + 2x_2y_1 - x_2y_3$.

2. Construct an example of a symmetric and a skew-symmetric bilinear form defined on $V_4(R)$.

3. Given the bilinear form in E_3

$$f(\alpha,\beta) = x_1y_1 - x_3y_1 + 2x_2y_1 + x_1y_2 - x_3y_2 + x_2y_2 + x_1y_3 - 2x_2y_3 + x_3y_3,$$

write it as the sum of a symmetric and a skew-symmetric bilinear form.

4. Prove that if A is any p by n matrix, then A^TA and AA^T are symmetric.

5. Prove that if A is skew-symmetric, then A^2 is symmetric.

6. Let A and B be symmetric matrices of the same size. Is AB symmetric?

7. If p is the characteristic of a field F, prove that p is a prime number.

8. Give two examples of a positive definite bilinear form and two examples of a form that is not positive definite.

*9. Given a bilinear form $f(\alpha,\beta)$ defined on a vector space V(R) and a basis for the space, suppose the matrix B is associated with the form and basis. If the basis of the space is changed, then f will have a new matrix, say B', relative to the new basis. It can be shown that the rank of B is equal to the rank of B', and more generally, that the rank of the matrix of a bilinear form is invariant under change of basis. Thus one defines the rank of the form to be the rank of a matrix of the form relative to some basis.

Now prove the following theorem: let f be a bilinear form defined on a vector space V of dimension n over the complex field F. There exists a basis $\{\alpha_1, \cdots, \alpha_n\}$ of V such that:

(i) $f(\alpha_i, \alpha_j) = 0$ for $i \neq j$;

(ii) $f(\alpha_i, \alpha_i) = 1$ for $1 \leqq i \leqq r$, where r = rank of f;

(iii) $f(\alpha_j, \alpha_j) = 0$ for $r + 1 \leqq j \leqq n$;

the associated matrix is of the form

$$\begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix} .$$

References: Nering [9], Chap. 4; Nomizu [10], Chap. 4; Halmos [6], Chap. 1.

10. Define two different inner product functions in E_n and prove that the sum of two inner product functions is an inner product function.

11. Is the difference of two inner product functions in E_n an inner product function in E_n? Is any positive scalar multiple of an inner product function in E_n an inner product function? If the answer is yes, prove it; if not, produce a counterexample.

1.3. Length, Distance, Orthogonality. We have various notations for inner product functions:

$$f(\alpha, \beta) = (\alpha, \beta) = \alpha \cdot \beta = \alpha\beta,$$

and if we are considering vectors which are expressed in terms of a fixed basis $\{\alpha_1, \cdots, \alpha_n\}$, say $\alpha = \sum_{i=1}^{n} x_i \alpha_i$ and $\beta = \sum_{j=1}^{n} y_i \alpha_i$, then the vectors can be unambiguously denoted by the coordinate vectors X and Y so we also have in this case the notation

$$f(\alpha, \beta) = f(X,Y) = X^TY$$

and

$$f(\alpha, \alpha) = (\alpha, \alpha) = \alpha \cdot \alpha = \alpha^2 = f(X,X) = X^TX.$$

From this point on we will deal with coordinate vectors.

Definition. Let f(X,Y) be an inner product function defined on E_n; if X is any vector in E_n, then the length of X is

$$||X|| = \sqrt{f(X,X)} = \sqrt{X^TX} .$$

Note that since $f(X,X) \geq 0$ this definition does not take us out of R, and also that for any scalar

$$||kX|| = \sqrt{f(kX,kX)} = \sqrt{k^2 f(X,X)} = |k|\sqrt{X^TX} = |k| \; ||X|| .$$

Definition. Let X be any nonzero vector in E_n; then the vector

$$X^* = \frac{1}{||X||}X$$

is called the normalized vector associated with X (note that $||X^*|| = 1$).

Definition. Let X and Y be any vectors in E_n and let d(X,Y) be a function defined on pairs of vectors X,Y such that

$$d(X,Y) = ||Y-X||.$$

Then d(X,Y) is a distance function in E_n and for a given pair of vectors X,Y we call d(X,Y) the distance from X to Y.

Various notations for a distance function are available; for example,

$$d(X,Y) = \sqrt{f(Y-X,\ Y-X)} = \sqrt{(Y-X)^T (Y-X)}\ .$$

Lemma. If X and Y are any vectors in E_n having length 1, then $|X^T Y| = ||X||\ ||Y||$.

Proof. Consider the vector X - Y; we have for an inner product function

$$\begin{aligned} f(X-Y,\ X-Y) &= f(X,X) - f(X,Y) - f(Y,X) + f(Y,Y) \\ &= ||X||^2 - 2X^T Y + ||Y||^2. \end{aligned}$$

Since $||X|| = ||Y|| = 1$, we have from the positive definiteness of f

$$\begin{aligned} 2 - 2X^T Y &\geqq 0, \\ X^T Y &\leqq 1\ . \end{aligned}$$

Similarly, from an examination of f(X+Y, X+Y) we get

$$- X^T Y \leqq 1$$

so we can write

$$|X^T Y| \leqq 1.$$

Theorem 4 (Schwartz inequality). If X and Y are any vectors in E_n, then

$$|X^T Y| = ||X||\ ||Y||.$$

Corollary (the triangle inequality). If X and Y are any vectors in E_n, then

$$||X+Y|| \leqq ||X|| + ||Y|| .$$

Theorem 5. If $d(X,Y)$ is any distance function in E_n, then

(i) $d(X,Y) = d(Y,X)$;

(ii) $d(X,Y) \geqq 0$ and $d(X,Y) = 0$ if and only if $X = Y$;

(iii) $d(X,Y) \leqq d(X,Z) + d(Z,Y)$.

Definition. The vectors X and Y in E_n are said to be orthogonal if $X^TY = 0$.

The zero vector in E_n is thus orthogonal to every vector in E_n.

Definition. Let $\{X_1, \cdots, X_n\}$ be a set of vectors in E_n; if $X_i^TX_j = 0$ for $i, j = 1, \cdots, n, i \neq j$, then the set is orthogonal. A set is said to be orthonormal if it is orthogonal and every vector in the set has length 1.

Theorem 6. Any orthogonal set of nonzero vectors in E_n is linearly independent.

We now introduce the concept of an orthogonal projection of a vector on another vector by examining some geometrical concepts in E_2. Consider two vectors in E_2 as in Figure 1. We

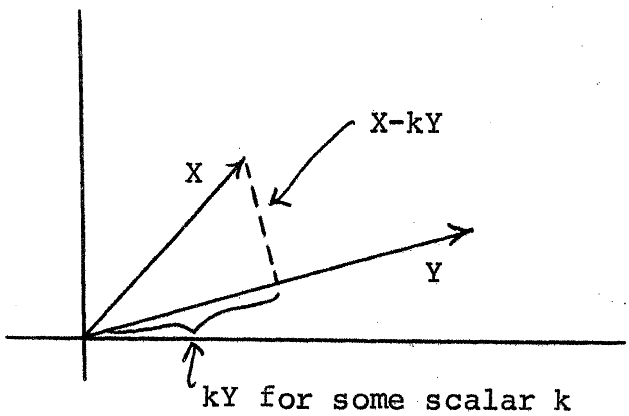


Figure 1

want to call the vector kY (k not yet specified) the orthogonal projection of X on Y. What is the value of the scalar k? We first observe that if a scalar k exists such that X-kY and Y are orthogonal, then

$$(X-kY)^T Y = 0$$

from which we easily obtain

$$X^T Y - kY^T Y = 0$$

and

$$k = \frac{X^T Y}{Y^T Y} = \frac{X^T Y}{||Y||^2} .$$

Conversely, if k is defined as above, then it is easy to show that X-kY and Y are orthogonal.

With these geometrical concepts in mind we introduce the following

Definition. If X and Y are any nonzero vectors in E_n, then the orthogonal projection of X on Y is the vector

$$\frac{X^T Y}{||Y||^2} Y .$$

Suppose we return to E_2 and that {X,Y} is a basis for the space as shown in Figure 2. Can we determine an orthonormal basis for E_2? The answer is clearly yes since we can make use of the orthogonal projection of X on Y by setting

$$Z_1 = Y$$

$$Z_2 = X - \frac{X^T Y}{||Y||^2} Y .$$

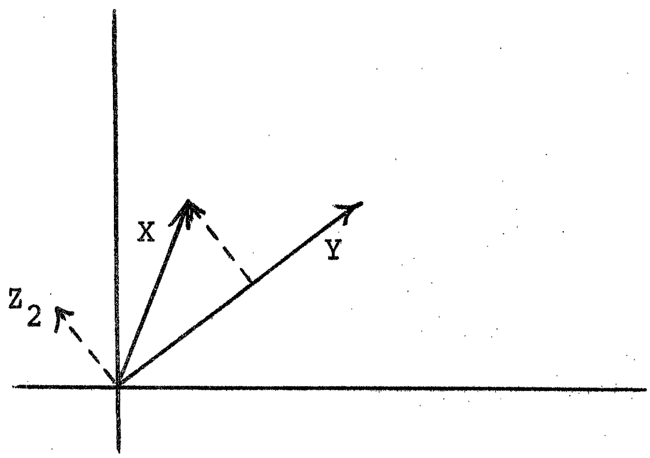


Figure 2

The set $\{Z_1, Z_2\}$ is orthogonal and since from the geometry they are each nonzero, they can each be normalized and the result is what we seek.

These ideas are generalized in the following result, which is referred to as the Gram-Schmidt orthogonalization process.

Theorem 7. Every finite dimensional Euclidean vector space contains an orthonormal basis.

Sketch of the proof. Every finite dimensional vector space contains a basis (from linear algebra -- see, for example, Nering [9], Chap. 1). Suppose this basis is the set $\{X_1, \cdots, X_n\}$. Let $Z_1 = X_1$,

$$Z_2 = X_2 - \frac{X_2^T X_1}{||X_1||^2} X_1$$

and

$$Z_k = X_k - \sum_{j=1}^{k-1} \frac{X_k^T Z_j}{||Z_j||^2} Z_j \ , \quad k = 3, \cdots, n.$$

Perform an induction to get an orthonormal set and normalize the vectors in the resulting set.

It is oftentimes helpful to use the concept of angle between vectors in E_n. Angle is defined in terms of its cosine and is a straightforward generalization of familiar concepts from E_2.

Definition. If X and Y are any two nonzero vectors in E_n, then the (unique) angle θ, $0 \leqq \theta \leqq \pi$, such that

$$\cos \theta = \frac{X^T Y}{||X|| \; ||Y||}$$

is called the angle between X and Y.

From the Schwartz inequality we have

$$\cos \theta = \left| \frac{X^T Y}{||X|| \; ||Y||} \right| \leqq 1$$

so the angle θ so defined retains features that are associated with angle in E_2.

We conclude this section with several useful theorems relating to orthogonality.

Definition. Let X be any vector in E_n and $A = \{X_1, \cdots, X_n\}$ be any set of vectors in E_n. If X is orthogonal to every vector in A we say that X is orthogonal to the set A. Moreover, if $A = \{X_1, \cdots, X_n\}$ and $B = \{Y_1, \cdots, Y_n\}$, then we say the sets A and B are orthogonal if for each $X_i \varepsilon A$ and $Y_i \varepsilon B$ we have $X_i^T Y_i = 0$.

Theorem 8. If X and $A = \{X_1, \cdots, X_n\}$ is orthogonal to the set $A = \{X_1, \cdots, X_n\}$, then X is orthogonal to the space spanned by A.

<u>Theorem 9</u>. If $A = \{X_1, \cdots, X_n\}$ is a set of vectors from E_n, then the set of all vectors orthogonal to A is a subspace of E_n.

<u>Definition</u>. Let S be any set of vectors from E_n; then the set of all vectors orthogonal to S, denoted $S^{\perp}$, is called the <u>orthogonal complement</u> of S in E_n.

<u>Exercises</u>.

1. Prove Theorem 4 using the preceding lemma.
2. Prove that if x and y are any nonnegative real numbers and if $x^2 \leqq y^2$, then $x \leqq y$.
3. Use Theorem 4 and the preceding exercise to prove the corollary to Theorem 4. Prove that if the Schwartz inequality reduces to an equality for vectors $X, Y \in E_n$, then $\{X,Y\}$ is linearly dependent.
4. Prove Theorem 5; in what case does (iii) of Theorem 5 hold as an equality?
5. Prove Theorem 6.
6. Given $X = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, $Y = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$, $Z = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$,

   calculate the following.

   (a) X^TX; X^TY; Y^TZ; Y^*; Z^*; $Y^{*T}X^*$; the orthogonal projection of X on Y and of X on Z.

   (b) Calculate the orthogonal projection of Z on X.

   (c) Construct a basis for E_3 that is not orthogonal and from this basis construct an orthonormal basis.

(d) Verify that $X^TY = Y^TX$; why do we have this symmetry?

7. Prove Theorem 7.

8. Prove Theorem 8.

9. Prove Theorem 9.

10. Apply the last definition in this section to $S^{\perp}$ and develop the set $(S^{\perp})^{\perp}$. Verify that $S\subset(S^{\perp})^{\perp} = S^{\perp\perp}$.

1.4. Orthogonality and Double Description of Vector Subspaces. We want to show how a vector space can be defined either in terms of a spanning set or as the solution space of a system of homogeneous linear equations. When both definitions can be specifically applied to a vector space we say that we have a double description for the space.

Theorem 10. If $R = \{X_1,\cdots,X_p\}$ is an orthonormal set in E_n and if Y is any vector in E_n, then the vector

$$Z = Y - \sum_{i=1}^{p} (Y^TX_i)X_i$$

is orthogonal to R (and hence to the space spanned by R).

Proof. Consider

$$Z^TX_j = [Y - \sum_{i=1}^{p} (Y^TX_i)X_i]^TX_j = Y^TX_j - \sum_{i=1}^{p} (Y^TX_i)X_i^TX_j.$$

Since R is orthonormal we have $X_i^TX_j = 0$ for $i\neq j$ and $X_i^TX_j = 1$ for $i=j$ so that

$$Z^TX_j = Y^TX_j - Y^TX_j = 0$$

hence Y is orthogonal to R.

<u>Theorem 11</u>. Let S be a subspace of E_n having dimension $m < n$ and let $X \varepsilon E_n$. Then X can be expressed as the sum Y_1+Y_2, where $Y_1 \varepsilon S$ and $Y_2 \varepsilon S^{\perp}$.

<u>Proof</u>. S has an orthonormal basis, say $\{X_1, \cdots, X_m\}$. If $X \varepsilon S$, the theorem is obvious. Suppose $X \not\varepsilon S$; let

$$X = Y_1 + Y_2$$

where

$$Y_1 = \sum_{i=1}^{m} (X^T X_i) X_i$$

Now $Y_1 \varepsilon S$ and $Y_2 = X - Y_1 = X - \sum_{i=1}^{p} (X^T X_i) X_i$. By Theorem 10, Y_2 is orthogonal to the set $\{X_1, \cdots, X_m\}$ and hence Y_2 is orthogonal to S.

<u>Theorem 12</u>. The decomposition $X = Y_1 + Y_2$ in Theorem 11 is unique.

<u>Theorem 13</u>. If S is any subspace of E_n, then

(i) $S \cap S^{\perp} = \{0\}$;

(ii) $E_n = S \oplus S^{\perp}$;

(iii) $S^{\perp\perp} = S$.

<u>Proof</u>. (i) If $X \varepsilon S \cap S^{\perp}$, then $X \varepsilon S$ and $X \varepsilon S^{\perp}$. This implies $X^T X = 0$ and $X = 0$.

(ii) The symbol $\oplus$ denotes the direct sum of the vector subspaces indicated. Let $A = \{X_1, \cdots, X_k\}$ be an orthonormal basis for S and let Y be any vector in E_n. Then

$$Z = Y - \sum_{i=1}^{k} (Y^T X_i) X_i$$

is orthogonal to A and hence to S. Thus $Z \varepsilon S^{\perp}$. Also,

$$Y = Z + \sum_{i=1}^{k} (Y^T X_i) X_i$$

is the sum of a vector $Z \varepsilon S^{\perp}$ and $\Sigma (Y^T X_i) X_i \varepsilon S$. Since $S \cap S^{\perp} = \{0\}$, we have expressed Y as the (unique) sum of a vector in S and a vector in $S^{\perp}$.

We now make use of results on sums and direct sums of vector subspaces [see Nering [9], p. 21-22; Halmos [6], p. 28-30]; in particular the theorem that if U and V are any subspaces of a vector space W, the following conditions are equivalent: (a) $W = U \oplus V$; (b) $U \cap V = \{0\}$ and $U + V = W$ (i.e., U and V are complementary subspaces); (c) any vector $Y \varepsilon W$ can be written in the form $X_1 + X_2$, where $X_1 \varepsilon U$ and $X_2 \varepsilon V$, in one and only one way.

We have shown that $Z \varepsilon E_n$ can be expressed uniquely as the sum of a vector in S and a vector in $S^{\perp}$. Since (c) implies (a) we have $S \oplus S^{\perp} = E_n$.

(iii) Consider the vector $Y = Z + \Sigma (Y^T X_i) X_i$ where $Z \varepsilon S^{\perp}$ and $\Sigma (Y^T X_i) X_i \varepsilon S$; we have

$$Z^T Y = Z^T Z + Z^T [\Sigma (Y^T X_i) X_i] = Z^T Z = ||Z||^2 .$$

Thus if $Y \varepsilon S^{\perp\perp}$, then $Z^T Y = 0$ implies $||Z||^2 = 0$ which means that $Y = \Sigma (Y^T X_i) X_i \varepsilon S$. This also implies that $S^{\perp\perp} \subset S$ and since we observed earlier that $S \subset S^{\perp\perp}$, we conclude that $S = S^{\perp\perp}$.

Finally, it should be clear that if S is a subspace of E_n with dimension r, then $S^{\perp}$ has dimension n-r. We can now establish the double description theorem for vector subspaces:

<u>Theorem 14</u>. Every r-dimensional subspace of E_n is the solution space of a system of n-r independent homogeneous linear equations in the variables $x_1, \cdots, x_n$.

<u>Proof</u>. Let S be any subspace of E_n having dimension r and let T be the orthogonal complement of S. T has dimension n-r; suppose we have a basis for T,

$$\begin{aligned} T_1 &= (b_{11}, b_{12}, \cdots, b_{1n}) , \\ &\vdots \\ T_{n-r} &= (b_{n-r,1}, \cdots, b_{n-r,n}) . \end{aligned}$$

Since T is the orthogonal complement of S, we know that S is the orthogonal complement of T and the set of all vectors orthogonal to S is T. Since the set of all vectors orthogonal to T is the set of all vectors orthogonal to a basis of T, the set of all vectors X satisfying

$$\begin{aligned} T_1X &= 0 \\ T_2X &= 0 \\ &\vdots \\ T_{n-r}X &= 0 \end{aligned}$$

is S. These equations can be written more fully as

$$\begin{aligned} b_{11}x_1 + \cdots + b_{1n}x_n &= 0 \\ \vdots \qquad\qquad & \\ b_{n-r,1}x_1 + \cdots + b_{n-r,n}x_n &= 0 \end{aligned}$$

and these are independent since the coefficient vectors are a basis for T.

Thus every subspace of E_n can be described in two ways: by providing a basis for the space or by providing a system of homogeneous linear equations whose solution set is the subspace.

Figure 3 displays a schematic diagram which illustrates the situation in detail.

Exercises

1. Let $\{X_1, \cdots, X_p\}$ be an orthonormal set in E_n where $p \leqq n$. Prove that if Y is any vector in E_n, then

$$\sum_{i=1}^{p} |Y^T X_i|^2 \leqq ||Y||^2 .$$

2. Given the vectors

$$X_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0 \\ 5 \\ 4 \end{bmatrix},$$

verify that $\{X_1, X_2, X_3\}$ is a basis for E_3 and then use the Gram-Schmidt orthogonalization process to construct an orthonormal basis for E_3.

3. Let S be a subspace of E_4 spanned by the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 1 \\ 3 \\ 0 \end{bmatrix};$$

describe $S^{\perp}$ in two ways.

4. Find a basis for the solution space of the linear equation

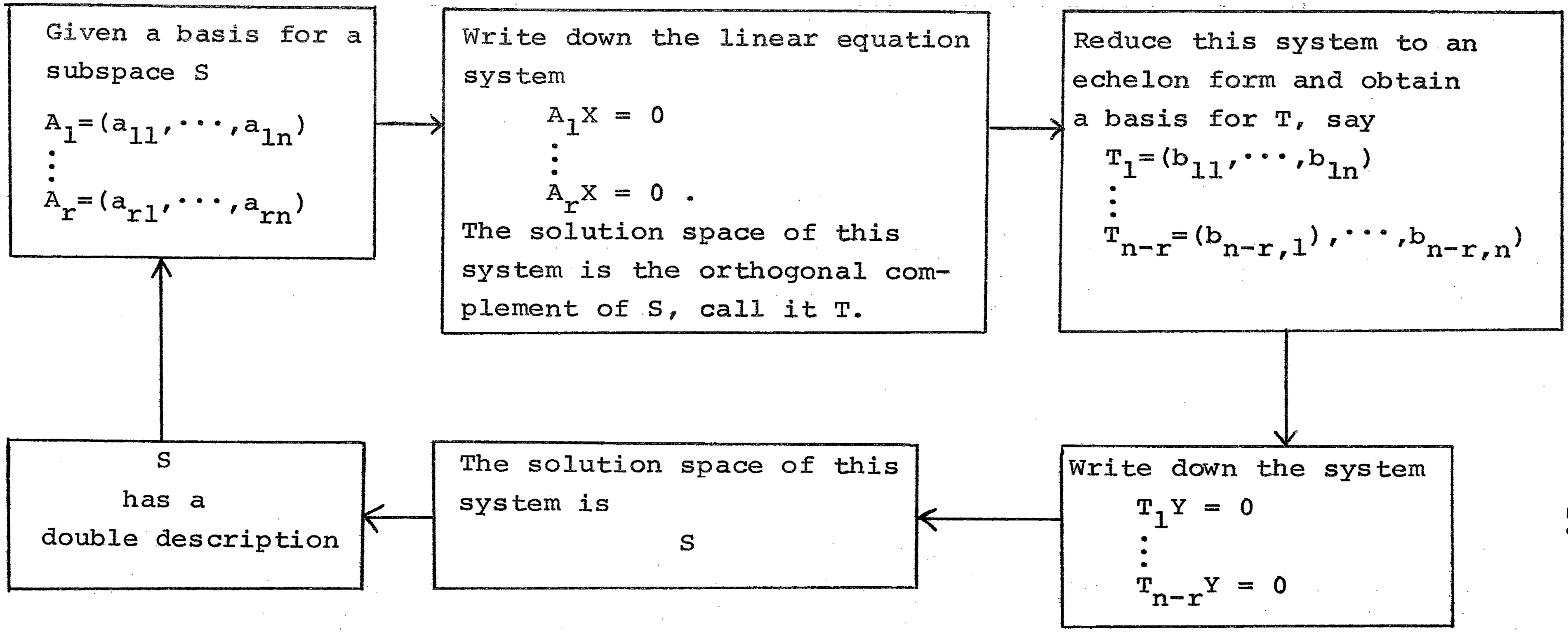


Figure 3. Double Description for Vector Subspaces.

system

$$\begin{aligned} x_1 + 2x_2 - x_3 - x_4 &= 0 \\ 3x_1 - x_2 + x_3 + 4x_4 &= 0. \end{aligned}$$

Also find a basis for the orthogonal complement of this space.

5. Suppose S is a vector space spanned by the vectors

$$\begin{bmatrix} 2 \\ 1 \\ 3 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 5 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -2 \\ 3 \\ 1 \\ 4 \end{bmatrix};$$

find a system of homogeneous linear equations whose solution set is S.

6. Prove Theorem 12.

7. Let W be the subspace of E_5 spanned by the set

$$X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad X_4 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix},$$

$$X_5 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

Find a basis for W and find a system of equations whose solution set is W.

1.5. Linear Manifolds. We will take the point of view that a linear manifold is a "displaced" vector subspace. This will enable us to utilize directly the previous work on subspaces.

<u>Definition</u>. A linear manifold is a set

$$L = X + S,$$

where X is a fixed vector in E_n and S is a subspace of E_n (note that the symbol "+" means the algebraic sum of the set consisting of the vector X and the set S).

An example from E_2 displays (Figure 4) the displacement aspect of this definition; the fixed vector X being a key element in the displacement.

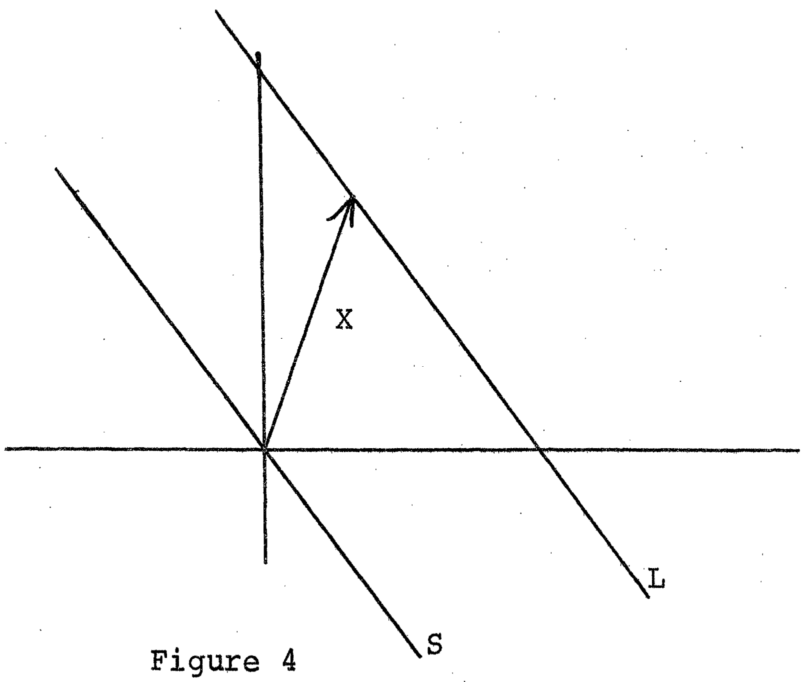


Figure 4

<u>Definition</u>. The manifold L is said to have dimension r if S has dimension r.

<u>Definition</u>. The linear manifolds $L_1 = X_1 + S_1$ and $L_2 = X_2 + S_2$ are said to be parallel if and only if either

$S_1 \subset S_2$ or $S_2 \subset S_1$. If S_1 and S_2 have the same dimension, they are parallel if and only if $S_1 = S_2$.

Now suppose that $L = X + S$ is a linear manifold and that $\{Y_1, \cdots, Y_r\}$ is a basis for S. Then every vector $Y \varepsilon L$ can be written in the form

$$Y = X + t_1 Y_1 + \cdots + t_r Y_r ,$$

$t_i \varepsilon R$. The expression above is sometimes called a parametric representation for L since X and a basis for S can be chosen in many ways.

Definition. Let $X_0, X_1, \cdots, X_p$ be any vectors in E_n; the smallest linear manifold containing these vectors is the linear manifold containing them which has smallest dimension.

Given any points or vectors $X_0, X_1, \cdots, X_p$ in E_n, we know that they are contained in some manifold -- the space E_n. The question is, how can we determine the smallest manifold containing these points?

Suppose we begin with the vectors $X_0, X_1, \cdots, X_p$; then we can define a manifold containing them as

$$L = X + S$$

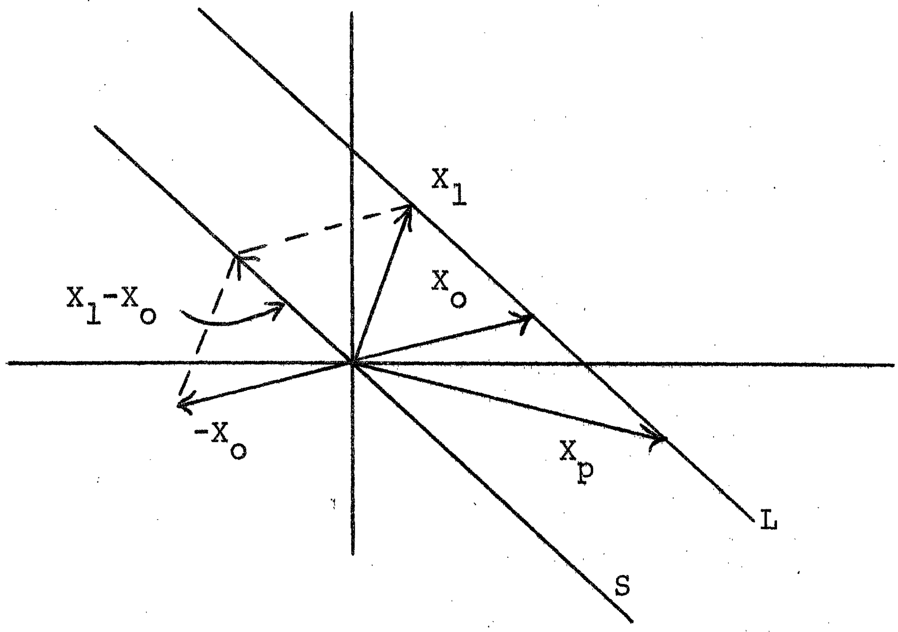
for some fixed $X \in L$ and some subspace S. Now X can be any vector whatever in L, so we choose $X = X_0$. Then

$$L = X_0 + S$$

and it is clear that

$$L + (-X_0) = S$$

so that S contains the vectors $X_i - X_0$ for $i=1, \cdots, p$.



On the other hand, if S contains $\{X_1 - X_0, \cdots, X_p - X_0\}$, then since

$$L = X_0 + S$$

L must contain $X_0 + (X_1-X_0) = X_1, \cdots, X_0 + (X_p-X_0) = X_p$. Finally, the span (see Nering [9], p. 20) of the vectors $X_1-X_0, \cdots, X_p-X_0$ is the smallest subspace containing these vectors. If we use the notation $< >$ to denote the span of a set of vectors, we can write the smallest manifold containing the vectors $X_0, X_1, \cdots, X_p$ as

$$L = X_0 + < X_1-X_0, \cdots, X_p-X_0 >.$$

When we refer to "the manifold containing a given set" we will understand that we mean the smallest manifold containing the set.

If $X_0, X_1, \cdots, X_r$ is any set of vectors in E_n, then the set $\{X_1-X_0, \cdots, X_p-X_0\}$ can be either dependent or independent. If this set is independent then L has dimension p, otherwise L has dimension $k<p$.

An arbitrary vector X is in a linear manifold L if and only if X can be written

$$X = X_0 + t_1(X_1-X_0) + \cdots + t_r(X_r-X_0)$$

for $t_i \varepsilon R$ $(i=1, \cdots, r)$ and $X_0, X_1, \cdots, X_r$ in L. Then

$$\begin{aligned} X &= X_0 + t_1X_1-t_1X_0+\cdots+t_rX_r-t_rX_0 \\ &= (X_0-t_1X_0-\cdots-t_rX_0)+t_1X_1+\cdots+t_rX_r \\ &= (1-t_1-\cdots-t_r)X_0+t_1X_1+\cdots+t_rX_r \ . \end{aligned}$$

Now let $t_0 = 1 - \sum_{i=1}^{r} t_i$ and then $\sum_{i=0}^{r} t_i=1$ and we can write

$$X = t_0X_0 + t_1X_1 + \cdots + t_rX_r \ ,$$

$$\sum_{i=0}^{r} t_i = 1 \ ,$$

$$t_i \varepsilon R \ .$$

Thus we see that the condition $\sum_{i=0}^{r} t_i = 1$ is a necessary and sufficient condition for $X \varepsilon L$.

<u>Definition</u>. If

$$X = k_1X_1 + \cdots + k_pX_p, \ k_i \varepsilon R, \ \sum_{i=1}^{p} k_i = 1, \ X_i \varepsilon E_n$$

then X is called a <u>weighted average</u> of the vectors $X_1, \cdots, X_p$.

Definition. If Y is any vector in a linear manifold L where

$$Y = t_0X_0 + t_1X_1 + \cdots + t_rX_r$$

$$\sum_{i=0}^{r} t_i = 1, \quad X_0, \cdots, X_r \varepsilon L ,$$

then L is said to be spanned by $X_0, X_1, \cdots, X_r$.

We observe that the set of all linear combinations of a set of vectors (unrestricted combinations) is a vector subspace, whereas the set of all weighted averages (restricted combinations) of a set of vectors is a linear manifold.

Theorem 15. (Double description theorem for linear manifolds). If L is a manifold of dimension r in E_n spanned by the vectors $X_0, X_1, \cdots, X_r$, then L is the solution set of a system of linear equations AX = B. Conversely, let AX = B be a solvable system of linear equations, where A has rank n-r; then the solution set of this system is a linear manifold of dimension r.

Definition.

A point in E_n is a linear manifold of dimension 0;

A line in E_n is a linear manifold of dimension 1;

A plane in E_n is a linear manifold of dimension 2;

A hyperplane in E_n is a linear manifold of dimension n-1.

As a special case of Theorem 15 we see that a hyperplane in E_n can be described as the solution set of a linear equation

$$a_1x_1 + \cdots + a_nx_n = b \quad (\text{not all } a_i = 0).$$

Definition. Let $X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ be vectors in E_n.

(i) $X > Y$ if and only if $x_i > y_i, \forall i$.

(ii) $X \geqq Y$ if and only if $x_i \geqq y_i, \forall i$.

(iii) $X \geq Y$ if and only if $X \geqq Y$ and $X \neq Y$.

(iv) X is strictly positive if $X > 0$.

(v) X is positive if $X \geq 0$.

(vi) If $X \geq 0$ and $\sum_{i=1}^{n} x_i = 1$, then X is a probability vector.

(vii) If $X \geq 0$ and $k = \frac{1}{\sum_{i=1}^{n} x_i}$, then $X^* = kX$ is a probability vector and $X = \frac{1}{k}X^* = \sum_{i=1}^{n} x_i X^*$.

Exercises

1. Prove Theorem 15. Hint: use Theorem 14 on the vector subspace S.

2. Given the system of linear equations defined by

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 4 \\ 1 & 1 & 0 & 3 & 9 \\ 2 & 0 & 1 & 5 & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_5 \end{bmatrix} = \begin{bmatrix} 7 \\ 15 \\ 23 \end{bmatrix}.$$

(a) Express the linear manifold of solution vectors X as the sum of a vector in the manifold and the span of basis vectors for some subspace S.

(b) Describe the subspace S in two ways.

(c) Describe the subspace $S^{\perp}$ in two ways.

3. For each of the following linear manifolds write down a parametric representation and a determining set of linear

equations.

(a) $L_1 = (3, 1, 1) + \langle(-2, 2, 1), (1, 2, -1)\rangle$

(b) $L_2 = (1, 0, 2) + \langle(2, 3, 0), (0, 1, 4)\rangle$

4. Prove that a vector X will be contained in a linear manifold L if and only if X can be expressed as

$$X = X_0 + t_1(X_1-X_0) + \cdots + t_r(X_r-X_0).$$

5. In E_3 find the smallest linear manifold L containing the vectors

$$\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}.$$

1.6. Hyperplanes and Orthogonality. Consider the hyperplane defined by the equation $2x_1 + 7x_2 = 0$. This can be written

$$C^T X = [2 \quad 7] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 .$$

If we graph the situation as in Figure 5 we see that the solution in set S of this equation consists of all vectors X orthogonal to the fixed vector C.

What is the situation for the equation $2x_1 + 7x_2 = 19$? Each vector $X \varepsilon H = \{X | C^T X = 19\}$ has the same orthogonal projection on the fixed vector C. This is illustrated in Figure 6.

From an earlier section the orthogonal projection of any vector $X \varepsilon H$ is given by

$$\frac{C^T X}{||C||^2} C .$$

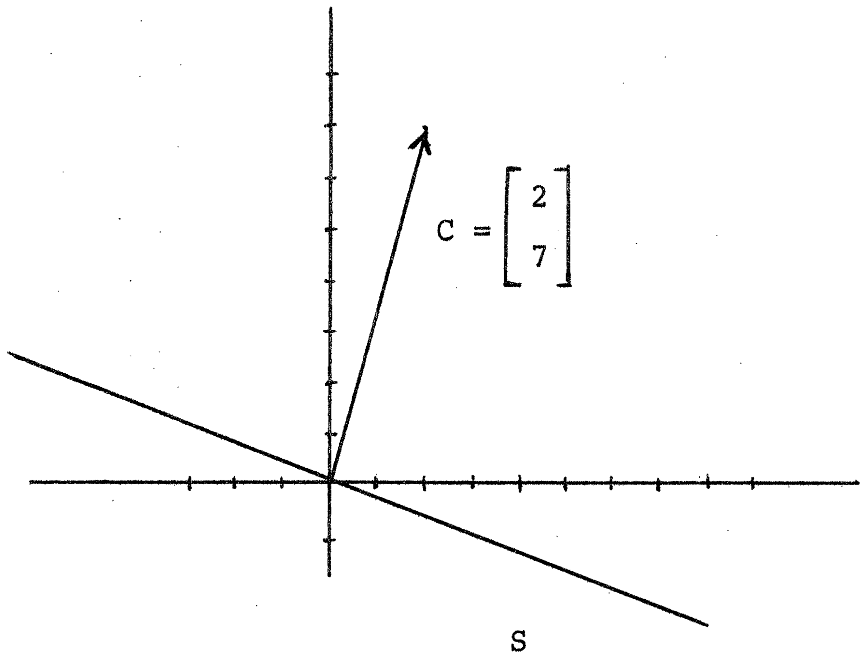


Figure 5

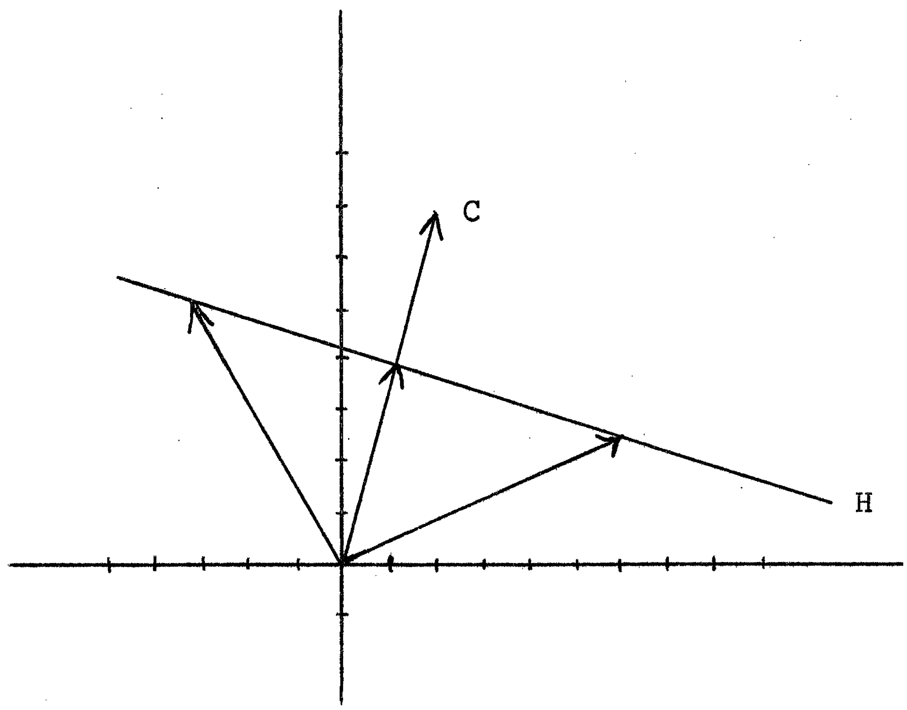


Figure 6

The length of this vector is

$$\left|\left| \frac{C^T X}{||C||^2} C \right|\right| = \frac{|C^T X|}{||C||^2} ||C|| = \frac{|C^T X|}{||C||} .$$

These comments are easily generalized; if H is a hyperplane in E_n defined by $C^T X = z$, then the length of the orthogonal projection is

$$\frac{|C^T X|}{||C||} = \frac{|z|}{||C||}$$

and this length is sometimes referred to as the normal distance of the hyperplane H from the origin. The fixed vector C is called a normal vector to H and it is clear that given a normal vector one can then unambiguously specify a hyperplane. In some of the discussions below it will be natural to speak of normals or normal vectors when we wish to refer to corresponding hyperplanes. Indeed, we will encounter situations in which we will locate normal vectors in certain sets so that corresponding hyperplanes will be located in desired ways.

Definition. If $H = \{X | C^T X = z,\ C \neq 0,\ X \epsilon E_n\}$, then the vectors

$$\pm \frac{1}{||C||} C$$

are called unit normals to the hyperplane H.

The geometrical concept of "moving a hyperplane parallel to itself" occurs often in optimization mathematics. Figure 7 illustrates the situation; $H_0 = \{X | C^T X = z_0\}$ and

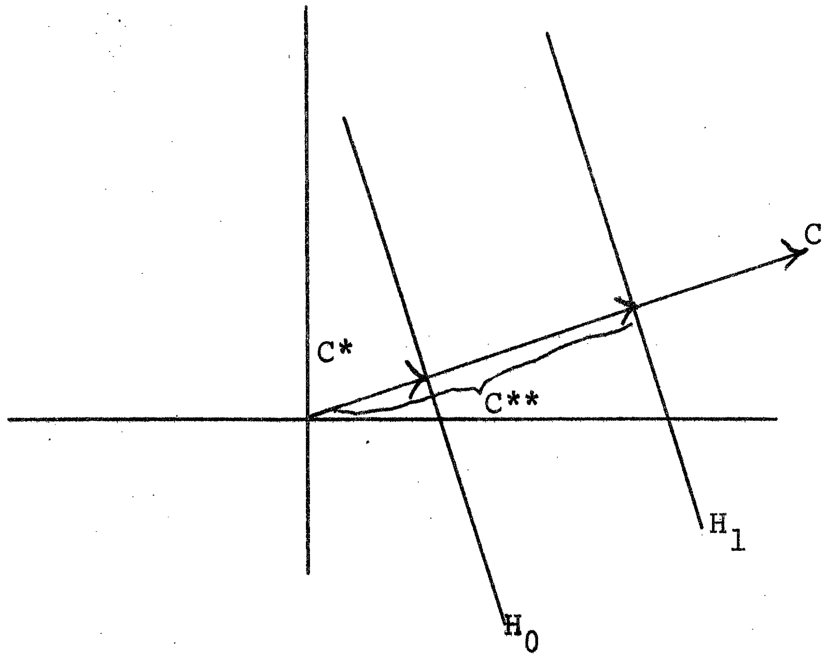


Figure 7

$H_1 = \{X|C^TX = z_1\}$ so that C* is the orthogonal projection of $X\varepsilon H_0$ on C and C** is the orthogonal projection of $X\varepsilon H_1$ on C. Since $C^{**}\varepsilon H_1$, we have $C^TC^{**} = z_1$ and for some scalar k

$$C^TC^{**} = C^T(C^* + kC) = C^TC^{**} + kC^TC$$
$$= z_0 + k||C||^2 = z_1$$

Then

$$k||C||^2 = z_1 - z_0;$$

since $||C||^2 > 0$, $k > 0$ implies that $z_1 - z_0 > 0$ or that $z_1 > z_0$. Thus when a hyperplane is "moved up" the value of $z_1 > z_0$. Likewise, $k < 0$ would mean that H_1 was "moved down" and $z_1 < z_0$.

Exercises

1. Given the equation $2x_1 - x_2 + 7x_3 = 13$, find the following.
   (a) The unit normal vectors to the hyperplane.
   (b) If X is any vector in the hyperplane defined by the

equation above, what is the orthogonal projection of X on the normal vector to the hyperplane?

(c) Write X in terms of a parametric representation in which the basis for an appropriate subspace appears in the parametric representation.

1.7. Convexity. Suppose the vectors X_1 and X_2 in E_n span a line L. Then if $Y \in L$ we know that

$$Y = t_1X_1 + t_2X_2, \; t_1 + t_2 = 1,$$

or

$$Y = tX_1 + (1-t)X_2, \; t \in R.$$

If $t = 0$ we have $Y = X_2$ and if $t = 1$, $Y = X_1$. We say that Y is between X_1 and X_2 if and only if $0 < t < 1$. The line segment joining X_1 and X_2 can be written in the form

$$Y = tX_1 + (1-t)X_2, \quad 0 \leqq t \leqq 1.$$

Definition. A subset C of E_n is convex if, whenever two points X_1 and X_2 are in C, the line segment joining X_1 and X_2 is also in C. I.e., C is convex if, for any X_1 and X_2 in C, the vector $Y \in C$ where

$$Y = t_1X_1 + t_2X_2, \; t_1 + t_2 = 1, \; t_1 \geqq 0, \; t_2 \geqq 0.$$

Definition. Let $X_1, \cdots, X_p$ be any vectors in E_n, the vector

$$Y = a_1X_1 + \cdots + a_pX_p, \; \sum_{i=1}^{p} a_i = 1, \; a_i \geqq 0, \; \forall_i,$$

is called a convex combination of the vectors $X_1, \cdots, X_p$.

A set consisting of exactly one point and the null set are regarded as convex since in these cases the definition of convexity is satisfied "vacuously."

Theorem 16. The intersection of any collection of convex sets is convex.

The definition of convex set is geometrically appealing but it can be difficult to apply directly if we have a set and wish to determine if it is convex. The following theorem is often useful.

Theorem 17. A set C is convex if and only if every convex combination of vectors in C is in C.

Proof. If every convex combination of vectors in C is in C, then every line segment joining a pair of vectors in C must be in C, so C is convex.

Suppose C is convex; we want to show that every convex combination of vectors in C is in C. Let

$$Y = \sum_{i=1}^{r} t_i X_i, \quad \sum_{i=1}^{r} t_i = 1, t_i \geq 0$$

be a convex combination in C. For $r = 1$ and $r = 2$ we have $Y \varepsilon C$; assume that a convex combination of fewer than r vectors of C is in C. The following is an identity

$$t_1 X_1 + \cdots + t_{r-1} X_{r-1} = (1-t_r)\left[\frac{t_i}{\sum_{i=1}^{r-1} t_i} X_1 + \cdots + \frac{t_{r-1}}{\sum_{i=1}^{r-1} t_i} X_{r-1}\right]$$

since $\sum_{j=1}^{r} t_j = 1$ implies $\sum_{j=1}^{r-1} t_j = 1 - t_r$. We then use this to write

$$Y = (1 - t_r)\left[\frac{t_1}{\sum_{i=1}^{r-1} t_i} X_1 + \cdots + \frac{t_{r-1}}{\sum_{i=1}^{r-1} t_i} X_{r-1}\right] + t_r X_r .$$

Without loss of generality we can assume that $t_r \neq 1$, otherwise we have $Y = X_r$. Now for $t_r \neq 1$ each coefficient of each vector in the sum in the brackets immediately above has the property that

$$\frac{t_j}{\sum_{i=1}^{r-1} t_i} \geqq 0 \quad (j=1, \cdots, r-1)$$

and

$$\sum_{j=1}^{r-1} \frac{t_j}{\sum_{i=1}^{r-1} t_i} = \frac{\sum_{j=1}^{r-1} t_j}{\sum_{i=1}^{r-1} t_i} = 1.$$

This means that the linear combination of vectors within the brackets above is a convex combination. By our induction hypothesis this convex combination is a vector in C. Then the equation above gives a convex combination of two vectors in C, so $Y \varepsilon C$ and the theorem is proved.

Definition. Let S be any subset of E_n; then the convex hull of S is the smallest convex set containing S. It is denoted H(S).

Since S is a subset of E_n and E_n is convex there is a convex set containing S. The intersection of all convex sets containing S is convex by Theorem 16, so such a smallest convex set exists.

Theorem 18. The convex hull of a set S is the set of all convex combinations of the vectors of S.

Proof. See Nering [9], Chap. 6.

Definition. If C is a nonempty set of vectors in E_n, then C is a cone if it is closed under multiplication by nonnegative scalars.

Definition. If C is a cone in E_n and is also closed under vector addition, then C is a convex set and is called a convex cone.

If C is a convex cone and if there is a set of vectors $\{X_1, \cdots, X_p\}$ of C such that every vector in C can be represented as a nonnegative linear combination of the vectors $X_1, \cdots, X_p$, then the set is called a spanning set (or a set of generators) for C and C is also called a finite cone (or a finitely generated cone).

The cone generated or spanned by a single nonzero vector is called a half line, and the convex hull of a finite set of vectors is called a convex polyhedron.

Definition. If S is a convex set, an extreme point of S is a point $X \epsilon S$ such that there are no other points $X_1, X_2 \epsilon S$, $X_1 \neq X_2$, such that $X = kX_1 + (1-k)X_2$, $0 < k < 1$.

Definition. Let f(X) be a function defined on a convex set C (i.e., the domain of f is the set C); then f is said to be a convex function if for any points X_1, $X_2 \varepsilon C$

$$f[kX_1+(1-k)X_2] \leqq kf(X_1) + (1-k)f(X_2) , \quad 0 \leqq k \leqq 1 .$$

Definition. Let f(X) be defined over a convex set S; then f(X) is a concave function if for any X_1, $X_2 \varepsilon S$,

$$f[kX_1+(1-k)X_2] \geqq kf(X_1) + (1-k)f(X_2) , \quad 0 \leqq k \leqq 1 .$$

Note: if f(X) is concave, then -f(X) is convex and conversely.

Theorem 19. If f(X) is both convex and concave over a convex set S, then f(X) is a linear function over S.

Exercises

1. Let

$$X_1 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 6 \\ 7 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 1 \\ 7 \end{bmatrix}, \quad X_4 = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$$

(a) What is the convex hull of the set $\{X_1, X_2, X_3, X_4\}$?

(b) Graph the set 1(a); what is the cone spanned by X_1 and X_3?

(c) Is X_1 an extreme point of the convex hull of $\{X_1, \cdots, X_4\}$?

2. Prove Theorem 16.

3. Prove Theorem 18.

4. Let $X_1 = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$, $X_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $X_3 = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$. Sketch a graph of the set of all nonnegative linear combinations of the vectors X_1, X_2, X_3. Call this set N; can N be expressed as the solution set of a system of homogeneous linear inequalities? If

so, write down such a system.

5. Return to the convex set in 1(a). Can you write out a set of linear inequalities whose solution set is equal to the convex hull in 1(a)?

6. Given the functions defined over all of the reals (unless indicated otherwise) as follows:

(a) $f(x) = x^2 - x+3$, (g) $H(x) = -4x^2+3$,

(b) $g(x) = x^3+x^2-x$, (h) $Q(x) = 3x^4-4x^3-2x^2+1$, $-2 \leqq x \leqq 3$,

(c) $h(x) = e^x$,

(d) $G(x) = \log_e x$, $x > 0$, (i) $R(x) = 3x+1$.

(e) $F(x) = \dfrac{x}{\sqrt{x-1}}$, $x > 1$,

Which are convex? Which are concave?

1.8. <u>More on Double Description</u>. We have shown in earlier sections that we have a double description for vector subspaces and for linear manifolds in E_n. We also have a double description for convex cones and for bounded convex sets (convex polyhedra). Given a convex cone, it can be regarded as the set of all nonnegative linear combinations of some spanning set or as the set of all solutions of a system of homogeneous linear inequalities $AX \geqq 0$. A convex polyhedron can be described as the set of all convex combinations of some finite set of vectors or as the solution set of a system of linear inequalities $AX \geqq B$.*

---

*The last two statements are not proved here since proofs are lengthy and complicated. See various papers in Kuhn and Tucker [7] for proofs and related discussions. See also Spivey and Thrall [12], p. 474-490.

Double description is important because it gives us conditions under which we can relate systems of linear equalities to spanning sets. From an applied point of view, problems frequently take the form of the former, whereas many theorems and operational results are stated in terms of the latter. Having double description theorems means that we can pass back and forth with ease.

Figure 8 displays in brief form the full range of double description theorems that are available.

1.9. Concepts from Point Set Theory. In the next chapter we will speak of open and closed sets and of the boundary of a set. We now introduce concepts which will make these terms clear.

Definition. Let S be any set in E_n and $x_o \varepsilon S$. Then a neighborhood N of x_o is the set

$$N(x_o,r) = \{x|d(x_o,x) < r, \quad r > 0\}.$$

Definition. If S is any set in E_n and $x \varepsilon S$, then x is an interior point of S if some neighborhood, perhaps a small one, is a subset of S.

Definition. If S is a set in E_n, then the interior of S is the set of all interior points of S.

Examples

(1) If $x_o \varepsilon E_1$, a neighborhood is the interior of an interval centered at x_o.

Set	Set of Linear Combinations or "Span"	System of Linear Equalities or Inequalities
Vector Subspace	S is the set of all linear combinations of a set of vectors	$S = \{X \mid AX=0 \text{ for some } A\}$.
Linear Manifold	L is the set of all weighted averages of a set of vectors	$L = \{X \mid AX=B \text{ for some } A\}$.
Convex Cone	C is the set of all nonnegative linear combinations of a set of vectors.	$C = \{X \mid AX \geqq 0 \text{ for some } A\}$.
Convex Polyhedron	K is the set of all convex combinations of a set of vectors.	$K = \{X \mid AX \geqq B \text{ for some } A\}$.

Figure 8

(2) If $x_o \varepsilon E_2$, a neighborhood is the interior of a circular disc with center at x_o and radius r as in Figure 9.

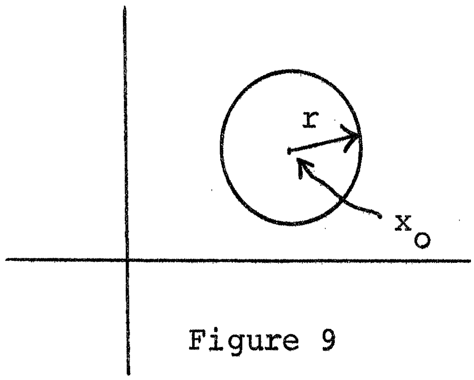


Figure 9

(3) If $x_o \varepsilon E_n$, a neighborhood is the interior of a "solid ball" centered at x_o and having radius r.

The concept of interior can correspond, as in the above examples, somewhat closely to the intuitive notion of interior, but the definition applies to cases which do not conform to the intuition. For example, consider the set of all rational points in E_1 (i.e., the set of all points on the real line that correspond to the rational numbers). This set has no interior at all. For another example, consider E_n and the distance function defined as follows: if x, y are any points in E_n, then $d(x,y) = 1$ if $x \neq y$, and $d(x,x) = 0$. Then every point x is an interior point of every set containing x.

Definition. A set is said to be open if each of its points is an interior point.

Theorem 20. A neighborhood is an open set.

Proof. Let $N(x_o,r)$ be a neighborhood in E_n,

$$N(x_o,r) = \{x | d(x_o,x) < r\},$$

and let x be any point in $N(x_o,r)$. We want to show that x is an interior point. Since $x \in N(x_o,r)$,

$$d(x_o,x) < r \text{ or } d(x_o,x) = r-h \quad \text{for some } h > 0.$$

Let y be any point in $N(x,h)$; then

$$d(x_o,y) \leqq d(x_o,x) + d(x,y) < (r-h) + h = r$$

or

$$d(x_o,y) < r. \text{ This implies } y \in N(x_o,r) \Rightarrow N(x,h) \subset N(x_o,r).$$

Definition. A point $x_o \in E_n$ is a limit point of the set S if every $N(x_o,r)$ contains a point of S different from x_o.

Examples

(1) Let $S = \{(x,y) | x^2 + y^2 \leqq 1\}$; every point on the circumference is a limit point of S. Suppose x^* is an interior point of S; is x^* a limit point?

(2) Let $T = \{x | 0 < x \leqq 1\}$. The point $x = 0$ is a limit point of T, but $x \notin T$. The point $x = 1$ is also a limit point of T.

Definition. A set is closed if it contains each of its limit points.

Theorem 21. If x_o is a limit point of S, then every $N(x_o,r)$ contains infinitely many points of S.

Proof. Suppose some $N(x_o,r)$ contains a finite number of points of S, say $p_1, \cdots, p_n$. Then there is some point p_i closest to x_o; for this point let $d(x_o,p_i) = r^*$. Then construct $N(x_o,r^*)$; this neighborhood contains none of the points $p_1, \cdots, p_n$, but this is a contradiction since x_o is a limit point of S.

Definition. Let $S \subset E_n$ and let $x_o \varepsilon E_n$; then if every $N(x_o, r)$ contains a point of S and a point not in S, x_o is a boundary point of S.

Note that a closed set contains all its boundary points because a boundary point either belongs to the set or is a limit point of the set (why?).

Definition. The boundary of a set is the set of all boundary points of the set.

Definition. A set in E_n is said to be bounded if it is a subset of some $N(0,r)$ for $r > 0$. A set is said to be unbounded if it is not bounded.

Theorem 22. The complement of an open set is closed. If a set is closed, its complement is open.

Theorem 23. The union of any collection of open sets is open.

Theorem 24. The intersection of any collection of closed sets is closed.

Discussions of these and related concepts can be found in many introductory books on analysis; two good sources are Boas [3] and Natanson [8].

We noted earlier that the set $L = \{X | AX = b\}$, where A is a nonzero vector and b is a scalar, is a hyperplane. The set $Q = \{X | AX \geqq b\}$ is called a halfspace in E_n. It is indicated in the exercises at the end of this section that $Q^* = \{X | AX > b\}$ is an open set or open half space and Q is a closed set or closed half space.

Exercises

1. The unit ball in E_n is defined by

$$B = \{(x_1, \cdots, x_n) \varepsilon E_n \Big| \sum_{i=1}^{n} x_i \leqq 1\}.$$

   Prove that B is a convex set.

2. Prove Theorem 21.

3. Prove Theorem 22.

4. Prove Theorem 23.

5. Prove Theorem 24.

6. Prove that the set $H = \{X | AX \geqq b\}$, A a nonzero vector in E_n and b a scalar, is closed.

7. Prove that the set $H^* = \{X | AX > b\}$ is open.

8. Prove that a hyperplane in E_n is a closed set.

9. Note distinctions between the concept of bounded set and boundary of a set. Give an illustration of an unbounded set which has a finite set of boundary points. Construct an example of an unbounded set that does not contain its boundary points.

10. Can a set be open and closed? If so, produce two examples.

1.10. Brief Review of Linear Transformations. Suppose we consider E_n, a basis for E_n consisting of the unit vectors $\{U_1, \cdots, U_n\}$, E_p, and a unit basis for E_p, $\{U'_1, \cdots, U'_p\}$. From linear algebra we know that if T is a linear transformation from E_n into E_p, then T is fully specified if we indicate the images of the basis vectors of E_n under T (see Nering [9], Chap. II). In colloquial language, we know where every vector in E_n goes (under T) if we know where the basis vectors in E_n

go. Suppose then for each basis vector $U_j \varepsilon E_n$ we have

$$T(U_j) = a_{1j}U'_1 + \cdots + a_{pj}U'_p$$
$$= \sum_{i=1}^{p} a_{ij}U'_j \ .$$

Now if X is any vector in E_n, consider its image vector under T,

$$T(X) = y_1U'_1 + \cdots + y_pU'_p \ .$$

We know that the representation of the vectors $T(U_j)$ and $T(X)$ in terms of the basis vectors U'_j is unique.

Since T is a linear transformation we have

$$T(X) = T(x_1U_1 + \cdots + x_nU_n)$$
$$= x_1T(U_1) + \cdots + x_nT(U_n)$$
$$= x_1\left(\sum_{i=1}^{p} a_{i1}U'_i\right) + \cdots + x_n\left(\sum_{i=1}^{p} a_{in}U'_i\right)$$
$$= \sum_{i=1}^{p} \left(\sum_{i=1}^{n} a_{ij}x_j\right) U'_i .$$

But since a vector has a unique representation in terms of a basis, this means that

$$y_i = \sum_{j=1}^{n} a_{ij}x_j$$

or, in matrix form

$$Y = AX.$$

Under these conditions it can be shown that the matrix $A = [a_{ij}]$ represents the linear transformation T unambiguously (relative to the chosen bases).

The matrix equation above indicates that the coordinate vector of the image of X under T is the product of the matrix of the transformation T and the coordinate vector X.

Moreover, note that the columns of the matrix A are the coordinates of the images under T of the basis vectors in E_n.

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & & a_{2j} & & a_{2n} \\ \vdots & & \vdots & & \vdots \\ a_{p1} & & a_{pj} & & a_{pn} \end{bmatrix}$$

↑ coordinate vector of $T(U_j)$

<u>Definition</u>. The set of all vectors $X \varepsilon E_n$ that are mapped by T on the Zero vector in E_p is called the <u>kernel</u> of the linear transformation T.

<u>Definition</u>. If X is any vector in E_n, then the set of all image vectors T(X) is called the range of T.

<u>Theorem 25</u>. Let T be a linear transformation from E_n into E_p; then

(i) the kernel of T is a subspace of E_n;

(ii) the range of T is a subspace of E_p;

(iii) dim (kernel) + dim (range) = n = dim (E_n).

Note that one can use a system of homogeneous linear equations to define the kernel of a linear transformation T.

For example, suppose A is the matrix of a linear transformation from E_n into E_p relative to a chosen pair of bases. Then

$$\text{Kernel of } T = \{X \mid AX = 0\}$$

A system of homogeneous linear equations always has a solution (the zero vector). Another way of saying this is: the kernel of a linear transformation always contains the zero vector (of E_n). Note that this is part of what is implied by Theorem 25(i) since a subspace must contain the zero vector. It may be the case that the kernel of T contains only the zero vector; then we know that T is a one-to-one transformation. On the other hand, if the kernel is a subspace other than the zero vector, there will be nonzero vectors in E_n which are mapped by T on the zero vector in E_p and the corresponding system of linear equations $AX = 0$ has a nontrivial solution.

We now consider a special case of the foregoing.

Question 1. Are there vectors in E_n such that $AX = 0$ and $X \geq 0$? That is, does the kernel of T contain a positive vector X?

To provide an answer, we normalize the vectors $X \geq 0$ (this is possible, if such vectors exist, since $X \geq 0$), and note that such a normalized vector is a probability vector. We consider

Question 1**. Does T map probability vectors in E_n on the zero vector in E_p?

Definition. Let $\{U_1, \cdots, U_n\}$ be the set of unit vectors in E_n; then the convex hull of these vectors, denoted $H_n(U_1, \cdots, U_n)$ is called the <u>fundamental probability simplex</u> (F.P.S.) in E_n.

We can now state Question 1 in yet another way:

<u>Question 1**</u>. Does the image of the F.P.S. under T contain the origin of E_p?

We will see in the next chapter that answers to these questions will lead us to theorems that enable us to prove the existence and duality theorems of linear programming as well as other important theorems in the theory of linear inequalities and in the theory of games.

<u>Theorem 26</u>. A linear transformation T maps subspaces into subspaces.

<u>Theorem 27</u>. A linear transformation T maps linear manifolds into linear manifolds.

<u>Theorem 28</u>. A linear transformation maps convex sets into convex sets. In particular, the image of the F.P.S. in E_n is a convex set in E_p.

<u>Exercises</u>

1. Prove Theorem 25.
2. Prove Theorem 26.
3. Prove that if the kernel of T consists only of the zero vector, then T is a one-to-one linear transformation.
4. Prove Theorem 27.
5. Prove Theorem 28.

Chapter 2. Linear Optimization

2.1. Proof of a Key Theorem. We return to a consideration of Question 1** of Section 1.10 and recall that the j^{th} column vector of the matrix A is the image of the vector $U_j \varepsilon E_n$. Therefore, the image of $H_n(U_1, \cdots, U_n)$ -- the fundamental probability simplex in E_n -- is the set $H_p(A_1, \cdots, A_n)$, where $H_p(A_1, \cdots, A_n)$ is the convex hull of the image vectors $A_1, \cdots, A_n$ in E_p. To put it another way, if we consider the set of all convex combinations of the unit vectors in E_n and then consider the image of this set in E_p, then the latter will consist of exactly the same convex combinations of the images of the unit vectors (Theorem 28).

Now if the zero vector is in the set $H_p(A_1, \cdots, A_n)$, there exists a probability vector in E_n satisfying $AX = 0$, $X \geq 0$, and we answer Question 1 (and Questions 1* and 1**) in the affirmative. On the other hand, if the zero vector in E_p is not contained in the set H_p, there is no probability vector in E_n satisfying $AX = 0$, $X \geq 0$.

Case I. Suppose $0 \notin H_p$; what happens in this case? In Figure 10 we show the set $H_p(A_1, \cdots, A_n)$ as the indicated convex hull of the vectors $A_1, \cdots, A_n$, and the origin (of E_p) is shown contained in a hyperplane L ($0 \notin H_p$).

Now since H_p is a bounded and closed set, there exists a point $p \varepsilon H_p$ that is closest to the origin (a basic result from the calculus). Let U be the vector having the origin as initial point and P as end point (we know that $U \neq 0$ because

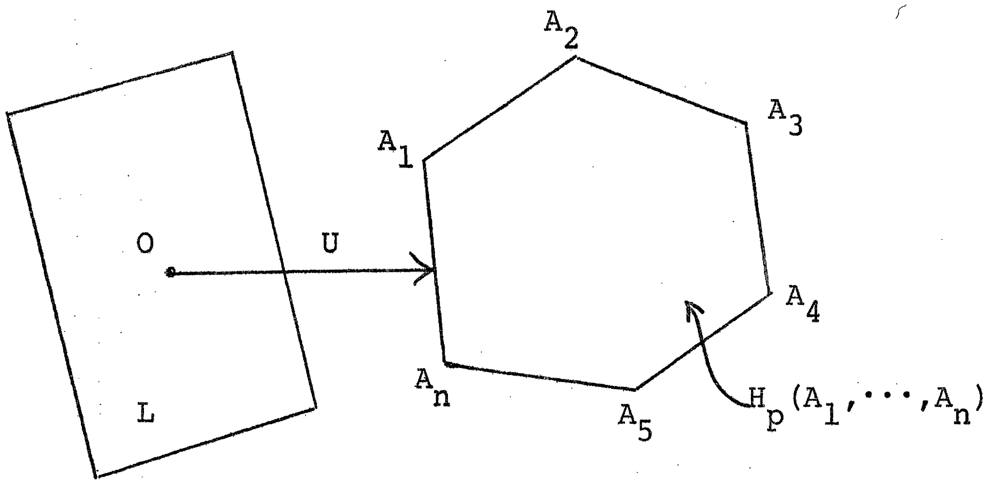


Figure 10

$0 \notin H_p$). Let U be a normal vector to a hyperplane L containing the origin (see Figure 10).

We first show that L does not intersect H_p for any point P chosen as above. Suppose $Q \in H_p$ is also in L and consider the line segment PQ in Figure 11. Since $Q \in L$, the vector OQ

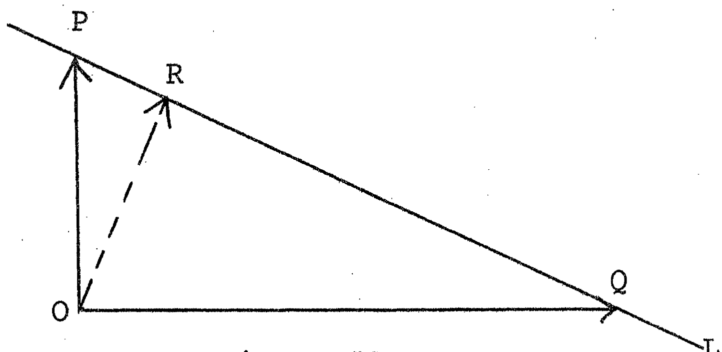


Figure 11

is orthogonal to OP, so the angles OPQ and OQP are acute and PQ is the hypotenuse of a right triangle. Construct an altitude of the triangle from O and let R be the point of intersection of this altitude with PQ. Since $P, Q \in H_p$ and H_p is convex, the line segment joining P and Q is in H_p, so $R \in H_p$. This means that $R \in H_p$ is closest to the origin. Hence we have a contradiction since we assumed that $P(\neq R)$ is closest to the origin. Thus L does not intersect H_p and H_p is in the interior of the half space bounded by the hyperplane L.

Furthermore, given H_p in the interior of the half space and the normal vector U, every vector in H_p has a positive inner product with U; in particular,

$$UA_i > 0 \quad (i=1, \cdots, n) .$$

This establishes the following result:

if $0 \notin H_p$ (i.e., there is no vector $X \geq 0$ in E_n such that $AX = 0$), then there exists a vector $U \in E_p$ such that

$$UA_i > 0 \quad (i=1, \cdots, n)$$

where A_i is the i^{th} column of the matrix A.

Conversely, if there exists a vector $U \in E_p$ and that $UA_i > 0$ for $i=1, \cdots, n$, then $0 \notin H_p(A_1, \cdots, A_n)$. To prove this, suppose that $UA_i > 0$ for $i=1, \cdots, n$ and $0 \in H_p$;

$$c_1A_1 + \cdots + c_nA_n = 0$$

for $c_i \geqq 0$, $\sum_{i=1}^{n} c_i = 1$. Moreover, some $c_i > 0$, since $0 \in H_p$. Now

$$U0 = U(c_1A_1 + \cdots + c_nA_n) = c_1UA_1 + \cdots + c_nUA_n .$$

Since $UA_i > 0$ for every i and $c_i > 0$ for at least one i, we have

$$c_1A_1 + \cdots + c_nA_n > 0 .$$

But this is a contradiction since $UO = 0$ for every U.

The situation $O \in H_p$ is characterized by the theorem of Gordan (of which the above remarks constitute a proof):

The system $X \geq 0$, $AX = 0$ has a solution if and only if there is not a vector $U \in E_p$ such that $UA > 0$.

Case II. If $O \in H_p$, then O is either on the boundary of H_p or in the interior of H_p. Suppose O is on the boundary of H_p and we have the situation shown in Figure 12. Here we have $UA_1 = 0$, $UA_n = 0$ and $UA_i > 0$ for $i=2, \cdots, n-1$. Now since

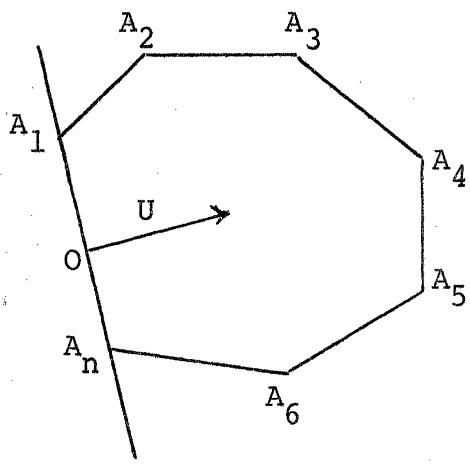


Figure 12

$O \in H_p$ we have

$$O = x_1A_1 + x_2A_2 + \cdots + x_{n-1}A_{n-1} + x_nA_n ,$$

where $x_i \geqq 0$, $\sum_{i=1}^{n} x_i = 1$. Also from the case in Figure 12 we see that, in particular, $x_1 > 0$, $x_n > 0$ and $x_i = 0$ for $i=2, \cdots, n-1$ (since O is on the line segment joining A_1 and A_n). The preimage X of $O \in E_p$ is the same linear combination of the basis vectors of E_n as O is of the vectors $A_1, \cdots, A_p$ in E_p, so

$$X = T^{-1}(x_1A_1 + \cdots + x_nA_n) = x_1U_1 + \cdots + x_nU_n$$
$$x_1 > 0, \quad x_n > 0$$
$$x_j = 0 \text{ for } j=2, \cdots, n-1.$$

Consider the vectors

$$UA = [UA_1, \cdots, UA_n] \text{ and } X^T = [x_1, \cdots, x_n];$$

whenever $UA_i = 0$, the corresponding $x_i > 0$ and whenever $UA_i > 0$, the corresponding $x_i = 0$. Thus in this case we have

$$X^T + UA > 0.$$

This is a proof for the special case in Figure 12 and contains the essential features of a proof which must be established for whatever "edge" the origin is contained in (see Good [5], p. 9-10).

Case III. If $O \varepsilon H_p$ and is an interior point of H_p, then there is a convex combination

$$O = x_1A_1 + \cdots + x_nA_n, \quad x_i > 0 \quad (i=1, \cdots, n).$$

This means that if X is a preimage of $O \varepsilon E_p$, then X is strictly positive. Now let U = 0 and again we have

$$X^T + U^TA > 0.$$

A concise statement of all three possibilities is contained in the following theorem (called the key theorem by Tucker, see Good [5], p. 7).

Theorem 29. The system of linear inequalities

$$AX = 0$$
$$X \geqq 0$$
$$UA \geqq 0$$

has a solution X* and U* such that $X^{*T} + U^*A > 0$.

Proof. Except for details relating to Case II, this is proved in the foregoing discussion.

It is perhaps more useful to state this theorem in the following way. Let $A_1, \cdots, A_n$ be any finite set of vectors in E_p; there exist scalars $x_1^*, \cdots, x_n^*$ and a vector $U^* \varepsilon E_p$ such that

$$A_1x_1^* + \cdots + A_nx_n^* = 0$$
$$x_j^* \geqq 0, \ (j=1, \cdots, n)$$
$$U^*A_j \geqq 0, \ (j=1, \cdots, n),$$

and

$$x_j^* + U^*A_j > 0, \ (j=1, \cdots, n).$$

Corollary to Key Theorem. Let B, $A_1, \cdots, A_n$ be n+1 vectors in E_p. If $UB \geqq 0$ for every U satisfying the inequalities $UA_i \geqq 0$, i=1, $\cdots$, n, then B is a nonnegative linear combination of the vectors $A_1, \cdots, A_n$.

Proof. Consider the vectors $-B, A_1, \cdots, A_n$; from the key theorem we know that there exist $x_i^* \geqq 0$, i=0, 1, $\cdots$, n, satisfying

$$(1) \qquad -Bx_0^* + A_1x_1^* + \cdots + A_nx_n^* = 0,$$

and a vector U^* satisfying $U^*A_i \geqq 0$, $i=1, \cdots, n$, and $U^*(-B) \geqq 0$, for which

(2) $\quad x_o^* + U(-B) > 0$

(3) $\quad x_i^* + UA_i > 0 \quad (i=1, \cdots, n)$.

Write (1) as

(4) $\quad A_1x_1^* + \cdots + A_nx_n^* = x_o^*B$.

If we can divide both sides of (4) by $x_o^* > 0$ the conclusion of the theorem would immediately result. Suppose $x_o^* = 0$; then (2) would require that $UB < 0$ and from (3) we have $UA_i \geqq 0$. This contradicts the hypothesis, so $x_o^* > 0$. Dividing (4) on both sides by x_o^* gives the desired result.

This corollary is known as the Farkes theorem or lemma in the literature on linear optimization and the theory of games.

<u>Theorem 30</u> (Theorem of the Alternative for Matrices). The dual systems

$$MX \leqq 0 \qquad\qquad M^TU \geqq 0$$
$$X \geqq 0 \qquad\qquad U \geqq 0$$

have solutions U^* and X^* such that $U^* - MX^* > 0$ and $M^TU^* - X^* > 0$.

<u>Proof</u>. Suppose the system $MX \leqq 0$, when written out more fully, appears as

$$
\begin{aligned}
&a_{11}x_1 + \cdots + a_{1n}x_n \leqq 0 \\
&\vdots \qquad\qquad\quad \vdots \\
&a_{p1}x_1 + \cdots + a_{pn}x_n \leqq 0.
\end{aligned}
$$

Insert slack variables $w_1, \cdots, w_p$ and get a linear equation system whose matrix is

$$A = [M \quad I] .$$

Let

$$U^* = \begin{bmatrix} u_1^* \\ \vdots \\ u_p^* \end{bmatrix}, \quad Y^* = \begin{bmatrix} x_1^* \\ \vdots \\ x_n^* \\ w_1^* \\ \vdots \\ w_p^* \end{bmatrix} ;$$

applying the key theorem to the matrix A we know that there exist vectors U* and Y* satisfying

$$AY^* = 0, \; Y \geqq 0, \; A^TU \geqq 0, \; Y^* + A^TU^* > 0.$$

Now consider $A^TU^* \geqq 0$; we get

$$A^TU^* = \begin{bmatrix} M^T \\ I \end{bmatrix} U^* \geqq 0$$

or $$M^TU \geqq 0 \text{ and } IU^* \geqq 0 \Rightarrow U^* \geqq 0.$$

Also from

$$[M \quad I] \begin{bmatrix} X^* \\ W^* \end{bmatrix} = 0 \qquad \begin{bmatrix} X^* \\ W^* \end{bmatrix} + \begin{bmatrix} M^T \\ I \end{bmatrix} U^* > 0$$

$$\begin{bmatrix} X^* \\ W^* \end{bmatrix} \geqq 0$$

we get

$$MX^* + IW = 0 \Rightarrow \quad MX^* = -W^* \leqq 0$$

and

$$X^* + M^T U^* > 0$$

$$W^* + \quad U^* > 0.$$

Since $-MX^* = X^*$, from the latter inequality we get

$$U^* - MX^* > 0.$$

Collecting results, we have

$$MX^* \leqq 0 \qquad M^T U^* \geqq 0$$

$$X^* \geqq 0 \qquad U^* \geqq 0$$

and $\quad X^* + M^T U^* > 0, \quad U^* - MX^* > 0,$

which was to be shown.

<u>Theorem 31</u>. (The Skew-Symmetric Matrix Theorem). Let K be a skew-symmetric matrix; the system $KW \geqq 0$, $W \geqq 0$ has a solution W^* such that $W^* + KW^* > 0$ (i.e., W^* has at least one positive component since $W^* + KW^* > 0$ implies that $W^* \neq 0$).

Proof. Let M be a skew-symmetric matrix. Applying Theorem 30 to M we get

$$(5)\quad \begin{matrix} MX \leqq 0 \\ X \geqq 0 \end{matrix} \qquad (6)\quad \begin{matrix} M^T U \geqq 0 \\ U \geqq 0 \end{matrix} \qquad \text{or } (7)\quad \begin{matrix} MU \geqq 0 \\ U \geqq 0. \end{matrix}$$

These systems have solutions X* and U* such that

$$(8)\quad \begin{matrix} U^* - MX^* > 0 \\ M^T U^* + X^* > 0 \end{matrix} \qquad \text{or} \quad (9)\quad \begin{matrix} U^* - MX^* > 0 \\ X^* - MU^* > 0. \end{matrix}$$

Combining the systems (5), (7) gives

$$(10)\quad \begin{matrix} M(X^* + U^*) \leqq 0 \\ (X^* + U^*) \geqq 0 \end{matrix}$$

and combining (9) gives

$$(11)\quad (X^* + U^*) - M(X^* + U^*) > 0.$$

Now let $K = -M$ and $W^* = X^* + U^*$; then (10) and (11) can be written

$$(12)\quad \begin{matrix} KW^* \geqq 0 \\ W^* \geqq 0 \\ W^* + KW^* > 0 \end{matrix}$$

as was to be shown.

Suppose, on the other hand, we wanted to find a W* satisfying (12), given a skew-symmetric matrix K. We take its negative, call it $C (C = -K)$. Now use Theorem 30 on the matrix C since C is skew-symmetric. Then there is a vector $W^* = X^* + U^*$ such that

$$W^* \geqq 0$$
$$CW^* \leqq 0$$
$$W^* - CW^* > 0.$$

This result can also be stated as follows. There exists a probability vector P such that $CP \geqq 0$, C skew-symmetric, such that $P + CP > 0$. Geometrically this means that the convex hull of the column vectors of the matrix C intersects the closed, nonnegative orthant in E_p.

Moreover, W^* has at least one component greater than zero, since $W^* + KW^* > 0$ means that $W^* \neq 0$. Finally, the vectors W^* and KW^* have the property that for each $j=1, \cdots, n$, the j^{th} component of one of the vectors W^*, KW^* is positive if and only if the j^{th} component of the other is zero. The reason is clear: the sum is positive but $W^{*T}KW^* = 0$ since $-(W^{*T}KW^*)^T = (W^{*T}KW^*)^T$ because of the skew-symmetric property of the matrix K.

2.2. Linear Optimization. In this section we will consider the following linear programming problem and its dual,

$$\text{maximize } f = c_1x_1 + \cdots + c_nx_n$$

subject to

$$\begin{array}{ccc} a_{11}x_1 + \cdots + a_{1n}x_n & \leqq & b_1 \\ \vdots \qquad\qquad \vdots & & \vdots \\ a_{p1}x_1 + \cdots + a_{pn}x_n & \leqq & b_p \\ x_j & \geqq & 0 \quad (j=1, \cdots, n), \end{array}$$

and

$$\text{minimize } g = b_1 u_1 + \cdots + b_p u_p$$

subject to

$$a_{11}u_1 + \cdots + a_{p1}u_p \geqq c_1$$
$$\vdots$$
$$a_{1n}u_1 + \cdots + a_{pn}u_p \geqq c_n$$
$$u_i \geqq 0 \quad (i=1, \cdots, p).$$

Lemma 1. If X and U are feasible programs, respectively, then $C^T X \leqq U^T B$.

Lemma 2. If X* and U* are feasible programs, respectively, and if $C^T X^* = U^{*T} B$, then X* and U* are optimal vectors.

Proof. Suppose X* and U* are feasible; from Lemma 1 we have for all feasible programs $C^T X \leqq U^T B$. In particular, $C^T X^* \leqq U^T B$. But $C^T X^* = U^{*T} B$ so $U^{*T} B \leqq U^T B$ for any feasible U. This means that U* is an optimal program for the minization problem. A similar argument establishes that X* is optimal for the maximization problem.

Now define a skew-symmetric matrix K to be

$$K = \begin{bmatrix} 0 & -A & B \\ A^T & 0 & -C \\ -B^T & C^T & 0 \end{bmatrix},$$

the vector $W = \begin{bmatrix} U \\ X \\ t \end{bmatrix}$, t a scalar. From Theorem 31 we know that

there exists a vector

$$W^* = \begin{bmatrix} U^* \\ X^* \\ t^* \end{bmatrix}$$

such that

$$(13) \qquad \begin{aligned} KW^* &\geqq 0 \\ W^* &\geqq 0 \end{aligned}$$

and

$$(14) \qquad KW^* + W^* > 0.$$

The inequalities (13) give , when written out,

(i) $AX^* \leqq Bt^*$

(ii) $A^TU^* \geqq Ct^*$

(iii) $C^TX^* \geqq B^TU^*$

and the inequality (14) gives

(iv) $AX^* < Bt^* + U^*$

(v) $A^TU^* + X^* > Ct^*$

(vi) $C^TX^* + t^* > B^TU^*$.

Lemma 3. Suppose $t^* > 0$; then there are optimal vectors X^o and U^o such that

$$C^TX^o = U^{oT}B, \; U^o + B > AX^o, \; U^{oT}A + X^{oT} > C^T.$$

Proof. Multiply W^* by the scalar $\frac{1}{t^*}$; then

$$W^o = \frac{1}{t^*} W^* = \begin{bmatrix} \frac{U^*}{t^*} \\ \frac{X^*}{t^*} \\ 1 \end{bmatrix} = \begin{bmatrix} U^o \\ X^o \\ t^o \end{bmatrix},$$

and X^o and U^o then satisfy (i) and (ii) and are feasible. Lemma 2 and (iii) show that X^o and U^0 are optimal, respectively, and Lemma 1 shows that $C^T X^o = U^{oT} B$. The inequalities in the statement of Lemma 3 follow from (iv) and (v) with $t^o = 1$.

Lemma 4. Let $t^* = 0$; then

(a) at least one of the LP problems has no feasible vector.

(b) If the maximization problem has a feasible vector, then the solution set is unbounded and $C^T X$ is unbounded over this solution set (a corresponding statement holds for the dual problem).

(c) Neither problem has an optimal solution.

Proof. Suppose X is a feasible vector for the maximization problem. Using (ii), $t^* = 0$ and the nonnegativity of X we get

$$U^{*T} AX \geqq 0.$$

This, together with (vi), $t^* = 0$, and the feasibility of X yields

$$(15) \qquad 0 \leqq U^{*T} AX \leqq U^{*T} B < C^T X^* .$$

To prove (a) we note that if the minimization problem has a feasible vector U, then we can get

$$(16) \qquad 0 \geqq U^T AX^* \geqq C^T X^*.$$

Now (16) contradicts (15), proving (a).

To prove (b), consider the vector $X + \lambda X^*$, $\lambda \geqq 0$. Now $X + \lambda X^* \geqq 0$. From (i) and $t^* = 0$ we have $A(X + \lambda X^*) \leqq AX \leqq B$. Therefore $X + \lambda X^*$ for any $\lambda \geqq 0$ is feasible. This proves the first assertion of (b). Moreover, from (15) we have $C^T X^* > 0$; then

$$C^T(X + \lambda X^*) = C^T X + \lambda C^T X^*$$

can be made arbitrarily large by choosing λ sufficiently large. This proves the second assertion of (b). Finally, (c) is an immediate consequence of (b) for the maximization and the minimization problem.

Lemma 5. Either both the maximization problem and the minimization problem have optimal vectors or neither does. In the first case the maximum and minimum values are equal; their common value is called the optimal value of the dual L.P. problems.

Proof. If one of the problems has optimal vectors, then (c) of Lemma 4 shows that $t^* > 0$. Then Lemma 3 shows that both problems have optimal vectors X^o and U^o such that $\max C^T X = \min U^T B$.

Theorem 32 (the Duality Theorem). A feasible X^o is optimal if and only if there is a feasible vector U^o with $C^T X^o = U^{oT} B$. A feasible U^o is optimal if and only if there is a feasible vector X^o with $C^T X^o = U^{oT} B$.

Proof. We prove only the first statement since the second has an analogous proof. Lemma 2 is a proof of the sufficiency of this theorem. To prove necessity, suppose that X^o is optimal. By Lemma 4(c) we must have $t^o > 0$. By Lemma 3 the minimization problem also has an optimal vector U^o, and by Lemma 5 max $C^T X$ = min $U^T B$.

Theorem 33 (Existence Theorem of Linear Programming). A necessary and sufficient condition that one (and therefore both) of the linear programming problems have optimal vectors is that both have feasible vectors.

Proof. The necessity is obvious. To prove sufficiency, suppose that both problems have feasible vectors. By Lemma 4(c) we have $t^* > 0$ and by Lemma 3 both problems have optimal vectors.

Theorem 34 (A Complementary Slackness Theorem). If both problems have feasible vectors, then they have optimal vectors X^o and U^o such that:

(1) if $A^i X^o = b_i$, A^i the i^{th} row vector of the matrix A, then $u_i^o > 0$ (the i^{th} dual variable is positive);

(2) if $U^{oT} A_j = c_j$, A_j the j^{th} column vector of A, then $x_j^o > 0$ (j^{th} primal variable is positive).

Proof. By the existence theorem both problems have optimal vectors, so Lemma 5(c) implies that $t^* > 0$. By Lemma 3 there are optimal vectors such that

$$(17) \qquad U^{oT} + B > AX^o, \quad U^{oT} A + X^{oT} > C^T.$$

The theorem is clearly proved, for consider the inequality on the left above. It can be written as

$$\begin{bmatrix} u_1^o \\ \vdots \\ u_i^o \\ \vdots \\ u_p^o \end{bmatrix} + \begin{bmatrix} b_1 - A^1X^o \\ \vdots \\ b_i - A^iX^o \\ \vdots \\ b_p - A^pX^o \end{bmatrix} > 0 .$$

Then if $A^iX^o - b_i = 0$, we must have $u_i^o > 0$. Now consider the inequality on the right in (17). This can be written as

$$[U^{oT}A_1 - c_1 \cdots U^{oT}A_j - c_j \cdots U^{oT}A_n - c_n] + [x_1^o \cdots x_j^o \cdots x_n^o] > [0 \cdots 0 \cdots 0].$$

Obviously if we have $U^{oT}A_j = c_j$, the strict inequality requires $x_j^o > 0$.

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