Setting Single-Period Optimal Capacity Levels and Prices for Substitutable Products

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Technical Report 97-03
April 1997

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Abstract

In this paper, we consider the determination of optimal capacity levels and prices for two substitutable products in a single-period problem. We first consider the case where the firm is a price taker but can determine optimal capacity levels for both products. We then consider the case where the firm can set the price for one product and the optimal capacity level for the other. Next, we consider the case where capacity is fixed for both products, but the firm can set prices. For each of these cases, we examine the sensitivity of optimal prices and capacities to problem parameters. Finally, we consider the case where each product is managed by a product manager trying to maximize individual product profits rather than overall firm profits and analyze how optimal price and capacity decisions are affected.

1 Introduction

Firms must continuously make pricing and capacity decisions to respond to market forces. Facing uncertain demand, firms must balance pricing and production decisions to respond to market demand. Many firms produce a variety of products, some of which may be substitutable by consumers. The fact that products are substitutable makes pricing and capacity decisions more difficult. This is due to the fact that the firm needs to consider the effect that a change in the price of one product is going to have on the demand level for another product. A good example is in electronics manufacturing where a firm might produce a variety of chips. The price of a faster chip affects the demand for a slower chip as well since if the prices for the two types of chips are sufficiently close, many customers might opt for the faster chip, thereby significantly decreasing demand for the slower chip.

In this paper, we study single-period pricing and capacity setting decisions for a firm that produces two substitutable products. Our aim is to build intuition on how pricing and capacity decisions change as a function of costs, demand functions, and preexisting capacity levels; we therefore focus on simple single-period models. This also enables us to contrast our results with the famous news vendor problem which is a single period problem where the capacity (production level) for only a single product is chosen. We also focus on the issue of centralized versus decentralized decision making and how this affects the nature of the decisions made. Many firms assign a product manager to each product that the firm produces, and the role of the product manager is to maximize the profits made by the product assigned to the manager. If all products are produced on separate production lines, and the products are unsubstitutable, the decisions made by each product manager trying to maximize product profits will correspond to decisions made by a centralized controller maximizing system-wide profits. The more interesting case is when products are substitutable; in that case, we analyze when the system-optimal decisions are the same as the individual product-optimal decisions.

The classic problem where price is known and capacity is uncertain is the news vendor problem. Following this classic problem, many extensions have been made. For example, Ismail and Louderback [8], Lau [11] and Kabak and Schiff [9] have studied a one product news vendor problem where the probability of achieving a pre-determined profit level is maximized. Ismail and Louderback[1], and Sankarasubramanian and Kumaraswamy [14] and Lau and Lau [12] have studied a single product news vendor problem where the demand level is dependent on the price set. Therefore, both optimal price and optimal order quantity are determined. Li et al. [13] focus on a two-product news vendor problem where the probability of achieving a profit target is maximized.

There have been a variety of attempts to introduce the effects of capacity constraints on the price and production decisions in the news vendor problem. Kreps and Scheinkman [10] examine a problem where two identical firms compete with products that are perfect substitutes. The demand for each product is a function of its own price. The two firms enter into a two-stage competition where they set capacities in the first stage and make pricing decisions independently in the second stage. Staiger and Wolak [15] focus on two firms who produce the same product and have the same capacity costs. They analyze an infinitely repeated game where prices are adjusted periodically, and examine the effects of having excess capacity.

Examples of research involving substitutable products includes Ignall and Veinott [7] who examine the optimality of myopic policies with several products. Bassok et al. [1], Bitran and Dasu [2], Hsu and Bassok [6], and Gerchak et al. [5] study ordering policies with substitutable products while Carmon and Nahmias [4] examine lot-sizing decisions in semiconductor manufacturing where the products are substitutable. In this line of research, it is assumed that the manufacturer is a price taker for all of its products.

We study the case of a firm producing two products with price dependent demands where the firm has the ability to make pricing and/or capacity decisions for one or both of its products. We begin in section two by discussing the case where a firm is a price-taker for both of its products but has control over the amount of capacity to install for each product. In section three, we analyze the situation of a firm which needs to decide on the amount of capacity to install for one of its products and the price to set for the other product. This is a situation facing firms who introduce a new and improved product which has the potential to cannibalize sales from its existing product and where the firm only has a limited amount of capacity online for this new product. An example of this would be a microchip manufacturer launching the next generation of microchips after its existing product has been cloned. In section four, we examine the case where both products have a given capacity constraint and the firm sets prices for both products. This situation is commonplace for firms who have no ability to increase capacity in the near term but can only control sales and profits by adjusting their prices.

We end in section five with a discussion on how the decisions made in the previous sections differ when decisions regarding price and/or capacity are made sequentially instead of simultaneously. These situations can arise when a firm has different brand managers making decisions to maximize each product's profit independently rather than maximizing system wide profits.

We use the following notation throughout the paper:

P_a: price of product A;

P_b: price of product B;

C_a: production capacity for product A;

C_b: production capacity for product B;

q_a: per unit variable cost of product A;

q_b: per unit variable cost of product B;

i_a : cost of adding one unit of dedicated capacity for product A;

i_b : cost of adding one unit of dedicated capacity for product B;

$u_a(P_a, P_b)$: mean demand for product A;

$u_b(P_a, P_b)$: mean demand for product B.

For analytical simplicity, we assume throughout the paper that:

(A1) the demands for product A and B are distributed uniformly over the intervals $[u_a(P_a, P_b) - r, u_a(P_a, P_b) + r]$ and $[u_b(P_a, P_b) - s, u_b(P_a, P_b) + s]$ where r and s are the ranges of realizable demands above/below their respective means.

We assume further that:

(A2) the unit investment costs plus variable production costs do not exceed the product price; $i_a + q_a < P_a$ and $i_b + q_b < P_b$ with $i_a, q_a, i_b, q_b > 0$;

(A3) the mean demand of product A, $u_a(P_a, P_b)$, is decreasing in P_a and increasing in P_b and the mean demand of product B is increasing in P_a and decreasing in P_b.

Note that (A3) immediately follows from the fact that the products are substitutable.

2 Capacity Decisions for a Price Taking Firm

We begin our discussion with a firm that manufactures two products and must decide on the amount of capacity to install to manufacture each product. In this situation the firm is a price-taker in both of the markets in which it competes. For this situation, Lau and Lau [7] provide a solution procedure for determining the optimal capacities to achieve a given probability of obtaining a profit target. In our model, we maximize the expected profit and derive the sensitivity of the optimal capacities to changes in key parameters.

The firm's profit function is defined as:

$$\begin{aligned} R(C_a, C_b) \quad = \quad & (P_a - q_a) \cdot \int_{u_a(P_a,P_b)-r}^{C_a} x_a \cdot f_a(P_a, P_b, x_a) dx_a \\ & + (P_a - q_a) \cdot C_a \cdot \int_{C_a}^{u_a(P_a,P_b)+r} f_a(P_a, P_b, x_a) dx_a \\ & + (P_b - q_b) \cdot \int_{u_b(P_a,P_b)-s}^{C_b} x_b \cdot f_b(P_a, P_b, x_b) dx_b \\ & + (P_b - q_b) \cdot C_b \cdot \int_{C_b}^{u_a(P_a,P_b)+s} f_b(P_a, P_b, x_b) dx_b \\ & - i_a \cdot C_a - i_b \cdot C_b, \end{aligned} \tag{2.1}$$

where

$f_a(P_a, P_b, x_a) = \frac{1}{2r}$ and x_a is the demand for product A and

$f_b(P_a, P_b, x_b) = \frac{1}{2s}$ and x_b is the demand for product B.

We first show that a unique set of capacities exists that maximizes (2.1).

Proposition 1 *If $P_a > q_a > 0$ and $P_b > q_b > 0$, then there exists a unique maximum of (2.1) over $C_a > 0$ and $C_b > 0$. The optimal capacities are:*

$C_a = (u_a(P_a, P_b) + r) - \frac{2 \cdot i_a \cdot r}{(P_a - q_a)}$ *and*

$C_b = (u_b(P_a, P_b) + s) - \frac{2 \cdot i_b \cdot s}{(P_b - q_b)}$.

Proof: To ensure that a unique maximum exists for positive values of C_a and C_b, a solution must exist to the first order conditions and all second order conditions must be satisfied. The first order conditions are:

$\frac{\partial R(C_a, C_b)}{\partial C_a} = -i_a - \frac{C_a \cdot (P_a - q_a)}{2r} + \frac{(u_a(P_a, P_b) + r) \cdot (P_a - q_a)}{2r} = 0$

$\frac{\partial R(C_a, C_b)}{\partial C_b} = -i_b - \frac{C_b \cdot (P_b - q_b)}{2s} + \frac{(u_b(P_a, P_b) + s) \cdot (P_b - q_b)}{2s} = 0.$

Rearranging the terms above, we obtain:

$$C_a = (u_a(P_a, P_b) + r) - \frac{2 \cdot i_a \cdot r}{(P_a - q_a)} \geq 2r\left(1 - \frac{i_a}{(P_a - q_a)}\right) \text{ and} \tag{2.2}$$

$$C_b = (u_b(P_a, P_b) + s) - \frac{2 \cdot i_b \cdot s}{(P_b - q_b)} \geq 2s\left(1 - \frac{i_b}{(P_b - q_b)}\right), \tag{2.3}$$

which are positive by assumption (A2) that some profit can be earned on each product.

The sufficient conditions for the existence of a maximum are:

$$\frac{\partial^2 R(C_a, C_b)}{\partial C_a^2} = -\frac{(P_a - q_a)}{2r} < 0 \tag{2.4}$$

and

$$\frac{\partial^2 R(C_a, C_b)}{\partial C_a^2} \cdot \frac{\partial^2 R(C_a, C_b)}{\partial C_b^2} - \left(\frac{\partial^2 R(C_a, C_b)}{\partial c_a \partial C_b}\right)^2 = \frac{(P_a - q_a)(P_b - q_b)}{4rs} > 0, \tag{2.5}$$

which both hold by assumption (A2), completing the proof. □

We now turn our attention to how the optimal capacity decisions vary with changes in the key parameters of the model.

Proposition 2 *The changes in the optimal capacity mix to changes in the parameters are:*

(a) if P_a increases, then C_a decreases, if $\left|\frac{\partial u_a(P_a,P_b)}{\partial P_a}\right| > \frac{2 \cdot i_a \cdot r}{(P_a-q_a)^2}$, *and C_b increases;*

(b) if P_b increases, then C_a increases and C_b decreases if $\left|\frac{\partial u_b(P_a,P_b)}{\partial P_b}\right| > \frac{2 \cdot i_b \cdot s}{(P_b-q_b)^2}$ *;*

(c) if q_a increases, then C_a decreases and C_b does not change;

(d) if q_b increases, then C_a does not change and C_b decreases;

(e) if i_a increases, then C_a decreases and C_b does not change;

(f) if i_b increases, then C_a does not change and C_b decreases;

(g) if r increases, then C_a increases, if $\frac{(P_a-q_a)-2i_a}{(P_a-q_a)} > 0$, *and C_b does not change;*

(h) if s increases, then C_a does not change and C_b increases if $\frac{(P_b-q_b)-2i_b}{(P_b-q_b)} > 0$.

Proof: We show below the methodology for (a). The proofs of the other cases are similar.

From Equation 2.2, we know that:

$C_a = (u_a(P_a, P_b) + r) - \frac{2 \cdot i_a \cdot r}{(P_a-q_a)}$

Taking the derivative with respect to P_a, we obtain

$\frac{\partial C_a}{\partial P_a} = \frac{\partial u_a(P_a,P_b)}{\partial P_a} + \frac{2 \cdot i_a \cdot r}{(P_a-q_a)^2}$.

Since $\frac{\partial u_a(P_a,P_b)}{\partial P_a} < 0$ by assumption(A3), we therefore conclude that C_a will decrease if $\left|\frac{\partial u_a(P_a,P_b)}{\partial P_a}\right| > \frac{2 \cdot i_a \cdot r}{(P_a-q_a)^2}$ when P_a increases.

Similarly, from Equation 2.3, we know that

$C_b = (u_b(P_a, P_b) + s) - \frac{2 \cdot i_b \cdot s}{(P_b-q_b)}$ and thus $\frac{\partial C_b}{\partial P_a} = \frac{\partial u_b(P_a,P_b)}{\partial P_a} > 0$ by assumption (A3). Therefore, we conclude that C_b will always increase as P_a increases. □

Parts (a) and (b) of Proposition 2 point to an interesting phenomenon; namely that an expectation of a price increase in a product which will result in a decrease in the mean demand for that product does not necessarily result in the firm decreasing the available capacity for that product. Essentially, the condition in part (a) shows that if the mean decrease in demand is not fast enough and the cost of capacity is not too expensive, it may in some cases be more profitable to increase capacity for that product. This is because even though on average there is less demand for that product, the unit profit may be higher and therefore more capacity might be profitable. The following example shows the possible behaviors for C_a mentioned above.

Example 1: Case (i): C_a and C_b both increase with an increase in P_a.

In this case, let $q_a = 3$, $q_b = 2$, C= 2000, $c_1 = 60$, $c_2 = 50$, r= 400, D= 3000, $d_1 = 100$, $d_2 = 19$, s= 250, $i_a = 1$, $i_b = 1$, $P_a = 6$, and $P_b = 10$. Here, the optimal solution is $C_a = 2273.33$, $C_b = 2301.5$, and $R(P_a, P_b) = 18592.58$. If we increase P_a from 6 to 7, we observe that C_a increases to 2280, C_b increases to 2320.5, and $R(P_a, C_b)$ increases to 20652.25.

Case (ii): C_a decreases and C_b increases with an increase in P_a . For this example, suppose $q_a = 3$, $q_b = 2$, C= 2000, $c_1 = 60$, $c_2 = 50$, r= 400, D= 3000, $d_1 = 100$, $d_2 = 19$, s= 250, $i_a = 1$, $i_b = 1$, $P_a = 10$, and $P_b = 10$. In this case, the optimal solution is $C_a = 2185.71$, $C_b = 2377.5$ and $R(P_a, P_b) = 26168.39$.

By increasing P_a to 11, we find that the optimal C_a decreases to 2140, the optimal C_b increases to 2396.5 and $R(P_a, P_b)$ increases to 27774.25. □.

Similarly, parts (g) and (h) of Proposition 2 show that an increase in the variability of demand for a product may result in an increase or a decrease of the capacity for that product. However, changes in variable production or capacity costs always have monotonic consequences on optimal capacity levels.

3 Setting the Price of A and the Capacity Level of B

In this section we suppose that a firm currently has fixed (limited) capacity for a new product it is introducing but must decide how much capacity to maintain for its existing product. For example, in electronics manufacturing, when a firm first introduces a new product, the capacity is extremely limited due to low yields for the new product and the necessary time for the factory to ramp up for production (e.g., building a new faster chip). On the other hand, the older product that the firm produces may already be a stable product in a market with active competition. For the older product, the firm might have the option to change its capacity; however, it is a price taker for it. Therefore, the firm faces the joint problem of setting a capacity level for one product and a price level for the other.

In this situation, the firm's profit function is defined as:

$$
\begin{aligned}
R(P_a, C_b) \quad = \quad & (P_a - q_a) \cdot \int_{u_a(P_a,P_b)-r}^{C_a} x_a \cdot f_a(P_a, P_b, x_a) dx_a \\
& + (P_a - q_a) \cdot C_a \cdot \int_{C_a}^{u_a(P_a,P_b)+r} f_a(P_a, P_b, x_a) dx_a \\
& + (P_b - q_b) \cdot \int_{u_b(P_a,P_b)-s}^{C_b} x_b \cdot f_b(P_a, P_b, x_b) dx_b \\
& + (P_b - q_b) \cdot C_b \cdot \int_{C_b}^{u_a(P_a,P_b)+s} f_b(P_a, P_b, x_b) dx_b - i_b \cdot C_b.
\end{aligned} \tag{3.6}
$$

In this section and the next, we assume linear mean demand functions. In particular, we assume that:

(A4) the mean demand of product A, $u_a(P_a, P_b) = C - c_1 P_a + c_2 P_b > 0$ and $u_b(P_a, P_b) = D - d_1 P_a + d_2 P_b > 0$.

The use of such linear functions to model product demands is widespread in the economics literature (see Bulow [3] and Stokey [16]). We also assume that:

(A5) $c_1 > c_2 > 0$, $d_1 > d_2 > 0$, $c_1 > d_2 > 0$, and $d_1 > c_2 > 0$.

Note that these assumptions are reasonable for the following reasons:

- the assumption that $c_1 > c_2$ and $d_1 > d_2$ means that the demand function for product A is more sensitive to changes in the price of product A than changes in the price of product B (and similarly for the demand function for product B), and
- the assumption that $c_1 > d_2$ (and $d_1 > c_2$) ensures that an increase in the price of a product cannot increase overall demand, since for example increasing the price of

product A would decrease the mean demand for A more than it would increase the mean demand for B (i.e., one cannot create extra demand by increasing the prices of both products).

Finally,

(A6) for products whose capacity cannot change, we assume that their capacity is above the minimum realizable demand.

Proposition 3 *If $u_a(P_a, P_b)$ and $u_b(P_a, P_b)$ are both linear in P_a and P_b and the capacity C_a for product a exceeds the lower bound on demand, $u_a(P_a, P_b) - r$, then the function $R(P_a, C_b)$ achieves a unique maximum over $P_a > q_a > 0$ and $P_b > q_b > 0$.*

Proof:

It is sufficient as in Proposition 1 to show existence of a solution to the first order conditions and to show that the Hessian, H, of $R(P_a, C_b)$ is negative definite over $P_a > q_a > 0$ and $P_b > q_b > 0$. The first order conditions are:

$$\begin{aligned} \frac{\partial R(P_a, C_b)}{\partial P_a} &= \frac{[C_a^2 - (u_a(P_a, P_b) - r)^2]}{2r} + \frac{C_a \cdot [u_a(P_a, P_b) + r - C_a]}{2r} \\ &+ \frac{(P_a - q_a) \cdot [C_a - (u_a(P_a, P_b) - r)]}{2r} \cdot \frac{\partial u_a(P_a, P_b)}{\partial P_a} \\ &- \frac{(P_b - q_b) \cdot [C_b - (u_b(P_a, P_b) - z)]}{2z} \cdot \frac{\partial u_b(P_a, P_b)}{\partial P_a} \\ &= 0, and \end{aligned} \tag{3.7}$$

$$\frac{\partial R(P_a, C_b)}{\partial C_b} = -i_b + \frac{(u_b(P_a, P_b) + s - C_b) \cdot (P_b - q_b)}{2s} = 0. \tag{3.8}$$

Solving the above equations, we can obtain the optimal values for P_a and C_b. For the second order conditions, the terms of H are:

$H_{11} = \frac{\partial^2 R(P_a, C_b)}{\partial P_a^2} = \frac{[C_a - (u_a(P_a, P_b) - r)]}{r} \cdot \frac{\partial u_a(P_a, P_b)}{\partial P_a}$

$$-\frac{(P_a-q_a)}{2r}\left[\left(\frac{\partial u_a(P_a,P_b)}{\partial P_a}\right)^2-(C_a-(u_a(P_a,P_b)-r))\cdot\frac{\partial^2 u_a(P_a,P_b)}{\partial P_a^2}\right]$$
$$-\frac{(P_b-q_b)}{2s}\left[\left(\frac{\partial u_b(P_a,P_b)}{\partial P_a}\right)^2-(C_b-(u_b(P_a,P_b)-s))\cdot\frac{\partial^2 u_b(P_a,P_b)}{\partial P_a^2}\right]$$
$$H_{21}=H_{12}=\frac{\partial^2 R(P_a,C_b)}{\partial P_a \partial C_b}=\frac{(P_b-q_b)}{2s}\cdot\frac{\partial u_b(P_a,P_b)}{\partial P_a}$$
$$H_{22}=\frac{\partial^2 R(P_a,C_b)}{\partial C_b^2}=-\frac{(P_b-q_b)}{2s}.$$

The sufficient conditions for concavity are:

$$H_{11}<0, H_{22}<0, \tag{3.9}$$

and

$$H_{11}H_{22}-H_{12}^2>0. \tag{3.10}$$

For H_{11}, notice that $\frac{\partial u_a(P_a,P_b)}{\partial P_a}<0$ by assumption (A3) and the capacity of A is greater than the minimum realizable demand by (A6). The first term is, therefore, negative. The second derivative elements in the next two terms of H_{11} vanish so that the two terms both become negative and yield $H_{11}<0$. For H_{22}, negativity follows by assumption (A2).

Inequality (3.10) follows by first noting that the $(\frac{(P_b-q_b)}{2s}\cdot\frac{\partial u_b(P_a,P_b)}{\partial P_a})^2$ terms from $H_{11}H_{22}$ and H_{12}^2 cancel. The remaining terms are

$$H_{11}H_{22}-H_{12}^2=\left(\left(\frac{[C_a-(u_a(P_a,P_b)-r)]}{r}\cdot\frac{\partial u_a(P_a,P_b)}{\partial P_a}\right)-\frac{(P_a-q_a)}{2r}\left(\frac{\partial u_a(P_a,P_b)}{\partial P_a}\right)^2\right)\left(-\frac{(P_b-q_b)}{2s}\right),$$

which is the product of two negatives by our previous arguments in the proof of (3.9). The result follows. □

Once again, we are interested in how the optimal decisions change as a function of the optimal parameters.

Proposition 4 *Assuming the conditions in Proposition 3,*

(a) if q_a increases, then P_a increases and C_b increases;

(b) if q_b increases, then P_a decreases and C_b decreases;

(c) if i_b increases, then P_a decreases and C_b decreases;

(d) if C increases, then P_a increases and C_b increases;

(e) if c_1 increases, then P_a decreases and C_b decreases;

(f) if c_2 increases, then P_a increases and C_b increases;

(g) if D increases, then P_a does not change and C_b increases;

(h) if d_1 increases, then P_a does not change and C_b decreases;

(i) if d_2 increases, then P_a increases and C_b increases;

(j) if C_a increases, P_a and C_b decrease.

Proof:

The goal here is to identify the effect of changes in one parameter on the optimal solution of the first order conditions in (3.7) and (3.8). To reflect changes in parameters other than the decision variables, we add extra terms to the definition of R so that $R(P_a, C_b)$ is also written as $R(P_a, C_b, q_a)$ when we explicitly consider changes in q_a.

In this case, the first order conditions in (3.7) and (3.8) are

$$\nabla_{C_b,P_a} R(P_a, C_b, q_a) = 0,$$

where ∇_{C_b,P_a} refers to the partial derivatives with respect to C_b and P_a alone. We suppose the solution to (3.7) and (3.8) is (C_b^*, P_a^*) when $q_a = q_a^*$.

In the following, we use the notation $\nabla^2_{x,y/z}$ for the partial differential operator given by

$$\begin{pmatrix} \frac{\partial^2}{\partial x \partial z} \\ \frac{\partial^2}{\partial y \partial z} \end{pmatrix}.$$

We next consider changes from q_a^* to $q_a = q_a^* + \delta_{q_a}$. The solutions of (3.7) and (3.8) are then $P_a = P_a^* + \delta_{P_a}$ and $C_b = C_b^* + \delta_{C_b}$. We wish to find the sign of δ_{P_a} and δ_{C_b} given $\delta_q > 0$.

First, using the first order properties, implicit function theorem, and that $\nabla_{C_b,P_a} R(P_a^*, C_b^*, q_a^*) = 0$, we must have

$$\begin{aligned} 0 &= \nabla_{C_b,P_a} R(P_a, C_b, q_a) \\ &= \nabla_{C_b,P_a/P_a} R(P_a^*, C_b^*, q_a^*)\delta_{P_a} + \nabla_{C_b,P_a/C_b} R(P_a^*, C_b^*, q_a^*)\delta_{C_b} + \nabla_{C_b,P_a/q_a} R(P_a^*, C_b^*, q_a^*)\delta_{q_a} \\ &\quad + \epsilon_{P_a}\delta_{P_a} + \epsilon_{C_b}\delta_{C_b} + \epsilon_{q_a}\delta_{q_a}, \end{aligned}$$

where ϵ_{P_a}, ϵ_{C_b}, and ϵ_{q_a} all approach zero as δ_{q_a} approaches zero. For small changes δ_{q_a}, we therefore seek δ_{P_a} and δ_{C_b} to solve

$$\nabla_{C_b,P_a/P_a} R(P_a^*, C_b^*, q_a^*)\delta_{P_a} + \nabla_{C_b,P_a/C_b} R(P_a^*, C_b^*, q_a^*)\delta_{C_b} = -\nabla_{C_b,P_a/q_a} R(P_a^*, C_b^*, q_a^*)\delta_{q_a}.$$

To simplify the notation, let:

$$\begin{aligned} H_{q_a} &= [\nabla^2_{C_b,P_a} R(P_a, C_b, q_a)] \\ &= \begin{bmatrix} \frac{\partial^2 R(P_a,C_b)}{\partial P_a^2} & \frac{\partial^2 R(P_a,C_b)}{\partial P_a \partial C_b} \\ \frac{\partial^2 R(P_a,C_b)}{\partial P_a \partial C_b} & \frac{\partial^2 R(P_a,C_b)}{\partial C_b^2} \end{bmatrix} \end{aligned}$$

It is straightforward to show that the determinant of the matrix H_{q_a} has the same sign as the Hessian of $R(P_a, C_b)$ in the proof of Proposition 3 since the only addition has been δ to the P_a terms of that matrix.

Thus, we have

$$\begin{aligned} \delta_{P_a} &= \frac{\det \begin{bmatrix} -\delta_{q_a} \frac{\partial^2 R(P_a,C_b)}{\partial P_a \partial q_a} & \frac{\partial^2 R(P_a,C_b)}{\partial P_a \partial C_b} \\ -\delta_{q_a} \frac{\partial^2 R(P_a,C_b)}{\partial C_b \partial q_a} & \frac{\partial^2 R(P_a,C_b)}{\partial C_b^2} \end{bmatrix}}{\det H_{q_a} > 0} \\ &= \frac{\det \begin{bmatrix} \delta_{q_a} \left[\frac{[C_a-(u_a(P_a,P_b)-r)]}{2r} \cdot \frac{\partial u_a(P_a,P_b)}{\partial P_a} \right] & \frac{(P_b-q_b)}{2s} \cdot \frac{\partial u_b(P_a,P_b)}{\partial P_a} \\ 0 & -\frac{(P_b-q_b)}{2s} \end{bmatrix} > 0}{\det H_{q_a} > 0}. \end{aligned}$$

Since $\frac{\partial u_a(P_a,P_b)}{\partial P_a} < 0$, the numerator above will always be positive. Therefore, as q_a increases, P_a increases.

The change in C_b is

$$\delta_{C_b} = \frac{\det \begin{bmatrix} \frac{\partial^2 R(P_a,C_b)}{\partial P_a^2} & -\delta_{q_a} \frac{\partial^2 R(P_a,C_b)}{\partial P_a \partial q_a} \\ \frac{\partial^2 R(P_a,C_b)}{\partial P_a \partial C_b} & -\delta_{q_a} \frac{\partial^2 R(P_a,C_b)}{\partial C_b \partial q_a} \end{bmatrix}}{\det H_{q_a} > 0}$$

$$
= \frac{\det \begin{bmatrix} \begin{array}{l} \frac{[C_a-(u_a(P_a,P_b)-r)]}{r} \cdot \frac{\partial u_a(P_a,P_b)}{\partial P_a} \\ -\frac{(P_a-q_a)}{2r}\left(\frac{\partial u_a(P_a,P_b)}{\partial P_a}\right)^2 \\ -\frac{(P_b-q_b)}{2s}\left(\frac{\partial u_b(P_a,P_b)}{\partial P_a}\right)^2 \end{array} & \delta_{q_a}\left[\frac{[C_a-(u_a(P_a,P_b)-r)]}{2r} \cdot \frac{\partial u_a(P_a,P_b)}{\partial P_a}\right] \\ \frac{(P_b-q_b)}{2s} \cdot \frac{\partial u_b(P_a,P_b)}{\partial P_a} & 0 \end{bmatrix}}{\det H_{q_a} > 0}.
$$

Since $\frac{\partial u_a(P_a,P_b)}{\partial P_a} < 0$ and $\frac{\partial u_a(P_a,P_b)}{\partial P_b} > 0$, the numerator above is again positive by assumption (A3). Therefore, as q_a increases, C_b increases .

Repeating the procedure above, we arrive at the other conclusions for (b) through (i) in Proposition 4.

(j) In the case where we increase C_a, we have four potential cases: (i) P_a and C_b increase, (ii) P_a increases and C_b decreases, (iii) P_a decreases and C_b increases, (iv) P_a and C_b decrease.

Cases (i) and (ii) are not possible, since given the added capacity, one would never increase price. If P_a increases, then demand will decrease for product A and increase for product B. If this strategy improved profits in case (i), it would have been done at the lower level of capacity for product A. In case (ii) if the price of A increased, we would not decrease the capacity of product B since it would be in greater demand. We may also discard (iii) as a plausible outcome given that when P_a decreases demand for product B will also decrease. Hence, we would not want to increase the capacity of product B. If this strategy would improve the overall profits, it would have been implemented under the initial capacity level of A. Therefore, we are left with the conclusion that as C_a increases, P_a and C_b decrease. □

Example 2: When P_b is changed, we may observe different behaviors in the decision variables. We now provide examples of the three possible behaviors that can occur when we increase P_b.

Case (i): P_a and C_b increase

In this case let $C_a = 500$, $q_a = 2$, $q_b = 2$, C= 2000, $c_1 = 100$, $c_2 = 99$, r= 420, D= 3000, $d_1 = 100$, $d_2 = 10$, s= 1000, $i_b = 1$ and $P_b = 5$. Here, the optimal solution is $P_a = 18.24$,

$C_b = 3015.8$ and $R(P_a, P_b) = 12218.8$.

If we increase P_b from 5 to 6, we observe that P_a increases to 19.16 ,C_b increases to 3091.628 and $R(P_a, C_b)$ increases to 15006.61.

Case (ii): P_a increases and C_b decreases

For this example suppose $C_a = 500$, $q_a = 2$, $q_b = 2$, C= 2000, $c_1 = 350$, $c_2 = 18$, r= 420, D= 3000, $d_1 = 100$, $d_2 = 99$, s= 1000, $i_b = 1$ and $P_b = 15$. We obtain an optimal solution of $P_a = 5.29$, $C_b = 2869.44$ and $R(P_a, C_b) = 24510.26$.

By increasing P_b to 16, we find that the optimal P_a increases to 5.34, the optimal C_b decreases to 2785.52 and $R(P_a, C_b)$ increases to 25312.3.

Case (iii): P_a decreases and C_b increases

In this example let: $C_a = 500$, $q_a = 2$, $q_b = 2$, C= 2000, $c_1 = 350$, $c_2 = 99$, r= 420, D= 3000, $d_1 = 100$, $d_2 = 5$, s= 1000, $i_b = 1$ and $P_b = 3$. The optimal solution is $P_a = 5.36$, $C_b = 1726.814$ and $R(P_a, C_b) = 1181.003$

By increasing P_b to 3.3, we see that P_a decreases to 5, C_b increases to 2156.558 and $R(P_a, C_b)$ increases to 1867.28. □.

In Proposition 4, we discover that, when a firm has control over the price of one of its products and the capacity level of the other, changes in the production costs of either product have an impact on both decision variables. If the production cost of the product which we have price control over increases, then we should raise the price on this product and also increase the capacity of the other product. On the other hand, if the production cost increases on the product whose price we cannot control, it is best to decrease its capacity and to increase the price of the other product. We find that if the capacity investment costs for product B increase, we should increase our price on product A and reduce the capacity for product B. We also discover that we will increase the price of product A and the capacity of product B if the mean demand for product A increases by a constant (i.e., the mean demand shifts upwards). However, if the mean demand for product B changes in the same fashion, we will keep the price of product A the same and increase the capacity for product B. In the case where the price elasticity for product A(c_1) increases, we will decrease both the price of product A and the capacity level for product B. When the price elasticity for product B increases however, we will not change the price of product A but we will decrease the capacity for product B. If the cross price elasticities increase, we should increase both the price of product A and the capacity for product B. When the capacity for product A is increased, then the price of product A and the capacity for product B will decrease. We also observe that when the price of the product in the market in which a firm is a price taker increases, the firm's optimal strategy will differ depending on the price elasticities of the firm's products.

4 Pricing Decisions: Two Products with Capacity Constraints

We now examine the case where both products being manufactured have a given capacity constraint and the firm must set prices for both of its products. The situation modeled here reflects the situation where the high investment costs for new capacity make it unprofitable to build new capacity (e.g., because it is forecast that the products are at the end of their lifecycles and therefore it is not possible to recoup the costs of new investment) and therefore the firm can only exercise pricing control. We will again assume that the demand for products A and B are distributed uniformly and that the mean demands are linear in prices. The profit function is now defined as:

$$\begin{aligned} R(P_a, P_b) = & (P_a - q_a) \cdot \int_{u_a(P_a,P_b)-r}^{C_a} x_a \cdot f_a(P_a, P_b, x_a) dx_a \\ & + (P_a - q_a) \cdot C_a \cdot \int_{C_a}^{u_a(P_a,P_b)+r} f_a(P_a, P_b, x_a) dx_a \\ & + (P_b - q_b) \cdot \int_{u_b(P_a,P_b)-s}^{C_b} x_b \cdot f_b(P_a, P_b, x_b) dx_b \end{aligned} \tag{4.11}$$

$$+(P_b - q_b) \cdot C_b \cdot \int_{C_b}^{u_a(P_a,P_b)+s} f_b(P_a, P_b, x_b) dx_b$$

Proposition 5 *Under assumptions (A1) through (A6), the function $R(P_a, P_b)$ is concave in P_a and P_b.*

Proof: The first order conditions for optimality are:

$$\begin{aligned} \frac{\partial R(P_a, P_b)}{\partial P_a} &= \frac{[C_a^2 - (u_a(P_a, P_b) - r)^2]}{2r} + \frac{C_a \cdot [u_a(P_a, P_b) + r - C_a]}{2r} \\ &\quad + \frac{(P_a - q_a) \cdot [C_a - (u_a(P_a, P_b) - r)]}{2r} \cdot \frac{\partial u_a(P_a, P_b)}{\partial P_a} \\ &\quad - \frac{(P_b - q_b) \cdot [C_b - (u_b(P_a, P_b) - s)]}{2s} \cdot \frac{\partial u_b(P_a, P_b)}{\partial P_a} \\ &= 0 \end{aligned} \tag{4.12}$$

$$\begin{aligned} \frac{\partial^2 R(P_a, P_b)}{\partial P_b} &= \frac{[C_b^2 - (u_b(P_a, P_b) - s)^2]}{2s} + \frac{C_b \cdot [u_b(P_a, P_b) + s - C_b]}{2s} \\ &\quad + \frac{(P_b - q_b) \cdot [C_b - (u_b(P_a, P_b) - s)]}{2s} \cdot \frac{\partial u_b(P_a, P_b)}{\partial P_b} \\ &\quad - \frac{(P_a - q_a) \cdot [C_a - (u_a(P_a, P_b) - r)]}{2r} \cdot \frac{\partial u_a(P_a, P_b)}{\partial P_b} \\ &= 0 \end{aligned} \tag{4.13}$$

The terms of the Hessian of the profit function are:

$$H_{11} = \frac{\partial^2 R(P_a,P_b)}{\partial P_a^2} = \frac{[C_a - (u_a(P_a,P_b) - r)]}{r} \cdot \frac{\partial u_a(P_a,P_b)}{\partial P_a}$$
$$- \frac{(P_a - q_a)}{2r} \left[\left(\frac{\partial u_a(P_a,P_b)}{\partial P_a} \right)^2 - (C_a - (u_a(P_a, P_b) - r)) \cdot \frac{\partial^2 u_a(P_a,P_b)}{\partial P_a^2} \right]$$
$$- \frac{(P_b - q_b)}{2s} \left[\left(\frac{\partial u_b(P_a,P_b)}{\partial P_a} \right)^2 - (C_b - (u_b(P_a, P_b) - s)) \cdot \frac{\partial^2 u_b(P_a,P_b)}{\partial P_a^2} \right]$$
$$H_{21} = H_{12} = \frac{\partial^2 R(P_a,P_b)}{\partial P_a \partial P_b} = \frac{[C_a - (u_a(P_a,P_b) - r)]}{2r} \cdot \frac{\partial u_a(P_a,P_b)}{\partial P_b} + \frac{[C_b - (u_b(P_a,P_b) - s)]}{2s} \cdot \frac{\partial u_b(P_a,P_b)}{\partial P_a}$$
$$- \frac{(P_a - q_a)}{2r} \left[\frac{\partial u_a(P_a,P_b)}{\partial P_a} \frac{\partial u_a(P_a,P_b)}{\partial P_b} - (C_a - (u_a(P_a, P_b) - r)) \cdot \frac{\partial^2 u_a(P_a,P_b)}{\partial P_a \partial P_b} \right]$$
$$- \frac{(P_b - q_b)}{2s} \left[\frac{\partial u_b(P_a,P_b)}{\partial P_a} \frac{\partial u_b(P_a,P_b)}{\partial P_b} - (C_b - (u_b(P_a, P_b) - s)) \cdot \frac{\partial^2 u_b(P_a,P_b)}{\partial P_a \partial P_b} \right]$$
$$H_{22} = \frac{\partial^2 R(P_a,P_b)}{\partial P_b^2} = \frac{[C_b - (u_b(P_a,P_b) - s)]}{s} \cdot \frac{\partial u_b(P_a,P_b)}{\partial P_b}$$
$$- \frac{(P_b - q_b)}{2s} \left[\left(\frac{\partial u_b(P_a,P_b)}{\partial P_b} \right)^2 - (C_b - (u_b(P_a, P_b) - s)) \cdot \frac{\partial^2 u_b(P_a,P_b)}{\partial P_b^2} \right]$$

$-\frac{(P_a-q_a)}{2r}\left[\left(\frac{\partial u_a(P_a,P_b)}{\partial P_b}\right)^2-(C_a-(u_a(P_a,P_b)-r))\cdot\frac{\partial^2 u_a(P_a,P_b)}{\partial P_b^2}\right].$

The sufficient conditions for concavity are:

$$H_{11}<0, H_{22}<0, \tag{4.14}$$

and

$$H_{11}H_{22}-H_{12}^2>0. \tag{4.15}$$

For H_{11} and H_{22}, notice that $\frac{\partial u_a(P_a,P_b)}{\partial P_a}<0$ and $\frac{\partial u_b(P_a,P_b)}{\partial P_b}<0$ by assumption (A3) and C_a and C_b are greater than their respective minimum realizable demands by assumption (A6). The first terms of H_{11} and H_{22} are, therefore, negative. The second derivative elements in H_{11} and H_{22} vanish so that the last two terms of both become negative and thus yield $H_{11}<0$ and $H_{22}<0$.

When we calculate the determinant of the Hessian and simplify, we obtain the following expression

$\frac{(C_a-(u_a(P_a,P_b)-r)(C_b-(u_b(P_a,P_b)-s)(c_1d_1-c_2d_2)}{2rs}+\frac{(C_a-(u_a(P_a,P_b)-r)(P_b-q_b)(c_1d_1^2-c_2d_1d_2)}{2rs}$
$+\frac{(C_b-(u_b(P_a,P_b)-s)(P_a-q_a)(c_1^2d_1-c_1c_2d_2)}{2rs}+\frac{(P_a-q_a)(P_b-q_b)(c_1^2d_1^2-c_1c_2d_1d_2)}{2rs}$
$+\frac{(P_a-q_a)(P_b-q_b)c_2^2d_2^2}{4rs}>0$

We know from assumption (A5) that $c_1>c_2$, $d_1>d_2$ and from assumption (A6) the capacities for each product will be greater than or equal to the lower bounds on their respective demands. In addition, as mentioned in Section 1, we assume that the price of each product is greater than the sum of the variable production costs and investment costs (A2). Consequently, all of the terms in the expression above will also be positive and thus the determinant of the Hessian is positive. □

Proposition 6 *Under the assumptions of Proposition 5,*

(a) if C increases, then P_a increases, P_b increases;

(b) if c_1 increases, then P_a decreases, P_b decreases;

(c) if c_2 increases, then P_a increases, P_b increases.

Given the symmetry of the problem, the parameters D, d_1 and d_2 will exhibit the same behavior as their counterparts above.

Proof: (a) Following the same methodology as in Section 3, we will now show the case where the demand parameter C is increased from C to $C+\delta$. The proofs of the other parameters are done similarly.

We again have two linear demand curves: $\mathrm{u}_a(\mathrm{P}_a, P_b) = C - c_1 P_a + c_2 P_b$ and $\mathrm{u}_b(\mathrm{P}_a, P_b) = D - d_1 P_b + d_2 P_a$.

We define H_C as:

$$H_C = \begin{bmatrix} \frac{\partial^2 R(P_a,P_b)}{\partial P_a^2} & \frac{\partial^2 R(P_a,P_b)}{\partial P_a \partial P_b} \\ \frac{\partial^2 R(P_a,P_b)}{\partial P_a \partial P_b} & \frac{\partial^2 R(P_a,P_b)}{\partial P_b^2} \end{bmatrix}$$

It is straightforward to show that the determinant of the matrix H_C has the same sign as the Hessian of $R(P_a, P_b)$ in the proof of Proposition 5 so that $\det H_C > 0$. We now examine the changes in the variables to an increase in the parameter C. The change in the price of product A when C increases is:

$$\begin{aligned}
\delta_{P_a} &= \frac{\det \begin{bmatrix} -\delta\frac{\partial^2 R(P_a,P_b)}{\partial P_a \partial C} & \begin{matrix} \frac{(C_a-(u_a(P_a,P_b)-r))c_2}{2r} + \frac{(C_b-(u_b(P_a,P_b)-s))d_2}{2s} \\ + \frac{(P_a-q_a)c_1c_2}{2r} + \frac{(P_b-q_b)d_1d_2}{2s} \end{matrix} \\ -\delta\frac{\partial^2 R(P_a,P_b)}{\partial P_b \partial C} & -\frac{(C_b-(u_b(P_a,P_b)-s))d_1}{s} - \frac{(P_a-q_a)c_2^2}{2r} - \frac{(P_b-q_b)d_1^2}{2s} \end{bmatrix}}{\det H_C > 0} \\
&= \frac{\det \begin{bmatrix} -\frac{\delta(C_a-(u_a(P_a,P_b)-r))}{2r} - \frac{\delta(P_a-q_a)c_1}{2r} & \begin{matrix} \frac{(C_a-(u_a(P_a,P_b)-r))c_2}{2r} + \frac{(C_b-(u_b(P_a,P_b)-s))d_2}{2s} \\ + \frac{(P_a-q_a)c_1c_2}{2r} + \frac{(P_b-q_b)d_1d_2}{2s} \end{matrix} \\ \frac{\delta c_2(P_a-q_a)}{2r} & -\frac{(C_b-(u_b(P_a,P_b)-s))d_1}{s} - \frac{(P_a-q_a)c_2^2}{2r} - \frac{(P_b-q_b)d_1^2}{2s} \end{bmatrix}}{\det H_C > 0}.
\end{aligned}$$

As discussed above, the denominator will always be positive. When we calculate the determinant in the numerator and simplify, we obtain

$$\frac{\delta(C_b-(u_b(P_a,P_b)-s)(P_a-q_a)(c_1d_2-c_2d_2)}{2rs} + \frac{\delta(P_a-q_a)(P_b-q_b)(c_1d_1^2-c_2d_1d_2)}{4rs}$$

$$+\frac{\delta(C_a-(u_a(P_a,P_b)-r)(C_b-(u_b(P_a,P_b)-s)d_1}{4rs}+\frac{(C_a-(u_a(P_a,P_b)-r)(P_b-q_b)d_1^2}{4rs}>0.$$

From assumption (A5) we know that $c_1 > c_2$, $d_1 > d_2$. From from assumption (A6), we know that the capacities for each product will be greater than or equal to the lower bounds on their respective demands. We also know from assumption (A2) that the price of each product must exceed the combined variable production costs and the per unit investment cost for capacity. Consequently, all of the terms in the expression above are positive. Therefore, we conclude that if C increases, the price of product A will increase.

The change in the price of product B when C increases is:

$$\delta_{P_b} = \frac{\det\begin{bmatrix} -\frac{(C_a-(u_a(P_a,P_b)-r))c_1}{r}-\frac{(P_a-q_a)c_1^2}{2r}-\frac{(P_b-q_b)d_2^2}{2s} & -\delta\frac{\partial^2 R(P_a,P_b)}{\partial P_a \partial C} \\ \frac{(C_a-(u_a(P_a,P_b)-r))c_2}{2r}+\frac{(C_b-(u_b(P_a,P_b)-s))d_2}{2s} & \\ +\frac{(P_a-q_a)c_1c_2}{2r}+\frac{(P_b-q_b)d_1d_2}{2s} & -\delta\frac{\partial^2 R(P_a,P_b)}{\partial P_b \partial C} \end{bmatrix}}{\det H_C > 0}$$

$$= \frac{\det\begin{bmatrix} -\frac{(C_a-(u_a(P_a,P_b)-s))c_1}{r}-\frac{(P_a-q_a)c_1^2}{2r}-\frac{(P_b-q_b)d_2^2}{2s} & -\frac{\delta(C_a-(u_a(P_a,P_b)-r))}{2r}-\frac{\delta(P_a-q_a)c_1}{2r} \\ \frac{(C_a-(u_a(P_a,P_b)-r))c_2}{2r}+\frac{(C_b-(u_b(P_a,P_b)-s))d_2}{2s} & \\ +\frac{(P_a-q_a)c_1c_2}{2r}+\frac{(P_b-q_b)d_1d_2}{2s} & \frac{\delta c_2(P_a-q_a)}{2r} \end{bmatrix}}{\det H_C > 0}.$$

For the numerator in this case we obtain the following expression:

$$-\frac{\delta(P_a-q_a)(P_b-q_b)(c_1d_1d_2-c_2d_2^2)}{4rs}+\frac{\delta(C_a-(u_a(P_a,P_b)-r))^2c_2}{4r^2}+\frac{\delta(P_a-q_a)(C_b-(u_b(P_a,P_b)-s))c_1d_2}{2rs}$$
$$+\frac{\delta(C_a-(u_a(P_a,P_b)-r))(C_b-(u_b(P_a,P_b)-s))d_2}{4rs}+\frac{\delta(P_b-q_b)(C_a-(u_a(P_a,P_b)-r))c_1d_2}{4rs}>0.$$

Since $c_1 > c_2$, $d_1 > d_2$ by assumption (A5) and the prices of each product are greater than the variable production costs as discussed in the proof of δ_{P_a} above, the first term of the expression will always be positive. All of the other terms will also be positive since the capacities of both products will be greater than or equal to their respective lower bounds by assumption (A6). Therefore, we conclude that if C increases, the price of product B will also increase. □

Example 3: If C_a increases, then any of the following cases are possible:

Case (i): $\mathbf{P}_a$ and $\mathbf{P}_b$ decrease

In this example let: $C_a = 1000$, $C_b = 1000$, $q_a = 2$, $q_b = 2$, C= 2000, $c_1 = 50$, $c_2 = 35$, r= 400, D= 3000, $d_1 = 50$, $d_2 = 35$ and s= 500. In this case the optimal solution is $P_a = 98.03$, $P_b = 109.28$ and $R(P_a, P_b) = 174435.5$.

If we increase C_a from 1000 to 1001, we observe that with the set of parameters above, P_a decreases to 98 ,P_b decreases to 109.27 and $R(P_a, P_b)$ increases to 174474.30.

Case (ii): $\mathbf{P}_a$ and $\mathbf{P}_b$ increase

In this case suppose: $C_a = 1000$, $C_b = 1000$, $q_a = 2$, $q_b = 2$, C= 2000, $c_1 = 1000$, $c_2 = 18$, r= 400, D= 3000, $d_1 = 21$, $d_2 = 19$ and s= 1000. In this case the optimal solution is $P_a = 2.24$, $P_b = 90.91$and $R(P_a, P_b) = 72452.92$.

In this case, increasing C_a to 1001, we find that the optimal P_a increases to 3.14, the optimal P_b increases to 92.22 and $R(P_a, P_b)$ increases to 73461.97.

Case (iii): $\mathbf{P}_a$ decreases and $\mathbf{P}_b$ increases

For this example let: $C_a = 1000$, $C_b = 1000$, $q_a = 2$, $q_b = 2$, C= 2000, $c_1 = 60$, $c_2 = 50$, r= 400, D= 3000, $d_1 = 1000$, $d_2 = 19$ and s= 250. In this case the optimal solution is $P_a = 21.71$, $P_b = 2.86$ and $R(P_a, P_b) = 16293.95$.

By increasing C_a to 1001, we see that P_a decreases to 21.70, P_b increases to 2.860694 and $R(P_a, P_b)$ increases to 16299.87.

Note that the case where P_a increases and P_b decreases is not possible because this strategy would adversely affect the number of product A being sold and thus would not help the firm take advantage of the additional capacity. In fact, if the firm were able to improve its profits using this strategy, it would have done so before adding more capacity. Since our problem is symmetric in A and B, we see that C_b can also exhibit the same behavior. □

Example 4: When the production costs of the products change, we have more than one potential outcome. We show below examples how changes in q_a affect our optimal pricing strategy. Given the symmetry of the problem, the same outcomes can occur for q_b.

As the product cost of product A (q_a) increases, we have the following possible cases:

Case (i): P_a and P_b increase

In this example let: $C_a = 1000$, $C_b = 1000$, $q_a = 2$, $q_b = 2$, C= 2000, $c_1 = 50$, $c_2 = 35$, r= 400, D= 3000, $d_1 = 50$, $d_2 = 35$ and s= 500. In this case the optimal solution is $P_a = 98.03$, $P_b = 109.28$ and $R(P_a, P_b) = 174435.5$.

If we increase q_a from 2 to 3, we observe that P_a increases to 98.15, P_b increases to 109.35 and $R(P_a, P_b)$ decreases to 173578.5.

Case (ii): P_a increases and P_b decreases

For this example let: $C_a = 1000$, $C_b = 500$, $q_a = 2$, $q_b = 1$, C= 2000, $c_1 = 50$, $c_2 = 49$, r= 400, D= 1000, $d_1 = 51$, $d_2 = 49$ and s= 250. In this case the optimal solution is $P_a = 588.41$, $P_b = 576.96$ and $R(P_a, P_b) = 696029.8$.

In this case, increasing q_a to 3, we find that the optimal P_a increases to 588.42, the optimal P_b decreases to 576.95 and $R(P_a, P_b)$ decreases to 695218.8.

It is clear that the case where both prices decrease is not possible since it would only adversely affect the profitability of both products. The case where P_a decreases and P_b increases is also not possible since this strategy would have been pursued before the increase in production costs if it were possible to increase the profit function's value. □

From proposition 6, we discover that changes in the price elasticities of the products have different effects on the optimal pricing strategy. When a product's demand becomes more sensitive to changes in its price, the optimal strategy is to decrease the price of both products. On the other hand, if the cross price elasticity increases, the optimal prices of both products increases. When production costs change, the optimal pricing scheme will be different depending on the parameters of the mean demand functions and the available capacities. We observe that when production costs increase a firm will either increase the price of the product whose cost has increased and decrease the price of the other or increase both products' prices. Similarly, the parameters of the model affect the optimal pricing strategy when the capacity of one the products is changed. In the case of an increase in capacity, we have shown cases where a firm may increase or decrease the price of both of its products or it may decrease the price on the product that has the additional capacity while increasing the price of the other.

5 Individual-Product Optimal Decisions

In the previous sections, we examined cases where a firm makes decisions about both of its products' capacities/prices simultaneously. We also assumed that the firm was trying to maximize overall profits (from both product lines). However, it is often the case that firms have separate brand managers for each product line and that these brand managers are evaluated based upon the profitability of their product line alone. In that case, the brand manager will make decisions in order to maximize profits of his or her own product line alone rather than to maximize overall profits for the firm. In this section, we analyze how each of the decisions analyzed in Sections 2-4 would be changed by the fact that they are made by managers trying to maximize individual-product optimal decisions rather than globally optimal decisions maximizing the sum of the profits from the two product lines. We therefore differentiate between the *globally optimal* decisions in Sections 2-4 and the *individually optimal* decisions in this section.

We also note that in this section we pay attention to the *order* in which decisions are made and announced. We will show below that this order is significant when both managers can only make pricing decisions but the order does not affect the eventual decisions otherwise. We explore various assumptions about the managers' behavior as in the classical Cournot, Stackelberg, and collusion models of duopoly.

5.1 Capacity decisions

We return to the case in Section 2 where the firm determines the optimal capacities for both of its products; however, we now assume that products A and B have their own brand managers. Regardless of whether manager A or B makes its capacity decision first; the profit function for the manager of product A is:

$$\mathrm{R}_1(\mathrm{C}_a) = (P_a - q_a) \cdot \int_{u_a(P_a,P_b)-r}^{C_a} x_a \cdot f_a(\mathrm{P}_a, P_b, x_a) dx_a$$

$+(P_a - q_a) \cdot C_a \cdot \int_{C_a}^{u_a(P_a,P_b)+r} f_a(P_a, P_b, x_a) dx_a - i_a \cdot C_a$ where $f_a(\mathrm{P}_a, P_b, x_a) = \frac{1}{2r}$ and x_a is the demand for product A.

Similarly, product B's profit function is defined as:

$\mathrm{R}_2(\mathrm{C}_b) = (P_b - q_b) \cdot \int_{u_b(P_a,P_b)-s}^{C_b} x_b \cdot f_b(\mathrm{P}_a, P_b, x_b) dx_b + (P_b - q_b) \cdot C_b \cdot \int_{C_b}^{u_a(P_a,P_b)+s} f_b(P_a, P_b, x_b) dx_b - i_b \cdot C_b$

$f_b(\mathrm{P}_a, P_b, x_b) = \frac{1}{2s}$ and x_b is the demand for product B.

Proposition 7 *The individually optimal C_a and C_b are the same as the globally optimal C_a and C_b of Proposition 1.*

Proof: The result directly follows from the fact that the first order conditions for C_a and C_b are the same as those in Section 2. □

Proposition 7 states that if brand managers can only make capacity decisions, even if they make individually optimal decisions, they will end up maximizing global profits as one manager's capacity decision does not affect the other's decision. In the next subsections, however, we show that when pricing decisions are involved (which affect the other product's demand as well), individually optimal decisions differ from globally optimal decisions.

5.2 Deciding on the price of A and the capacity level of B

We now examine the case where the optimal price of A, P_a^*, and the optimal capacity for B, C_b^*, is determined.

The manager maximizing product A's profits has the following objective function:

$$R_1(P_a) = (P_a - q_a) \cdot \int_{u_a(P_a,P_b)-r}^{C_a} x_a \cdot f_a(\mathrm{P}_a, P_b, x_a) dx_a$$

$$+(P_a - q_a) \cdot C_a \cdot \int_{C_a}^{u_a(P_a,P_b)+r} f_a(P_a, P_b, x_a) dx_a,$$

where: $f_a(\mathrm{P}_a, P_b, x_a) = \frac{1}{2r}$ and x_a is the demand for product A.

Note that the manager of product A does not care about the decision that the manager of product B makes about the capacity for product B. However, the manager of product B does care about the price of product A as this affects the demand for product B. Therefore, the optimal C_b is a function of the optimal P_a which is obtained by solving the first order condition above. Product B's profit function is defined the same as $R_2(C_b)$ in the previous subsection.

The first order condition of $R_1(P_a)$ is:
$\frac{\partial R_1(P_a)}{\partial P_a} = \frac{[C_a^2-(u_a(P_a,P_b)-r)^2]}{2r} + \frac{(P_a-q_a)\cdot[C_a-(u_a(P_a,P_b)-r)]}{2r} \cdot \frac{\partial u_a(P_a,P_b)}{\partial P_a}$
$+\frac{C_a\cdot[u_a(P_a,P_b)+r-C_a]}{2r} = 0$. Rearranging terms and substituting in the functions for $u_a(P_a, P_b)$ and $u_b(P_a, P_b)$ we can obtain the optimal value of P_a.

Comparing the optimal P_a in this case with the optimal P_a derived in Section 3, we clearly see that the optimal pricing strategy for product A differs depending on whether individually optimal or globally optimal decisions are being made. The following example shows globally optimal price and capacity levels can be significantly different than individually optimal levels.

Example 5: Suppose that $C_a = 1700$, $q_a = 3$, $q_b = 2$, C= 2000, $c_1 = 60$, $c_2 = 50$, r= 400, D= 3000, $d_1 = 55$, $d_2 = 40$, s= 250, $i_b = 1$ and $P_b = 77.98$. We obtain a globally optimal solution of $P_a = 76.38$, $C_b = 2009.886$ and $R(P_a, C_b) = 228761.18$. Product A's profits in this case are 94560.83. For the same parameters, the individually optimal solutions are: $P_a = 68.82$ with a profit for product A of $R(P_a) = 105701.13$ and consequently $C_b = 1707.557$ with $R(C_b) = 111224.07$ resulting in a total profit of 216925.20. We see in this case that the manager of product A can improve the profitability of product A alone by decreasing its price compared to the globally optimal price of product A; however this decreases the sum of the profits from both products.

5.3 Pricing decisions

In Section 4, we discussed the case where a firm simultaneously sets the prices for both of its products. We now examine the case where managers for products A and B set prices in order to optimize individual product profits. Let us assume that the price of product A is set first. The profit function for product A will be the same as the previous subsection $\mathrm{R}_1(P_a)$. However, as we can see, the optimal price for A depends on what price will be chosen for product B. Since we assume that all pricing and capacity information is known, the decision maker for the price of product A can predict what product B's optimal pricing strategy will be after the decision is made regarding the price of A. Product B's profit function is defined as:

$$\mathrm{R}_2(\mathrm{P}_b) = (P_b - q_b) \cdot \int_{u_b(P_a,P_b)-s}^{C_b} x_b \cdot f_b(\mathrm{P}_a, P_b, x_b) dx_b$$

$$+(P_b - q_b) \cdot C_b \cdot \int_{C_b}^{u_a(P_a,P_b)+s} f_b(P_a, P_b, x_b) dx_b - i_b \cdot C_b,$$

where $f_b(\mathrm{P}_a, P_b, x_b) = \frac{1}{2s}$ and x_b is the demand for product B. Therefore, before determining product A's optimal price, we solve product B's problem.

The first order condition of $\mathrm{R}_2(P_b)$ is:

$$\frac{\partial R_2(P_b)}{\partial P_b} = \frac{[C_b^2-(u_b(P_a,P_b)-s)^2]}{2s} + \frac{(P_b-q_b)\cdot[C_b-(u_b(P_a,P_b)-s)]}{2s} \cdot \frac{\partial u_b(P_a,P_b)}{\partial P_b}$$

$$+\frac{C_b\cdot[u_b(P_a,P_b)+s-C_b]}{2s} = 0.$$

From the equation above, we can obtain an expression for the optimal price for product B, which may then be used to find the optimal price for product A. To find the optimal price for product A, we have three different approaches available: (i) the Cournot model, (ii) the Stackelberg model, and (iii) the collusion model. In the Cournot model, each product manager assumes that the price of her or his product does not affect the price of the other. In the Stackelberg, one manager follows the other's lead in setting price. In the collusion model, both managers act together to establish best prices for overall profit.

(i) Cournot model

In this approach, the manager of product A assumes an optimal price of product B, P_b^*, as a parameter in the first order condition of $\mathrm{R}_1(P_a)$ to obtain:

$$\frac{\partial R_1(P_a)}{\partial P_a} = \frac{[C_a^2-(u_a(P_a,P_b^*)-r)^2]}{2r} + \frac{(P_a-q_a)\cdot[C_a-(u_a(P_a,P_b^*)-r)]}{2r} \cdot \frac{\partial u_a(P_a,P_b^*)}{\partial P_a}$$

$$+\frac{C_a\cdot[u_a(P_a,P_b^*)+r-C_a]}{2r} = 0.$$

The optimal solution, P_a^*, was shown in Subsection 5.2 where P_b is now substituted with P_b^*. The manager of product B follows the same procedure to obtain a price with P_a^* as a parameter. Solving the simultaneous equations for the two prices produces the result.

(ii) Stackelberg model

For this approach, the manager of product A assumes that the manager of product B is the price setter. We substitute the optimal P_b^* *as a function of* P_a directly into the function $R_1(P_a)$ to obtain:.

$$R_1(P_a) = (P_a - q_a) \cdot \int_{u_a(P_a,P_b)-r}^{C_a} x_a \cdot f_a(P_a, P_b^*, x_a)dx_a$$

$$+(P_a - q_a) \cdot C_a \cdot \int_{C_a}^{u_a(P_a,P_b)+r} f_a(P_a, P_b^*, x_a)dx_a.$$

The first order condition for this function clearly differs from the Cournot approach shown above since P_b^* is dependent on P_a. Therefore, the optimal price for A differs from the price obtained in approach (i).

We now illustrate the above approaches. For analytical simplicity, we examine the case where both products have unlimited capacities and the mean demand functions for products A and B are $u_a(P_a, P_b) = C - c_1P_a + c_2P_b$ and $u_b(P_a, P_b) = D - d_1P_b + d_2P_a$ respectively and where $c_1 > c_2$, $d_1 > d_2, c_1 > d_2$, and $d_1 > c_2$.

(i) Cournot model

We first begin by finding the optimal price of product B. In this case, product B's profit function is defined as:

$$R_2(P_b) = (P_b - q_b) \cdot u_b(P_a, P_b).$$

The first order condition is:

$$u_b(P_a, P_b) + (P_b - q_b) \cdot \frac{\partial u_b(P_a, P_b)}{\partial P_b} = 0.$$

Rearranging terms, we obtain:

$$P_b^* = \frac{D + d_2P_a + q_bd_1}{2d_1}.$$

We now solve for the optimal price for product A. The profit function for product A is:

$$R_1(P_a) = (P_a - q_a) \cdot u_a(P_a, P_b).$$

The first order condition is then:

$$u_a(P_a, P_b^*) + (P_a - q_a) \cdot \frac{\partial u_a(P_a, P_b^*)}{\partial P_a} = 0.$$

Substituting P_b^* into the equation above and solving for P_a we obtain the optimal price of A to be:

$$P_a^* = \frac{Dc_2 + 2Cd_1 + 2c_1d_1q_a + c_2d_1q_b}{4c_1d_1 - c_2d_2}.$$

(ii) Stackelberg model

We again begin by finding the optimal price of product B. The profit function for product B is the same as in the Cournot model above and hence the optimal price of B, P_b^*, will be same expression.

The profit function for product A is now defined as:

$$R_1(P_a) = (P_a - q_a) \cdot u_a(P_a, P_b^*).$$

Substituting P_b^* into the profit function and solving for P_a in the first order condition, we now have the optimal price of A to be:

$$P_a^* = \frac{Dc_2 + 2Cd_1 + 2c_1d_1q_a + c_2d_1q_b - c_2d_2q_a}{4c_1d_1 - 2c_2d_2}.$$

(iii) Setting both prices simultaneously(collusion)

We now return to the model presented in Section 4, where the product managers agree to set both prices simultaneously. In other words, there is collusion in setting the prices for both products in order to optimize system wide profits. We present below the optimal prices for the products where both products have no capacity constraints.

The profit function in this case is:

$$R(P_a, P_b) = (P_a - q_a) \cdot u_a(P_a, P_b) + (P_b - q_b) \cdot u_b(P_a, P_b).$$

The first order conditions are:

$$
\begin{aligned}
u_a(P_a, P_b) + (P_a - q_a) \cdot \frac{\partial u_a(P_a, P_b)}{\partial P_a} &= 0 \\
u_b(P_a, P_b) + (P_b - q_b) \cdot \frac{\partial u_b(P_a, P_b)}{\partial P_b} &= 0.
\end{aligned}
$$

Solving the equations for P_a and P_b we obtain:

$$P_a^* = \frac{C + c_1 q_a - d_2 q_b}{2c_1} + \frac{(c_2 + d_2) \cdot (2c_1(D - c_2 q_a + d_1 q_b) + (c_2 + d_2)(C + c_1 q_a - d_2 q_b))}{2c_1(4c_1 d_1 - (c_2 + d_2)^2)}$$

and

$$P_b^* = \frac{(c_2 + d_2) \cdot (2c_1(D - c_2 q_a + d_1 q_b) + (c_2 + d_2)(C + c_1 q_a - d_2 q_b))}{4c_1 d_1 - (c_2 + d_2)^2}.$$

From the expressions in (i), (ii), and (iii) we observe that the optimal prices obtained from the model depend on the price elasticity and cost parameters. We provide below examples where the optimal prices are different relative to each other based on these parameters.

Example 6: Collusion prices > Stackelberg prices > Cournot prices.

Suppose $q_a = 2$, $q_b = 2$, C$= 2000$, $c_1 = 60$, $c_2 = 30$, D$= 2000$, $d_1 = 60$ and $d_2 = 20$. In this case we obtain the following results:

	Collusion	Stackelberg	Cournot
$P_a =$	29.63	24	23.04
$P_b =$	29.51	21.67	21.51

Example 7: Stackelberg prices > Cournot prices > Collusion prices.

In this case let $q_a = 2$, $q_b = 200$, C$= 2500$, $c_1 = 175$, $c_2 = 2$, D$= 2000$, $d_1 = 79$ and $d_2 = 10$. The resulting prices are:

	Collusion	Stackelberg	Cournot
$P_a =$	8.52	11.109	11.107
$P_b =$	117.08	117.1296	117.1295

Example 8: (Collusion price of A > Cournot price of A > Stackelberg price of A) and (Cournot price of B > Stackelberg price of B > Collusion price of B).

For this example let $q_a = 200$, $q_b = 2$, C= 2500, $c_1 = 175$, $c_2 = 2$, D= 2000, $d_1 = 79$ and $d_2 = 10$. The optimal prices are:

	Collusion	Stackelberg	Cournot
$P_a =$	107.75	107.23	107.26
$P_b =$	19.31	20.444	20.447

From these examples, we may conclude that the timing of pricing decisions, the optimization of individual versus system wide profits along with the relative sizes of the price elasticities, and costs can have different impacts on a firm's optimal pricing strategy.

6 Conclusions and Further Research

In this paper, we addressed joint capacity and price decisions for substituable products. We have shown that pricing and capacity decisions are highly affected by the actual parameters that the decision makers can control. The presence of decision makers who optimize individual channel profits versus optimizing system wide has a significant impact on the decisons made.

Many research questions on pricing and capacity setting for substitutable products remain open. For example, the case where a firm has control over both products' prices and capacities remains open. (Preliminary research shows this problem to be extremely challanging.) Furthermore, we have only considered a single period problem in this paper. Further research should consider multiple period problems.

Bibliography

[1] Bassok, Y., R. Anupindi, and R. Akella, "Multiperiod Inventory Models with Substitution,"

Technical Report, Department of Industrial Engineering and Management Sciences, Northwestern University, Evanston, IL, 1993.

[2] Bitran, G.R. and Dasu, S., "Ordering policies in an environment of stochastic yields and substitutable demands," *Operations Research*, Vol. 42 (1992), 999-1017.

[3] Bulow, J. I., "Durable-goods monopolists," *Journal of Political Economy*, Vol. 90 (1982) 314-332.

[4] Carmon, T. and Nahmias, S., "A preliminary model for lot sizing in semiconductor manufacture," *Proceedings of the Seventh International Conference on Inventories*, (1994).

[5] Gerchak, Y., A. Tripathy, and K. Wang, "Co-Production Models with Random Functionality Yields," To appear in *IIE Transactions*, 1993.

[6] Hsu, A., and Bassok, Y., "Random Yield and Random Demand in a Production System with Downward Substitution," Technical Report, Industrial Engineering and Management Sciences Department, Northwestern University, Evanston, IL, 1994.

[7] Ignall, E.J. and Veinott, A.F. Jr., "Optimality of myopic policies for several substitute products," *Management Science*, Vol. 15 (1969) 284-304.

[8] Ismail, B. and Louderback, J., "Optimizing and satisficing in stochastic cost-volume-profit analysis," *Decision Sciences*, Vol. 10 (1979) 205-217.

[9] Kabak, I. and Schiff, A., "Inventory Models and Management objectives," *Sloan Management Review*, Vol. 10 (1978)

[10] Kreps, D.M. and Scheinkman, J.A., "Quantity precommitment and Bertrand competition yield Cournot outcomes," *Bell Journal of Economics*, Vol. 14 (1983) 326-337.

[11] Lau, H., "The newsboy problem under alternative optimization objectives," *Journal of Operational Research Society*, Vol. 31 (1980), 525-535

[12] Lau, A. and Lau, H., "The newsboy problem with price-dependent demand distribution," *IIE Transactions*, Vol. 20, No. 2 (1988) 168-175.

[13] Li, J., Lau, A. and Lau, H., "Some analytical results for a two-product newsboy problem," *Decision Sciences*, Vol. 21 (1990) 710-726.

[14] Sankarasubramanian, E. and Kumaraswamy, S., "Optimal ordering quantity for pre-determined level of profit," *Management Science*, Vol. 29 (1983) 512-514

[15] Staiger, R.W. and Wolak, F.A., "Collusive pricing with capacity constraints in the presence of demand uncertainty," *RAND Journal of Economics*, Vol. 23, No. 2 (1992) 203-220.

[16] Stokey, N. L., "Rational expectations and durable goods pricing," *Bell Journal of Economics*, Vol. 12 (1981) 112-128.