OPTIMAL ESTIMATION IN UNCERTAIN-STOCHASTIC DISCRETE-TIME SYSTEMS

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Abstract

In this paper we consider estimation problems in uncertain-stochastic discrete-time dynamic systems. To solve these problems, we prove necessary and sufficient conditions for the identifiability and optimality of parameter estimates in uncertain-stochastic linear regression. Using these results we derive optimal filtering and smoothing algorithms. We also present a suboptimal filtering algorithm for nonlinear uncertain-stochastic systems.

Keywords: uncertain-stochastic system, filtering, two-filter smoothing.

Section 1. Introduction.

The Kalman filter is known as the best mean square state estimator if true values of the input signal characteristics are known. These estimates are used widely in the such applied areas as a navigation, radar tracking, data processing in complicated electronic systems, numerous control problems etc. [1], [2],

[3],[4],[5].

In many practical situations *a priori* information about the input signal characteristics is incomplete. In order to overcome this serious obstacle one may use adaptive or robust filters [6], [7] , [8]. All of these filters use a limited amount of available *a priori* information. So the corresponding estimates may be very inaccurate if the lack of *a priori* information is serious. It is however well-known that very often it is possible to observe some components of the input vector. Our aim is to derive the recursive filtering and smoothing algorithms which allow us to obtain acceptable results even in the case when we know nothing about the characteristics of the input signal. That algorithm was presented in [9]. The optimal filtering algorithm described in this paper is a generalization of the previous one in the case of complex observations containing the information about both the state vector and the input signals.

The results below make the construction of an optimal two-filter smoothing algorithm for uncertain system processes possible. Similar results for stochastic systems are given in [10]. The suboptimal recursive filter for the uncertain-stochastic nonlinear system is also considered .

The results of the numerical experiments are given. Using input signal observations in the presented algorithm allows us to obtain accuracy of the new filtering estimates under *a priori* input uncertainty close to ones of the Kalman filter with known input signal characteristics. It also turns out that the smoothing estimates are significantly more accurate than the filtering ones. Thus, smoothing is preferable than filtering in the case of postexperimental data processing.

Section 2. Problem formulation.

Consider the dynamic system given by the following difference equation

$$x_t = a_t x_{t-h} + b_t u_t + \xi_t \quad , \quad t = h,2h,\ldots;\ x_0 = \nu \qquad (1)$$

where $x_t \in \mathbf{R}^n$ is a state vector; $\nu \in \mathbf{R}^n$ is an uncertain initial condition; $u_t \in R^m$ is an uncertain input vector, $\xi_t \in R^n$ is a random input vector with known characteristics $E\{\xi_t\} = 0$; $cov(\xi_t, \xi_s) = \delta_t^s C_t$; $h > 0$ is a time increment. Here and further we suppose ν and u_t to be unrestricted and nonrandom.

Let us consider the following observation model

$$\begin{cases} y_0 = \varphi_0 \nu + \omega_0 , \\ y_t = \varphi_t u_t + \psi_t x_t + \eta_t \xi_t + \omega_t , \ t = h, 2h, \ldots, \end{cases} \qquad (2)$$

where $\omega_t \in R^n$ is a vector of random disturbances with known characteristics $E\{\omega_t\} = 0$; $cov(\omega_t, \omega_s) = \delta_t^s Q_t$,$Q_t > 0$ for all $t \geq 0$. Matrices a_t, b_t, φ_t, ψ_t, η_t of appropriate dimensions are known. We suppose $\{\xi_t\}$, $\{\omega_t\}$ to be independent.

Our aim is to construct a linear estimate $\hat{x}_t$ given observations $\{ y_\tau, 0 \leq \tau \leq t \}$, which minimizes the following criterion

$$J_t = \sup_{\{u_\tau\}_0^t, \nu} E\{ (x_t - \hat{x}_t)^T \Sigma (x_t - \hat{x}_t) \} , \qquad (3)$$

where $\Sigma \geq 0$ is known symmetric weight matrix.

Section 3. Parameter identifiability of the uncertain-stochastic linear regression model.

In this section we state some preliminary results concerning the general type of the linear regression model.

Consider the multidimension linear regression

$$Y = \Phi U + H \Xi + \Omega , \qquad (4)$$

where $U \in R^q$ is an uncertain (nonrandom) vector; $\Xi \in R^s$ $\Omega \in R^l$ are independent random vectors with known characteristics

$E\{\Xi\} = 0$; $E\{\Omega\} = 0$; $cov(\Xi, \Xi) = C$; $cov(\Omega, \Omega) = Q > 0$.

Our aim is to find the best linear minimax estimate (BLME) $\hat{x} = \alpha Y$ of $x = A U + D \Xi$, which minimizes the following criterion

$$\tilde{J}(\hat{x}) = \sup_U E\{ (x - \hat{x})^T \Sigma (x - \hat{x}) \} , \qquad (5)$$

Let us call x identifiable if there exists at least one estimate

$\tilde{x} = \varphi(Y)$ ($\varphi(.)$ is some measurable function of Y) for which $\tilde{J}(\tilde{x}) < \infty$.

Proposition 1. i) If x is identifiable then

$$\Sigma A \Phi^{+}\Phi = \Sigma A \qquad (6)$$

where Φ^{+} is the matrix Φ pseudoinverse;

ii) if x is identifiable then $\hat{x}$ is BLME if and only if

$$\Sigma \operatorname{cov}(x - \hat{x} , Y - \Phi \hat{U}) = 0 \qquad (7)$$

where $\hat{U}$ is an arbitrary estimate of U , which satisfies

$$\Sigma E \{ x - A \hat{U} \} = 0 ;$$

iii) BLME $\hat{x}$ is also the best estimate within the class of $\tilde{J}$ -optimal unbiased estimates, if Ξ and Ω are Gaussian random vectors.

The proof is given in the **Appendix** .

Section 4. Optimal filtering algorithm.

First we find the conditionally-optimal solution within the class of all linear *unbiased recursive* filters

$$\hat{x}_t = \alpha_t \hat{x}_{t-h} + \beta_t y_t \quad , \quad E\{ x_t - \hat{x}_t \} = 0.$$

Denote $\mu_t = \psi_t b_t + \varphi_t$, $\kappa_t = \psi_t + \eta_t$, and let $\Delta_t = x_t - \hat{x}_t$ be the estimate error, where

$$\Delta_t = x_t - \hat{x}_t = (a_t - \alpha_t - \beta_t \psi_t a_t) x_{t-h} + \xi_t + \alpha_t \Delta_{t-h} + (b_t - \beta_t \mu_t) u_t - \beta_t (\kappa_t \xi_t + \omega_t)$$

The following conditions are necessary and sufficient for $\hat{x}_t$ to be unbiased

$$\begin{cases} \alpha_t = (I - \beta_t \psi_t) a_t \\ \beta_t = b_t \mu_t^{+} + z_t (I - \mu_t \mu_t^{+}) \\ b_t \mu_t \mu_t^{+} = b_t \end{cases} \qquad (8)$$

where z_t is an arbitrary matrix of appropriate dimension. Then

$\Delta_t = \varepsilon_t - z_t (I - \mu_t \mu_t^{+}) \nu_t$, where

$\nu_t = \kappa_t \xi_t + \omega_t + \psi_t a_t \Delta_{t-h}$, $\varepsilon_t = a_t \Delta_{t-h} + \xi_t - b_t \mu_t^{+} \nu_t$, $\operatorname{cov}(\nu_t, \nu_t) = \lambda_t$.

Minimizing criterion J_t and taking into account (8) we obtain

$$z_t = (a_t k_{t-h} a_t^T \psi_t^T + C_t \kappa_t^T - b_t \mu_t^+ \lambda_t)\ (I-\mu_t\mu_t^+)^T\ ((I-\mu_t\mu_t^+)\lambda_t(I-\mu_t\mu_t^+)^T)^+ ,$$

$$k_t = cov(\ \Delta_t,\Delta_t\ ) = cov(\ \varepsilon_t,\varepsilon_t\ ) - cov(\ \varepsilon_t,\nu_t\ )(I-\mu_t\mu_t^+)^T \times$$
$$((I-\mu_t\mu_t^+)\lambda_t(I-\mu_t\mu_t^+)^T)^+(I-\mu_t\mu_t^+)\,cov(\ \nu_t,\varepsilon_t\ ) \qquad (9)$$

From (9) and the matrix Schwartz inequality [7] it follows that

$$\mu_t^+ = (\ \mu_t^T\lambda_t^{-1}\mu_t\ )^+\mu_t^T\lambda_t^{-1} .$$

In the case when $\varphi_0^T\varphi_0 > 0$, the initial conditions are

$$\begin{cases} \hat{x}_0 = (\ \varphi_0^T Q_0^{-1}\varphi_0\ )^{-1}\varphi_0^T Q_0^{-1} y_0 , \\ k_0 = (\ \varphi_0^T Q_0^{-1}\varphi_0\ )^{-1}. \end{cases} \qquad (10)$$

Hence the filtering algorithm is as follows

$$\begin{cases} \bar{x}_t = a_t\hat{x}_{t-h}\ ;\ \bar{k}_t = a_t k_{t-h} a_t^T + C_t\ ;\ \mu_t = \psi_t b_t + \varphi_t\ ,\ \kappa_t = \psi_t + \eta_t, \\ \rho_t = \bar{k}_t\psi_t^T + C_t\eta_t^T\ ;\ \lambda_t = \psi_t a_t k_{t-h} a_t^T\psi_t^T + \kappa_t C_t \kappa_t^T + Q_t\ , \\ \mu_t^+ = (\ \mu_t^T\lambda_t^{-1}\mu_t\ )^+\mu_t^T\lambda_t^{-1}\ , \qquad (11) \\ \hat{x}_t = \bar{x}_t + [\ b_t\mu_t^+ + \rho_t\lambda_t^{-1}(\ I - \mu_t\mu_t^+\ )\ ]\ [\ y_t - \psi_t\bar{x}_t\ ]\ , \\ k_t = \bar{k}_t + b_t\mu_t^+\ \lambda_t(\ b_t\mu_t^+)^T - b_t\mu_t^+\ \rho_t^T - (\ b_t\mu_t^+\ \rho_t^T\ )^T - \rho_t\lambda_t^{-1}(I-\mu_t\mu_t^+)\rho_t^T \end{cases}$$

Proposition 2. i) The following conditions are sufficient for $\hat{x}_t$ to be the J_t- optimal estimate

$$\begin{cases} b_\tau\mu_\tau\mu_\tau^+ = b_\tau\ ,\ \text{for all}\ \tau \in [\ 0,t\ ], \\ \varphi_0^T\varphi_0 > 0\ ; \end{cases}$$

ii) the estimate $\hat{x}_t$ given by (10),(11) is the solution of the filtering problem for (1)-(3) not only within the recursive estimates, but also within all linear estimates,

iii) $\hat{x}_t$ is the best unbiased estimate if $\{\ \xi_t\},\{\ \omega_t\}$ and ω_0

are Gaussian.

The proof immediately follows from the fact, that $\hat{x}_t$ given by (10),(11) satisfies the conditions (6),(7) of Proposition 1.

Section 5. Fixed interval smoothing algorithm.

Consider the dynamic system (1) without the random component $\{ \xi_t \}$ of the input signal

$$x_t = a_t x_{t-h} + b_t u_t \quad , \quad t = h,2h,\ldots; \; x_0 = \nu \qquad (12)$$

In this case (12) defines a pure uncertain system. Let us also consider an observation model of the special type

$$\begin{cases} y_0 = \varphi_0 \nu + \omega_0 , \\ y_t = \varphi_t u_t + \omega_t^1 , \quad t = h,2h,\ldots,T-h ,T. \\ z_t = \psi_t x_t + \omega_t^2 , \\ y_T = \varphi_T x_T + \omega_T , \end{cases} \qquad (13)$$

where $\varphi_\tau^T \varphi_\tau > 0$, $t = 0,\ldots,T$; $\mathrm{cov}(\omega_t^1,\omega_s^1) = \delta_t^s Q_t$; $Q_t > 0$; $\mathrm{cov}(\omega_t^2,\omega_s^2) = \delta_t^s P_t$; $P_t > 0$; $\mathrm{cov}(\omega_t^2,\omega_s^2) = 0$.

Consider the problem of finding a J_t^s-optimal linear estimate $\hat{x}_t^s$ of x_t, $t \in [0,T]$ given all observations (13) where

$$J_t^s = \sup_{\{ u_\tau \}_0^T,\nu} E \{ (x_t - \hat{x}_t^s)^T \Sigma (x_t - \hat{x}_t^s) \}$$

Using (12) we may easily derive the reversed-time dynamic system

$$x_{t-h}^r = a_t^{-1} x_t^r - a_t^{-1} b_t u_t \; , \; x_T^r = x_T \; , \; t = T, T-h,\ldots,0 \qquad (14)$$

which is pathwise-equivalent to (12).

By applying (10),(11) we obtain from (12),(13) the forward-time estimate

$$\begin{cases} \hat{x}_0 = (\varphi_0^T Q_0^{-1} \varphi_0)^{-1} \varphi_0^T Q_0^{-1} y_0 , \\ k_0 = (\varphi_0^T Q_0^{-1} \varphi_0)^{-1}, \end{cases} \qquad (15)$$

$$\begin{cases} \bar{x}_t = a_t \hat{x}_{t-h} + b_t(\varphi_t^T Q_t^{-1} \varphi_t)^{-1} \varphi_t^T Q_t^{-1} y_t , \\ \bar{k}_t = a_t k_{t-h} a_t^T + b_t (\varphi_t^T Q_t^{-1} \varphi_t)^{-1} b_t^T , \end{cases} \qquad (16)$$

$$\begin{cases} \hat{x}_t = \bar{x}_t + \bar{k}_t \psi_t^T (\psi_t \bar{k}_t \psi_t^T + P_t)^{-1} [z_t - \psi_t \bar{x}_t] , \\ k_t = \bar{k}_t - \bar{k}_t \psi_t^T (\psi_t \bar{k}_t \psi_t^T + P_t)^{-1} \psi_t \bar{k}_t , \end{cases} \qquad (17)$$

Analogously from (14) it follows that the reversed-time estimate $\bar{x}_t^r$ is given by the equations

$$\begin{cases} \hat{x}_T^r = (\varphi_T^T Q_T^{-1} \varphi_T)^{-1} \varphi_T^T Q_T^{-1} y_t , \\ k_T^r = (\varphi_T^T Q_T^{-1} \varphi_T)^{-1}, \end{cases} \qquad (18)$$

$$\begin{cases} \bar{x}_{t-h}^r = a_t^{-1} \hat{x}_t^r - a_t^{-1} b_t (\varphi_t^T Q_t^{-1} \varphi_t)^{-1} \varphi_t^T Q_t^{-1} y_t , \\ \bar{k}_{t-h}^r = a_t^{-1} k_t^r (a_t^{-1})^T + a_t^{-1} b_t (\varphi_t^T Q_t^{-1} \varphi_t)^{-1} b_t^T (a_t^{-1})^T , \end{cases} \qquad (19)$$

$$\begin{cases} \hat{x}_{t-h}^r = \bar{x}_{t-h}^r + \bar{k}_{t-h}^r \psi_{t-h}^T (\psi_{t-h} \bar{k}_{t-h}^r \psi_{t-h}^T + P_{t-h})^{-1} [z_{t-h} - \psi_{t-h} \bar{x}_{t-h}^r] , \\ k_t^r = \bar{k}_t - \bar{k}_{t-h}^r \psi_{t-h}^T (\psi_{t-h} \bar{k}_{t-h}^r \psi_{t-h}^T + P_{t-h})^{-1} \psi_{t-h} \bar{k}_{t-h}^r . \end{cases} \qquad (20)$$

From (15)-(20) it follows that $\bar{x}_t^r$ is the BLME of x_t given all " past " observations and $\bar{x}_t^r$ is the BLME of x_t given all " future " observations, i.e. all observations obtained after the moment t up to **T** .

The following proposition describes the connection between $\hat{x}_t$, $\bar{x}_t^r$ and the best linear smoothing estimate $\hat{x}_t^s$.

Proposition 3.

$$\begin{cases} \hat{x}_t^s = k_t^s [(k_t)^{-1} \hat{x}_t + (\bar{k}_t^r)^{-1} \bar{x}_t^r], \\ k_t^s = [(k_t)^{-1} + (\bar{k}_t^r)^{-1}]^{-1} \end{cases} \qquad (21)$$

where k_t, $\bar{k}_t^r$ and k_t^s are error covariance matrices of $\hat{x}_t$, $\bar{x}_t^r$ and $\hat{x}_t^s$ respectively.
The proof of Proposition 3 also follows from Proposition 1.

Section 5. Nonlinear filtering algorithm.

Consider the nonlinear discrete-time uncertain-stochastic system

$$x_t = a(x_{t-h},t) + b(t)u_t + \xi_t\ ,\ x_0 = \nu \qquad (22)$$

where $a(x,t)$ is a known differentiable function with respect to x; ν and u_t are the uncertain initial condition and input signal respectively; $\{\xi_t\}$ is the same as in (1).

The observation model is given by

$$\begin{cases} y_0 = \varphi_0 \nu + \omega_0\ , \\ y_t = \varphi_t u_t + \omega_t^1\ ,\quad t = h,2h,\ldots \\ z_t = \psi(x_t,t) + \omega_t^2\ , \end{cases} \qquad (23)$$

where $\varphi_\tau^T \varphi_\tau > 0$, $t = 0,h,2h,\ldots$; $\psi(x,t)$ is a known differentiable function with respect to t .

We consider the problem of suboptimal estimation of the state vector in (22) given observations (23).

Using the linearization and the optimal linear uncertain-stochastic filtering equations (10),(11) we obtain the following suboptimal nonlinear recursive filter

$$\begin{cases} \hat{x}_0 = (\varphi_0^T Q_0^{-1} \varphi_0)^{-1} \varphi_0^T Q_0^{-1} y_0\ , \\ k_0 = (\varphi_0^T Q_0^{-1} \varphi_0)^{-1}, \end{cases} \qquad (24)$$

$$\begin{cases} \bar{x}_t = a(\hat{x}_{t-h},t) + b(t)(\varphi_t^T Q_t^{-1} \varphi_t)^{-1} \varphi_t^T Q_t^{-1} y_t\ , \\ \bar{k}_t = A_t k_{t-h} A_t^T + b(t)(\varphi_t^T Q_t^{-1} \varphi_t)^{-1} b^T(t) + C_t\ , \end{cases} \qquad (25)$$

$$\begin{cases} \hat{x}_t = \bar{x}_t + \bar{k}_t \Psi_t^T (\Psi_t \bar{k}_t \Psi_t^T + P_t)^{-1} [\, z_t - \psi(\bar{x}_t,t) - \Psi_t \bar{x}_t\,], \\ k_t = \bar{k}_t - \bar{k}_t \Psi_t^T (\Psi_t \bar{k}_t \Psi_t^T + P_t)^{-1} \Psi_t \bar{k}_t \end{cases} \qquad (26)$$

where $A_t = (\partial a/\partial x)|_{x=\hat{x}_{t-h}}$, $\Psi_t = (\partial \psi/\partial x)|_{x=\bar{x}_t}$

Equations (24)-(26) may be considered as the analog of the extended Kalman filter for the nonlinear difference dynamic system (22),(23) with the uncertain-stochastic input signals.

Section 7. Numerical example.

1. Let us consider a motion of the aircraft mass center

$$\begin{bmatrix} x_t^1 \\ x_t^2 \end{bmatrix} = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{t-h}^1 \\ x_{t-h}^2 \end{bmatrix} + \begin{bmatrix} 0 \\ u_t \end{bmatrix} + \begin{bmatrix} \xi_t^1 \\ \xi_t^2 \end{bmatrix}, \quad \begin{bmatrix} x_0^1 \\ x_0^2 \end{bmatrix} = \begin{bmatrix} \nu^1 \\ \nu^2 \end{bmatrix}, \qquad (27)$$

where x_t^1 is a distance between the aircraft and the origin of coordinates , x_t^2 is a speed, $\mathbf{u}_t$ is an unknown component of the acceleration which may be interpreted as uncertain input signal (control). Unobservable random disturbances ξ_t we assume to be a zero-mean discrete-time white noise with the covariance matrix **C**.

Information concerning { x_t^1 },{ x_t^2 } and { $\mathbf{u}_t$ } is given by the observations:

$$\begin{aligned} y_0^1 &= \nu^1 + \omega_0^1 \;; \quad y_0^2 = \nu^2 + \omega_0^2 \\ y_t^1 &= u_t + \omega_t \;; \quad y_t^2 = x_t^1 + \eta_t \end{aligned} \qquad (28)$$

where { ω_t } and { η_t } are zero-mean discrete white noises with the covariances **Q** and **P** respectively, $\omega_0 = (\omega_0^1, \omega_0^2)^T$ is a zero-mean Gaussian vector with the covariance matrix $\mathbf{Q}_0 = \mathbf{diag}\,(q_0^1, q_0^2)$, h is a time increment. ω_0 , { ξ_t }, { ω_t } and { η_t } are independent. The filtering problem is to calculate recursively the $\mathbf{J}_t$-optimal estimate of the vector $(x_t^1, x_t^2)^T$ given observations (28) . The following parameter values were selected for the numerical tests:

$\mathbf{Q} = 225$; $\mathbf{P} = 400$; $q_0^1 = 10^8$; $q_0^2 = 10^8$; $\mathbf{T} = 20$; $\mathbf{t} = 0.1$;
$x_0^1 = 5000$; $x_0^2 - 100$; $\mathbf{u(t)} = 20 + 10\,\mathbf{sin}(0.1\pi t)$;

$$\mathbf{C} = \begin{bmatrix} 33.3 & 50.0 \\ 50.0 & 100.0 \end{bmatrix}.$$

Selected parameter values **Q** and **P** correspond to accuracy of the on board acceleration sensor and radar respectively, noise ξ_t simulates the atmospheric disturbances, values x_0^1 , x_0^2 and function **u(t)** describe one of the possible modern aircraft manoeuvres.
We use the algorithm (10),(11). Table 1 gives the estimation

results: $\hat{\sigma}_1$ and $\hat{\sigma}_2$ are the estimation error standard deviations of the distance and the speed respectively. σ_1 and σ_2 are the estimation error standard deviations obtained by the Kalman filter in the ideal situation when the full information about u_t is *a priori* available. From the Table 1 it follows that the estimates given by the algorithm (10),(11) are close enough to those of the Kalman filter and may be used in the case of unknown but observable input signal when the Kalman filter is useless.

Time	$\hat{\sigma}_1$	σ_1	$\hat{\sigma}_2$	σ_2
0	10^4	10^4	10^4	10^4
1	20.0	20.0	56.85	57.08
3	15.12	15.49	14.63	18.82
5	14.39	15.17	14.09	18.70
7	14.39	15.17	14.07	18.68
9	14.38	15.17	14.06	18.68
15	14.38	15.17	14.06	18.68
20	14.38	15.17	14.06	18.68

Table 1. Standard deviations of the distance ($\hat{\sigma}_1$) and speed ($\hat{\sigma}_2$) estimation errors for the filter (10),(A1) and the corresponding values σ_1 and σ_2 for the Kalman filter.

2. Let us now consider some control system given by

$$\ddot{x} - 0.4\dot{x} + 0.16x = u_1 + 0.4\, u_1 + u_2 \;, \quad x(0) = x_0 \;, \dot{x}(0) = \dot{x}_0 \qquad (29)$$

where $\{ u_1 \}$,$\{ u_2 \}$,x_0 and $\dot{x}_0$ are unknown input signals and initial conditions. Equation (29) may be rewritten as the first order differential system

$$\left.\begin{aligned} \dot{x}_1 &= x_2 + u_1 \\ \dot{x}_2 &= -0,16x_1 - 0,4x_2 + u_2 \end{aligned}\right\} \qquad (30)$$

where $x(t) = x_1(t)$.
The observations are given by

$$\left.\begin{aligned} \dot{y}_1 &= u_1 + u_2 + \dot{\omega}_1 \\ \dot{y}_2 &= u_1 - u_2 + \dot{\omega}_2 \\ \dot{z} &= x_1 + \dot{\eta} \end{aligned}\right\} \quad t \in [\,0,3\,] \qquad (31)$$

where $\{\ \eta\ \}$ is the standard Wiener process, $\{\ (\omega_1,\omega_2)\ \}$ is the zero-mean Wiener process with the differential covariance

$$\mathrm{cov}(\ \omega,\omega\ ) = \mu \begin{bmatrix} 1 & 0,7 \\ 0,7 & 1 \end{bmatrix}$$

The known parameter μ characterizes the accuracy of the observations.

The fixed-interval smoothing problem is to calculate J_t^s-optimal estimate $\hat{x}_t^s$ for all $t \in [\ 0,3\ ]$. First we discretize the model (30),(31) with some small time increment h and then use the algorithm (21). Table 2 gives the estimation results: $\hat{\sigma}_1,\hat{\sigma}_2,\hat{\sigma}_3$ are the filter estimation error standard deviations obtained by $\mu = 0,1$; $\mu = 0,5$; $\mu = 4$ and $\sigma_1^s,\sigma_2^s,\sigma_3^s$ are the smoothing estimation error standard deviations respectively. From the Table 2 it follows that the smoothing estimate is significantly accurate than the corresponding filter estimate.

Time	μ = 0,1		μ = 0,5		μ = 4	
	$\hat{\sigma}_1$	σ_1^s	$\hat{\sigma}_2$	σ_2^s	$\hat{\sigma}_3$	σ_3^s
0.0	∞	1.305	∞	1.426	∞	1.736
0.3	3.511	1.047	3.515	1.098	3.532	1.236
0.6	2.474	0.843	2.486	0.897	2.536	1.076
0.9	1.995	0.704	2.017	0.796	2.108	1.041
1.2	1.701	0.637	1.736	0.759	1.874	1.036
1.5	1.496	0.637	1.546	0.760	1.732	1.037
1.8	1.340	0.681	1.409	0.783	1.641	1.038
2.1	1.218	0.747	1.306	0.822	1.582	1.045
2.4	1.118	0.820	1.227	0.884	1.542	1.075
2.7	1.030	0.891	1.165	0.978	1.515	1.182
3.0	0.957	0.957	1.117	1.117	1.497	1.497

Table 2. Standard deviations of the filtering estimate error ($\hat{\sigma}_1$, $\hat{\sigma}_2, \hat{\sigma}_3$) and the corresponding values $\sigma_1^s, \sigma_2^s, \sigma_3^s$ for the smoothing.

Appendix . Proof of the Proposition 1.

i) Let x be identifiable, i.e. there exists an estimate $\hat{x} = \varphi(Y)$:

$$\tilde{J} = \sup_{U} E\{ \| x - \varphi(Y) \|^2 \} = J^0 < \infty ,$$

where $\| a \|^2 = a^T \Sigma a$. Then the corresponding error Δ is

$$\Delta = x - \varphi(Y) = A U + D \Xi - \varphi(Y).$$

Then

$$\tilde{J} = \sup_{U} E\{ \| m \|^2 + \| \Delta - m \|^2 \},$$ where $m = E\{ \Delta \}$ is the biase of $\varphi(Y)$. Obviously $E\{ \varphi(Y) \} = \theta(V)$, where $V = E\{ Y \} = \Phi U$ and $\theta(V)$ is some nonrandom function. Then $m = A U - \theta(V)$.

Denote $R(\Phi)$ the linear span of the vector-columns of Φ. Suppose $V^* \in R(\Phi)$ and $\mathcal{U} = \{ U : \Phi U = V^* \}$. Then $U = U^* + (I - \Phi^+ \Phi) Z$

where $U^* \in U$ and Z is an arbitrary vector from R^q. Let $m^* = A U^* - \theta(V^*)$ and $G = A(I - \Phi^+\Phi)$, then for every $U \in U$ we have

$m = A(U^* + (I - \Phi^+\Phi)Z) - \theta(V^*) = m^* - GZ.$

$E\{\| x - \varphi(Y) \|^2\} \geq \| m \|^2 = \| m^* - GZ \|^2$ for any $Z \in R^q$. If $\Sigma G \neq 0$ then $f(Z) = \| m^* - GZ \|^2 = Z^T(G \Sigma G^T)Z - 2(m^*)^T \Sigma G Z + (m^*)^T \Sigma m^*$ is unrestricted from above. So there exists Z^0: $\| m^* - G Z^0 \|^2 > J^0$.

Thus, for $U^0 = U^* + (I - \Phi^+\Phi)Z^0$ and $Y^0 = \Phi U^0 + H \Xi + \Omega$ we have

$E\{\| x - \varphi(Y^0) \|^2\} \geq \| m^* - G Z^0 \|^2 > J^0$, which is impossible. From this contradiction, it follows that $\Sigma G = \Sigma A(I - \Phi^+\Phi) = 0$.

ii) Suppose that Z^* is the optimal matrix coefficient, i.e. $x = A\Phi^+Y + Z^*(I - \Phi\Phi^+)Y$ is the BLME. For arbitrary $Z = Z^* + \delta Z$ we have the corresponding estimate

$\tilde{x} = A\Phi^+Y + Z(I - \Phi\Phi^+)Y = \hat{x} + \delta Z(I - \Phi\Phi^+)Y.$

$\tilde{J}(\tilde{x}) = J_1(Z^*) + J_2(\delta Z) - J_3(Z^*, \delta Z)$, where

$J_1(Z^*) = E\{\| x - \hat{x} \|^2\} = \tilde{J}(\hat{x}),$

$J_2(\delta Z) = tr\{ cov(\delta Z(I - \Phi\Phi^+)Y, \delta Z(I - \Phi\Phi^+)Y)\Sigma \} \geq 0$ (A1)

$J_3(Z^*, \delta Z) = 2tr\{ cov((I - \Phi\Phi^+)Y, x - \hat{x})\Sigma \delta Z \}.$

Note that $(I - \Phi\Phi^+)Y = Y - \Phi^+\hat{U}$, where $\hat{U} = (\Phi^+ + F(I - \Phi\Phi^+))Y$ is the estimate of U and $\Sigma E\{ x - A\hat{U} \} = 0$, F is an arbitrary matrix. Then

$J_3(Z^*, \delta Z) = 2tr\{ cov(Y - \Phi\hat{U}, x - \hat{x})\Sigma \delta Z \}$ (A2)

If $\Sigma cov(Y - \Phi\hat{U}, x - \hat{x}) \neq 0$ then from (A2),(A3) it follows that there exists δZ^0:

$\delta\tilde{J} = J_2(\delta Z^0) - J_3(Z^*, \delta Z^0) < 0$. Hence $\tilde{J}(\tilde{x}) = \tilde{J}(\hat{x}) + \delta\tilde{J} < \tilde{J}(\hat{x})$

which means that $\hat{x}$ is not optimal. So $\Sigma cov(Y - \Phi\hat{U}, x - \hat{x}) = 0$ is necessary for $\hat{x}$ to be BLME.

Now suppose that for some Z^* $\mathrm{cov}(\ Y - \Phi\ \hat{U}\ ,\ x - \hat{x}\ )\Sigma = 0$, then $J_3(\ Z^*,\delta Z\ ) = 0$ for all δZ, and for $x = \hat{x} + \delta Z(\ I - \Phi\Phi^+)Y$

$$\tilde{J}(\tilde{x}) = J_1(Z^*) + J_2(\delta Z) \geq J_1(Z^*) = \tilde{J}_1(\hat{x}).$$

Hence $\hat{x}$ is the BLME.

Now it is easy to check that (7) is fulfilled if

$$\hat{x} = (\ A\ \Phi^+Y + D\ C\ H^TP^{-1}(\ I - \Phi\Phi^+)\ )Y \qquad (\ A3\ )$$

where $P = \mathrm{cov}(Y,Y) = H\ C\ H^T + Q$.

iii) Let the unbiased estimate $\tilde{x} = \psi(Y)$ be given by $\psi(Y) = \hat{x} + \varphi(Y)$, where $E\ \{\ \varphi(Y)\ \} = 0$.

Denote $\nu = D\ \Xi - [\ A\ \Phi^+ + D\ C\ H^TP^{-1}(\ I - \Phi\Phi^+)\ ][\ H\ \Xi + \Omega\ ]$,

$w = [\ A\ \Phi^+ + D\ C\ H^TP^{-1}(\ I - \Phi\Phi^+)\ ]\ y$,

then $\Delta = x - \hat{x} = \nu - \varphi(Y)$. The bias is

$$E(\varphi(Y)) = \int_{R^1} \varphi(y)[\ (2\pi)^1\det(P)\ ]^{1/2}\exp(\ -1/2(\ y-\Phi w)^TP^{-1}(\ y-\Phi w)\ )dy = 0$$

for all $w \in R^q$.Then the function $\varphi(Y)$ has the following property:

$$\varphi(Y^0) = \varphi(\ Y^0 + \Phi\ w\ )\ ,\ w \in R^q,\ Y^0 \in R^1 \qquad (\ A4\ )$$

Let $Y = (\ I-\Phi\ \Phi^+)Y + \Phi\ \Phi^+Y,\ \Phi\ \Phi^+Y \in R(\ \Phi\ )$. Using (A4) it may be shown that $\varphi(Y) = \varphi(\ (\ I-\Phi\ \Phi^+)Y\ )$.

So $\Delta = \nu - \varphi(\ (\ I-\Phi\ \Phi^+)Y\ )$ and the optimal nonlinear x estimation problem is equivalent to the vector ν estimation problem given the observations $(\ I-\Phi\ \Phi^+)Y$, i.e. $\varphi(Y) = E\ \{\ \nu\ |\ (\ I-\Phi\Phi^+)Y\ \}$. But $\mathrm{cov}(\ \nu\ ,\ (\ I-\Phi\Phi^+)Y\ ) = 0$, hence ν and $(\ I-\Phi\Phi^+)Y$ being Gaussian are independent and $E\ \{\ \nu\ |\ (\ I-\Phi\Phi^+\ )Y\ \} = E\ \{\ \nu\ \} = 0$. So there is no unbiased estimate that is more accurate than the linear one in the case of Gaussian disturbances.

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