ENGINEERING RESEARCH INSTITUTE
UNIVERSITY OF MICHIGAN
ANN ARBOR

THE PROPAGATION OF ELECTROMAGNETIC WAVES IN A MAGNETRON-TYPE SPACE CHARGE

Technical Report No. 8
Electron Tube Laboratory
Department of Electrical Engineering

BY

GEORGE R. BREWER

Approved by: W. G. DOW

G. HOK

Project M921

CONTRACT NO. DA-36-039 sc-5423
SIGNAL CORPS, DEPARTMENT OF THE ARMY
DEPARTMENT OF ARMY PROJECT NO. 3-99-13-022
SIGNAL CORPS PROJECT 27-112B-0

Submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy in the University of Michigan.

July, 1951

ABSTRACT

The propagation of electromagnetic waves in a magnetron type space charge is studied by using small signal, non-relativistic approximations. The following cases are analyzed:

1. Plane Magnetron -

a. Propagation of a plane electromagnetic wave in a direction parallel to the applied magnetic field.

b. Propagation of a plane electromagnetic wave in a direction perpendicular to the applied magnetic field and normal to the anode and cathode.

c. Propagation of a plane electromagnetic wave of phase velocity slow compared to that of light in a direction perpendicular to the applied magnetic field and parallel to the electron drift motion.

2. Cylindrical Magnetron -

a. Propagation of a TEM-type electromagnetic wave in a cylindrical space chargo in a direction parallel to an axially applied magnetic field.

b. Radial propagation of a cylindrical electromagnetic wave in a cylindrical space charge.

The analysis yields values for the propagation constant of the wave in the space charge, expressed in terms of an effective dielectric constant, which depends on the ratio of the signal radian frequency ω to the cyclotron radian frequency ω_c ($= eB_0/m$). It is

found that this effective dielectric constant can assume any real value, positive or negative. For given ω/ω_c, this knowledge of the effective dielectric constant makes possible the determination of the reactive effects of the space charge on a confining circuit.

The influence of the space charge on the frequency of a multi-anode magnetron is discussed qualitatively, as is the possibility of amplification of an electromagnetic wave along the plane magnetron space charge. Several experiments, conducted to determine the validity of the theory, are described. The results of these experiments appear to confirm certain critical parts of the theory.

ACKNOWLEDGMENTS

The author wishes to aknowledge with gratitude the assistance, during the course of the research which is reported herein, of all of the members of the University of Michigan Electron Tube Laboratory. He is most particularly indebted to Mr. Gunnar Hok for the numerous valuable suggestions obtained in the course of many hours of patient consultation. He is also appreciative of Mr. H. W. Welch's encouragement and advice, and of Mr. John W. Van Natter's conscientious efforts in the design and construction of the experimental tube models.

Personnel of the Laboratory

H. W. Welch J. R. Black G. Hok	Research Engineers
J. S. Needle	Instructor of Electrical Engineering
S. Ruthberg	Research Associate
V. R. Burris	Machine Shop Foreman
R. F. Steiner J. W. Van Natter	Assembly Technicians
R. F. Denning D. L. McCormick T. G. Keith E. A. Kayser	Laboratory Machinists
N. Navarre	Draftsman
S. Spiegelman J. Long	Stenographers

TABLE OF CONTENTS

TABLE OF CONTENTS (Cont'd)

APPENDICES

LIST OF ILLUSTRATIONS

LIST OF ILLUSTRATIONS (Cont'd)

I. INTRODUCTION

The propagation of electromagnetic waves in ionized media has been treated in a large number of papers, particularly with reference to the ionosphere. However, only a very small number of these papers[1] are applicable to the type of space charge region which is presumed to exist in a magnetron.

In this paper the propagation of electromagnetic waves in the magnetron space charge is studied, together with the effect of the space charge on an r-f circuit in which it is placed.

It is well known that the problem of the interaction between the fields in an oscillating multi-anode magnetron and the rotating space charge cloud is sufficiently complex to have allowed, to date, only solutions containing several restrictive approximations. In order to obviate mathematical entanglements as much as possible (and thus avoid the type of solutions requiring numerical integration), this analysis will be concerned with the small signal interaction of waves and electrons in a specified space charge cloud with certain uniform and simple types of electromagnetic fields. It is hoped that this presentation will allow the desired physical principles to be brought out without requiring extended mathematical treatment. The results are believed

---

1 See for example:

Welch, H. W., Jr., "Space Charge Effects and Frequency Characteristics of CW Magnetrons", Univ. of Mich. Electron Tube Laboratory Technical Report No. 1, November 15, 1948.

Blewett, J. P. and Ramo, S., "High Frequency Behavior of a Space Charge Rotating in a Magnetic Field", Phys. Rev., V57, pages 635-641, April, 1940.

Lamb, W. E. and Phillips, M., "Space Charge Frequency Dependence of a Magnetron Cavity", J. Appl. Phys., V18, pages 230-238, February, 1947.

Welch, H. W., Jr., "Effects of Space Charge on Frequency Characteristics of Magnetrons", Proc. I.R.E., page 1434, December, 1950.

to be applicable, insofar as the small signal analysis will allow, in the case of space charge clouds used for frequency modulation which are usually placed in a structure of such geometry that the simple field analysis is valid. It is also hoped that from these results one may be able to deduce qualitatively or semi-quantitatively the effects in the case of the more complicated fields of a multi-anode magnetron and other structures in which this type of space charge cloud could be used.

This analysis is an extension of that reported by Welch[1] and is specifically an attempt to determine the effective index of refraction of the space charge region as experienced by an electromagnetic wave propagating into or through this region. A knowledge of the index of refraction, and thus the dielectric constant, as a function of the frequency of the wave and the magnetic field, will enable the calculation of the reactive (and in some cases also resistive) effects of the space charge on the microwave circuit. Welch treats this problem under the assumption that the space charge swarm moves with constant linear velocity independent of position, so that the second term on the left side of Eq. II-1 below was not included. The present work is an extension and refinement on the previous treatment in that the variation of the electron velocity with position in the magnetron is considered.

Since the most general type of plane (or cylindrical) wave can be considered to be resolved into plane (or cylindrical) waves travelling along the coordinate axes, in this report the propagation of electromagnetic waves in the magnetron space charge will be idealized by considering the wave to be plane (or cylindrical), propagating along one

---

1 Welch, H. W., Jr., Loc. cit.

of the coordinate axes.

The electric field of the propagating wave will cause the electrons to undergo perturbations about their steady or equilibrium paths. The electrons will be acted upon by other forces also, including the applied magnetic field, and the motion of the electrons subject to these forces represents a current associated with the propagating wave so that its velocity of propagation is affected. In addition the electrons can collide with other particles in the space or with the electrodes, thus losing some of their energy; as a result, the wave will be diminished in amplitude as it progresses through the medium. These effects resulting from the electron motion are the subject of this study.

The results of this analysis are presented in a form enabling predictions to be made of the effects of the space charge cloud on an r-f circuit. From this information, one could design structures for frequency modulation, amplitude modulation, etc., of a microwave signal, using this type of space charge.

In what follows, the basic equations to be used in the analysis are discussed in Chapter II, followed in Chapter III by a derivation of the index of refraction of the plane and cylindrical space charges. Chapter IV contains interpretations of the results of Chapter III; Chapter V the effect of the space charge on its associated r-f circuit. In Chapter VI, the results of experiments conducted to verify the theory are presented. The more detailed mathematical treatments are included in the appendices so that this material can be omitted in reading with no loss of continuity.

The MKS rationalized system of units is used throughout.

II. THE BASIC RELATIONS

A magnetron space charge is created between two parallel plane or concentric cylindrical electrodes, one an electron emitter, by the application of a d-c electric potential between the electrodes and a steady magnetic field parallel to the electrodes in the plane case and along the axis in the cylindrical case. The electrons will possess a drift velocity normal to both the magnetic and electric fields.

In this section the equation of motion of the electrons in this space charge, under the influence of the electric and magnetic fields, is derived using perturbation methods, and is discussed briefly. The form of the space charge density distribution in the static magnetron is not as yet known with certainty; therefore the various distributions obtained by several workers are presented and discussed.

The shape of the magnetron space charges for the plane and cylindrical geometries are represented in Figs. 2-1 and 2-2. These figures also show the coordinate systems and field vectors to be used in the analysis which is to follow.

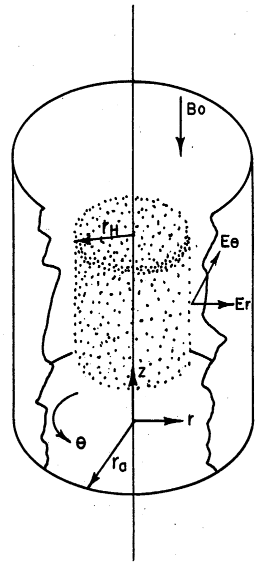


FIG. 2.1 COORDINATE SYSTEM AND FIELD VECTORS OF CYLINDRICAL MAGNETRON

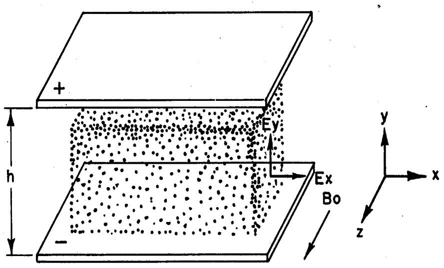


FIG. 2.2 COORDINATE SYSTEM AND FIELD VECTORS OF PLANE MAGNETRON

1. The Equation of Motion.

The equation of motion of the electrons, subject to the forces of the applied magnetic field and the electric field of the electromagnetic wave will be discussed initially in terms of the Euler Hydrodynamical equation. In order to provide a firm basis for consideration of the Euler equation as the equation of motion in this non-relativistic treatment of electron-wave interaction, this equation is shown, in Appendix 1, to be derivable from the Boltzmann Transport equation

$$\frac{\partial f}{\partial t} + \vec{c}\,\frac{\partial f}{\partial \vec{r}} + \vec{F}\,\frac{\partial f}{\partial \vec{c}} = \left(\frac{\partial f}{\partial t}\right)_{coll.}$$

with no knowledge of the exact form of the velocity distribution function. In this equation f represents the electron velocity distribution function, $\vec{r}$ is the position vector of the group of electrons under consideration, $\vec{c}$ the vector describing the velocity and $\vec{F}$ the vector force field acting on the electrons. The equation relates the change in the occupation of a cell of velocity space due to the action of the fields etc. to the change due to encounters with other electrons of the gas.

However, since the Boltzmann equation is valid only under conditions of approximate thermal equilibrium in the gas, the a priori assumption must be made that the wave propagating through the eleorron gas produces only a small perturbation on this equilibrium. That is, the energy of the random motion tending to maintain thermal equilibrium is assumed[1] large in comparison with the energy imparted to the particles by the wave. This thermal equilibrium does not exist near the boundaries

[1] In order to obtain some idea of the conditions imposed by this assumption, consider that the mean random energy of the electrons is 3/2 kT per electron. Then this assumption can be written as $W_W \ll 3/2\ kT$ where W_W is the mean vibrational energy imparted to the electrons by the wave. Since the applied magnetic field does not affect the energy of the electrons, W_W can be written for purposes of illustration as:

of the space charge, unless rather artificial boundary conditions are imposed, possibly raising a question as to the validity of the application of the Boltzmann equation to such a medium.

Consistent with the non-relativistic case, the Lorentz force due to the magnetic field of the wave is neglected, so that the Euler Hydrodynamical equation is:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\,\vec{v} = -\frac{e}{m}\,\vec{E} + \vec{v} \times \vec{B}_0 - \frac{1}{nm}\nabla \vec{p} \qquad \text{II-1}$$

where $\vec{v}$ = linear velocity of a group of electrons contained in elemental volume $d\tau$[1]

B_0 = applied constant magnetic field

p = electron gas pressure

n = number density of electrons

In a number of papers treating the same general subject as this report, the pressure gradient term in the equation of motion is neglected. In so doing, no account is taken of any effects due to the random or thermal motions of the electrons. In order to be able to give a complete treatment of the effect of random motions, the form of the

---

$$W_W = \frac{1}{2} mv^2 = m \frac{e}{m} \frac{E}{i\omega}^2$$

so that

$$E \ll \sqrt{\frac{3kT\,\omega^2\, m}{e}}$$

If $T = 10^4 {}^\circ K$ and $\omega = 6\pi \times 10^9$ it is found that the field strength of the wave must be much less than 230 volt/cm.

1 The volume $d\tau$ must be of dimensions very much less than a wavelength of the propagating wave but must contain a sufficiently large number of electrons that a statistical mean value of their behavior can be obtained. Due to the high electron density, both of these conditions can be satisfied. There will be random fluctuations with time in the number of electrons in $d\tau$, causing fluctuations in the effects of these electrons but this will cause no appreciable change in the propagating characteristics of the wave.

electron velocity distribution function would be required. This determination usually involves mathematical complications unjustified for the purpose of this report.[1] However, a first-order approximation can be made by considering the electron gas as exhibiting ideal behavior so that $p = nkT$. This is admittedly a rather severe assumption but is believed to be at least somewhat closer to the actual state of affairs than the complete neglect of the pressure term. The temperature T referred to here is a measure of the mean random energy of translation of the electrons. There is reason to believe[2] that by some mechanism of electron interaction this electron temperature can achieve values as high as 10^5 °K, greatly in excess of the cathode temperature.

In Appendix 2, this substitution $p = nkT$ is made into Eq. II-1. The solution for the wave velocity for a typical case is carried through in the same manner as in Chapter III. The resulting equations show that the inclusion of the pressure gradient term does not affect the propagation characteristics of the electromagnetic wave but causes to appear another wave similar to the plasma oscillations found in gases.

The pressure gradient term will therefore be omitted and

---

1 See for example - Cohen, Spitzer, and Routley, "The Electrical Conductivity of an Ionized Gas", Phys. Rev. 80, 2, October 15, 1950.

2 The reasons for the belief in the existence of this large electron temperature are: (a) the relatively large current collected by the anode in a cutoff smooth-bore magnetron, and (b) the experimental measurements of Linder.

Linder, E. G., "Excess Energy Electrons and Electron Motion in High Vacuum Tubes", Proc. I.R.E. 26, page 346, 1938.

Linder, E. G., "Effect of High Energy Electron Random Motion upon the Shape of the Magnetron Cutoff Curve", J. Appl. Phys. 9, page 331, 1938.

Eq. II-1 becomes

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\, \vec{v} = -\frac{e}{m}\, \vec{E} + \vec{v} \times \vec{B}_o \qquad \text{II-2}$$

which is the usual equation describing electron motions in a field in which the velocity varies with both position and time.

It is shown in Appendix 3 that the effect of collisions between electrons and atoms or ions in the space charge, in which the electrons lose some of their translational energy, can be represented by a frictional type force proportional to the velocity, i.e. by gv, where g is the inverse mean time between collisions.

Another effect of the collisions between electrons and ions or molecules, can be seen from a consideration of the force acting on the electrons due to the positive ions present in the space charge. Lorentz[1], in his treatment of wave propagation in material media, uses as the total force on a charged particle

$$F = E + \frac{1}{3\varepsilon_o}\, P$$

where P is the polarization of the medium and E the electric field of the electromagnetic wave. Darwin[2] shows, by a consideration of the electron orbits near idealized positive charges, that the average effect of electron-ion collisions is to produce an acceleration

$$-\frac{1}{3\varepsilon_o}\, N \frac{e}{m}\, \xi$$

where ξ is the perturbed electron position. The equation of motion of such an electron is then, since $Ne\,\xi = P$:

$$\ddot{\xi} = \frac{eF}{m} - \frac{1}{3\varepsilon_o}\, \frac{e}{m}\, P$$

---

1 Lorentz, H. A., "The Theory of Electrons", B. G. Teubner, Leipzig - 1909, Chapter IV.

2 Darwin, Chas., "The Refractive Index of an Ionized Medium" II, Proc. Roy. Soc. London V182, page 152, 1944.

and, using :

$$F = E + \frac{1}{3\varepsilon_0} P \quad , \quad \ddot{\xi} = \frac{e}{m} E \ .$$

He concludes that the "process of collision produces dynamically a depolarizing effect, reducing the effective average force on an electron from F to E". The force on an electron due to the electric field is then -eE.

It is shown in Appendix 4 that electron-electron collisions in the gas do not change the total dipole moment of the space charge and thus do not affect the propagation of waves in the medium.

Eq. II-2 becomes, with the addition of the frictional force representing collisions:

$$\frac{\partial \vec{v}}{\partial t} + g\vec{v} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{e}{m} \quad \vec{E} + \vec{v} \times \vec{B}_0 \qquad \text{II-3}$$

As mentioned before, in the absence of the wave the electrons are presumed to pursue steady orbits under the influence of the applied magnetic field and d-c electric field. In the presence of the wave the electrons will be periodically perturbed from this steady or time invariant motion. (The word perturbed as used here is meant to imply smallness of the magnitude of the deviation from the steady orbit except where specifically noted.) The total electron velocity will then be represented by an ordered or d-c term and a perturbed or a-c term as

$$v = v_0 + v_1 \qquad \text{II-4a}$$

Likewise small volumes of space charge will be moved periodically from their mean positions, so let the total space charge density likewise be represented by a d-c term and a perturbed term as:

$$\rho = \rho_0 + \rho_1 \qquad \text{II-4b}$$

Substituting these relations into Eq. II-3 and keeping only perturbation terms:

$$\frac{\partial v_1}{\partial t} + gv_1 + (\vec{v}_0\cdot\nabla)\vec{v}_1 + (\vec{v}_1\cdot\nabla)\vec{v}_0 + (\vec{v}_0\cdot\nabla)\vec{v}_1 = -\frac{e}{m}\vec{E} + \vec{v}_1 \times \vec{B}_0 \qquad \text{II-5}$$

Since this report is concerned with the steady state propagation of waves in a particular type of medium; assuming the wave will propagate, it appears reasonable to suppose that the state of the medium will be perturbed by the moving wave. Consequently the velocity and space charge density will be assumed to vary as

$$v_1 e^{i\omega t - \gamma s} \qquad\qquad \rho_1 e^{i\omega t - \gamma s}$$

where ω is the angular frequency of the impressed wave, γ the propagation constant, and s a **length unit** along the direction of propagation.

The equations of motion will be linear only if the term $(v_1\cdot\nabla)v_1$ vanishes, either as a result of other assumptions or by specification that the magnitude of v_1 is so small that this product is comparatively negligible. This small signal assumption will not always be necessary and this will be pointed out in each of the several cases to be treated in the next chapter.

The convection current density, when represented in the form,

$$J = (\rho_0 + \rho_1)\ (v_o + v_1)$$

can be separated into zero and first order terms, the perturbation or a-c part being

$$J_1 = \rho_0 v_1 + v_0 \rho_1 \qquad \text{II-6}$$

where the term $\rho_1 v_1$ has been neglected. Again this may be due to the fact that $\rho_1 = 0$ or it may be necessary to assume the smallness of both v_1 and ρ_1. Which of these reasons is responsible for the vanishing of the $\rho_1 v_1$ term will be pointed out in each case separately.

Eq. II-5 will now be reduced to its form appropriate to the plane and cylindrical cases to be considered.

a. Plane Magnetron. In this case the steady velocity is entirely in the x direction[1] so that Eq. II-5 becomes in component form:

$$\vec{x}: \frac{\partial v_x}{\partial t} + g v_x + v_o \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_o}{\partial y} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} = -\frac{e}{m} E_x - \omega_c v_x \qquad \text{II-7}$$

$$\vec{y}: \frac{\partial v_y}{\partial t} + g v_y + v_o \frac{\partial v_y}{\partial x} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} = -\frac{e}{m} E_y + \omega_c v_x$$

$$\vec{z}: \frac{\partial v_z}{\partial t} + g v_z + v_o \frac{\partial v_z}{\partial x} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} = -\frac{e}{m} E_z$$

b. Cylindrical Magnetron. In this case the steady electron velocity is considered as entirely in the Θ direction[1] so that Eq. II-5 becomes in component form:

$$\vec{r}: \frac{\partial v_r}{\partial t} + g v_r + \frac{v_o}{r} \frac{\partial v_r}{\partial \Theta} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\Theta}{r} \frac{\partial v_r}{\partial \Theta} + v_z \frac{\partial v_r}{\partial z} = -\frac{e}{m} E_r - \omega_c v_\Theta \qquad \text{II-8}$$

[1] In the case of the so-called "double stream" solution for electron motions, for a magnetron in the cut-off condition, that is no net electron current toward the anode, there will be as many electrons passing through the volume $d\tau$ toward the anode as away from it so that the net steady electron motion is still tangential.

$$\vec{\theta}:\ \frac{\partial v_\theta}{\partial t} + g v_\theta + \frac{v_\Omega}{r}\frac{\partial v_\theta}{\partial \theta} + v_r \frac{\partial v_\Omega}{\partial r} + v_r \frac{\partial v_\theta}{\partial r} + \qquad \text{II-8}$$

$$\frac{v_\theta}{r}\ \frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} = - \frac{e}{m} E_\theta \ + \omega_c v_r$$

$$\vec{z}:\ \frac{\partial v_z}{\partial t} + g v_z + \frac{v_\Omega}{r}\ \frac{\partial v_z}{\partial \theta} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} = - \frac{e}{m} E_z$$

In the next chapter the Eqs. II-7 and II-8 will be combined with the Maxwell field equations to enable a determination of the complex index of refraction of electromagnetic waves through this space charge medium. In these next sections the propagating wave will be considered as either plane or cylindrical, whose phase is invariant in the plane (or cylinder) perpendicular to the direction of propagation, thus neglecting consideration of any boundary conditions imposed by the circuit in which the space charge is placed. In later sections the effect of the boundary conditions imposed by the circuit will be treated.

2. The Space Charge Distributions.

In the next chapter the space charge density will be kept as an unknown parameter, until the final relation for the index of refraction is obtained. However for illustrative purposes and for numerical calculations, the Hull-Brillouin values of charge density will be used. Therefore, in order to justify this substitution, the various solutions of the static magnetron space charge will be examined briefly.

A number of papers have been published describing the theoretical steady-state space charge distribution in plane and cylindrical non-oscillating magnetrons. Figs. 2.3, 2.4 and 2.5 allow a comparison of the distributions obtained by various workers.

Fig. 2.3 shows the distribution of space charge density for the plane magnetron as given by Hull[1] and Brillouin[2], Page and Adams[3], and Twiss[4]. It is seen that the work of Twiss (when he considers only initial normal velocities of emission), Hull and Brillouin agree very closely, while that of Page and Adams is lower by as much as a factor of two. The curve reported by Twiss when both initial tangential and normal emission velocities are considered is also shown, the solid curve being derived from his equations and the dashed curve from the results of his qualitative reasoning. The Page and Adams solution is seen to differ from the other cases. The sharp rises at the cathode and at the edge of the space charge cloud are due to the assumption of zero velocity of emission and zero escape current from the cloud boundary. For the actual case of finite initial electron velocity and non-zero escape current these singularities will disappear, the curves intersecting the cathode and space charge outer boundary with finite values. Thus if the Page and Adams solution contained these boundary conditions the resulting curves would appear in somewhat better agreement with the others.

In the Page and Adams distribution the electrons are presumed to execute approximately cycloidal orbits from the cathode to the edge

---

1 Hull, A. W., "The Effect of a Uniform Magnetic Field on the Motion of Electrons Between Coaxial Cylinders", Phys. Rev. V18, page 31, 1921.

2 Brillouin, L., Journal de Physique, 1940.

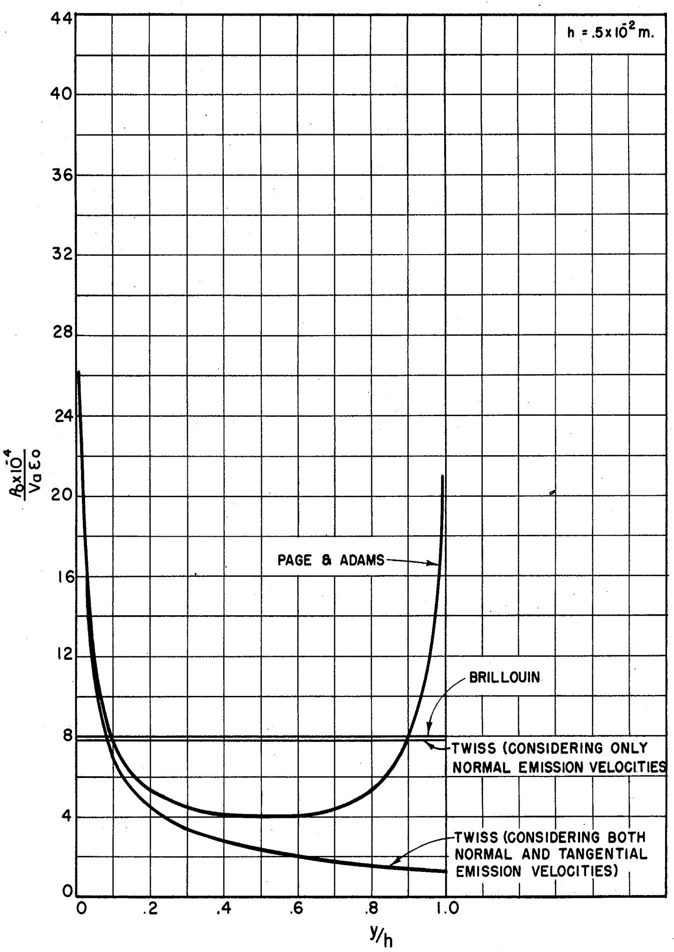
3 Page and Adams, "Space Charge in Plane Magnetron", Phys. Rev. V69, page 492, 1946.

4 Twiss, R. Q., "On the Steady State and Noise Properties of Linear and Cylindrical Magnetrons", M.I.T. - Ph.D. Thesis, 1950.

of the cloud and return, the so-called double stream motion. Brillouin considers that the electrons move parallel to the electrodes, the so-called single stream motion.

FIG. 2.3
COMPARISON OF SPACE CHARGE DENSITY DISTRIBUTIONS
OBTAINED BY VARIOUS WORKERS

PLANE MAGNETRON



In an unpublished report[1] Brillouin discusses in detail the types of distributions possible in a plane magnetron. He points out that either a single stream or double stream electron motion is possible; in either case the total charge within the cloud will be the same, as

1 Brillouin, L., "Electronic Theory of the Plane Magnetron", Columbia University, AMP Report 129.1R - OSRD 4510, to be published in part in the third volume of "Advances in Electronics".

will be the total electron energy. Thus, one cannot make a choice between the two possible solutions on the basis of energy, but he shows that the double stream motion leads to a volt-ampere characteristic with a negative resistance region so that this distribution should be unstable, with a tendency to degenerate into sustained oscillations. He also shows that for a space charge limited cathode, the electron trajectories do not cross, indicating single stream motion. Brillouin concludes from this that the stable steady-state space charge distribution in a space charge limited plane magnetron is that corresponding to single stream motion.

The Page and Adams double stream motion and the Brillouin single stream motion are the limiting cases of the possible types of electron orbit possible in the magnetron space charge. More generally, the electrons can be thought of as executing approximately cycloidal orbits whose maximum excursion is less than the thickness of the space charge cloud so that a number of these swarms can lie parallel to each other, each farther from the cathode; thus there appear a number of "virtual cathodes" between each of which exists one swarm. This is the so-called "multiple swarm motion". For a single swarm the Page and Adams solution is valid and for an infinite number of swarms the Brillouin distribution is more nearly correct. It would therefore be expected that the space charge density distribution corresponding to the actual electron motion lie between those corresponding to these two limiting cases.

In Figs. 2.4 and 2.5 are shown the space charge density distributions for the cylindrical magnetron, due to Brillouin[1],

[1] Brillouin, L., "Theory of the Magnetron - I", Phys. Rev. V60, page 385, 1941.

FIG. 2.5

COMPARISON OF SPACE CHARGE DENSITY DISTRIBUTIONS OBTAINED BY VARIOUS WORKERS

CYLINDRICAL MAGNETRON

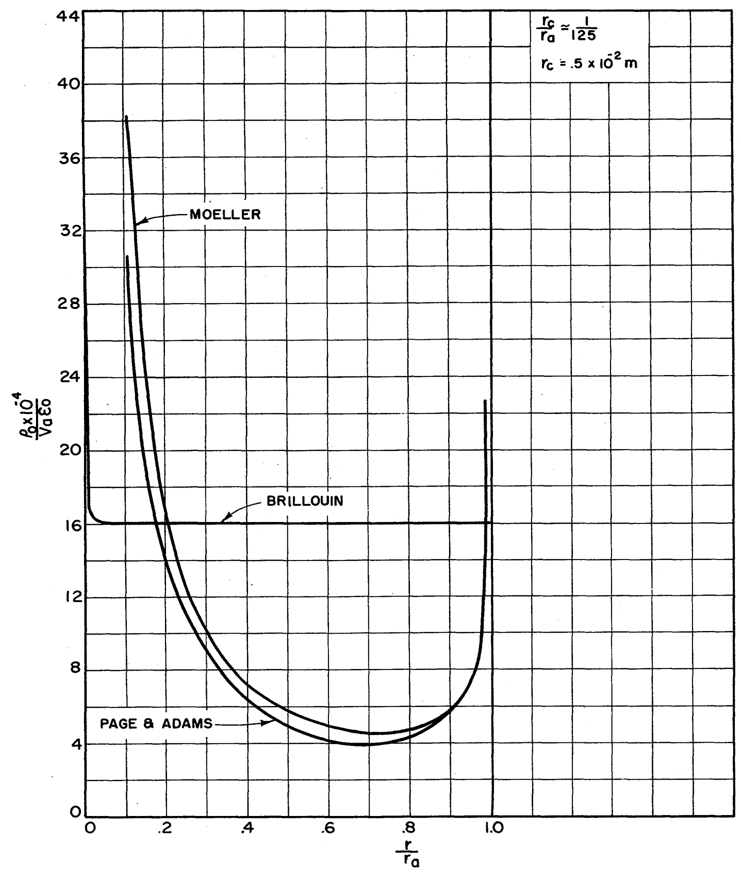
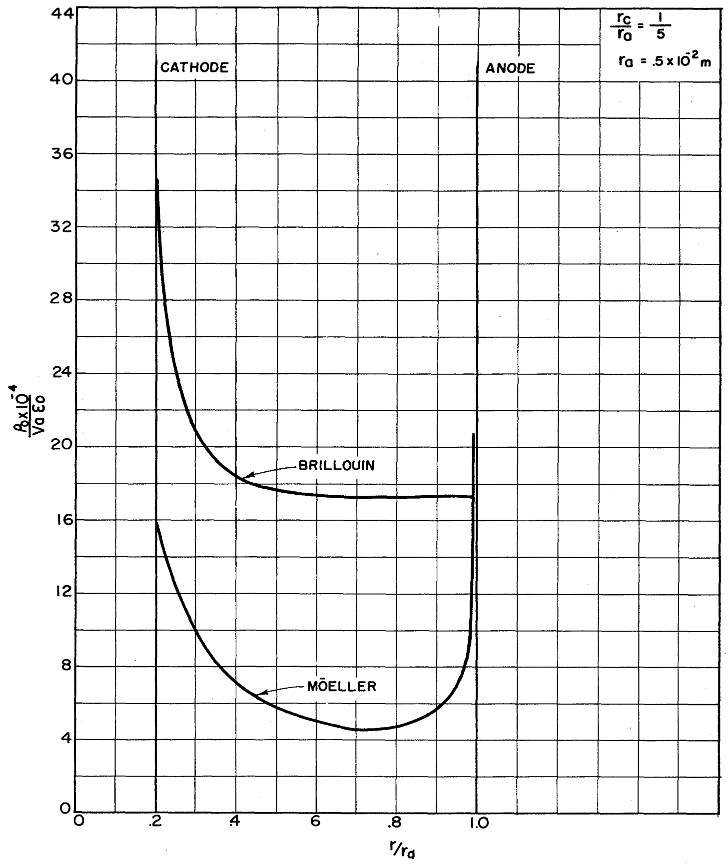


FIG. 2.4

COMPARISON OF SPACE CHARGE DENSITY DISTRIBUTIONS OBTAINED BY VARIOUS WORKERS

CYLINDRICAL MAGNETRON



Page and Adams[1], and Moeller[2]. These latter two are seen to differ rather markedly from the distribution of Brillouin. The Brillouin solution, it is remembered, results from the assumption of zero radial electron acceleration, that is, the electrons are considered as travelling in circles concentric with the cathode (single stream motion). The Page and Adams and the Moeller solutions are obtained by the use of mathematical series and involve no such assumption. As in the plane case, the infinite values of space charge density at the cathode and boundary, obtained by these latter workers, will be reduced to finite values upon inclusion of the initial velocity of emission and escape current.

Glagolev[3] has succeeded in integrating the equation of motion, by a method of successive approximations, to obtain the potential distribution of the cylindrical magnetron space charge. He finds that the potential differs only slightly from the Langmuir distribution in a space charge limited diode without magnetic field; the maximum deviation occuring at the edge of the space charge where it is 9% greater than the Langmuir value. From this one can conclude that the space charge density is only slightly greater than that derived from the Langmuir potential distribution. This latter yields a space charge density distribution closely resembling the Hull-Brillouin case.

1 Page and Adams, "Space Charge in Cylindrical Magnetron", Phys. Rev. V69, page 494, 1946.

2 Moeller, H. G., "Elektronenbannen und Mechanismus der Schwingungserregung in Schlitzanodenmagnetron", Hochfrequenztechnik und Elak., V47, page 115, 1936.

3 Glagolev, V. M., "The Passage of Steady Current in a Cylindrical Non-slit Magnetron" - Zhur. Tekh. Fiz. USSR - 19 - page 943, August 1949 - Translated by Naval Research Laboratory, Washington, D. C. NRL Translation No. 318.

With regard to the choice between these various space charge distributions; the problem is such in the cylindrical case as to make this more difficult than in the plane case. Allis[1] maintains that the double stream solution is the stable one, based on the observation that since anode current is observed in a non-oscillating cut-off magnetron, and in the single stream case (his B_0 solution) radial current is impossible, the double stream motion must take place. However, he seems to neglect the fact that the ratio of rotating space charge current to anode current in a cut-off magnetron is large, suggesting the possibility that most of the electrons are moving more or less in circles concentric with the cathode and the anode current be due to those relatively few electrons which, by a process not yet known, have lost some of their ordered or rotational energy. Allis shows that the single stream motion is possible for any radius of the space charge cloud. However for a ratio of space charge cloud radius to cathode radius greater than 2.023 the double-stream motion is also possible, and in view of the previous reasoning, Allis believes this latter type of motion probable.

Brillouin[2,3] presents a criterion to enable a decision to be made between double and single stream motion, based on whether the electron trajectories do or do not cross. He finds that under space charge limited conditions, for ratios of space charge cloud radius to cathode radius less than 2.273 (for small oscillations) the trajectories do not cross, indicating single-stream motion. However, for ratios greater

---

1 Allis, W. P., "Electronic Orbits in the Cylindrical Magnetron with Static Fields", Radiation Laboratory Special Report 9S, Section V, R.L. Report 122, October 1941.

2 Brillouin, L., loc. cit.

3 Brillouin, L., "The Influence of Space Charge on Electron Bunching", Phys. Rev. V70, page 187, August 1946.

than this, the trajectories <u>can</u> cross so that one cannot decide, for this case, between the possible motions on this basis.

On the basis of statistical considerations of electron motion, Hok[1] has recently demonstrated that the space charge distribution may be appreciably modified by the random interaction, however weak, between the discrete electrons forming the space charge.

Wasserman[2] has conducted an experiment in an attempt to determine which of the distributions, Brillouin or Moeller, is more nearly correct. His method involved measurement of the magnetic flux associated with the total space charge current rotating around the cathode. Fig. 2.6 shows his results, in which curve 1 represents the rotating current calculated from Moeller, and curve 2 the rotating current deduced from experimental measurement of the magnetic flux. Wasserman claims agreement within fifteen per cent of the Brillouin case and forty-five per cent with the Moeller case, from which he concludes that the "steady state cylindrical space charge distribution is best represented by the Brillouin relation". It is believed that his experimental method is somewhat inaccurate, however, and should yield values of rotating current lower than the actual values so that perhaps the agreement is better than shown by Fig. 2.6.

Similar measurements have been made by Möller[3] who claims

---

1 Hok, G., To appear in a forthcoming report from the University of Michigan Electron Tube Laboratory.

2 Wasserman, I. I., "Rotating Space Charge in a Magnetron with Solid Anode", <u>J. Tech. Phys. (USSR)</u> V18, page 785, 1948.

3 Möller, J., "Measurement of the Circulating Electron Current in a Magnetron", <u>Hochfrequenztechnik und Elak.,</u> V47, page 141, July 1936.

agreement within nine per cent of the value of magnetic flux determined from the Hull-Brillouin space charge distribution. However, this writer believes Möller's experimental technique to be susceptible to inaccuracies also.

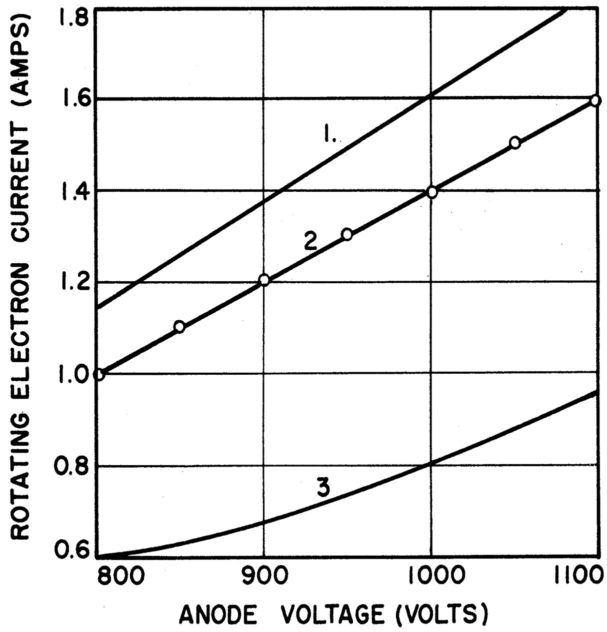


FIG. 2.6

The proper choice of the distribution of space charge will influence the results to be derived here to a certain extent; however, as will be seen later, the character of the solutions is determined mainly by the functional variation of electron velocity with distance from the cathode. Fortunately the angular velocity in the cylindrical case and the steady linear velocity in the plane case result from integrals of the equations of motion without resort to any assumptions as to the trajectories. That is, while one must make an assumption as to the electron orbits (as did Brillouin) or use a series method (as Möller or Page and Adams) to obtain a solution for the potential and space charge variation, the velocity is independent of these difficulties.

The Hull-Brillouin expressions for the velocity and space charge distribution in the plane and cylindrical magnetrons are given below.

Plane Magnetron

$$\rho_0 = -\omega_c^2 \varepsilon_0 \frac{m}{e}$$

$$v_{ox} = -\omega_c y \qquad \text{II-9}$$

Cylindrical Magnetron

$$\rho_0 = -\frac{m}{e} \frac{\omega_c^2}{2} \left[1 + \left(\frac{r_c}{r}\right)^4\right]$$

$$v_0 = r\xi = \frac{r\omega_c}{2}\left[1 - \frac{r_c^2}{r^2}\right] \qquad \text{II-10}$$

where v_0 is the steady velocity in the x and Θ directions and ξ is the angular velocity. $\omega_c = \frac{eB_0}{m}$ is the cyclotron angular velocity and r_0 is the cathode radius.

III. DETERMINATION OF THE COMPLEX INDEX OF REFRACTION

In this chapter the complex index of refraction of a magnetron space charge will be determined for various directions of propagation of the wave. In the case of a plane magnetron the waves are assumed to be plane; that is, invariant in phase in the plane normal to the direction of propagation.[1] The following cases are considered:

(a) propagation in the direction of the applied magnetic field (z),

(b) propagation in the direction normal to anode and cathode (y), and

(c) propagation parallel to the steady electron motion (x).

In the case of the cylindrical magnetron the waves are in each case assumed invariant in phase with angle around the cathode (θ) and the following cases are considered:

(a) a plane TEM wave propagating in the direction of the applied magnetic field(z), and

(b) propagation of a cylindrical wave in the direction normal to anode and cathode (r).

An electromagnetic wave of specified characteristics is considered to impinge on the space charge along the desired direction. Part of this wave will be transmitted into the space charge and set the electrons in motion. The magnetic field will cause the electrons to have additional components of velocity than those given them by the electric field of the impinging wave, necessitating additional field

---

[1] Insofar as the space charge can be considered as a linear medium, waves which do not fulfill this assumption can be formed by suitable superposition of these plane waves.

components associated with the wave in the space charge. This can be expressed by thinking of the wave in the space charge as made up of an inducing field, and a scattered or radiation field due to the motion of the electrons. This electron motion set up by the inducing wave is equivalent to an elementary current ev so that the coherent scattering or radiation due to the motion of all charges is equivalent to a current density distribution J = nev. This equivalent current distribution is the only radiation source in the space charge and since it is due to the inducing field, this and the radiation field must be identical and satisfy the field equations:

$$\nabla \times H = J + i\omega \varepsilon_0 E$$

$$\nabla \times E = - i\omega \mu_0 H$$

$$\nabla \cdot H = 0$$

from which can be derived the wave equation:

$$\nabla^2 E + \frac{\eta^2 \omega^2}{c^2} E = 0 .$$

η, the steady state complex index of refraction of the medium, is the quantity sought in this analysis. The index of refraction is a function of the ratio ω/ω_c of the signal frequency to the cyclotron frequency. The space charge can, of course, be thought of as an electronic plasma and this relation between η and ω as analogous to the usual dispersion equation for waves in a dispersive medium.

Since to the extent that the space charge can be considered as a linear medium, any wave motion in the magnetron can be made up of monochromatic waves propagating along the specified coordinate axes, this chapter will be concerned only with waves propagating along these axes and varying with time and space as $e^{j\omega t - \gamma s}$ where s is the unit of length measured along the direction of propagation.

The electron velocities are obtained in terms of the field components by the use of the equations of motion Eqs. II-7 and II-8. From the velocity equations the current is determined, which when substituted into the field equations allows solution for the propagation constant γ and therefore η .

In the cylindrical case explicit solutions for η as function of ω/ω_c are obtained only for the limiting case of no variation of ρ_0 with r; corresponding, for the Hull-Brillouin space charge distribution, to a vanishingly small cathode. Some qualitative arguments are advanced allowing an approximate interpolation of the properties of the space charge to be made between the limiting cases of plane magnetron and cylindrical magnetron with very small cathode.

The orientation of the field vectors to be used in the following developments are shown in Figs. 3.1 and 3.2.

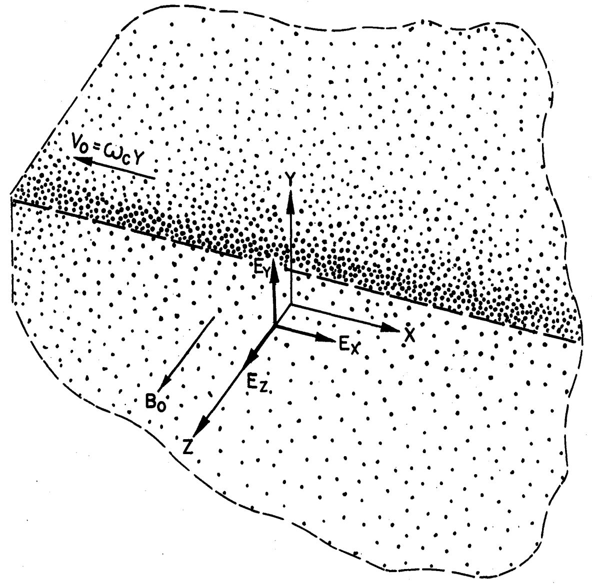


FIG. 3.1 ORIENTATION OF FIELD VECTORS ASSUMED FOR DEVELOPMENT OF WAVE PROPAGATION IN PLANE MAGNETRON

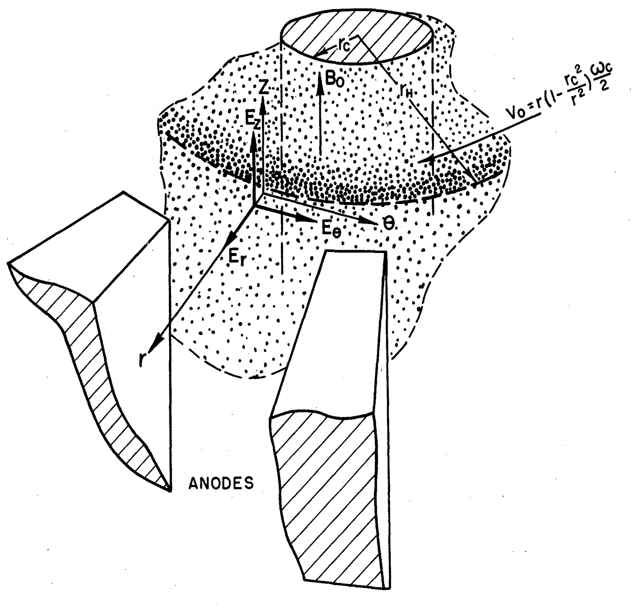


FIG. 3.2 ORIENTATION OF FIELD VECTORS ASSUMED FOR DEVELOPMENT OF WAVE PROPAGATION IN CYLINDRICAL MAGNETRON

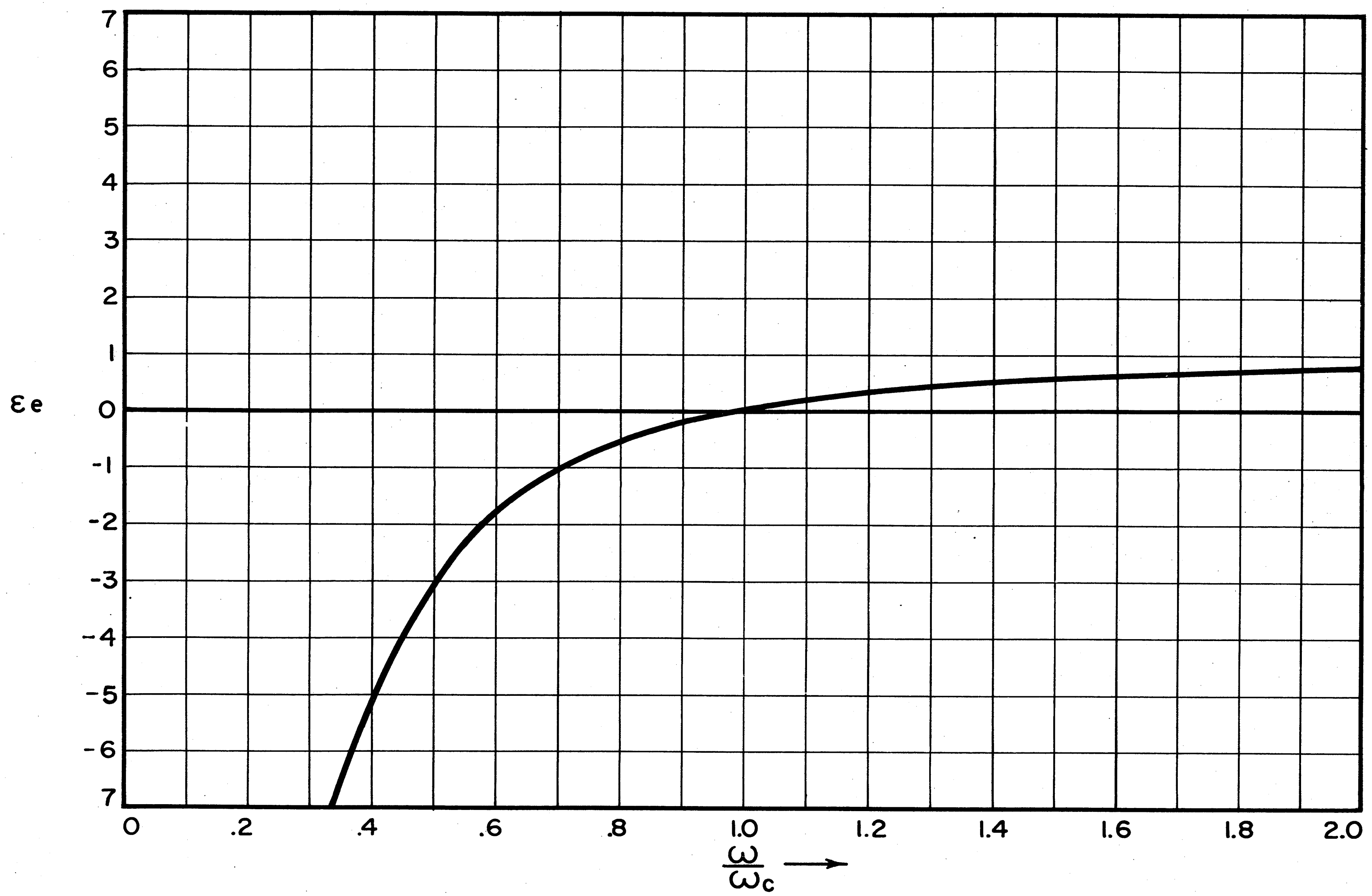


FIG. 3.3 ε_e FOR PLANE MAGNETRON

1. Propagation in the Direction of the Applied Magnetic Field.

A. Plane Magnetron. The propagation in the z direction of a plane wave ($\partial/\partial x = \partial/\partial y = 0$) with field and velocity components v_x v_y E_x E_y H_y H_x varying with z as $e^{-\gamma z}$ is considered in this section. Then Eqs. II-7 become:

$$\begin{aligned} i\omega\, v_x + g\, v_x - \omega_0\, v_y &= -e/m\, E_x - \omega_c\, v_y \\ i\omega\, v_y + g\, v_y &= -e/m\, E_y + \omega_c\, v_x \end{aligned} \qquad \text{III-1}$$

It is noted that in this case, since $E_z = 0$, $v_z = 0$ and the above equations are exact, requiring no assumption as to the magnitude of the velocities to remove the non-linear $(v_1 \cdot \nabla)\, v_1$ term.

Solving for the velocities:

$$v_x = \frac{-e/m\, E_x}{i\omega + g} \quad , \qquad v_y = \frac{-e/m\left[E_y + \dfrac{\omega_c}{i\omega + g}\, E_x\right]}{i\omega + g} \quad .$$

Since $\nabla \cdot J = \dfrac{\partial J_z}{\partial z} = -i\omega\rho_1 = 0$ the currents are:

$$J_x = \rho_0\, v_x \quad , \qquad J_y = \rho_0\, v_y \quad .$$

Combining these expressions with the field equations, one obtains:

$$\gamma H_y = \left[-\frac{\rho_0\, e/m}{i\omega + g} + i\omega\, \varepsilon_0\right] E_x \qquad \text{(a)}$$

III-2

$$-\gamma H_x = \left[-\frac{\rho_0\, e/m}{i\omega + g} + i\omega\, \varepsilon_0\right] E_y - \frac{\rho_0\, e/m\, \omega_0\, E_x}{(i\omega + g)^2} \qquad \text{(b)}$$

$$\gamma E_y = -i\omega \mu_0\, H_x \qquad \text{(c)}$$

$$\gamma E_x = -i\omega \mu_0\, H_y \qquad \text{(d)}$$

These equations can be solved by evaluating the determinant of the coefficients of E_x and E_y to yield:

$$\left[\frac{\gamma^2}{i\omega\mu_o} + \frac{\rho_o e/m}{i\omega + g} - i\omega\,\varepsilon_o\right]\left[\frac{\gamma^2}{i\omega\mu_o} + \frac{\rho_o e/m}{i\omega + g} - i\omega\,\varepsilon_o\right] = 0 \;,$$

so

$$\gamma^2 = -\frac{\omega^2\eta^2}{c^2} = i\omega\mu_o\left[-\frac{\rho_o e/m}{i\omega + g} + i\omega\,\varepsilon_o\right] \;,$$

and, since $g << \omega$:

$$\eta^2 = 1 + \frac{\rho_o e}{\varepsilon_o m\omega^2} + \frac{i\,\rho_o\,eg}{m\,\varepsilon_o\,\omega^3}$$

$$= \varepsilon_e - i\,\frac{\sigma_e}{\omega\,\varepsilon_o}\,.$$

Making the substitution $\rho_o\,e/m = -\,\omega_o^2\,\varepsilon_o$ from Eqs. II-9:

$$\varepsilon_e = 1 - \frac{1}{(\omega/\omega_c)^2} \qquad \text{III-3}$$

and

$$\sigma_e = \frac{g\,\varepsilon_o}{(\omega/\omega_c)^2} \;,$$

where η is the complex index of refraction $\eta = \eta_r + i\eta_i$ and ε_e the effective dielectric constant, and σ_e the effective conductivity of the electron gas. It can be shown from the well known equation that:

$$\eta_r = \sqrt{\frac{\varepsilon_e}{2}\left[\sqrt{1 + \frac{g^2\,\omega_c^2}{\omega^6(1 - w_c^2/\omega^2)^2}} + 1\right]} \approx \varepsilon_e$$

$$\eta_i = \sqrt{\frac{\varepsilon_e}{2}\left[\sqrt{1 + \frac{g^2\,\omega_c^2}{\omega^6(1 - \omega_c^2/\omega^2)^2}} - 1\right]} \approx -\frac{g^2\,\omega_c^2}{\omega^3\,2\,\sqrt{\varepsilon_e}}$$

The quantity ε_e is plotted in Fig. 3.3 as function of ω/ω_c. It can be shown that a wave polarized with its electric field component in the z direction will experience the same effective dielectric constant as found above.

B. Cylindrical Magnetron. Propagation in the z direction of a plane wave ($\partial/\partial_\theta = 0$) with field and velocity components v_r v_θ E_r E_θ H_r H_θ varying as $e^{-\gamma z}$ is considered in this section. This corresponds to a TEM wave propagating along a coaxial line, the components E_θ H_r being necessary to account for the tangential electron motions in the space charge. The equations of motion Eq. II-8 become:

$$i\omega\, v_r + g\, v_r + v_r \frac{\partial v_r}{\partial r} = -e/m\, E_r - \omega_c\, v_\theta$$

III-4

$$i\omega\, v_\theta + g\, v_\theta + v_r \frac{\partial v_o}{\partial r} + v_r \frac{\partial v_\theta}{\partial r} = -e/m\, E_\theta + \omega_c\, v_r$$

In order to linearize the equations it is necessary to assume the perturbation velocities so small as to make any cross product term negligible in comparison with the other terms in Eq. III-4. Solving these equations for the velocity components, after dropping the terms $v_r \frac{\partial v_\theta}{\partial r}$, and $v_r \frac{\partial v_r}{\partial r}$ (assuming the derivatives to be continuous and bounded):

$$v_r = \frac{-e/m\left[E_\theta - \frac{i\omega + g}{\omega_c} \quad E_r\right]}{-\frac{(i\omega + g)^2}{\omega_c} + \frac{\partial v_o}{\partial r} - \omega_c}$$

$$v_\theta = \frac{e/m\left[E_r + \frac{i\omega + g}{\omega_c - \partial v_o/\partial r} \quad E_\theta\right]}{-\frac{(i\omega + g)^2}{\omega_c - \partial v_o/\partial r} - \omega_c}\ .$$

From Eq. II-10:

$$\frac{\partial v_o}{\partial r} = \frac{\omega_c}{2}\left(1 + \frac{r_o^2}{r^2}\right)$$

so that the above equations become:

$$v_r = \frac{\omega_c \, e/m \left[E_\theta - \frac{i\omega + g}{\omega_c} \; E_r \right]}{\omega_c \xi \; + \; (i\omega + g)^2} \qquad \text{(a)}$$

III-5

$$v_\theta = \frac{\xi \, e/m \left[E_r + \frac{i\omega + g}{\xi} \; E_\theta \right]}{- \, (i\omega + g)^2 - \omega_c} \qquad \text{(b)}$$

where

$$\xi = \frac{\omega_c}{2} \left(1 - \frac{r_c^2}{r^2} \right) . \qquad \text{(c)}$$

Since, in this case $\nabla \cdot E = \frac{1}{r} \frac{\partial}{\partial r} (r E_r) = -i\omega \, \rho_1 = 0$,

the currents are

$$J_r = \rho_o \, v_r \qquad\qquad J_\theta = \rho_o \, v_\theta \; .$$

The field equations then become:

$$\gamma H_\theta = i\omega \, \varepsilon_o \, Er \quad + \quad \frac{\rho_o \, \omega_c \, e/m \left[E_\theta - \frac{i\omega + g}{\omega_c} \; E_r \right]}{\omega_c \xi \; + \; (i\omega + g)^2} \qquad \text{(a)}$$

$$-\gamma \, Hr - \frac{\partial Hz}{\partial r} = i\omega \, \varepsilon_o \, E_\theta \quad + \quad \frac{\rho_o \xi \; e/m \left[E_r + \frac{i\omega + g}{\xi} \; E_\theta \right]}{\omega_c \xi \; + \; (i\omega + g)^2} \qquad \text{(b)}$$

$$\frac{1}{r} \frac{\partial}{\partial r} \; (r \, H_\theta) = 0 \qquad \text{(c)}$$

$$\gamma E_\theta \; = \; - \, i\omega \, \mu_o \, H_r \qquad \text{(d)}$$

III-6

$$-\gamma E_r \; = \; - \, i\omega \, \mu_o \, H_\theta \qquad \text{(e)}$$

$$\frac{1}{r} \frac{\partial}{\partial r} \; (r \, E_\theta) = -i\omega \, \mu_o \, H_z \qquad \text{(f)}$$

Differentiating Eq. III-6(f) with respect to r:

$$-i\omega\mu_o \frac{\partial H_z}{\partial r} = -\frac{1}{r^2}\frac{\partial}{\partial r}(r\,E_\theta) + \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\,E_\theta)$$

From Eqs. III-6(b) and (d) above:

$$-\frac{i\gamma^2}{\omega\mu_o}E_\theta - \frac{1}{i\omega\mu_o}\frac{1}{r^2}\frac{\partial}{\partial r}(r\,E_\theta) + \frac{1}{i\omega\mu_o}\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\,E_\theta) = i\omega\,\varepsilon_o\,E_\theta - \xi\,X\,E_r - (i\omega + g)\,X\,E_\theta$$

where $X = \dfrac{\rho_o\,e/m}{\omega_c\xi + (i\omega + g)^2}$,

and from Eqs. III-6(a) and (c):

$$\frac{\gamma^2}{i\omega\mu_o}E_r = i\omega\,\varepsilon_o\,E_r + \omega_c\,X\,E_\theta - (i\omega + g)\,X\,E_r \ .$$

From this last equation

$$E_\theta = \Lambda\ E_r$$

where

$$\Lambda = \frac{\gamma^2/i\omega\mu_o - i\omega\,\varepsilon_o + (i\omega + g)\,X}{\omega_c\,X} \ .$$

Therefore:

$$\frac{\partial}{\partial r}(r\,E_\theta) = \frac{\partial}{\partial r}(r\,E_r) = rE_r\frac{\partial\Lambda}{\partial r} + r\Lambda\frac{\partial E_r}{\partial r} + E_r\Lambda$$

and

$$\frac{\partial^2}{\partial r^2}(r\,E_\theta) = \left[2\frac{\partial\Lambda}{\partial r} + r\frac{\partial^2\Lambda}{\partial r^2}\right]E_r + \left[2r\frac{\partial\Lambda}{\partial r} + 2\Lambda\right]\frac{\partial E_r}{\partial r} + r\Lambda\frac{\partial^2 E_r}{\partial r^2} \ .$$

Substituting these expressions into Eq. III-6(g):

$$\frac{\partial^2 E_r}{\partial r^2} + \frac{1}{r}\left[\frac{2r}{\Lambda}\frac{\partial\Lambda}{\partial r} + 1\right]\frac{\partial E_r}{\partial r} + \left[i\omega\mu_o\left(\omega_c\Lambda X + \frac{\xi X}{\Lambda}\right) - \frac{1}{r^2}\left(\frac{r}{\Lambda}\frac{\partial\Lambda}{\partial r} + 1\right) + \frac{2}{\Lambda}\frac{\partial\Lambda}{\partial r} + \frac{r}{\Lambda}\frac{\partial^2\Lambda}{\partial r^2}\right]E_r = 0 \qquad \text{III-7}$$

The general solution to the differential equation III-7 would be rather formidable, unless X does not vary with r. In the case of the Hull-Brillouin space-charge solution this condition can be satisfied if the cathode is infinitely thin.

In this special case Eq. III-7 reduces to:

$$\frac{\partial^2 E_r}{\partial r^2} + \frac{1}{r}\frac{\partial E_r}{\partial r} + \left[i\omega\mu_o\left(\omega_o \Lambda X + \frac{\xi X}{\Lambda}\right) - \frac{1}{r^2}\right] E_r = 0 \qquad \text{III-8}$$

which is simply the Bessel equation.

Comparing Eq. III-8 with the similar equations for wave transmission in a cylindrical dielectric-filled waveguide it is seen that

$$\gamma^2 - \omega^2/c^2 = i\omega\mu_o\left(\omega_c \Lambda X + \frac{\xi X}{\Lambda}\right) \qquad \text{(a)}$$

or III-9

$$\gamma^2 = -\omega^2/c2 + \left[\frac{\omega^2/c^2 \quad \rho_o e/m\varepsilon_o}{\omega_c\xi \ + (i\omega + g)^2} + \xi\ \omega_c\ X\mu_o\right]\cdot\left(1 - \frac{ig}{\omega}\right) . \qquad \text{(b)}$$

Letting $\gamma^2 = -\omega^2/c^2\ \eta^2$ and using the Hull-Brillouin value for the space charge density (with vanishingly small cathode) from Eq. II-10, $\frac{\rho_o e}{m\ \varepsilon_o} = -\frac{\omega_c^2}{2}$ and the corresponding angular velocity $\xi = \frac{\omega_c}{2}$.

This equation then becomes:

$$\varepsilon_e = \mathrm{Re}\,\eta^2 = 1 + \frac{1 + \frac{1}{2\ \omega^2/\omega_c^2}}{1 - 2\ \omega^2/\omega_c^2} \qquad \text{(a)}$$

III-10

$$\frac{\sigma_e}{\omega\ \varepsilon_o} = -\mathrm{I}\,\eta^2 = \omega g\,\frac{1 + \frac{1}{2\omega^2/\omega_c^2}}{1 - 2\omega^2/\omega_c^2} \qquad \text{(b)}$$

Eqs. III-10 are plotted in Fig. 3.4. It is seen that, depending on the value of ω/ω_c, ε_e can take on all real values; positive and negative. The conductivity resulting from energy loss by the electrons due to collisions with the heavy particles in the space charge is seen

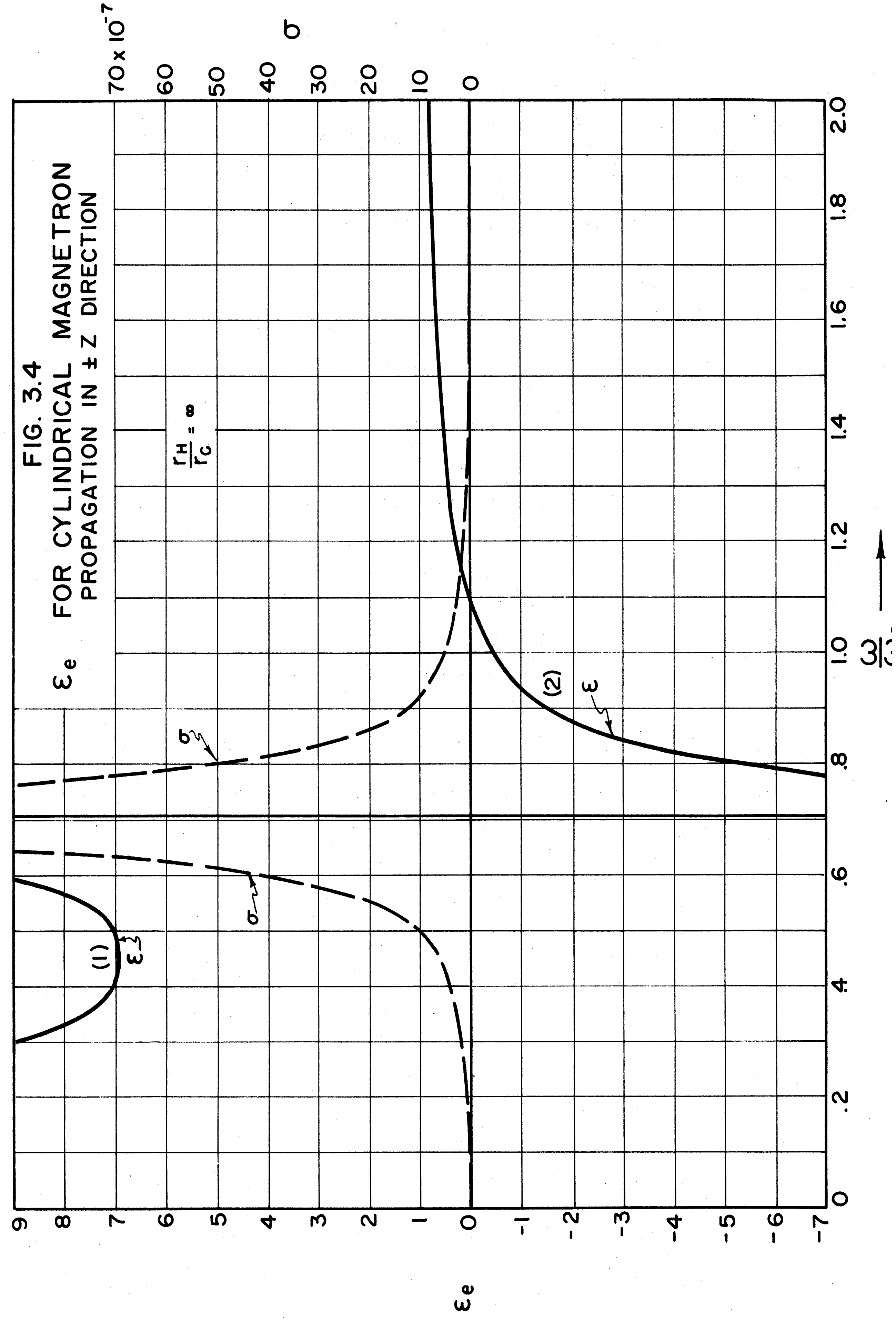


FIG. 3.4
ε_e FOR CYLINDRICAL MAGNETRON
PROPAGATION IN ± Z DIRECTION

to reach a maximum value for $\omega/\omega_c = 1/\sqrt{2}$ as could be predicted from the velocity Eqs. III-5 since the velocities are maximum at this value of ω/ω_c.

C. Discussion of Variation of ε_θ with r. The effective dielectric constant of a magnetron space charge has been determined exactly, (for the small signal case) for the plane magnetron and the cylindrical magnetron with vanishingly small filament, in the above sections (A) and (B). Since neither of these structures is used in practice, it becomes very desirable to be able to determine the dielectric properties of the space charge as a function of the radius. Unfortunately the complete solution of Eq. III-7 is very laborious, even if the Hull-Brillouin relation for ρ_0 is used. Perturbation, or approximate methods do not appear to simplify the problem appreciably. Therefore until such time as the need for a complete solution to Eq. III-7 arises, thus justifying the time required for its solution, one must be content with an interpolation based on the limiting solutions already obtained and a knowledge of the variation of the space charge density ρ_0 and angular velocity ξ as functions of r.

Eq. III-7 is seen to contain terms $\frac{1}{\Lambda}\frac{d\Lambda}{dr} = \frac{d\ \ln\Lambda}{dr}$ which is, except for a constant, equal to $\frac{d}{dr}\ln X$. For r_H/r_c greater than about three, the quantity $\ln X$ does not vary more than 25% from its value at $r_H/r_c = \infty$. For this reason it does not appear to be a prohibitively bad approximation to neglect the terms in $\frac{1}{\Lambda}\frac{d\Lambda}{dr}$ in Eq. III-7 for $r_H/r_c > 3$. This equation is then reduced to Eq. III-7.

The critical points of the first term in the brackets in Eq. III-8 will be examined qualitatively as a function of r to attempt an interpolarion between the values $r_H/r_c = 1$ and $r_H/r_c = \infty$.

Eq. III-9(b) can be written in the form

$$\varepsilon_\theta = 1 - \frac{\frac{\rho_0 e}{m\varepsilon_0}\left[1 + \xi\frac{\omega_c}{\omega^2}\right]}{\omega_c\xi - \omega^2}$$

Upon substitution of the relations Eqs. II-10, it can be shown that the variation of the above fraction with r is negligible for $r_H/r_c > 2$ so that for larger values of r_H/r_c the value of ω/ω_c at which $\varepsilon_\theta = 0$ does not change appreciably from its value for $r_H/r_c = \infty$. This slow variation with r results from the mathematical form of the various functions of r involved in the expressions for ρ_0 and ξ.

However in the case of the infinite singularity, the value of ω/ω_c is determined almost completely by $2\frac{\omega^2}{\omega_c^2} = 1 - \frac{r_c^2}{r^2}$ which is a fairly rapidly varying function of r for $r_H/r_c < 5$. Therefore this critical point can be expected to vary more, for large values of r_H/r_c, than the point $\varepsilon_\theta = 0$.

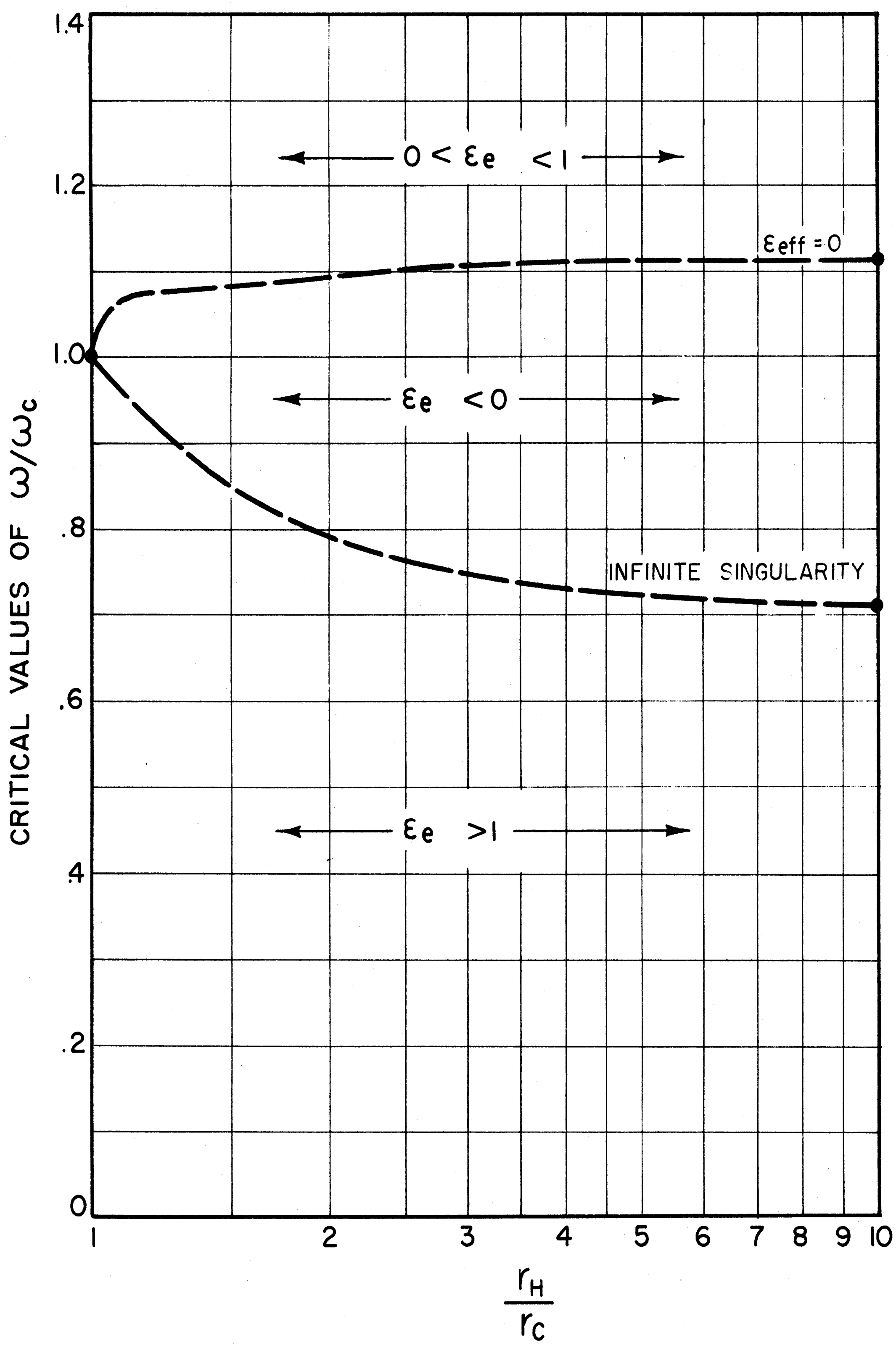
From these qualitative considerations the curves of Fig. 3.5 were interpolated from the values determined from the analysis (shown as heavy dots on the ordinates). It is to be emphasized that these curves are only the result of some judicious guessing and do not result directly from the solution of Eq. III-7. In a strict mathematical sense one cannot draw such conclusions as these, since they necessitate the dependence on r of the propagation constant in Eq. III-8, which violates the assumption made there. However it is believed that these approximations are reasonably valid so long as the radial propagation constant is only a slowly varying function of r, i.e. for $r_H/r_c > 3$.

A partial justification for these approximations, made in violation of the condition that γ be independent of r, lies in the fact that the energy density varies logarithmically with r, so that in the region where ρ_0 and ξ vary most rapidly, there is relatively low energy density.

FIG. 3.5

EFFECT OF CLOUD RADIUS ON CRITICAL VALUES OF ω/ω_c - APPROXIMATE INTERPOLATION

PROPAGATION IN Z DIRECTION



2. Propagation in the Direction Normal to the Anode and Cathode.

A. Plane Magnetron. In this section propagation in the $\pm$ y direction of a plane wave ($\partial/\partial x = \partial/\partial z = 0$) with field components $E_x\ E_y\ H_z\ v_x\ v_y$ varying as $e^{-\gamma y}$ will be considered. The force equations in this case are identical to Eqs. III-1 so that the velocity components are:

$$v_x = \frac{-e/m\ E_x}{i\omega + g} \qquad v_y = \frac{-\frac{e}{m}\left[E_y + \frac{\omega_c}{i\omega + g} E_x\right]}{i\omega + g} , \qquad \text{III-11}$$

where it has been necessary to assume the perturbation velocities small so that their product and therefore the non-linear term $(v_1 \nabla)v_1$ can be neglected.

The perturbed charge density is found, using the continuity relation:

$$\nabla \cdot J = -i\omega\, \rho_1 = \frac{\partial J_y}{\partial y} = -\gamma J_y$$

so:

$$\rho_1 = \frac{\gamma}{i\omega} J_y ,$$

but there is assumed to be no component of steady electron velocity in the y direction; so that, since ρ_1 does not vary with x, the currents are given by:

$$J_x = \rho_0 v_x \qquad J_y = \rho_0 v_y .$$

Use of the velocity equations above yields:

$$J_x = \frac{-\frac{\rho_0 e}{m} E_x}{i\omega + g} \qquad J_y = \frac{-\frac{\rho_0 e}{m}\left[E_y + \frac{\omega_c}{i\omega + g} E_x\right]}{i\omega + g} . \qquad \text{III-12}$$

Substituting these relations into the field equations, there result:

$$-\gamma H_z = \left[-\frac{\rho_0\ e/m}{i\omega + g} + i\omega\, \varepsilon_0\right] E_x \qquad \text{(a)}$$

III-13

$$0 = \left[- \frac{\rho_0 \, e/m}{i\omega + g} + i\omega \, \varepsilon_0 \right] E_y - \frac{\rho_0 \, e/m \, \omega_c}{(i\omega + g)^2} E_x \qquad \text{(b)}$$

III-13

$$\gamma E_x = - i\omega\mu_0 H_z \qquad \text{(c)}$$

Eliminating H_z from the first two by the use of Eq. III-13(c):

$$\left[\frac{\gamma^2}{i\omega\mu_0} + \frac{\rho_0 \, e/m}{i\omega + g} - i\omega \, \varepsilon_0 \right] E_x + 0 = 0$$

$$\left[\frac{\rho_0 \, e/m \, \omega_c}{(i\omega + g)^2} \right] E_x + \left[\frac{\rho_0 \, e/m}{i\omega + g} - i\omega \, \varepsilon_0 \right] E_y = 0$$

so that:

$$\left[\frac{\rho_0 \, e/m}{i\omega + g} - i\omega \, \varepsilon_0 \right] \left[\frac{\gamma^2}{i\omega\mu_0} + \frac{\rho_0 \, e/m}{i\omega + g} - i\omega \, \varepsilon_0 \right] = 0 \qquad \text{III-14}$$

and

$$\gamma^2 = - \frac{\omega^2}{c^2} \eta^2 = - i\omega\mu_0 \left[\frac{\rho_0 \, e/m}{i\omega + g} - i\omega \, \varepsilon_0 \right]$$

III-15

$$\eta^2 = \left[1 - \frac{\rho_0 e}{m\varepsilon_0 \, \omega^2} \right] + i \, \frac{\rho_0 e}{m\varepsilon_0} \, \frac{g}{\omega^3} \; .$$

Using Eq. II-9:

$$\eta^2 = 1 - \frac{1}{(\omega/\omega_0)^2} - i \, \frac{\omega_0^2 \, g}{\omega^3} \; ,$$

$$\varepsilon_e = 1 - \frac{1}{(\omega/\omega_0)^2} \; ,$$

III-16

$$\sigma_e = \frac{g \, \varepsilon_0}{(\omega/\omega_0)^2} \; ,$$

which are the same as Eqs. III-3. These relations are plotted in Fig. 3.3.

It can be shown that a wave polarized with its electric field in the z direction, will propagate with the same velocity as the wave considered above.

B. Cylindrical Magnetron. In the case of the cylindrical magnetron, propagation normal to the anode is considered as in the radial direction, with field components E_r E_θ H_z v_r v_θ and with $\partial/\partial\theta = \partial/\partial z = 0$.

The force equations are then identical to Eqs. III-4 so that the velocity components are:

$$v_r = \frac{\omega_c \frac{e}{m}\left[E_\theta - \frac{i\omega + g}{\omega_c} E_r\right]}{\omega_c \xi + (i\omega + g)^2}$$

$$v_\theta = \frac{-\xi \frac{e}{m}\left[E_r + \frac{i\omega + g}{\xi} E_\theta\right]}{\omega_c \xi + (i\omega + g)^2} ,$$

where again it has been necessary to assume the perturbation velocities small so that the non-linear product term can be neglected.

The charge density ρ_1 can be found from:

$$\nabla \cdot J = \frac{\partial J_r}{\partial r} = - i\omega\rho_1 \quad .$$

However, since we have assumed the steady electron flow to be entirely tangential, the currents are simply

$$J_r = \rho_0 v_r$$

$$J_\theta = \rho_0 v_\theta \quad .$$

The field equations are then:

$$\left[-\frac{\rho_0 \, e/m \, (i\omega + g)}{\omega_c \xi + (i\omega + g)^2} + i\omega \varepsilon_0\right] E_r + \frac{\omega_c \rho_0 \, e/m}{\omega_c \xi + (i\omega + g)^2} E_\theta = 0 \qquad \text{(a)}$$

III-17

$$-\frac{\partial H_z}{\partial r} = \left[-\frac{\rho_0 \, e/m \, (i\omega + g)}{\omega_c \xi + (i\omega + g)^2} + i\omega \varepsilon_0\right] E_\theta - \frac{\rho_0 \, e/m}{\omega_c \xi + (i\omega + g)^2} E_r \qquad \text{(b)}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r E_\theta) = - i\omega\mu_0 H_z \qquad \text{(c)}$$

Differentiating Eq. III-17(c) with respect to r and substituting into Eq. III-17(b):

$$\frac{\partial^2 E_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial E_\theta}{\partial r} - \frac{E_\theta}{r^2} =$$

$$i\omega\mu_o\left[\frac{-\rho_o\ e/m\ (i\omega + g)}{\omega_c\xi + (i\omega + g)^2} + i\omega\ \varepsilon_o\right]E_\theta - \frac{i\omega\mu_o\ \rho_o\ e/m\ \xi}{\omega_c\xi + (i\omega + g)^2}\ E_r\ .$$

By solving Eq. III-17(a) for E_r

$$E_r = \frac{-\ \omega_c\ \rho_o\ e/m\ \ E_\theta}{-\ \rho_o\ e/m\ (i\omega + g) + i\omega\ \varepsilon_o\left[\omega_c\xi + (i\omega + g)^2\right]}\ ,$$

and substituting into the differential equation for E_θ one obtains:

$$\frac{\partial^2 E_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial E_\theta}{\partial r} - \frac{1}{r^2}\ E_\theta \qquad \text{III-18}$$

$$-\ \omega\mu_o\left[-\ \omega\varepsilon_o - \frac{i\rho_o\ e/m\ (i\omega + g)}{\omega_c\xi + (i\omega + g)^2} + \right.$$

$$\left.\frac{i\omega_c\xi(\rho_o\ e/m)^2\ \dfrac{1}{\omega_c\xi + (i\omega + g)^2}}{-\rho_o\ e/m\ (i\omega + g) + i\omega\ \varepsilon_o\left[\omega_c\xi + (i\omega + g)^2\right]}\right]E_\theta = 0\ .$$

So long as the part in brackets is independent of r, it can be shown to be equal to the square of the radial propagation constant, γ^2.[1]

Separating the bracketed expression into real and imaginary parts and substituting $\eta^2 = -c^2/\omega^2\ \gamma^2$:

$$\text{Re}\eta^2 = 1 + \frac{\rho_o e}{m\varepsilon_o\omega^2}\left[\frac{\dfrac{\rho_o e}{m\varepsilon_o} + \omega^2}{\dfrac{\rho_o e}{m\varepsilon_o} - (\omega_c\xi - \omega^2)}\right] \qquad \text{III-19 (a)}$$

---

1 For a clarification of this analogy and a general discussion of cylindrical waves, see e.g. Schelkunoff, "Electromagnetic Waves", D. Van Nostrand and Co., page 406, 1943.

$$\text{Im}\,\eta^2 = \rho_o \frac{e}{m}\frac{g}{\omega}\left[2 + \frac{\omega_c\xi - \omega^2}{\omega^2}\right]\left[1 - \frac{\left(\rho_o \frac{e}{m}\right)^2 \frac{\omega_c \xi}{\omega^2}}{\left(-\rho_o \frac{e}{m} + \epsilon_o (\omega_c\xi - \omega^2)\right)^2}\right] \qquad \text{III-19(b)}$$

where as before terms in g^2 are neglected in comparison with ω^2 or $\omega_c{}^2$.

These expressions are plotted in Fig. 3.6.

Using the Hull-Brillouin relations Eq. II-10, for a vanishingly small cathode, the above equations become:

$$\varepsilon_e = 1 - \frac{1}{4\,\omega^2/\omega_c{}^2}\left[\frac{2\,\omega^2/\omega_c{}^2 - 1}{\omega^2/\omega_c{}^2 - 1}\right]$$

$$\frac{\sigma_e}{\omega\,\epsilon_o} = g\,\frac{\omega_c{}^2}{2\omega^3}\left[1 + \frac{1}{2\omega^2/\omega_c{}^2}\right]\left[1 - \frac{\omega_c{}^4/8\omega^2}{(\omega_c{}^2 - \omega^2)^2}\right]. \qquad \text{III-20}$$

By using the same type of qualitative arguments as mentioned in Section 1-C above, the variation in the critical points of Eq. III-20 can be interpolated between the $r_H/r_c = 1$ and $r_H/r_c = \infty$ values. These curves are shown dashed in Fig. 3.7.

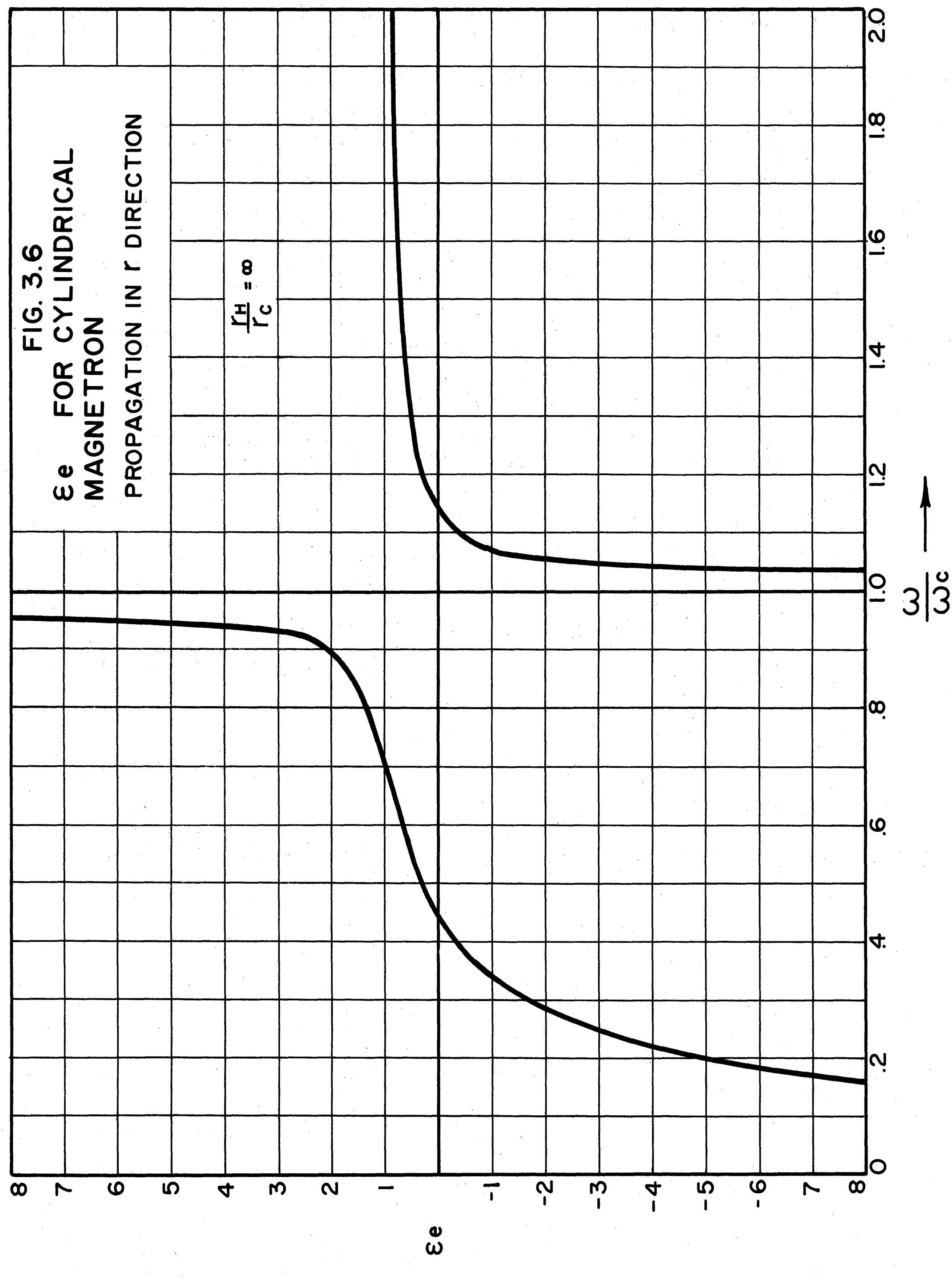
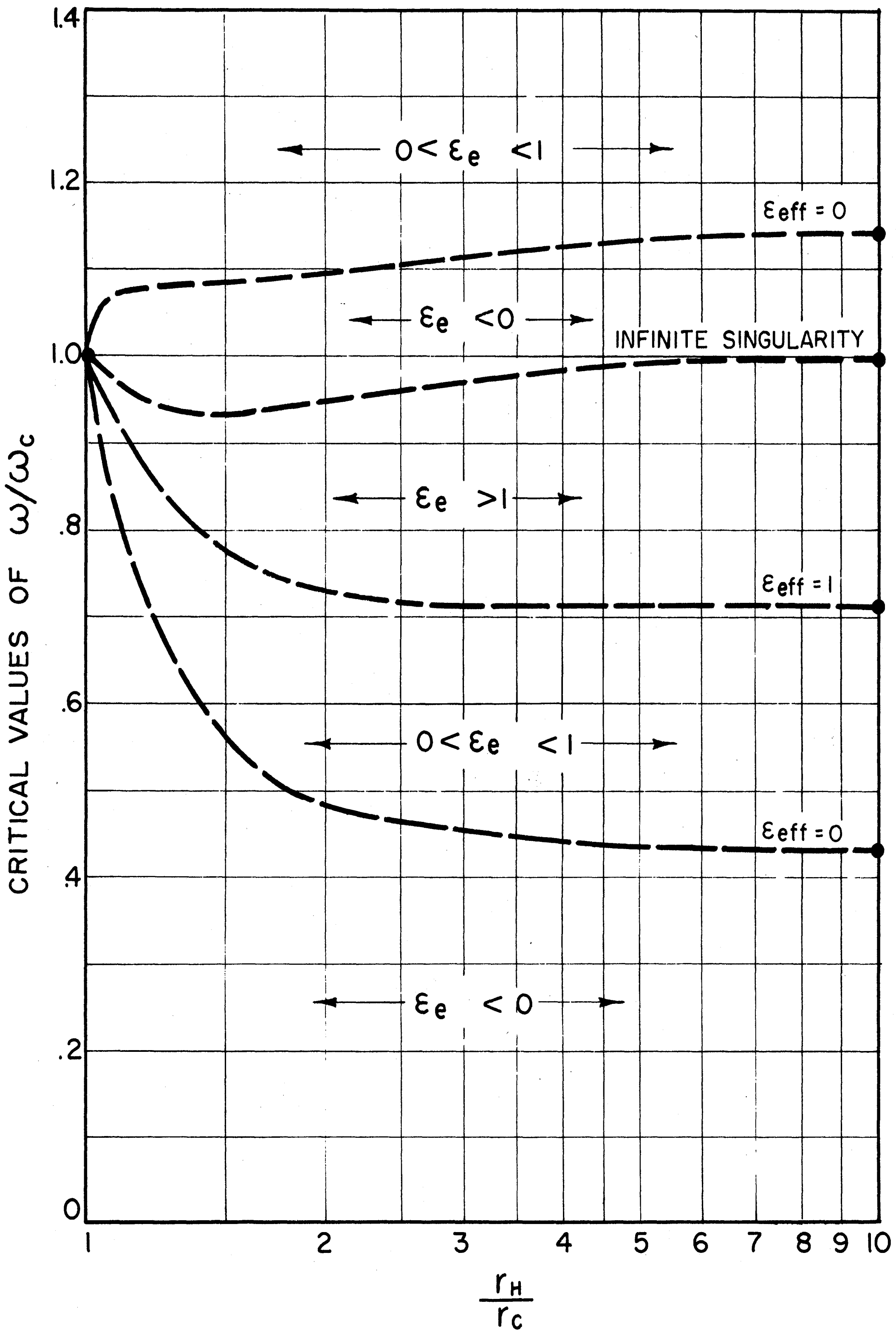


FIG. 3.6
εe FOR CYLINDRICAL
MAGNETRON
PROPAGATION IN r DIRECTION
$\frac{r_H}{r_c} = \infty$
εe
8
7
6
5
4
3
2
1
-1
-2
-3
-4
-5
-6
7
8
0
.2
.4
.6
.8
1.0
1.2
1.4
1.6
1.8
2.0
$\frac{\omega}{\omega_c}$

FIG. 3.7

EFFECT OF CLOUD RADIUS ON CRITICAL VALUES OF ω/ω_c – APPROXIMATE INTERPOLATION

PROPAGATION IN RADIAL DIRECTION



3. Propagation in the Direction Parallel to the Steady Electron Motion.

A. Plane Magnetron. The propagation of electromagnetic waves in the direction parallel to the steady electron velocity in the magnetron should prove of interest from two points of view. While this analysis is carried out only for the plane magnetron, the general results should be applicable at least qualitatively to considerations involving the cylindrical magnetron, and might be of assistance in explaining certain frequency effects of the space charge on the resonant circuit, such as frequency pushing and voltage tuning. Secondly, it is believed (as has been suggested by a number of people) that a magnetron structure should be capable of providing amplification of electromagnetic waves. This possibility was suggested earlier in the course of this work[1], the idea being based partly on an analysis carried out at that time and partly on intuitive considerations, extrapolating from the successful Electron Wave Tube[2] due to Haeff. The analysis presented in the aforementioned report (reference 1 on this page) has since been found to be incorrect, however the possibility of amplification has not been abandoned.

The problem of wave propagation in the direction of electron motion in a plane magnetron space charge has been treated in a

---

1 Quarterly Progress Report No. 2, Electron Tube Laboratory, University of Michigan, July 1950.

2 Haeff, A. V., "The Electron Wave Tube - A Novel Method of Generation and Amplification of Microwave Energy", Proc. I.R.E. 37, pages 4-10, 1949.

Labus, J., "High Frequency Amplification by Means of the Interaction Effect between Electron Streams", Arch. Elekt. Ubertragung 4, pages 353-360, 1950.

comprehensive manner by Macfarlane and Hay[1]. However, while they apparently found suitable mathematical expansions enabling rather general solution of the equations, some of their most interesting results are not presented for the case of the magnetron (their case $a = 1$). That is, they consider interaction with a beam of electrons injected between two parallel structures, the electron velocity varying linearly with distance normal to these structures, but not necessarily vanishing at one of them as in the magnetron with a cathode as one element of the delay line.

This problem of amplification in a plane magnetron structure has also been mentioned in a note by Buneman[2].

Therefore, while the present treatment of this problem will necessarily be of more limited scope than that of Macfarlane and Hay, it is hoped that the results can be applied profitably to the magnetron.

For consideration of electromagnetic wave propagation in the direction parallel to the steady or drift velocity of the electrons, two types of waves must be distinguished. The first wave will propagate with a velocity near that of light, being determined by the dielectric properties of the space charge, as well as the boundary conditions imposed by the confining circuit. A possible example of this case would be wave propagation along a plane parallel (without loading) transmission line, one of whose elements is an emitter, giving rise to a space charge with drift velocity along the length of the line. Since this case is of

---

1 Macfarlane, G. G. and Hay, H. G., "Wave Propagation in a Slipping Stream of Electrons: Small Amplitude Theory", Proc. Phys. Soc., Lond. B, LXIII, pages 409-427, 1950.

2 Buneman, O., "Generation and Amplification of Waves in Dense Charged Beams under Crossed Fields", Nature, V165, page 474, March 1950.

little practical interest, it will not be treated here.

The second type of wave involves propagation along a structure (usually periodic in space) such that the phase velocity of the wave is considerably less than the velocity of light. This structure is usually made as some type of periodically loaded transmission line, such as a loaded waveguide, (the side opposite the "slow wave" structure being an emitter) and the wave velocity can be made (within reason) to conform to the designer's wishes, usually one-tenth or less of the velocity of light. This low value of wave velocity allows certain simplifications to be made in the equations, as will be seen later.

The essential differences between the "field wave" and the "space charge wave" can then be summarized as follows. In the field wave, which is propagated with a phase velocity near to that of light, the space charge density exerts relatively little influence on the fields; that is, there is relatively little wave energy stored in the electron motions, so that this wave is characterized by $\nabla \cdot E = 0$. The wave energy in this case is stored alternately in the electric and magnetic fields. The space charge wave, propagating with a phase velocity small compared with that of light, is influenced to a great extent by the space charge; in fact the wave energy is stored alternately in the electric field of the wave and in the kinetic energy of the electrons. Since in this case there is relatively little energy stored in the magnetic fields, this wave type can be characterized by $\nabla \times E = 0$.

Propagation along a "Slow Wave" Structure

A schematic drawing of the structure to be considered in the following analysis is shown in Fig. 3.8. As in other sections of this report it is assumed that the electron velocity is slow compared to the velocity of light so that the treatment is non-relativistic and also the usual small signal method is used. In addition it is assumed that the wave velocity is small compared to the velocity of light so that the time rate of change of magnetic field can be neglected, and the electric field derived from a potential function. Then $\nabla \times E = 0$[1] and the field components present are E_x, E_y, and H_z.

The equations of motion are, from Eq. II-7:

$$i\omega v_x - \gamma v_o v_x = - e/m\, E_x$$

$$i\omega v_y - \gamma v_o v_y = - e/m\, E_y - \omega_c v_x \qquad \text{III-21}$$

---

1 The degree of approximation of this customary simplification can be seen by writing the electric field in terms of the vector potential A and the scalar potential ϕ as:

$$E = -\nabla\phi - \frac{\partial A}{\partial t}$$

and using the supplementary condition on A:

$$\nabla \cdot A = -\mu_o\varepsilon_o \frac{\partial \phi}{\partial t}$$

Then:

$$\nabla \cdot E = -\nabla^2\phi + \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$$

Considering the potential to represent a wave motion in the x direction so that $\phi = \phi_o e^{i\omega t - \gamma x}$

$$E = -\gamma^2\phi + \frac{\omega^2}{c^2}\phi$$

so that when $\omega^2/c^2 \ll \gamma^2$ this last term can be neglected and $E = -\nabla\phi$.

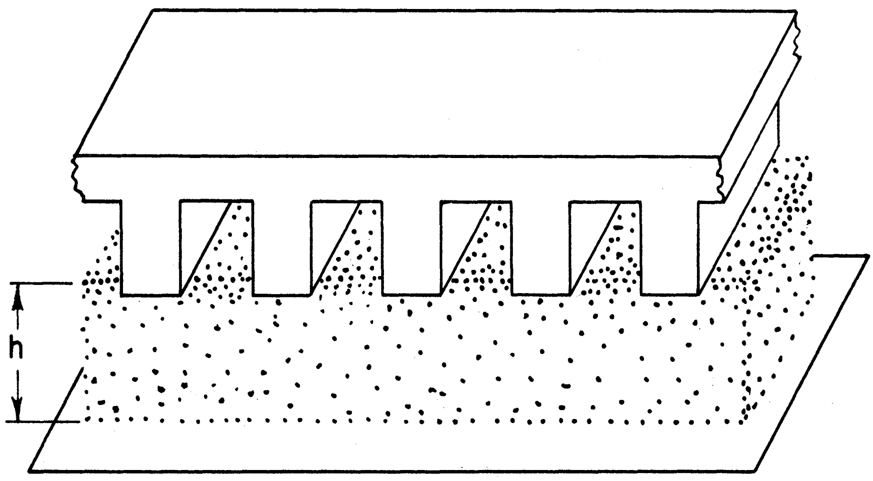


FIG. 3.8
IDEALIZED SPACE CHARGE IN PLANE MAGNETRON WITH PERIODIC ANODE

and from these the electron velocities are:

$$v_x = \frac{-e/m\, E_x}{i\omega + \gamma\,\omega_c y} \qquad v_y = \frac{-e/m\, E_y}{i\omega + \gamma\,\omega_c y} - \frac{e/m\,\omega_c\, E_x}{(i\omega + \gamma\,\omega_c y)^2}\,, \qquad \text{III-22}$$

where the substitution $v_o = -\omega_c y$ has been made. By this substitution it is implicitly assumed that the cathode is located at $y = 0$. The field equations are:

$$-\gamma E_y = \frac{\partial E_x}{\partial y} \qquad \text{(a)}$$

$$\frac{\partial H_z}{\partial y} = \rho_o v_x + v_o \rho_1 + i\omega\,\varepsilon_o E_x \qquad \text{(b)}$$

$$\text{III-23}$$

$$\gamma H_z = \rho_o v_y + i\omega\,\varepsilon_o E_y \qquad \text{(c)}$$

$$\rho_1 = \varepsilon_o\left[-\gamma E_x + \frac{\partial E_y}{\partial y}\right] \qquad \text{(d)}$$

Substituting Eq. III-22 into Eq. III-23, the following equations are obtained:

$$\frac{\partial H_z}{\partial y} = (A - \gamma \varepsilon_o v_o + i\omega \varepsilon_o) E_x + \varepsilon_o v_o \frac{\partial E_y}{\partial y} \qquad \text{(a)}$$

III-24

and

$$\gamma H_z = (A + i\omega \varepsilon_o) E_y - CE_x \qquad \text{(b)}$$

where:

$$A = \frac{-\rho_o e/m}{i\omega + \gamma\omega_c y} \qquad C = \frac{\rho_o e/m\, \omega_c}{(i\omega + \gamma\omega_c y)^2} .$$

Differentiating Eq. III-24(b), it follows that:

$$\frac{\partial H_z}{\partial y} = \frac{A + i\omega \varepsilon_o}{\gamma} \frac{\partial E_y}{\partial y} + \frac{E_y}{\gamma} \frac{\partial A}{\partial y} - \frac{C}{\gamma} \frac{\partial E_x}{\partial y} - \frac{E_x}{\gamma} \frac{\partial C}{\partial y} .$$

Equating this to Eq. III-24(a):

$$[A + (i\omega + \gamma\omega_c y)\varepsilon_o] \frac{\partial E_y}{\partial y} + E_y \frac{\partial A}{\partial y} - C \frac{\partial E_x}{\partial y} \qquad \text{III-25}$$

$$- \gamma \left[\frac{1}{\gamma} \frac{\partial C}{\partial y} + A + \varepsilon_o (i\omega + \gamma\omega_c y) \right] E_x = 0 .$$

From Eq. III-23:

$$E_y = -\frac{1}{\gamma} \frac{\partial E_x}{\partial y} \qquad \frac{\partial E_y}{\partial y} = -\frac{1}{\gamma} \frac{\partial^2 E_x}{\partial y^2}$$

so that Eq. III-25 becomes:

$$[A + (i\omega + \gamma\omega_c y)\varepsilon_o] \frac{1}{\gamma} \frac{\partial^2 E_x}{\partial y^2} + \left[\frac{1}{\gamma} \frac{\partial A}{\partial y} + C \right] \frac{\partial E_x}{\partial y}$$

III-26

$$+ \left[\frac{1}{\gamma} \frac{\partial C}{\partial y} + A + (i\omega + \gamma\omega_c y)\varepsilon_o \right] \gamma E_x = 0 .$$

The equations will be simplified by substitution of the new variable:

$$\ell = \frac{\omega}{\omega_c} \left[1 + \frac{\gamma\omega_c y}{i\omega} \right] \qquad \text{III-27}$$

Then Eq. III-2(b) becomes:

$$\left[\ell - \frac{1}{\ell}\right]\frac{\partial^2 E_x}{\partial \ell^2} + \frac{2}{\ell^2}\frac{\partial E_x}{\partial \ell} - \left[\frac{2}{\ell^3} - \frac{1}{\ell} + \ell\right] E_x = 0 \quad , \qquad \text{III-28}$$

This is the basic differential equation representing the electric field in the space charge region and presupposes only that the velocity of the wave in the space charge is small compared with the velocity of light. In what follows the exact nature of the external circuit will not be specified, it being presumed that a circuit of the characteristics desired to achieve certain performance can be constructed. It is seen from Eq. III-27 that Re $\ell = 0$ corresponds to synchronism between a layer of electrons at distance y above the cathode and a wave travelling in the -x direction.

Since the most interesting interaction effects take place for velocities near this synchronism condition, in what follows the attention will be confined to small values of ℓ. Therefore the solution to Eq. III-28 will be found in terms of a power series expansion in ℓ. However, first the equation will be examined briefly to demonstrate that this expansion is possible.

Eq. III-28 can be written in the form:

$$\frac{\partial^2 E_x}{\partial \ell^2} + \frac{1}{\ell}\left[\frac{2}{\ell^2 - 1}\right]\frac{\partial E_x}{\partial \ell} - \frac{1}{\ell^2}\left[\frac{2 - \ell^2 + \ell^4}{\ell^2 - 1}\right] E_x = 0$$

The terms $\frac{2}{\ell^2 - 1}$ and $\frac{2 - \ell^2 + \ell^4}{\ell^2 - 1}$ are both analytic at $\ell = 0$ and the equation has a regular singular point at $\ell = 0$. A power series expansion is therefore possible[1] about $\ell = 0$ (at least in the region $-1 < \ell < 1$).

1 See e.g. Rainville, E. D., "Intermediate Differential Equations", John Wiley and Sons, page 90, 1943.

Therefore letting:

$$E_x = \sum_{0}^{\infty} a_n \ell^n \qquad \text{III-29}$$

it follows that:

$$\frac{\partial E_x}{\partial \ell} = \sum_{0}^{\infty} n a_n \ell^{\,n-1}$$

and

$$\frac{\partial^2 E_x}{\partial \ell^2} = \sum_{0}^{\infty} n(n-1)\, a_n \ell^{\,n-2}$$

Substituting these series into Eq. III-28, the following recursion relation is found:

$$-\sum_{0}^{\infty} \left[n\ (n-3) + 2\right] a_n \ell^{\,n-3} + \sum_{0}^{\infty} \left[n\ (n-1) + 1\right] a_n \ell^{\,n-1} - \sum_{0}^{\infty} a_n \ell^{\,n+1} = 0$$

so that the first few coefficients are:

$a_o = 0$	$a_1 =$ arbitrary
$a_2 =$ arbitrary	$a_3 = a_1/2$
$a_4 = a_2/2$	$a_5 = 5/24\ a_1$
$a_6 = 11/40\ a_2$	$a_7 = 93/720\ a_1$
$a_8 = 321/1680\ a_2$	$a_9 = 3848/40320\ a_1$
$a_{10} = .1475\ a_2$	$a_{11} = .0760\ a_1$

The series solution for E_x is then:

$$E_x = a_2\left[\frac{a_1}{a_2}\ell + \ell^2 + \frac{1}{2}\,\frac{a_1}{a_2}\,\ell^3 + \frac{1}{2}\ell^4 + \frac{5}{24}\,\frac{a_1}{a_2}\,\ell^5 + \frac{11}{40}\ell^6 + \ldots\right] \qquad \text{III-30}$$

It is seen that two independent series are obtained, one including the even and the other the odd powers of the variable quantity.

The boundary condition to be imposed on the electric field E_x in the space charge is that it must vanish at the cathode, i.e. $E_x = 0$ when $y = 0$ ($\ell = \frac{\omega}{\omega_c}$). Under this condition Eq. III-30 becomes:

$$0 = \frac{a_1}{a_2}\frac{\omega}{\omega_c}\left[1 + \frac{1}{2}\frac{\omega^2}{\omega_c^2} + 0.21\frac{\omega^4}{\omega_c^4} + \dots\right] + \frac{\omega^2}{\omega_c^2}\left[1 + \frac{1}{2}\frac{\omega^2}{\omega_c^2} + 0.275\frac{\omega^4}{\omega^4} + \dots\right]$$

so that for $\omega/\omega_c < 0.8$ the relation

$$a_1/a_2 = -\,\omega/\omega_c \qquad \text{III-31}$$

is valid and is a very close approximation for $\omega/\omega_c < 1$. Eq. III-30 then becomes:

$$\frac{E_x}{a_2} = -\frac{\omega}{\omega_c}\ell\left[1 + \frac{1}{2}\ell^2 + 0.21\,\ell^4 + 0.129\,\ell^6 + 0.0966\,\ell^8 + \dots\right]$$

III-32

$$+\,\ell^2\left[1 + \frac{1}{2}\ell^2 + 0.275\,\ell^4 + 0.191\,\ell^6 + \dots\right]$$

The electric field E_x in the space charge is plotted in Fig. 3.9, vs Re ℓ from Eq. III-32 using the boundary condition Eq. III-31. Certain qualitative information concerning the space charge can be obtained from a study of these curves.

The cathode of the magnetron is, of course, represented by the right hand intersection with the abscissa of the curve corresponding to the particular value of ω/ω_c under consideration. The intersection at $\ell = 0$ corresponds to synchronism between the layer of electrons at a given value of y and the travelling electromagnetic wave.

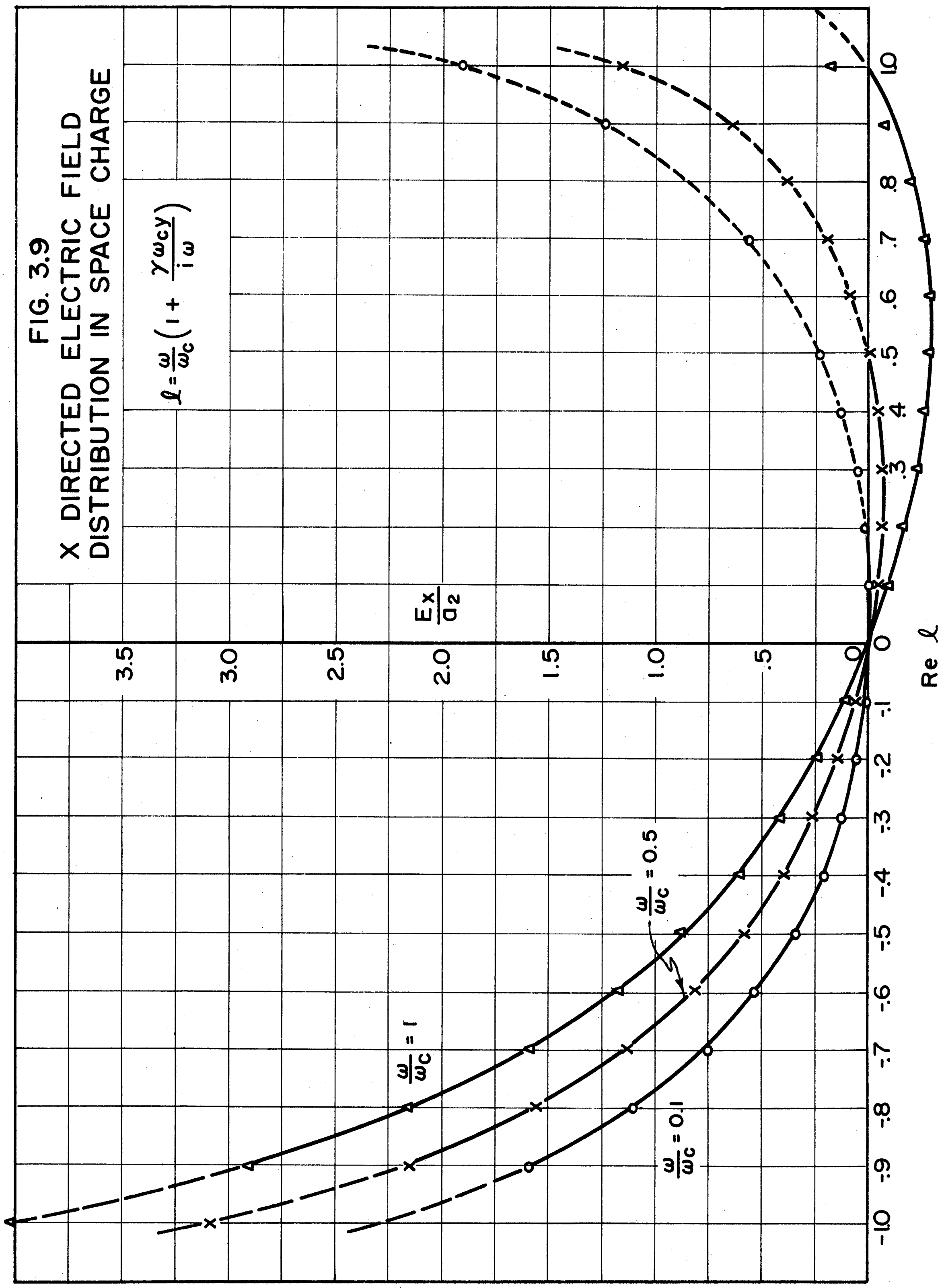


FIG. 3.9
X DIRECTED ELECTRIC FIELD
DISTRIBUTION IN SPACE CHARGE

It is seen that this synchronous layer of electrons becomes an infinite admittance sheet in the space charge. The portion of the curves to the right of $\ell = 0$ represent the field in the region in which the electrons are moving slower than the wave and the part to the left of $\ell = 0$ the region in which the electrons are moving faster than the wave. From this it appears that the interaction space in a magnetron is divided into two regions by this admittance sheet; the region between cathode and the infinite admittance sheet and the region between this sheet and the anode. As the electrons are caused to increase in velocity (e.g. by increasing the magnetic field) this sheet will be displaced toward the cathode.

Examination of Eq. III-28 reveals that in addition to the regular singular point at $\ell = 0$, this equation contains a second regular singular point at $\ell = \pm 1$. It will prove interesting to examine breifly the physical nature of these singularities also. For the value $\ell = -1$, the real part of Eq. III-27 becomes:

$$\omega_c = \omega \left(\frac{v_o}{v_p} - 1 \right)$$

where v_p is the phase velocity of the wave propagating with the electron stream, i.e. $v_p = - \omega/\mathrm{Im}\gamma$. The right side of this relation is seen to be the frequency of the wave whose forces are acting on the electrons, as seen by the moving electrons. That is, while a stationary observer (an electron) experiences a force due to the fields of frequency ω, an observer moving with velocity v_o experiences a force of frequency $\omega\ (v_o/v_p - 1)$. Therefore at the value of v_o/v_p for which this Doppler frequency is equal to the cyclotron frequency, the layer of electrons for which Re $\ell = \pm 1$ experiences a resonance effect between the wave and the applied magnetic field. From this it would be expected that the electric fields of the wave have a singularity[1] at the value Re $\ell = -1$. That this is indeed the case can be seen from Figs. 3.9 and 3.10.

The singularity at the point $\ell = +1$ is of less interest since this interaction (for $\omega/\omega_c < 1$) takes place below the cathode. However this singularity corresponds to the same type of phenomenon, with electrons moving in the + x direction interacting with fields of Doppler frequency $\omega(1 - \frac{v_o}{v_p})$ of a wave travelling in the + x direction.

Evaluation of the Electronic Admittance

Using the series expression (Eq. III-32) for the x directed electric field in the space charge region, together with Eq. III-23-a, the y directed electric field distribution can be determined.

$$E_y = \frac{1}{\gamma}\frac{\partial Ex}{\partial y} = i\frac{\partial Ex}{\partial \ell}$$

$$= ia_2\left[-\frac{\omega}{\omega_c}\left[1 + \frac{3}{2}\ell^2 + 1.05\,\ell^4 + 0.9\,\ell^6 + 0.87\,\ell^8 + \ldots\right] + \left[2\,\ell + 2\,\ell^3 + 1.65\,\ell^5 + 1.53\,\ell^7 + 1.47\,\ell^9 + \ldots\right]\right] \qquad \text{III-33}$$

This series converges even less rapidly than Eq. III-32 so that values were not obtained above $\ell = 0.9$; the curves representing this series solution are shown in Fig. 3.10. It is seen from this figure that the y directed electric field also appears to increase without limit near $\ell = \pm 1$.

1. It can be shown (Whittaker and Watson, "A Course of Modern Analysis", Cambridge, page 201, 1946) that in such a case as considered here, the function will be either analytic or have a logarithmic singularity at the regular singular point. In view of the above-mentioned account of the nature of the interaction at the singularity, the latter possibility appears more probable.

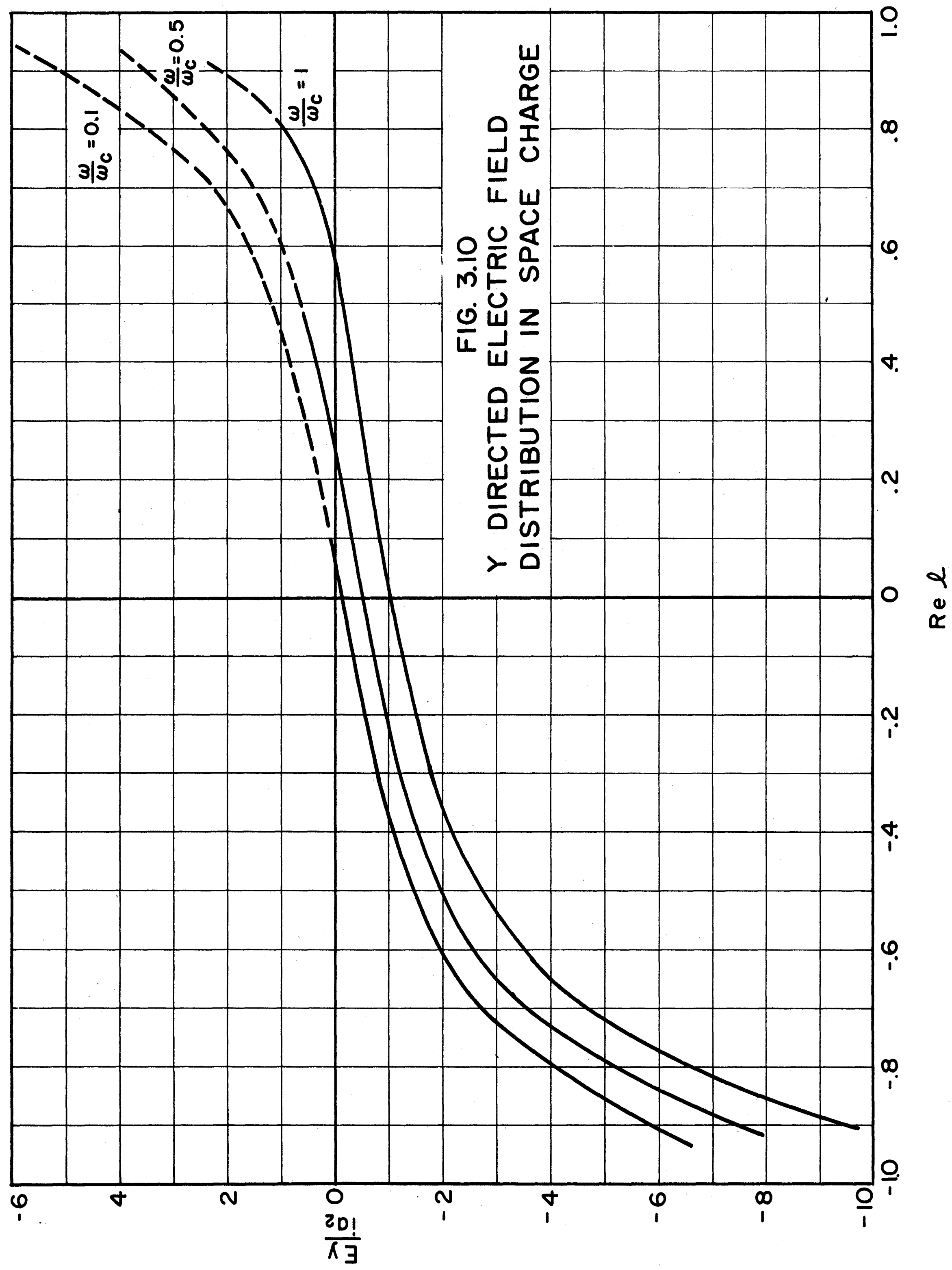


FIG. 3.10
Y DIRECTED ELECTRIC FIELD
DISTRIBUTION IN SPACE CHARGE

Reference is now made to section V-1 where the boundary conditions at the edge of the space charge cloud are discussed. Eq. V-8 of that section can be used to evaluate the admittance Y_e seen looking into the space charge cloud from the anode. Applying Eq. III-27, Eq. V-8 can be written in the form:

$$Y_e(h) = \frac{H_z(h)}{E_x(h)} = \frac{i\omega_0\, h\, \varepsilon_0}{\ell - \omega/\omega_c}\left[\left[\omega/\omega_c - \frac{2\ell - \omega/\omega_c}{\ell^2}\right]\frac{1}{i}\,\frac{E_y(h)}{E_x(h)} + \frac{2\ell - \omega/\omega_c}{\ell^3}\right] \qquad \text{III-34}$$

The fraction $\frac{1}{i}\,\frac{E_y(h)}{E_x(h)}$ can be determined from the quotient of Eqs. III-33 and III-32. However since ℓ will in general be complex, this expression will become quite long. Therefore a simplification will be effected by restricting the consideration to values of electron velocity near the synchronism condition. That is, by restricting ℓ to small values ($\ell < 0.4$) a simple equation for $E_y(h)/E_x(h)$ can be obtained, allowing an analytical expression for Y_e to be found. By neglecting powers of $\ell(h)$ higher than the first in Eqs. III-33 and III-34 there results:

$$\frac{1}{i}\,\frac{E_y(h)}{E_x(h)} \cong \frac{2\ell(h) - \omega/\omega_c}{-\frac{\omega}{\omega_c}\,\ell(h)}$$

so that the electronic admittance becomes:

$$Y_e(h) \quad - \frac{i\,\omega_0\,\varepsilon_0\,h}{\ell(h) - \omega/\omega_c}\left[-2 + \frac{1}{\ell(h)}\left(\omega/\omega_c + \frac{4}{\omega/\omega_c}\right) - \frac{2}{\ell^2(h)}\right] \qquad \text{III-35}$$

Letting

$$\ell = \ell_r + i\ell i$$

$$\gamma = \alpha + i\beta$$

Eq. III-27 shows that the phase constant of the travelling wave is given by:

$$\beta = \frac{1}{y}\ (\ell_r - \omega/\omega_c)$$

and the attenuation constant by:

$$\alpha = -\frac{\ell_i}{y}\ .$$

The real and imaginary parts of the electronic admittance are therefore given by:

$$\mathrm{Im}\ Y_e(h) = \frac{\omega_c\ \varepsilon_o\ h}{(\ell_r-\omega/\omega_c)^2 + \ell_i^2}\ \times$$

$$\left[(\ell_r-\omega/\omega_c)\left[-2 + \frac{\ell_r}{\ell_r^2 + \ell_i^2}\left(\frac{\omega}{\omega_c} + \frac{4}{\omega/\omega_c}\right) - \frac{2(\ell_r^2 - \ell_i^2)}{(\ell_r^2-\ell_i^2)^2+4\ \ell_i^2\ \ell_r^2}\right]\right.$$

$$\left. + \ell_i^2\left[\frac{-1}{\ell_r^2 + \ell_i^2}\left(\frac{\omega}{\omega_c} + \frac{4}{\omega/\omega_c}\right) + \frac{4\ \ell_r}{(\ell_r^2-\ell_i^2)^2 + 4\ \ell_i^2\ \ell_r^2}\right]\right] \quad \text{(a)}$$

III-36

$$\mathrm{Re}\ Y_e(h) = \frac{\ell_i\ \omega_c\ \varepsilon_o\ h}{(\ell_r - \frac{\omega}{\omega_c})^2 + \ell_i^2}\ \times$$

$$\left[(\ell_r-\omega/\omega_c)\left[\frac{1}{\ell_r^2 + \ell_i^2}\left(\frac{\omega}{\omega_c} + \frac{4}{\omega/\omega_c}\right) - \frac{4\ \ell_r}{(\ell_r^2-\ell_i^2)^2 + 4\ \ell_i^2\ \ell_r^2}\right]\right. \quad \text{(b)}$$

$$\left. + \left[-2 + \frac{\ell_r}{\ell_r^2 + \ell_i^2}\left(\frac{\omega}{\omega_c} + \frac{4}{\omega/\omega_c}\right) - \frac{2\ (\ell_r^2 - \ell_i^2)}{(\ell_r^2-\ell_i^2)^2 + 4\ \ell_i^2\ \ell_r^2}\right]\right]$$

In the above equations it is understood that the values of ℓ_r and ℓ_i to be used are those at the boundary $y = h$.

In order to illustrate the manner of variation of Im $Y_e(h)$, Eq. III-36-a is shown plotted in Fig. 3.11 for the simple case of no time average energy exchange between electrons and fields ($\ell_i = 0$).

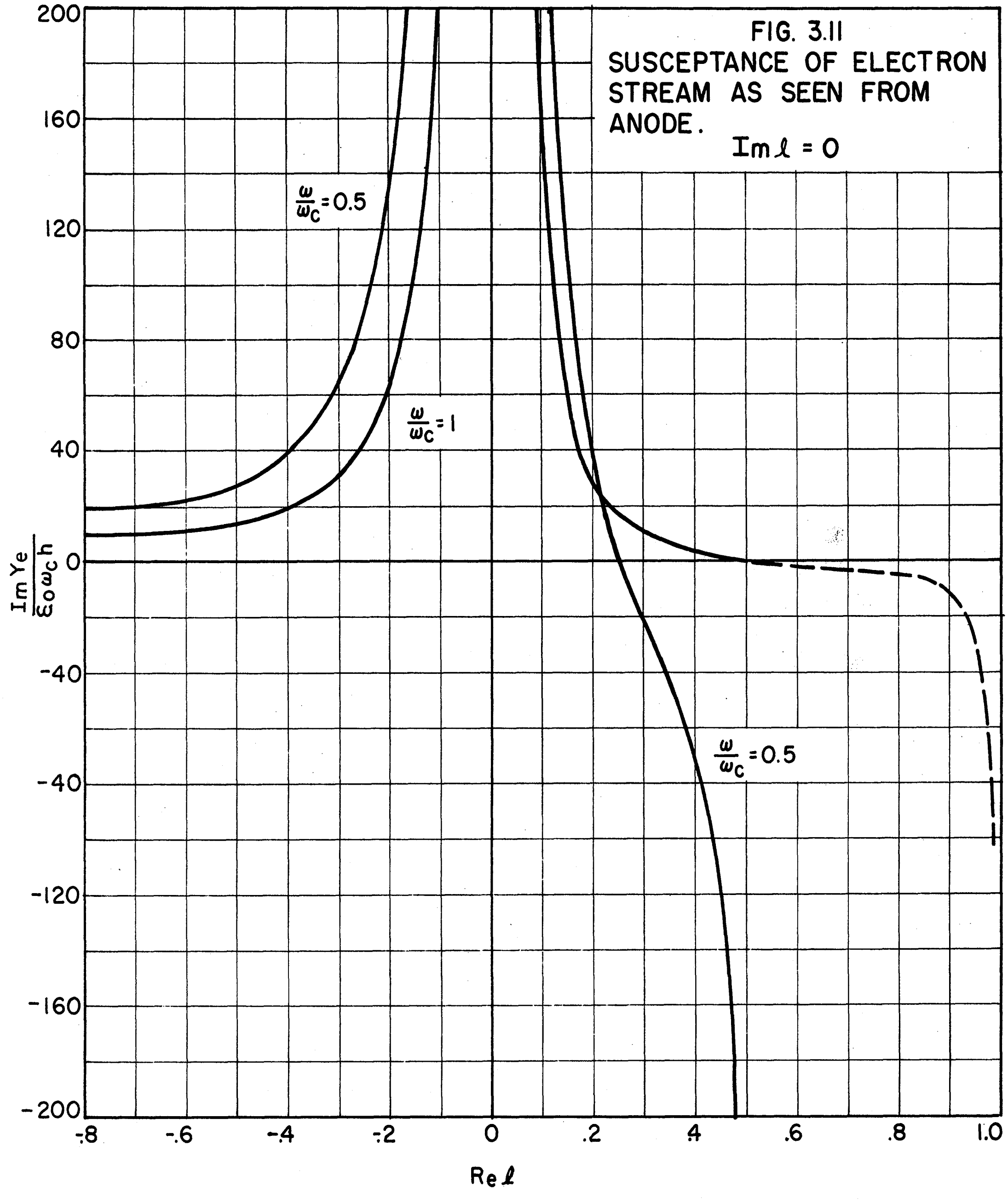


FIG. 3.11
SUSCEPTANCE OF ELECTRON STREAM AS SEEN FROM ANODE.
Im ℓ = 0
ω/ωc = 0.5
ω/ωc = 1
ω/ωc = 0.5
Im Ye / ε0 ωc h
Re ℓ
200
160
120
80
40
0
-40
-40
-120
-160
-200
-.8
-.6
-.4
-.2
0
.2
.4
.6
.8
1.0

These curves show the existance of the zero reactance sheet at $\ell = 0$, mentioned previously. They also show that the space charge appears inductive to the anode circuit for $\ell > \frac{1}{2}\,\frac{\omega}{\omega_c}$ and capacitive for $\ell < \frac{1}{2}\,\frac{\omega}{\omega_c}$. The value of electron velocity at which the electronic susceptance vanishes thus corresponds to one-half of the phase velocity of the wave, and the (Doppler) frequency of the force experienced by the electrons due to the travelling wave corresponds to the Larmor frequency ($\omega_c/2$). Electrons with velocity greater than that of the travelling wave appear capacitive but not so much so as the synchronous electrons. Therefore as the space charge cloud is expanded by increasing the d-c anode voltage, the boundary of the cloud appears inductive for $\ell > \omega/2\omega_c$ and capacitive for $\ell < \omega/2\omega_c$.

These curves or any of the other theoretical results derived here are, of course, not valid under conditions at or near oscillation, in which the cloud surface and the motions of the electrons can be greatly changed by the large fields present in the interaction space.

It is seen from Eq. III-36-b that for $\ell_r < 0$, corresponding to electrons moving faster than the travelling wave ($\beta<0$), $\ell_i > 0$ results in a negative electronic conductance. Since the wave has been assumed to vary as $e^{i\omega t - \gamma x}$, positive ℓ_i corresponds to a growing wave. This result is at least consistant and it appears that the magnetron space charge is capable of delivering energy to the fields of an anode structure of the proper characteristics. Of course, a definite answer to this question of amplification can be obtained only from a complete solution including the boundary conditions imposed by the confining circuit[1]. That is, one must demand that $Y_e + Y_c = 0$ where Y_c is the circuit admittance. Lack of time prevents the completion of this study.

Amplification by means of the magnetron space charge should be characterized by greater efficiency and power output capabilities than the travelling wave or electron wave tube. This follows from consideration of the phase focussing effect of the magnetic field (that is, removal from the interaction space of out-of-phase electrons) and from the fact that an electron bunch after having given energy to the fields of the wave, can move toward the anode, thus abstracting additional energy from the d-c field which can in turn be transferred to the r-f wave.

As has been suggested before[2], the existance of such amplification due to a magnetron space charge may aid in the explanation of the large observed noise output of magnetrons. That is, most considerations of possible noise mechanisms in the magnetron result in the conclusion that some amplification of the noise must take place after its generation (probably in the immediate vicinity of the cathode) until it is delivered to the fields of the resonators. Twiss[3] proposes a mechanism of radial amplification, involving the coherent constructive interference of the radially moving electrons in the multiple stream type space

1 Macfarlane and Hay found that amplification along the stream was in fact possible, but it is believed that the boundary conditions they used are open to question.

2 This possibility has been suggested by a number of authors, one of the first was probably Haeff, Phys. Rev. 75, page 1546, 1949.

3 Twiss, R. Q., Loc. cit.

charge. However it is at least conceivable that all or part of the amplification take place in the tangential direction, the waves representing the noise energy travelling with the electrons and interacting with the fields of the resonators.

IV. DISCUSSION OF THE RESULTS OF CHAPTER III

In the previous chapter the complex index of refraction of the magnetron space charge was determined as a function of the applied magnetic field and the frequency of the wave. By considering the medium as a nearly perfect dielectric, the index of refraction was expressed in terms of an effective relative dielectric constant. The results of this analysis are contained in the curves of ε_e vs ω/ω_c.

Considerable care must be exercised in the interpretation of the dielectric constant ε_e. The only safe interpretation that can be given to ε_e is as an expression of the phase velocity of the travelling wave. That is, a wave propagating through the space charge region which exhibits dielectric constant ε_e will travel with the same phase velocity as a wave in a real perfect dielectric of relative dielectric constant ε_e. In the region $\varepsilon_e > 1$ the space charge can be thought of as behaving as a real dielectric. In the region $\varepsilon_e < 0$ the space charge can be thought of as analogous to a conducting material in which the wave is attenuated. However in the region $0 < \varepsilon_e < 1$ no such comparisons can be made since real dielectrics exhibiting this property are not known. Similar concepts to this are encountered in ionospheric propagation problems.

In this chapter an attempt will be made to explain the mechanism and results of the electron-wave interaction and to give as good a physical picture of these various values of dielectric constant as possible.

1. Discussion of the Electron-Wave Interaction.

An electromagnetic wave, polarized along one of the coordinate axes, impinging from free space onto a magnetron space charge will, in general, be partially transmitted into the medium. In the space charge the electrons will be accelerated by the electric field of the impinging wave; however due to the applied magnetic field the electrons will have a velocity component normal to the direction of polarization of this wave, necessitating an additional component of electric field (and therefore magnetic field). The electrons thus set into motion by this "inducing field" will radiate (as small doublets) and can be thought of as coherent sources of a "radiation field" in the space charge. The inducing and radiation fields must be one and the same and constitute the wave whose propagation characteristics were found in Chapter III.

In the present analysis, the space charge is thought of as exhibiting medium-like behavior. That is, the motions of the electrons are considered to be influenced by and interlocked with the motions of the surrounding electrons. This type of interaction would be characteristic of a region of high electron density. This behavior is to be contrasted with that believed to exist in a region of relatively low electron density in which the effect of the electrons is just the summation of the effects of the individual electrons acting independently. That is, while the r-f fields in the electron cloud, in the latter case, will be essentially the same as those in the absence of the electrons; in the former case, in which the space charge exhibits medium-like properties, the total fields will be those dictated by the electron motions in the space charge.

The electrons in the space charge will be accelerated, thus abstracting energy from the wave, during one-half cycle and will, if they suffer no energy loss, return exactly this amount of energy to the wave during the next half cycle. This is shown in Section 4 of this chapter. The electrons therefore contribute primarily a reactive effect. If this reactive electron current leads, in time phase, the electric field of the wave the effect will be capacitive ($\varepsilon_e > 1$) and the wave will be propagated with phase velocity less than that of light in vacuo. If the electron current lags the field, the effect will be analogous to an inductance ($0 < \varepsilon_e < 1$) and the phase velocity of the wave will exceed that of light. At the value of ω/ω_0 at which $\varepsilon_e = 0$ the wave will be totally reflected from the space charge boundary. The behavior of the electrons at the boundary of the space charge at the value of ω/ω_0 at which $\varepsilon_e = 0$ has been discussed in a brief note by Forsterling and Wuster[1] in connection with the ionosphere.

It is noted that only those solutions have been obtained, for wave propagation in the space charge, for which the electric fields of the wave are not parallel to the applied magnetic field. The solutions for those cases in which the electric field is polarized in the direction of the applied magnetic field are immediately separable from those found above. These latter solutions are well known[2] and are trivial from the point of view of this analysis since the magnetic field does not influence the motion of the electrons, its effect on wave propagation being

---

1 Forsterling, K. and Wuster, H. O., "On the Origin of Harmonics in the Ionosphere - At Points where the Dielectric Constant is Zero", Comptes Rendus - Acadamie des Sciences - 231 - No. 17, page 831, October 23, 1950.

2 See for example: Nichols, H. W. and Schelleng, J. C., "On the Propagation of Electromagnetic Waves Through an Atmosphere Containing Free Electrons", Bell Syst. Tech. J. - 4, page 215, 1925.

felt only through the space charge density. It is the motion of the electrons due to the various forces (including the applied magnetic field) that causes the interesting effects of the space charge on wave propagation found here.

2. Illustration of Dielectric Constant by its Effect on a Resonant Circuit.

A magnetron space charge placed in a resonant circuit in such a way that it can interact with the electric fields, will have an effect on the resonant frequency of the circuit. Reference to the curves of ε_θ as a function of ω/ω_0 will show the nature of this effect. It is seen that ε_θ can assume values which are positive and greater than unity, in which case its effect on the circuit will be similar to that of an ordinary dielectric; it can be negative, in which case the effect will be similar to that of a conducting material. Finally, values of ε_θ positive but less than one can be obtained; in this case the phase velocity of the wave will exceed the velocity of light and in certain simple cases this effect on the circuit can be considered as similar to that of an inductance though this concept must be used with caution.

In order to illustrate the effect of a dielectric on an associated circuit, the resonant wavelength of a coaxial line one-quarter wavelength long, containing a dielectric material, will be computed below.

Consider a coaxial cavity containing a cylindrically symmetric dielectric material located in one end of the cavity. The electric fields of the TEM mode are

in air, region 1: $$E_{r1} = A_1 e^{\frac{i\omega x}{c}} + B_1 e^{\frac{-i\omega x}{c}} \qquad \text{IV-1}$$

in dielectric, region 2: $$E_{r2} = A_2 e^{\frac{i\omega x}{v_2}} + B_2 e^{\frac{-i\omega x}{v_2}} \qquad \text{IV-2}$$

subject to the boundary conditions (see Fig. 4.1):

$$E_{r1} = E_{r2} \qquad x = a$$

$$E_{r1} = 0 \qquad x = 0$$

$$I_{x2} = 0 \qquad x = b$$

Therefore

$$z_1 = \tan \frac{\omega x}{c} \qquad z_2 = \frac{\cot \frac{\omega(b-a)}{v2}}{\sqrt{\varepsilon_r}}$$

The condition for resonance is then

$$\cot \frac{2\pi\,(b-a)\,\sqrt{\varepsilon_r}}{\lambda_o} = \sqrt{\varepsilon_r}\ \tan \frac{\omega a}{c} = \sqrt{\varepsilon_r}\ \tan \frac{2\pi a}{\lambda_o} \qquad \text{IV-3}$$

The curve of Fig. 4.1 shows the resonant wavelength of this cavity as a function of ε_r.

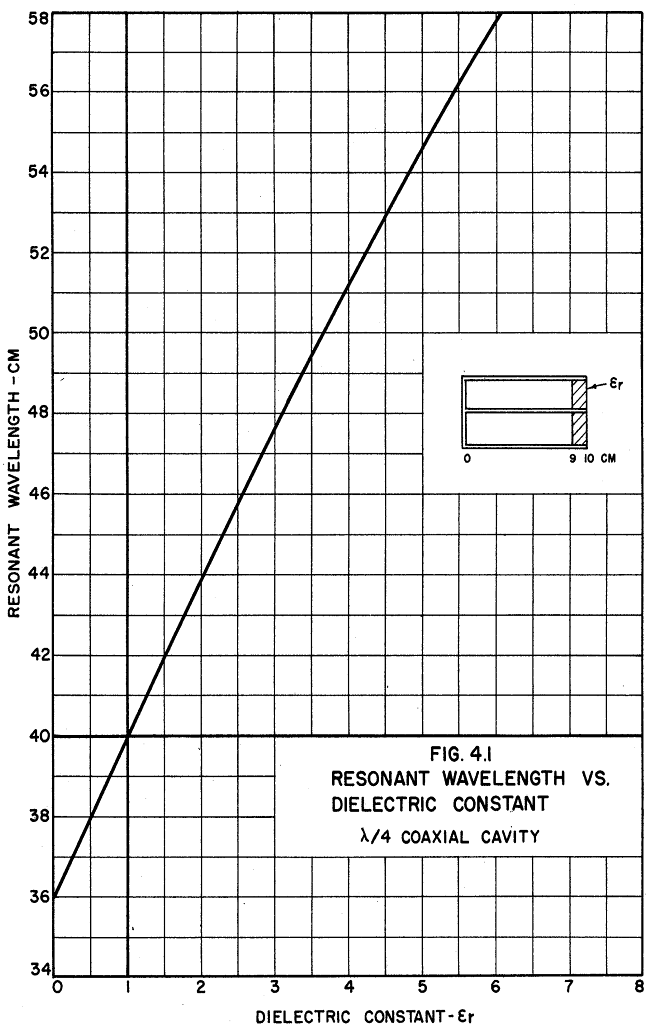


FIG. 4.1
RESONANT WAVELENGTH VS. DIELECTRIC CONSTANT
λ/4 COAXIAL CAVITY

It is seen that when $\varepsilon_r = 1$ the cavity is merely a $\lambda/4$ open line with resonant wavelength $\lambda = 4b = 40$ cm. An increase in ε_r has the effect of increasing a shunt capacitance across the open end, thus increasing the resonant wavelength. A decrease in ε_r below unity has the effect of decreasing the normal shunt capacitance of that portion of the line occupied by the dielectric, thus reducing the resonant wavelength. From Eq. IV-3 above, as $\varepsilon_r \rightarrow 0$ $\lambda_0 \rightarrow 4a = 36$ cm, so that the line is effectively opened at the boundary (a) of the dielectric. This is, of course, because the waves cannot penetrate the dielectric whose $\varepsilon_r = 0$ and are thereby totally reflected.

3. Discussion of the ε_e vs. ω/ω_c Curves.

An attempt will be made in this section to explain certain features of the curves showing the variation in the effective dielectric constant. It has been mentioned previously that the effective dielectric constant can take on values which are positive or negative. The effect on the propagating wave of the space charge, exhibiting properties in these various regions of ε_e, will be discussed together with a possible physical explanation of the zeros and infinities in these curves.

The region $\varepsilon_e > 1$ needs no explanation since here the wave behaves as if it were propagating through a real, perfect dielectric with relative dielectric constant ε_e. Waves impinging from free space onto the space charge will be partially reflected from the boundary of the space charge, due to the discontinuity in properties of the media.

In the region $0 < \varepsilon_e < 1$ the wave travels with a phase velocity greater than that of light in vacuo. For a space charge cloud of dimensions small compared with the wavelength of the impinging wave, the properties of the cloud can be explained with the aid of lumped circuit analogies. Thus in the region $0 < \varepsilon_e < 1$, the capacitance between boundary planes (normal to the electric field) of the space charge will be less than the corresponding capacitance with only free space between the planes. This decreased value of capacitance can be thought of as the parallel combination of a capacitor ($\varepsilon_r = 1$) and an inductance. The inductance seems to represent the effect of the space charge cloud, thus leading to the consideration of the space charge as exhibiting inductive properties in the range $0 < \varepsilon_e < 1$. This concept must be used with caution however.

At the critical value of ω/ω_c for which $\varepsilon_e = 0$, it is seen that at the boundary of the cloud (or at the surface in the cloud at which $\varepsilon_e = 0$) the normal component of electric field of the impinging wave must vanish. However, the remaining fields will completely penetrate the space charge, and because their phase velocities are infinite, the electric field will be in time phase at all points throughout the space charge region.

In the region of negative effective dielectric constant, it is seen that the wave is attenuated in the space charge (no propagation occurs). This attenuation is similar to that experienced by a wave impinging upon a conducting material, except that in the space charge there is no energy loss as a result of the attenuation, i.e. it is purely reactive in nature. A better analogy for the purpose of explaining the phenomenon of wave reflection in this region if that of a wave guide beyond cutoff. In this latter case, the propagation constant becomes imaginary so that the magnitude of the Poynting vector decreases exponentially along the direction of propagation, the energy of the wave being reflected continuously from various points along the line. This same type of continuous reflection process is believed to occur in the space charge with negative dielectric constant.

The concept of a skin depth can be applied to the space charge. In the region in which $\varepsilon_e < 0$, the wave will vary along its direction of propagation as $\exp - \frac{\omega}{c}\sqrt{|\varepsilon_e|}\, s$ so that the skin depth can be written:

$$d = \frac{c}{\omega\sqrt{|\varepsilon_e|}} = \frac{\lambda}{2\pi\sqrt{|\varepsilon_e|}}$$

where λ is the free space wavelength of the propagating wave. This skin depth relation will be used later to assist in an explanation of experimentally observed results of tests on this space charge.

Some additional information on the behavior of the electromagnetic waves propagating in the space charge can be obtained from computation of the group velocity in this medium. The most convenient expression for the group velocity for this purpose is:

$$v_g = \frac{d\omega}{dk}$$

where $k = \omega/v_p$ = circular wave number.

The phase velocity of a wave propagating in any direction normal to the steady electron motion in a plane magnetron is given by (Eq. III-3 or III-16):

$$\frac{c^2}{v_p^2} = \frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_0{}^2}{\omega^2}$$

so

$$v_g = \frac{k}{\omega}\frac{d\omega^2}{dk^2} = \frac{k^2}{\omega}c^2 = \frac{c^2}{v_p} = c\sqrt{1 - \frac{1}{\omega^2/\omega_0{}^2}}$$

It is seen that in this case $\eta\eta_g = 1$ where η and η_g are the phase and group refractive indices respectively, defined as $\eta = \frac{c}{v_p}$, $\eta_g = \frac{c}{v_g}$. This relation $\eta\eta_g = 1$ is normally satisfied in dielectric media and in regions containing electrons not under the influence of a magnetic field.[1] It is not ordinarily satisfied in an electron atmosphere under the influence of a magnetic field. However it is seen that in the case of the plane magnetron space charge this relation is satisfied even

---

[1] See e.g. Wale, H. A. and Stanley, J. P., "Group and Phase Velocities from the Magneto-Ionic Theory", J. Atmos. Terr. Phys. VI - No. 2, page 82, 1950.

though a magnetic field is present. This is a result of the singular variation of steady electron velocity with distance from the cathode, which just counterbalances the x directed Lorentz force.

In the case of z-directed propagation in the cylindrical space charge (Eq. III-10):

$$\frac{c^2}{v_p^2} = \frac{c^2k^2}{\omega^2} = 1 + \frac{1 + \frac{1}{2\,\omega^2/\omega_c^2}}{1 - 2\,\omega^2/\omega_c^2}$$

$$\frac{1}{v_g} = \frac{\omega}{k}\,\frac{dk^2}{d\omega^2} = \frac{v_p}{c^2}\left[1 + \frac{2}{1 - 2\,\omega^2/\omega_c^2}\right]$$

so that the relation $\eta\,\eta_g = 1$ is not satisfied in this case.

For a radially propagated wave in the cylindrical space charge (Eq. III-19):

$$\frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_c^2}{2\omega^2}\left[\frac{\omega^2 - \omega_c^2}{\omega^2 - \omega_c^2}\right]$$

so

$$\frac{1}{v_g} = \frac{v_p}{c^2}\left[1 + \frac{1}{4}\quad\frac{1}{(\omega^2/\omega_c^2 - 1)^2}\right]$$

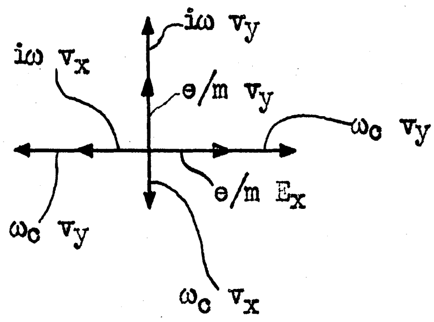
In this latter case also $\eta\,\eta_g \neq 1$.

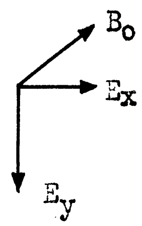
It is seen that in the above three cases the group velocity is equal to the product of the phase velocity and a positive quantity which depends on ω/ω_c. The group velocity therefore behaves similarly to the phase velocity insofar as the regions of negative dielectric constant are concerned. That is, the group velocity is imaginary where the phase velocity is imaginary ($\varepsilon_e < 0$), indicating no propagation in these regions of ω/ω_c.

In order to attempt a physical explanation of the singularities in ε_θ curves, examination of the equations of motion Eqs. II-7 and II-8 reveal that a group of electrons in the space charge is subjected to three forces; the electric field of the travelling wave, the Lorentz force due to the perturbed velocity, and the acceleration suffered by an electron moving to a point in space at which the steady electron velocity is different from the velocity at the point from which the electron came. For example, in the plane magnetron the equations of motion are given below and illustrated by the diagram.

$$i\omega\, v_x - \omega_c\, v_y = - e/m\, E_x - \omega_c\, v_y$$

$$-i\omega\, v_y = - e/m\, E_y + \omega_c\, v_x$$



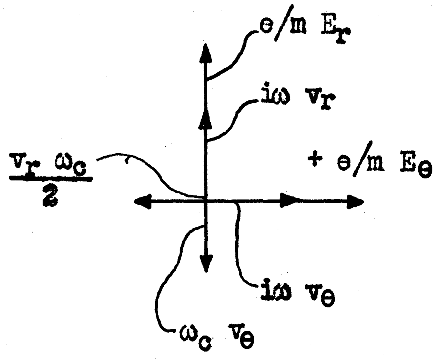


It is seen that the steady electron velocity is just such a function of the y coordinate that the x directed acceleration due to the motion of an electron from one place to another $(v_y \cdot \nabla)\, v_o$ is just balanced by the Lorentz force $-\omega_c y$. Solution for the electron velocities then shows no resonance effect.

In the cylindrical case, considering for example propagation in the z direction so that the fields are invariant in phase in an $r\theta$ plane, the equations are (for $r/r_0 >> 1$)

$$-i\omega\, v_r = -\, e/m\, E_r - \omega_c\, v_\theta$$

$$-i\omega\, v_\theta + v_r\, \omega_c/2 = -\, e/m\, E_\theta + \omega_c\, v_r$$



It is seen from these that the acceleration $(v_r \cdot \nabla)\, v_0$ does not exactly counterbalance the Lorentz force $\omega_c\, v_r$ so that solution for the radial electron velocity shows a resonance effect.

$$v_r \left[-i\omega - \frac{\omega_c^2}{2i\omega} \right] = -\, e/m\, E_r - \frac{\omega_c}{i\omega}\, e/m\, E_\theta$$

That is, for values of ω^2 near to $\omega_c^2/2$, v_r can take on very large values with finite fields. This "resonance" between the forces acting on the electron accounts for the singular value of ε_e observed for example at $\omega/\omega_c = 1/\sqrt{2}$ of Fig. 3.4.

One additional point, which should be mentioned before consideration of the effective dielectric constant curves is completed, is in relation to the existance of double refraction of the waves propagating through the space charge. In most treatments of the interaction of an electromagnetic wave with an electron stream under the influence of a static magnetic field, it is found that the wave exhibits double refraction. It has been shown in Chapter III that no such phenomenon occurs in the case of propagation in the magnetron space charge. In the plane magnetron space charge this is a result of the singular manner of variation of steady electron velocity with distance from the cathode, which just counter-balances the effect of the x directed Lorentz force.

The absence of double refraction in the cylindrical space charge must be explained separately for the two directions of propagation. In an electron atmosphere under the influence of a magnetic field, such as the ionosphere, the electric field of an electromagnetic wave propagating in a direction normal to the magnetic field will be split into two components. The wave whose electric field is parallel to the magnetic field (ordinary wave) will travel with a different velocity from the wave whose electric field is normal to the magnetic field (extraordinary wave). In the present treatment there was considered to be no electric field parallel to the magnetic field so that in section III-2 only the propagation characteristics of the wave corresponding to the extraordinary ray were found. In the cylindrical space charge no double refraction was found for a wave propagating along the magnetic field because of the geometry chosen. That is, only propagation of a TEM wave in a coaxial structure was considered, thus allowing no possibility of circular polarizations of different rotations.

4. Electron-Wave Energy Exchange.

The time rate of absorption of energy by the electrons from the electromagnetic field of the wave is:

$$W = \int_V J \cdot E^* \, dV \qquad \text{IV-4}$$

where the integration is extended over the space charge. For illustration, consider the case of propagation in a plane magnetron space charge in the y or z direction. Then from Eq. III-1:

$$J_x = \frac{-\rho_0 \, e/m \, E_x}{i\omega + g} \qquad J_y = \frac{-\rho_0 \, e/m \left[E_y + \frac{\omega_c}{i\omega + g} E_x\right]}{i\omega + g} \qquad \text{IV-5}$$

so that the energy absorption rate is:

$$\mathrm{Re}\, W = -\rho_0 \, e/m \ \ \mathrm{Re} \int \left[\frac{E_x E_x^*}{i\omega + g} + \frac{E_y E_y^*}{i\omega + g} + \frac{\omega_c}{(i\omega + g)^2} E_x E_y^*\right] dV \qquad \text{IV-6}$$

which becomes, since $g^2 \ll \omega^2$:

$$\mathrm{Re}\, W = -\rho_0 \, e/m \int \left[\frac{g \, E_x^2}{\omega^2} + \frac{g \, E_y^2}{\omega^2} - \frac{\omega_c}{\omega^2} E_x E_y^*\right] dV \, . \qquad \text{IV-7}$$

However, it can be shown, for example from Eqs. III-2, that E_x and E_y are 90° out of time phase so that when the consideration of the energy absorption is extended over a complete cycle this last term vanishes, leaving:

$$\mathrm{Re}\, W = -\frac{\rho_0 e}{2m} \frac{g}{\omega^2} \int \left[|E_x|^2 + |E_y|^2 \right] dV \quad . \qquad \text{IV-8}$$

In the case of no collisional energy loss by the electrons $g = 0$ and it is seen that there is no net interchange of energy between the electrons and the wave. Since the fields vary as $e^{i\omega t}$ the electrons gain energy from the wave during one-half cycle and give energy to the wave during the other half cycle so that the net energy change is zero. That the effect is therefore entirely reactive in the case of no collisional loss can be seen from Eq. IV-9.

Another interesting result can be derived from these energy considerations. The Poynting theorem can be expressed in the form[1]

$$-\nabla\cdot S = J\cdot E^* + i\omega\left[\varepsilon_0 |E|^2 - \mu_0 |H|^2\right] \qquad \text{IV-9}$$

where S is the complex Poynting vector. Then

$$-\nabla\cdot \mathrm{Re}\, S = \mathrm{Re}\, J\cdot E^* \qquad \text{IV-10}$$

which is the relation used above. Also

$$-\nabla\cdot \mathrm{Im}\, S = \mathrm{Im}\, J\cdot E^* = i\omega\left[\varepsilon_0 |E|^2 - \mu_0 |H|^2\right] \qquad \text{IV-11}$$

The collision-free space charge is a non-conducting medium, so that Im S = 0, from which it follows that Im $J\cdot E^*$ represents the difference between the mean values of electric and magnetic energy densities in the wave. From Eqs. IV-4 and IV-5:

$$\mathrm{Im}\, W = -\frac{\rho_0 e}{2m}\int\left[\frac{-i\omega|E_x|^2}{\omega^2} - \frac{i\omega|E_y|^2}{\omega^2}\right] dV = \mathrm{Im}\int J\cdot E^*\, dV$$

Therefore, since $|E|^2 = |E_x|^2 + |E_y|^2$, if one takes the time average of the energy density the following expression is obtained:

$$0 = +\frac{\rho_0 e}{2m}\frac{i\omega}{\omega^2}\int\left[|E_x|^2 + |E_y|^2\right] dV + \frac{i\omega}{2}\int\left[\varepsilon_0|E_x|^2 + \varepsilon_0|E_y|^2 - \mu_0|H|^2\right] dV$$

Using Eq. II-9 to eliminate ρ_0; this becomes:

$$0 = \frac{-\omega_0^2\,\varepsilon_0}{2\omega^2}\int\left[|E_x|^2 + |E_y|^2\right] dV + 1/2\int\left[\varepsilon_0\left(|E_x|^2 + |E_y|^2\right) - \mu_0|H|^2\right] dV$$

This equation becomes, since the integrations are extended over the same volume:

$$\left[-\frac{\omega_0^2}{\omega^2} + 1\right]\varepsilon_0\left[|E_x|^2 + |E_y|^2\right] - \mu_0 H^2 = 0 \qquad \text{IV-12}$$

---

[1] Stratton, J. A., "Electromagnetic Theory", McGraw-Hill, page 137, 1941.

From this relation it is seen that the electrons contribute the term $-\frac{\omega_o^2}{\omega^2}\left[|E_x|^2 + |E_y|^2\right]$ to the electric energy storage. The effect of the space charge can therefore be considered as equivalent to a change in the relative dielectric constant by the factor $1 - \omega_o^2/\omega^2$ which is the same as derived in a previous section.

Eq. IV-12 illustrates the condition existing when $\omega/\omega_c < 1$. In this region of ω/ω_c Eq. IV-12 cannot be satisfied by real E and H fields so that wave propagation per se cannot occur.

5. Discussion of Energy Loss in the Space Charge.

In general, as an electromagnetic wave is propagated through the magnetron space charge, it will be diminished in amplitude as the result of a real power loss by the wave to the surroundings. In the analysis of Chapter III the only source of loss mentioned was that due to collisions between the electrons and the heavy particles present in the space charge. In this section several additional sources of energy loss by the electrons will be considered. This loss of energy by the perturbed electrons is the same as energy loss by the propagating wave, since an electron will return to the wave all of the energy imparted to it during one-half cycle only if it is has lost no energy during the cycle.

This diminution of the amplitude of the wave with distance is due to a real energy loss and is entirely different from the corresponding phenomenon occuring when the effective dielectric constant is negative. In this latter case, as mentioned before, the amplitude is diminished because of continuous reflection as it progresses into the space charge.

The sources of energy loss by the electrons to the surroundings to be discussed here are:

(a) loss due to collisions with gas particles

(b) loss due to energy stored and dissipated in harmonic fields

(c) radiation loss

(d) loss due to collisions with the cathode

(e) loss due to collisions near a region where $\varepsilon_e = 0$.

a. Loss Due to Collisions with Gas Particles. As mentioned previously, when the amplitude of the propagating wave is small, the electrons in the space charge are linearly perturbed from their steady motion and during these perturbations can collide with heavy, relatively fixed atoms or ions. In the process of these collisions the electrons will transfer energy to the atoms, thus increasing the temperature of the gas atoms present, at the expense of the energy of the wave. The gross effect of these collisions was shown to be qualitatively similar to a frictional force proportional to the perturbed electron velocity.

As shown, e.g. in Fig. 3.4, the effective conductivity resulting from these collisions is of the order of 10^{-6} mhos/meter in the neighborhood of the singularity (i.e. near the value of ω/ω_c corresponding to maximum perturbed velocity), for typical gas pressures. This is of the same order as the conductivity of, e.g. Bakelite. This mechanism of energy loss can explain a decrease in Q of a high Q circuit in which the space charge is placed, but will not account for any magnitude of power loss such as might be observed at high r-f voltage levels.

b. Loss Due to Energy Stored and Dissipated in Harmonic Fields. Reference to the equations of motion Eqs. II-7 and II-8 or to Appendix 5 will show that when the term $(v_1 \cdot \nabla)v_1$ is included in the equations, the perturbed electron motion will no longer be sinusoidal with time but will contain components at harmonics of the fundamental frequency. These harmonics in the electron current will, of course, necessitate corresponding harmonic components of electric and magnetic fields, so that there will be additional waves propagated in the space charge. Some energy from the fundamental frequency wave will be transferred into these harmonic fields, which will in turn beat together giving rise to additional harmonics plus a wave of fundamental frequency.

Two cases can be considered. If the circuit in which the space charge is placed can absorb energy at any of these harmonic frequencies, there will be a net transfer of energy from the fundamental wave to this harmonic and will be lost from the system. If the circuit cannot absorb power from any of the harmonics (such as e.g. an idealized case of wave propagation in a space charge of infinite extent) there must exist a type of equilibrium between the rate of energy transfer from the fundamental to the harmonic fields and by means of the beating phenomena from the harmonic fields back to the fundamental wave. In a cavity not especially designed for the purpose, it would be highly coincidental if a resonance occurred at any of the lower harmonics, so that in general it would not be expected that appreciable energy could be transferred from these harmonic fields to a surrounding circuit; since little energy could be absorbed if the electrons had no large electric fields with which to interact.

c. Radiation Loss. The electrons, in their periodic motion, will be accelerated by the electric field of the propagating wave. It is well known that an accelerated electron will radiate energy. The total energy so radiated per second from n electrons per unit volume is[1]

$$R = \frac{2}{3}\frac{e^2 a^2}{c^2} n \ \frac{\text{ergs}}{\text{cm}^3\text{sec}} \qquad \frac{20\, e^2 a^2}{c^2} n \ \frac{\text{joules}}{\text{m}^3\text{sec}} \qquad \text{IV-13}$$

where e is the electronic charge, a the acceleration and c the velocity of light. Neglecting for the moment the effect of the magnetic field,

---

1 See e.g. Page and Adams, "Electrodynamics", Van Nostrand, page 328, 1940. Alfven, H., "Cosmical Electrodynamics", Oxford, page 35, 1950.

the acceleration can be written $a = (e/m)\,E$.

The energy stored in the field of the wave per unit volume is

$$W = \frac{\varepsilon_0 E^2}{2} \ \frac{\text{joules}}{m^3} \ .$$

Thus the "Q" or 2π times the ratio of peak energy stored to average energy dissipated per unit volume per cycle, due to radiation will be:

$$Q = 2\pi \frac{W}{R} = \frac{2\pi\ \varepsilon_0\ c^2\ E^2}{2x20\ n\ e^2 a^2} = \frac{2\pi\ \varepsilon_0\ c^2\ m^2}{20\ n\ e^4} \cong 6300$$

for $n = 5 \times 10^{10}$ electrons per cm^3. (Corresponding to a magnetic field of about 1000 gauss.)

These considerations of the energy radiated by a periodically accelerated electron assume, of course, that each electron can radiate independently of the surrounding electrons and that all of the energy so radiated is absorbed by the surroundings. These assumptions are believed to be fulfilled in practice.

The Q value of 6300 found above, for a space charge density corresponding to a magnetic field of 1000 gauss, will be increased or decreased as the space charge density is increased or decreased. This rate of energy loss by the space charge is not considered appreciable, so that it can be neglected while considering loss effects at large values of r-f signal strength.

<u>d. Loss Due to Collisions with the Cathode.</u> In this section the electrons will be considered to execute a double stream type of motion in the space charge, so that there is current both toward and away from the cathode. Then, in the absence of any r-f fields, an electron leaving the cathode will move out to a maximum distance from the cathode

and return to the cathode, arriving with just zero energy. If during its path, an electron loses some of its ordered energy, e.g. to noise, it will not be able to return to the cathode. On the other hand if an electron acquires additional energy during its excursion, e.g. from r-f fields in the region, it can arrive at the cathode with non-zero energy. In this way some of the energy of the r-f wave present in the space charge can be lost to dissipation at the cathode. It would be desirable to determine the magnitude of this loss as a function of the applied fields and frequency, etc.

A comprehensive treatment of the cathode bombardment energy loss by the r-f wave in the space charge would, unfortunately, be quite laborious. Therefore this section will be restricted to a consideration of a plane, temperature limited magnetron space charge and the order of magnitude of the loss determined for these conditions. It is hoped that from this calculation some useful information as to the loss in the space charge limited cylindrical space charge (with small r_H/r_o) can be inferred.

Using the notation of Fig. 2.2, for the temperature limited case the equation of motion of an electron in the interaction space of a plane magnetron can be written:

$$\begin{aligned} \ddot{y} &= -e/m\, E_y + \omega_c \dot{x} \\ \ddot{x} &= -\omega_c \dot{y} \end{aligned} \qquad \text{IV-14}$$

The anode voltage is considered as the sum of a d-c and an a-c term so that the fields are:

$$E_y = E_o + E_m \operatorname{Sin} (\omega t + \psi)$$

where $E_o = Vo/h$ and $E_m = V_m/h$, V_o and V_m representing the d-c and peak a-c anode voltages respectively.

Then if the x directed velocity is zero at the cathode $\dot{x} = -\omega_c y$ and

$$\ddot{y} = -e/m\ E_o - e/m\ E_m \sin(\omega t + \psi) - \omega_c^2 y\ .$$

This equation can be solved very easily for y as function of t by the use of the Laplace Transform to yield the following expression (the electron being assumed to leave the cathode with zero y directed velocity at t = 0):

$$y(t, \psi) = \frac{-eE_o}{m\,\omega_c^2}(1-\cos\omega_c t) - \frac{e\,E_m\,\omega\cos\psi}{\omega_c^2 - \omega^2}\left[\frac{\sin\omega t}{\omega} - \frac{\sin\omega_c t}{\omega_c}\right] \qquad \text{IV-15}$$

$$- \frac{e\,E_m \sin\psi}{\omega_c^2 - \omega^2}\left[\cos\omega t - \cos\omega_c t\right]$$

The y directed velocity is then:

$$\dot{y}(t, \psi) = -(e/m)(E_o/\omega_c)\sin\omega_c t - \frac{e\,E_m\,\omega\cos\psi}{\omega_c^2 - \omega^2}(\cos\omega t - \cos\omega_c t) \qquad \text{IV-16}$$

$$+ \frac{e\,E_m \sin\psi}{\omega_c^2 - \omega^2}(\omega\sin\omega t - \omega_c \sin\omega_c t)$$

The total electron velocity is $v^2 = \dot{x}^2 + \dot{y}^2$ but at the cathode $\dot{x} = 0$ so that for $y = 0$ $v = \dot{y}$ and the energy of an electron upon reaching the cathode is:

$$\frac{1}{2}m\dot{y}^2 = \frac{1}{2}\frac{e}{m^2}\left[-\frac{E_o}{\omega_c}\sin\omega_c t_1 - \frac{E_m}{\omega_c^2-\omega^2}\left[(\cos\omega t_1 - \cos\omega_c t_1)\,\omega\cos\psi\right.\right. \qquad \text{IV-17}$$

$$\left.\left. + \sin\psi\,(\omega\sin\omega t_1 - \omega_c \sin\omega_c t_1)\right]\right]^2$$

where t_1 is the time at which the electron strikes the cathode. That is, t_1 is such as to satisfy the equation, from Eq. IV-15:

$$y(t_1, \psi) = 0 \qquad \text{IV-18}$$

The energy of arrival of an electron at the cathode is considered as energy which has been imparted to it by the r-f fields. In the absence of r-f fields the electron arrives at the cathode with zero energy. In the presence of the r-f fields the electron, in its complete trajectory from emission to capture by the cathode, will effect a zero net energy exchange with the d-c fields so that the "excess" energy at the cathode must be due to the r-f fields. This assumes, of course, no capture of the electrons by the anode.

Eq. IV-18 was solved for the special case $E_0 = E_m$, $\omega/\omega_0 = 1/2$, for several values of ψ from 0 to 2π, allowing determination of t_1. These values were in turn used in Eq. IV-17 to calculate the energy of arrival at the cathode. This latter information was plotted vs ψ and integrated graphically. The total loss can then be found, by letting the loss due to $d\beta$ electrons striking the cathode be

$$dW = 1/2\ m\ \dot{y}^2\ (t_1, \psi)\ d\beta$$

and

$$d\beta = f\ d\psi$$

where f is the number of electrons emitted (and therefore captured) per cycle (ω) per unit angle (ψ), per unit cathode area. Then if β is the number of electrons emitted per cycle (ω) per unit cathode area:

$$\beta = \int_0^{2\pi} f\ d\psi$$

Assuming f to be independent of ψ, (which should be realized in practice for this temperature limited case):

$$\beta = 2\pi f$$

and

$$W = \frac{m\beta}{4\pi} \int_0^{2\pi} \dot{y}^2\ d\psi$$

but $\dot{y}^2 (t_1, \psi) = \left[\frac{e}{m} \frac{E_o}{\omega_c}\right]^2$ x Area under curve of $\frac{y^2}{\left[\frac{e}{m} \frac{E_o}{\omega_c}\right]^2}$ vs ψ.

This area was found to be approximately 4π so that

$$W = \frac{m\beta}{4\pi} 4\pi \left[\frac{e}{m} \frac{E_o}{\omega_c}\right]^2 = m\beta \left[\frac{e}{m} \frac{E_o}{\omega_c}\right]^2 \frac{\text{joules}}{m^2 \text{ cycle}} \quad . \qquad \text{IV-19}$$

The value of β can be found from the Dushmann equation:

$$\frac{\beta\omega}{2\pi} e = A_o T^2 \; e^{-\frac{\phi}{T} 11600}$$

which in the case of Tungsten at 2500°K is:

$$\beta = \frac{2\pi \times 0.75}{e\omega} \times 10^4 \quad \frac{\text{electrons}}{m^2 \text{ cycle}}$$

If the magnetron is visualized as composed of two parallel plates acting as a transmission line, the energy stored in the region between plates is, per unit area:

$$1/2 \; \varepsilon_0 \, E^2_m \, h = 1/2 \; \varepsilon_0 \frac{V_m^2}{h}$$

where h is the distance between plates.

The Q of the region between plates is then:

$$Q = 2\pi \frac{\text{Stored Energy Density}}{\text{Dissipated Energy Density per cycle}}$$

$$= 2\pi \frac{1/2 \; \varepsilon_0 \, V^2_m/h}{\frac{2\pi \times 0.75}{e\omega} \times 10^4 m \; \frac{e}{m}^2 \frac{V_o^2}{h^2} \frac{1}{\omega_c^2}}$$

Using the aforementioned values $V_o = V_m$, $\omega_c = 2\omega$ and letting $\lambda = 10$ cm. h = 1 cm. this becomes:

$$Q = \frac{4m \; \varepsilon_0 \, \omega^3 \, h}{1.5 \; e}$$

$$= 900$$

To anyone familiar with the back-bombardment capabilities of an oscillating multicavity magnetron this value of Q for the interaction space at first probably appears astonishingly high. However, it is interesting to compute a corresponding value of Q for such an oscillating tube. Typical values will be taken as follows:

$$Q_L = 60$$

Back-bombardment Power = 5% Power Output

Energy stored in resonator = Q_L Power Output

Then

$$Q_{BB} = \frac{Q_L\ PO}{.05\ PO} = 1200$$

which agrees surprisingly well with the value previously calculated. Since the energy density stored in a transmission line (in which might be placed a space-charge cloud for frequency-modulation purposes) is much less than in the multicavity magnetron, it appears that the energy loss in the space charge is not a very serious consideration. Of course, a space-charge cloud intended for switching or modulation purposes would not be subjected to the tangential r-f electric fields which contribute substantially to the electron energy at the cathode of an oscillating magnetron. That is, while the back-bombardment losses in an oscillating tube are much greater than those obtained in a simple modulation structure, the energy stored in the former is correspondingly larger so that the Q values appear in agreement as to order of magnitude. This would appear to lend a greater feeling of confidence in the value 900 obtained above.

<u>e. Loss due to collisions near a region where $\varepsilon_e = 0$.</u> There is another possible source of energy loss by the electrons, which involves collisions with the heavy particles in the space, but which was considered a bit too speculative to be included under the previous discussion of collisional losses.

Let it be supposed that the boundary or any other surface in the space charge cloud has the property that along this surface $\varepsilon_e = 0$. The continuity of the normal electric displacement across this surface is, of course, demanded. In the case of a wave impinging normally onto such a surface it would be expected that this boundary condition would result in dissappearance of the normal electric field at this surface ($\varepsilon_e = 0$). However other conditions can be imagined in which this situation might not prevail. For example, for a TEM wave striking the cylindrical space charge in a coaxial line in which the surface $\varepsilon_e = 0$ would probably be a cylinder coaxial with the line, the electric field on the outside of this surface might be reluctant to vanish.

Under this condition the other possibility arises, upon application of the continuous normal displacement boundary condition; namely, the possibility of extremely high (theoretically infinite) electric field at the surface $\varepsilon_e = 0$. If such a condition could exist, electrons traversing this surface might well acquire a very considerable energy which would be dissipated upon collision (either elastic or inelastic) with a gas particle. In this way it is at least conceivable that an appreciable amount of energy could be lost by the propagating wave in the neighborhood of a surface at which $\varepsilon_e = 0$.

V. BOUNDARY CONDITIONS AND EFFECT ON RESONANT CIRCUIT

In Chapter III the properties of a magnetron type space charge as a medium for the propagation of electromagnetic waves were determined under the condition that the fields are invariant in the plane normal to the direction of propagation. This condition can be met in some very simple practical cases, however in general it is invalid, so that the results of Chapter III will be re-evaluated in a qualitative way. The solution of the force and field equations becomes considerably more difficult under these more general conditions and will not be solved here. In any event the results of Chapter III provide at least qualitative indication of the properties of the space charge and are particularly applicable when the space charge is of dimensions small compared with a wavelength so that the fields can be considered invariant over the swarm.

In order to determine the effect of the space charge cloud on a microwave circuit, the appropriate field equations must be solved, subject to the boundary conditions of the confining circuit and the edge of the space charge. Since this solution obviously depends on the shape of the confining circuit, and the possibilities of variations in this shape are practically limitless, no attempt will be made here to obtain a solution for the exact effect of the space charge. However, using the equations from Chapter III the influence of the space charge on two typical circuits will be determined qualitatively.

1. Boundary Conditions at the Edge of the Space Charge.

The conditions to be met by the field components of an electromagnetic wave at the boundary between the space charge and free space are, with one exception, identical to the boundary conditions at any surface of discontinuity. That is, the continuity of the tangential E and H fields and of the normal B and D fields must be insured.

If the surface of the space charge is perturbed periodically in space by the propagating wave, and if there is a steady electron velocity parallel to the direction of propagation, an additional boundary condition becomes necessary. Fig. 5-1 shows a cross section, parallel to the direction of propagation, of such a perturbed surface. This condition will prevail in the case of propagation parallel to the steady electron motion. It is seen that these "ripples" constitute an r-f surface current. Therefore in addition to the usual boundary conditions, the discontinuity in the H_z fields at the boundary must be equal to this surface current. This boundary condition can be evaluated in the following approximate manner.

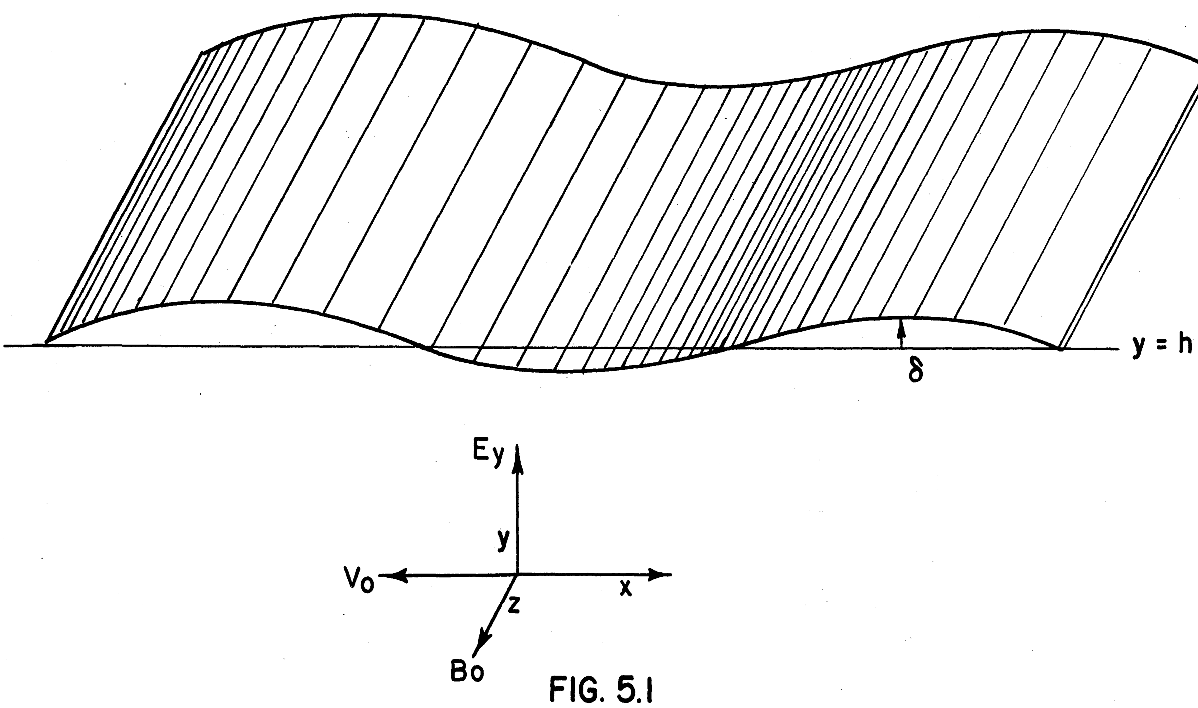


FIG. 5.1
PERTURBED SURFACE OF PLANE MAGNETRON SPACE CHARGE

The magnetic field, of a wave propagating in the x direction, at the surface of the space charge of a plane magnetron can be considered as replaced by the value of the magnetic field at the unperturbed boundary plus that due to a surface current[1]. That is, referring to Fig. 5.1, at the boundary:

$$\int \nabla \times H \cdot dA = \int_c H \cdot dS = \int J_{xs} \cdot dA \qquad \text{V-1}$$

where J_{xs} is the surface current due to the -x directed motion of the perturbations in the surface of the space charge. Then if the boundary is y = h:

$$H_z\,(h + \varepsilon) - H_z\,(h - \varepsilon) = \mp\ \rho_0 |v_{oh}|\, \delta \qquad \text{V-2}$$

where $|v_{oh}|$ is the magnitude of the steady electron velocity at the boundary and ε is a vanishingly small quantity. The plus sign corresponds to wave propagation in the -x direction and the minus sign to propagation in the +x direction.

The equation of this perturbed surface is:

$$F\,(x,y,t) = (y - h) - \delta_0 e^{i\omega t - \gamma x} = 0\ . \qquad \text{V-3}$$

Since the surface of any moving fluid coincides with a stream line, and along this line D/Dt = 0, if F = 0 represents the surface, the following relation is valid:

$$DF/Dt = 0\ .$$

This becomes, with the substitution of V-3:

$$\frac{\partial F}{\partial t} + (v \cdot \nabla)F = -\delta\, i\omega + v_{oh}\gamma\delta + v_y = 0 \qquad \text{V-4}$$

so that

$$\delta = \frac{-v_y}{\gamma v_{oh} - i\omega} \qquad \text{V-5}$$

---

1 Hahn, W. C., "Small Signal Theory of Velocity Modulated Electron Beams", G. E. Rev. 42, No. 6, page 258, June 1939.

Therefore from Eq. V-2:

$$H_z(h + \varepsilon) = \frac{\mp \rho_0 |v_{oh}| v_y}{\gamma v_{oh} - i\omega} + H_z(h) \qquad \text{V-6}$$

The magnetic field at the boundary can be found from the relations valid inside the space charge: (from Eq. III-23):

$$-\frac{\partial H_z}{\partial x} = i\omega \varepsilon_0 E_y + \rho_0 v_y$$

so that

$$H_z(h-\varepsilon) = \frac{i\omega \varepsilon_0 E_y(h) + \rho_0 v_y(h)}{\gamma} \qquad \text{V-7}$$

From Eqs. V-6 and III-22:

$$H_z(h + \varepsilon) = \left[\frac{1}{\gamma} + \frac{|v_{oh}|}{v_{oh} - i\omega}\right]\left[\frac{-\rho_0 \, e/m \, E_y}{i\omega + \gamma\omega_c \, yh} - \frac{\rho_0 \, e/m \, \omega_c \, E_x}{(i\omega + \omega_c \gamma yh)^2}\right] + \frac{i\omega \varepsilon_0 E_y}{\gamma}$$

The admittance, at the boundary, of the wave propagating in the -x direction is, using $|v_{oh}| = \omega_0 h$:

$$\frac{H_z(h)}{E_x(h)} = \left[-\frac{i\omega + 2\gamma\omega_{oh}}{\gamma(\omega_o h\gamma + i\omega)} \frac{\rho_0 \, e/m}{i\omega + \omega_o h} + \frac{i\omega \varepsilon_0}{\gamma}\right] \frac{E_y(h)}{E_x(h)} \qquad \text{V-8}$$

$$- \frac{i\omega + 2\gamma \omega_o h}{(\gamma\omega_o h + i\omega)\gamma} \frac{\omega_0 \rho_0 \, e/m}{(i\omega + \gamma\omega_o h)^2} .$$

For a direction of propagation normal to the steady electron motion there will be no perturbation as considered above. For example, in the case of propagation in the y direction, from Eq. III-13 the admittance at the boundary is:

$$\frac{H_z}{E_x} = -\frac{1}{\gamma}\left[-\frac{\rho_0 \, e/m}{i\omega + g} + i\omega \varepsilon_o\right] .$$

2. Effect of Space Charge on Its Associated Circuit.

A. In a Situation Suitable for Frequency Modulation of a Coupled Circuit. One of the principal applications of a magnetron type space charge (other than in an oscillating tube) is in its use as a frequency modulating element. The space charge can be incorporated, for example, in a coaxial line which is coupled into a resonant circuit, the frequency of which it is desired to change. Under proper conditions of applied magnetic field and signal wavelength the space charge can be made to exhibit the desired dielectric properties. Then a change in the size of the space charge cloud will result in a change in the reactance coupled into the resonant circuit, and therefore a change in its frequency. (See Section IV-2)

In this case, since the axial length of the cloud will usually be small compared with a wavelength, if ω/ω_0 is adjusted, using Figs. 3.4 and 3.5 to yield a value $\varepsilon_\theta \ll 1$ or $\varepsilon_\theta > 1$, the effect of the modulating structure will, in first approximation, be the same as if a capacitance were connected across the line at the midpoint of the space charge. The value of this capacitance will change with diameter of the space charge. This principle was used as the basis of a frequency modulated magnetron in the University of Michigan Electron Tube Laboratory.

B. Space Charge in a Multicavity Magnetron. The electric flux lines in the interaction space of a multicavity magnetron are shown (at a fixed instant of time) schematically in Fig. 5.2.

The electric and magnetic fields in the interaction space are represented mathematically by equations of the following form:[1]

---

1 Microwave Magnetrons, Radiation Laboratory Series No. 6, McGraw-Hill, page 65, 1948.

$$E_\theta = \frac{N\phi}{\pi} \sum_{m=-\infty}^{\infty} \frac{\sin\gamma\phi}{\gamma\phi} \frac{Z'_\gamma(kr)}{Z'_\gamma(kr_a)} e^{j\gamma\theta}$$

$$E_r = -j \frac{N\phi}{\pi kr} \sum_{m=-\infty}^{\infty} \gamma \frac{\sin\gamma\phi}{\gamma\phi} \frac{Z_\gamma(kr)}{Z'_\gamma(kr_c)} e^{j\gamma\theta}$$

$$H_z = -j\sqrt{\frac{\varepsilon_0}{\mu_0}} \frac{N\phi}{\pi} \sum_{m=-\infty}^{\infty} \frac{\sin\gamma\phi}{\gamma\phi} \frac{Z_\gamma(kr)}{Z'_\gamma(kr_a)} e^{j\gamma\theta}$$

where

$$Z_\gamma = J_\gamma(kr) - \frac{J_\gamma'(kr_c)}{N_\gamma'(kr_c)} N_\gamma(kr)$$

γ = $n + mN$

m = summation index, an integer

n = mode number

N = number of anode segments

2ϕ = angle subtended by the gap between anode segments

r_a = anode radius

r_c = cathode radius

It is seen that the electric fields in the interaction space have both r and θ components while the magnetic field is entirely z directed. These fields may be thought of as due to waves propagating in the $\pm$ r and $\pm\theta$ directions. One of these waves is qualitatively similar to the type of field considered in Section III-b-2, namely that due to a radially propagating transverse magnetic wave with field components E_r, E_θ. Therefore the characteristics of the inner sub-synchronous (so-called Hull-Brillouin) electron cloud should be given approximately by the curve in Figs. 3.6 or 3.7 with suitable modification for the effect of the radius of the cloud. That is, for example, with $\omega/\omega_c < .4$ the

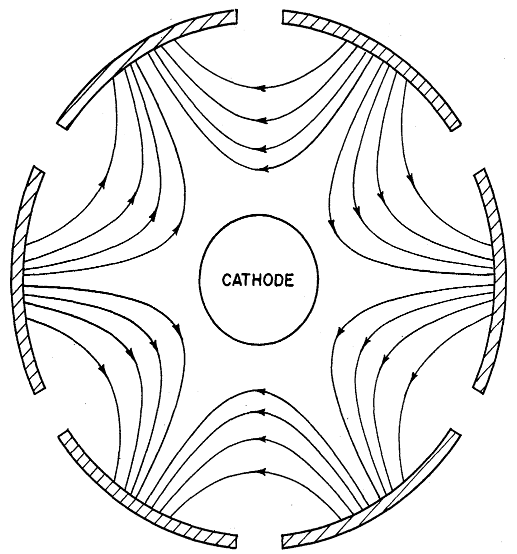


FIG. 5.2
QUALITATIVE CONFIGURATION OF ELECTRIC FIELD LINES IN INTERACTION SPACE OF MULTIANODE MAGNETRON

cloud should appear as a conducting surface, so that as it is expanded the result will be an increased capacitance between vanes with corresponding reduction in resonant wavelength. Experimental investigations conducted to check these points are reported in Section VI-3. The other wave type, involving propagation in the $\pm\,\theta$ direction are similar to those used in the analysis in section III-3 above.

These effects of the inner space charge cloud on the resonant frequency of a magnetron are present even in a non-oscillating tube and are not to be confused with synchronous effects of the space charge in an oscillating magnetron, such as pushing and voltage tuning[1]. These effects have been considered in more detail by Welch[2].

1 Pushing is the change in resonant frequency with anode current and is believed to be at least partly due to a change in the phase between the rotating synchronous space charge "spokes" and the interaction fields. Voltage tuning is an effect noticed in a magnetron with an extremely low Q resonant circuit having very low r-f voltage between segments. Under these conditions the frequency of oscillation is affected very considerably by the change in d-c voltage.

2 Univ. of Mich. Electron Tube Lab. Tech. Report No. 5. Also to be treated in a forthcoming report by H. W. Welch.

VI. EXPERIMENTAL RESULTS

In order to provide verification of the theoretical results and their interpretation presented in previous chapters, several experiments were performed and will be described in this chapter. In each of these experiments a microwave resonant cavity was constructed to include a cylindrical magnetron type space charge in a part of the cavity, so that the space charge could interact with the electric fields. In order to duplicate as closely as possible the conditions under which the analysis was made, the cavities were designed to present to the space charge only fields of a simple and symmetric geometry. That is, the fields corresponded to one of the fundamental modes of the cavity, as contrasted with the relatively more complex fields in the interaction space of a multicavity magnetron. In addition, the effect of the space charge on the resonant wavelength of a multicavity magnetron was studied.

The space charge, presenting a reactive impedance to the cavity, will affect the resonant wavelength of the cavity. This change in resonant wavelength as a function of the magnetic field is used as an indication of the dielectric properties of the space charge in relation to ω/ω_c.

1. Propagation Parallel to the Applied Magnetic Field.

To duplicate the field configuration considered in section 1-b of Chapter III for propagation parallel to the applied magnetic field; namely, an electric field with cylindrical symmetry, of the TEM type, a coaxial cavity was constructed as shown in Fig. 6.1. This cavity has a filament as part of its center conductor so that a space charge cloud is created in the center (longitudinally), where the electric fields

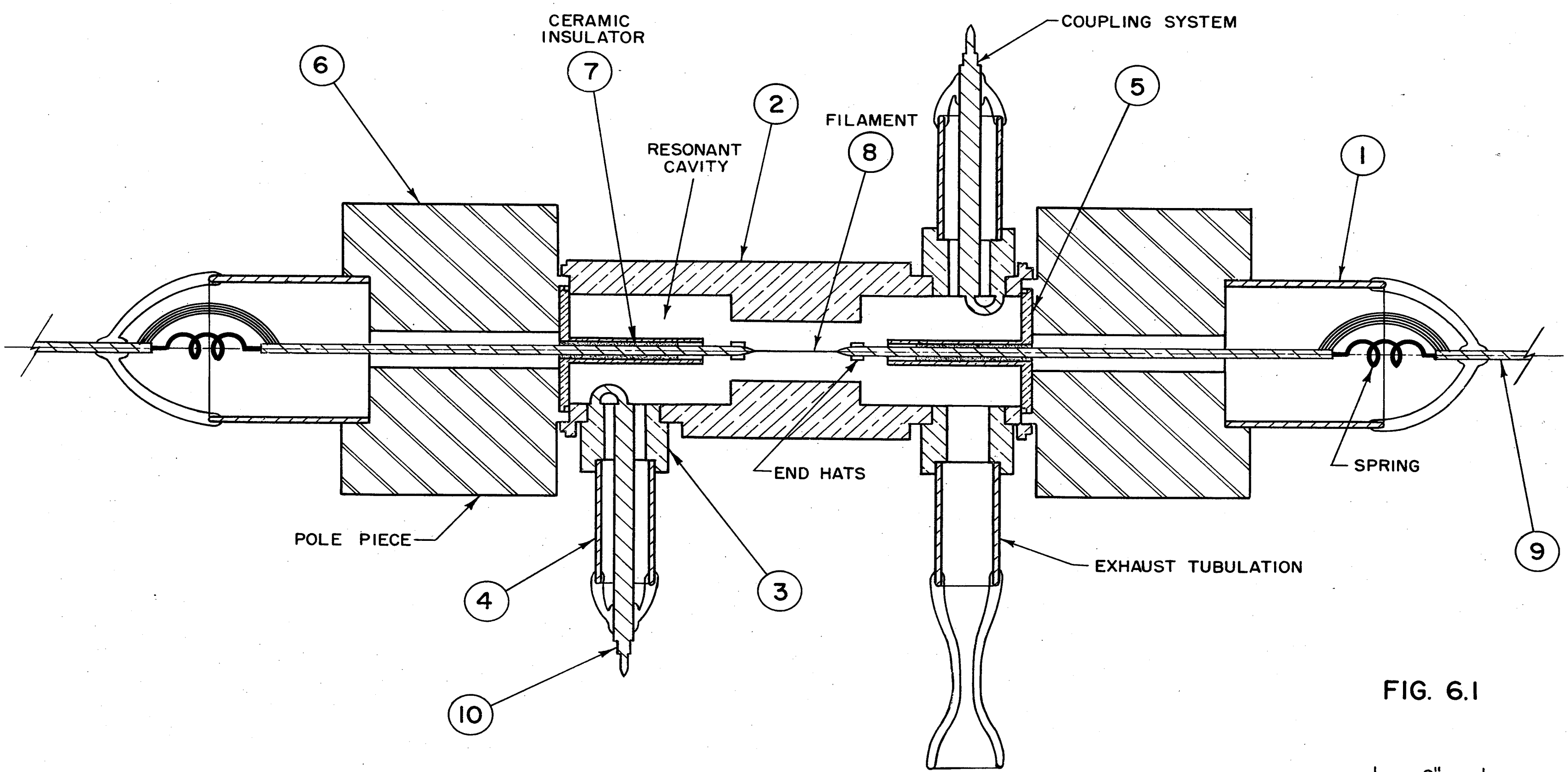


FIG. 6.1

ALL DIMENSIONS UNLESS OTHERWISE SPECIFIED MUST BE HELD TO A TOLERANCE - FRACTIONAL ± 1/64" DECIMAL ± .005," ANGULAR ± 1/2°

ENGINEERING RESEARCH INSTITUTE UNIVERSITY OF MICHIGAN ANN ARBOR MICHIGAN	DESIGNED BY GRB DRAWN BY MM CHECKED BY VRB	APPROVED BY SCALE DATE 4-21-50
PROJECT M-762	TITLE 10 CM MAGNETRON DIODE (EXPERIMENTAL) MODEL 3	
CLASSIFICATION	DWG. NO. B-11,003	
ISSUE / DATE		

DWG. NO. B

are of maximum amplitude. The resonant wavelength of this tube is about 10 cm. The cavity is provided with two coupling loops so that the cavity resonance can be located easily be detecting the r-f signal transmitted through the cavity. A photograph of this tube is shown in Fig. 6.2.

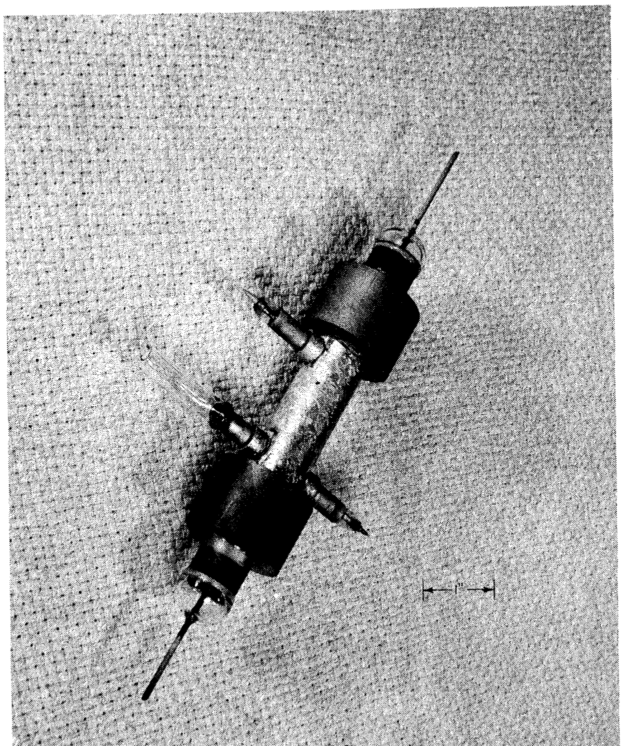


FIG. 6.2
PHOTOGRAPH OF 10 CM EXPERIMENTAL DIODE

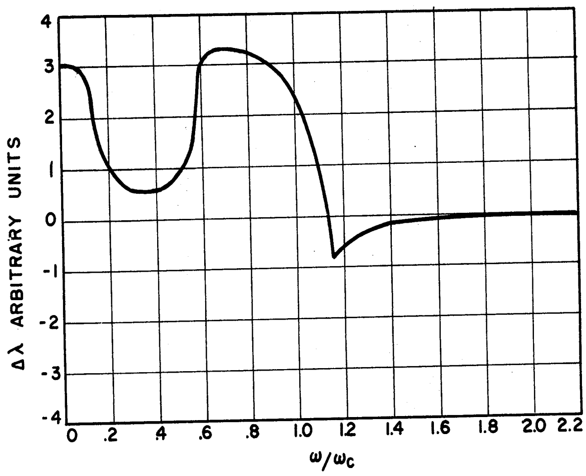


FIG. 6.3
WAVELENGTH SHIFT IN COAXIAL CAVITY PREDICTED FROM THEORY - USING HULL-BRILLOUIN VALUE OF SPACE-CHARGE DENSITY

Before considering the experimental data obtained with this tube, an examination of Fig. 3.4 and the interpolated curves of Fig. 3.5 for the value of r_H/r_c used in the experiment, will enable a qualitative prediction to be made of the expected resonant wavelength shift of the cavity. The experiment was conducted maintaining V_a/B_o^2 constant, as B_o is varied, so that the cloud radius was presumably constant with a value $r_H = 3\ r_c$. Then as the applied magnetic field is increased from zero (decrease in ω/ω_c) Figs. 3.4 and 3.5 show that the cloud should exhibit a positive dielectric constant less than unity, so that the resonant wavelength would be expected to decrease slightly from the value for $B_o = V_a = 0$. When the value $\omega/\omega_c = 1.12$ is reached, $\varepsilon_o = 0$, the space charge cloud begins to behave as a conducting material, increasing the resonant wavelength. That is, as ε_e becomes negative, and the cloud begins to behave more and more as a conductor, its effect on the resonant wavelength should be similar to that of a solid conductor of approximately equal volume. Therefore, after a first abrupt increase in wavelength at $\omega/\omega_c = 1.12$, λ_o should continue to increase[1] slowly with decreasing ω/ω_c. For ω/ω_c slightly less than the singular value (.68) ε_e

---

1 The reason for the continued increase in λ_o as ε_e becomes more negative lies in the existance of a "skin depth" associated with the space charge, as mentioned previously. That is, when $\varepsilon_e < 0$ the waves will not be completely reflected from the boundary of the swarm but will diminish in amplitude at an exponential rate, being reduced to 1/e of their value at the boundary after penetrating a distance $\lambda/2\pi\sqrt{|\varepsilon_e|}$. Therefore as ε_e becomes more negative, the "virtual boundary" approaches the actual boundary of the space charge so that the effective volume of the space charge is increased. This increases the resonant wavelength. For example, with $\lambda = 10$ cm, $\varepsilon_e = -3$ ($\omega/\omega_c = .82$), the "skin depth" is approximately equal to the length of the space charge cloud used in this experiment.

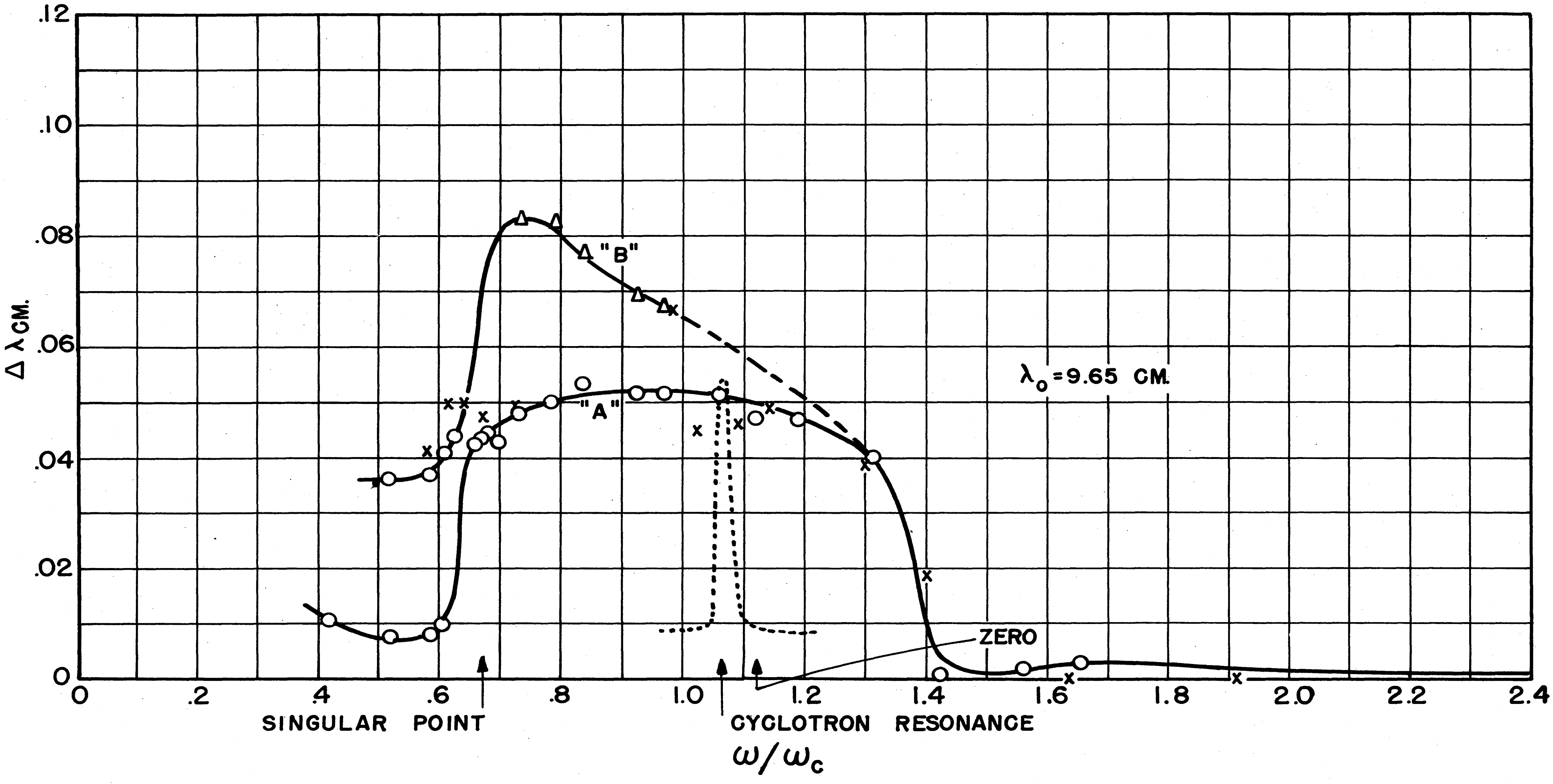
In the region $0 < \varepsilon_e < 1$, the waves will not be totally reflected from the space charge; however, that reflection which does occur will do so from the physical boundaries of the space charge.

changes to a very large positive value but no change in resonant wavelength should be noted until ε_θ begins to decrease sharply (around $\omega/\omega_c = 0.6$) when it should decrease. The resonant wavelength in the region $0.1 < \omega/\omega_c < 0.55$ should be relatively constant, less than the value in the range $0.6 < \omega/\omega_c < 1.12$ but greater than the value for $\omega/\omega_c > 1.12$. For $\omega/\omega_c < 0.1$ there should again be an abrupt increase in λ_0. These expected wavelength shifts are shown in Fig. 6.3.

The data obtained from this tube are shown in Fig. 6.4. These data were obtained by measuring the shift in resonant wavelength of the cavity, (the resonant wavelength was determined as that which gave maximum signal amplitude transmitted through the cavity) due to the presence of the space charge, as a function of the strength of the applied magnetic field. It is seen from Fig. 6.4 that the resonant wavelength of the cavity remains constant as the applied magnetic field is increased (ω/ω_c decreased) until $\omega/\omega_c = 1.4$, when it rises; one mode reaching a reasonably constant value and the other continuing to increase, both dropping again very abruptly at $\omega/\omega_c = 0.63$. The two curves shown are believed to be due to two resonances in the cavity, probably because of a longitudinal asymmetry in the formation of the cloud. However both curves exhibit the same general behavior as far as the discontinuties are concerned.

The sharp drop in λ_0 at $\omega/\omega_c = 0.63$ is regarded as confirmation of the analytical method followed. This follows from the fact that the angular velocity of an electron in a magnetron results from the integral of the angular equation of motion with only the assumption of zero angular velocity at the cathode. The value of the angular velocity, unlike the potential or space charge density, does not depend on any

FIG. 6.4
CHANGE IN RESONANT WAVELENGTH OF 10 CM. CAVITY vs ω/ω_c



PG. 89
BK. 6
G.R.B

choice or assumption regarding the electronic orbit, or on a series solution. Therefore, the angular velocity is believed to be invariant under any changes in emission, charge density, voltage, etc., and since the position of the singular value of ω/ω_c is, from the above theory, governed entirely by the functional dependence of the angular velocity on radius, this point of agreement between Figs. 6.4 and 6.3 is thought to be significant.

The situation regarding the increase in $\Delta\lambda$ at $\omega/\omega_c = 1.4$ does not agree with that predicted from Fig. 3.4. This will be discussed below and some possible explanations of the disagreement advanced.

From Eq. III-9 it is seen that the value of ω/ω_c for which $\varepsilon_\theta = 0$ is a function and the space charge density at the edge of the cloud. Thus, if the abrupt increase in λ_0 at $\omega/\omega_c = 1.4$ is interpreted as the point $\varepsilon_\theta = 0$, Eq. III-10 can be solved for $\rho_0(r_H)$ to obtain ($\omega/\omega_c = 1.4$)

$$\frac{\rho_0 e}{m\varepsilon_0} \cong -\omega_c^2$$

whereas the Hull-Brillouin relation would give

$$\frac{\rho_0 e}{m\varepsilon_0} = -\omega_c^2/2 .$$

This value, greater than that given by the Hull-Brillouin solution, is slightly surprising and reminds one of the solutions of Page and Adams[1] and Moeller[2] whose theoretical analyses indicated that ρ_0 increases abruptly at the edge of the space charge swarm. Also the measurements of Reverdin[3] yielded a space charge distribution (No. 1) with a peak at

[1] Page and Adams, loc. cit

[2] Moeller, H. G., loc. cit

[3] Reverdin, D. L., "Electron Optical Exploration of Space Charge in a Cut-off Magnetron", J. App. Phys. 22, page 257, March 1951.

the boundary of the cloud. However this peak was shown as lower than the Hull-Brillouin value.

The above interpretation of the increase in $\Delta\lambda$ at $\omega/\omega_c = 1.4$ of Fig. 6.4 should not be considered as final, however, since the experimental conditions of the space charge cloud were far from ideal. The principal possible source of error lies in the formation of the space charge cloud. The magnetic circuit used apparently made the production of an absolutely uniform magnetic field impossible, so that there will be a longitudinal force[1]

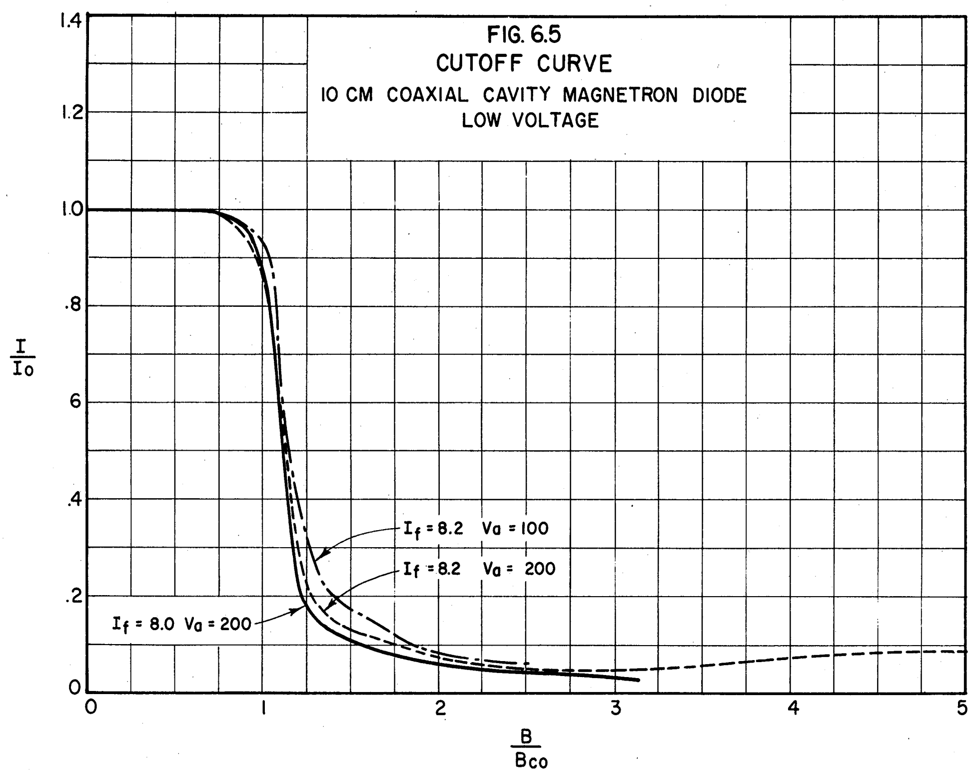
$$F_z \propto \frac{\partial H}{\partial z}$$

proportional to the z component of ∇H. Thus electrons can move axially from the filament under the influence of this force. In operation, such current was observed, constituting up to one-tenth of the Allis current for high anode voltages, as shown in the cut-off curve of Fig. 6.5. It would be expected that this drain of current would affect the composition of the space charge cloud. A calculation of the wavelength shift[2] for the case $\varepsilon_e = 8$ ($\omega/\omega_c = 0.4$) yielded a value greater than the observed by a factor of four, indicating that the cloud is probably much smaller (probably shorter) than believed, but the sharp rise and fall of the $\Delta\lambda$ characteristic is an indication that at least part of the cloud is of the form expected.

The shift in resonant wavelength of the cavity was observed as a function of magnetic field in the immediate vicinity of the cyclotron field (calculated 1060 gauss) and is shown in Figs. 6.4 and 6.6.

1 See, for example, Alfven, "Cosmical Electrodynamics", Oxford, page 19, 1950.

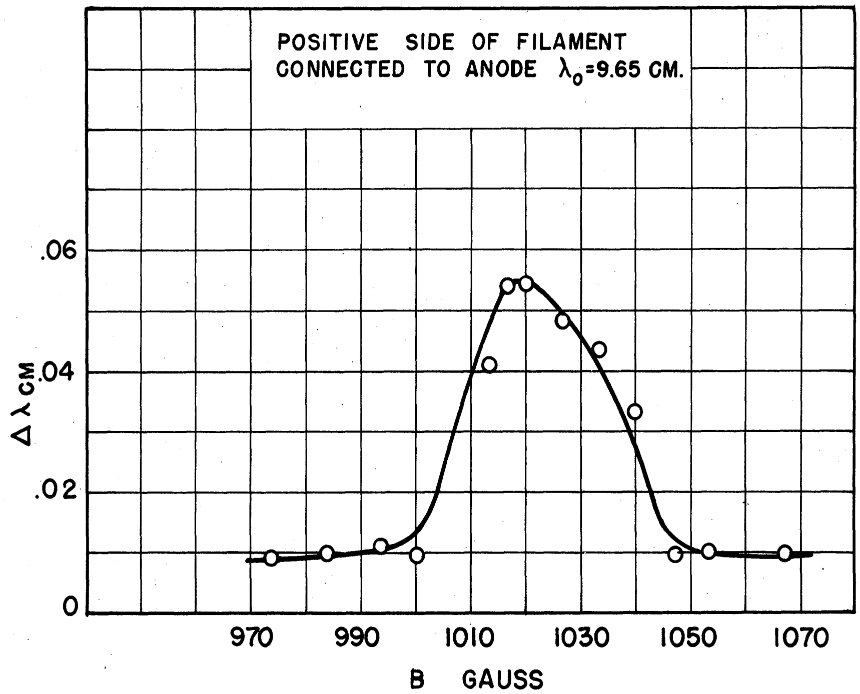
2 See Appendix 6



This observation was made with the positive side of the filament connected to the anode, no shift in wavelength being observed when the negative side of the filament was connected to the anode. It is noted that the wavelength shift maximum occurs at a value of ω/ω_c about seven percent greater than unity, indicating a probable error in magnet calibration or residual magnetism effect in the cyclotron resonance test. A search was made to try to detect any shift in the negative direction (decrease in λ_0) without success.

The tube on which the data shown in Figs. 6.4, 6.5, and 6.6 were taken was the fourth tube constructed for this purpose. The first of these proved unsatisfactory due to low Q caused by improper design of the resonant system. The second was accidentally melted in process of assembly. The third tube was constructed without end hats on the filament, so that the longitudinal leakage current was so large as to prevent electrons from reaching the anode cylinder surrounding the cathode. Under this condition no cut-off could be observed and a satisfactory space charge cloud could not be formed.

FIG. 6.6
CHANGE IN RESONANT WAVELENGTH OF 10 CM. CAVITY vs. MAGNETIC FIELD-SHOWING THE CYCLOTRON RESONANCE



PG. 91
BK. 6
G.R.B

2. Propagation in the Direction Normal to Anode and Cathode.

As a further study of the r-f properties of the magnetron space charge and to provide additional information as to the validity of the theoretical analysis in Chapter III, a suitable electron tube allowing experimental verification of Eq. III-20 was sought. This experimental electron tube would be placed in a resonant cavity of such configuration that the r-f electric and magnetic fields with which the electrons could interact would be as similar as possible to those assumed in the analysis of section III-2-a. That is, a tangential electric field (E_θ) and longitudinal magnetic field (H_z) were desired, with $\partial/\partial\theta = 0$. The only feasible circuit capable of producing these field configurations is a cylindrical cavity resonating in the TE_{011} mode. The space charge was to be placed along the axis of such a cavity. A tube and cavity of this design were constructed for the purpose of investigating the r-f properties of the space charge as seen by a radially propagating wave. A schematic drawing of this tube and resonant cavity is shown in Fig. 6.7. A photograph of the tube is shown in Fig. 6.8 and an assembly drawing of the tube in Fig. 6.9.

However, a cavity designed to resonate at 10 centimeters in this mode will have a radius of approximately 7 centimeters, far too great to allow the outer wall to be used as the anode for the space charge. That is, for magnetic fields in the desired range (2000 gauss) an astronomical d-c voltage would be required to expand the space charge to a small fraction of this anode radius. Therefore a series of longitudinal rods spaced on a circle concentric with the cathode were used as the d-c anode. If the rods are small enough and few in number, there would be relatively little metallic surface parallel to the θ

directed electric field so that there should be only small perturbation of the r-f fields at the cathode. This arrangement is shown schematically in Fig. 6.7.

The question naturally arises as to the ability of such an anode to produce d-c equipotential lines which are approximately circles concentric with the cathode. A field map of a section of such a (space charge free) structure was made by Mr. J. S. Needle, from which it was found that this anode approximates a solid cylindrical anode very closely for any radius less than about 0.7 of the anode radius. This type of experimental tube has the advantage that only the longitudinal bars and the cathode need be included in the vacuum envelope; the cavity may be external to the vacuum. These bars should interfere very little with the r-f fields in the cavity so that inside the anode bars the electron-wave interaction can take place.

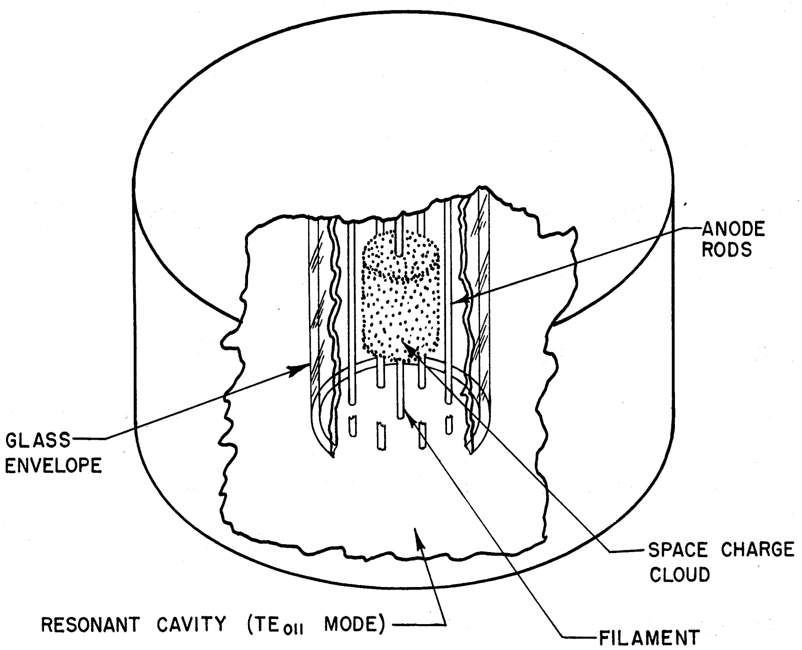


FIG. 6.7 TE_{011} RESONANT CAVITY FOR SPACE CHARGE STUDY

The shift in resonant wavelength of this cavity due to the presence of the space charge was calculated (Appendix 7). This calculation showed that a shift of the order of two percent could be expected. To check this, a conducting rod, of approximately the diameter of the space charge, was inserted in the center of the cavity; the wavelength shift was observed to be approximately one and one-half percent. To test the shielding effect of the rods in isolating the center from the outer region of the cavity, the conducting rod was inserted with the small rods in place to duplicate the rods in the actual tube (see Fig. 6.7). In this latter case the wavelength was increased by about three-tenths of a percent, still a measurable amount.

However, with the tube in its place in the cavity, no shift in resonant wavelength, due to the space charge, could be detected, even with the aid of several elaborate methods of detection. Two possible explanations can be advanced for this lack of effect of the space charge. In the first place, since the electric field must vanish at the cathode surface, the proportion of the total r-f energy stored in the region near the cathode, that is in the region occupied by the space charge cloud, is quite small. On this basis alone it would be expected that the wavelength shift be small, and it was in fact calculated to be at most two percent (in the absence of the longitudinal bars). However since a wavelength shift was observed in the preliminary test using a conducting rod to represent the space charge, it can be concluded only that the presence of the tube in the cavity so disturbed the fields as to reduce the effect of the space charge. The presence of the tube (with its Kovar-glass seals, etc.) would undoubtedly reduce the Q_0 of the cavity (usually very high in the TE_{01} mode), possibly enough to cause the bandwidth to exceed the wavelength shift caused by the space charge. A second possibility is that there might exist in the cavity certain extraneous fields which can interact with the electrons in such a manner as to reduce the effect of the electrons on the fields of the desired mode.

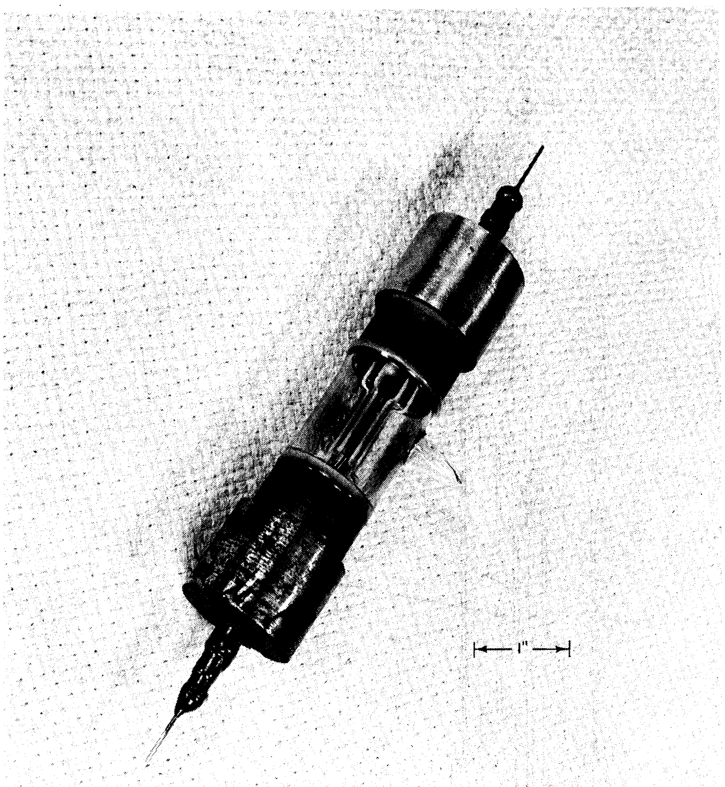
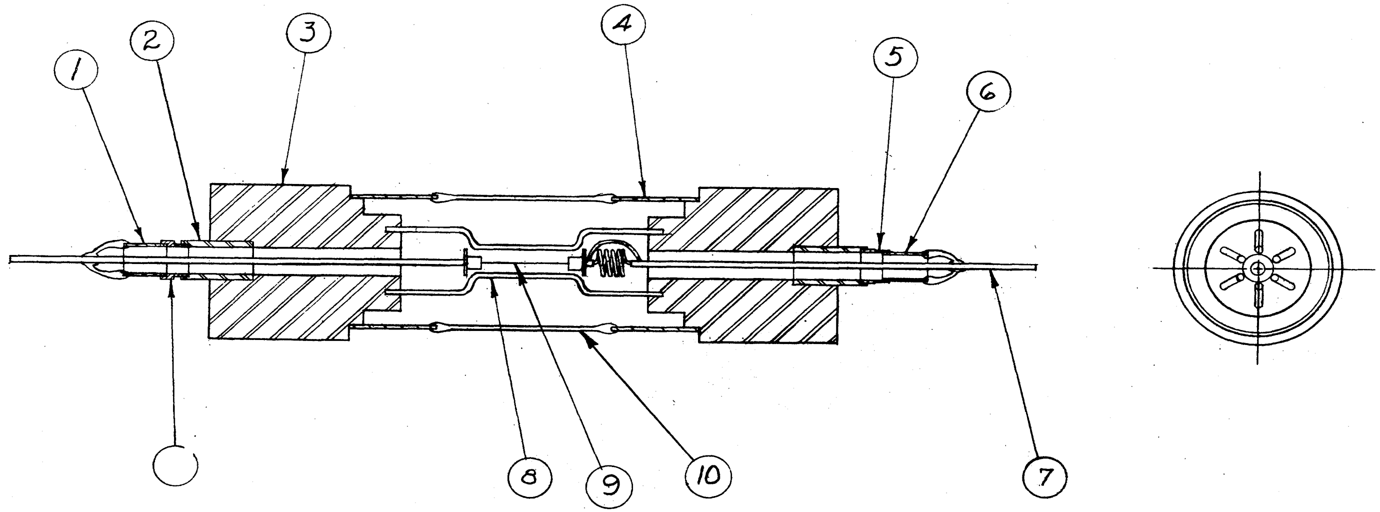


FIG. 6.8
PHOTOGRAPH OF EXPERIMENTAL TUBE

One effect of the space charge which could be detected was the change in the amplitude of the transmitted signal vs frequency characteristic of the cavity as a function of the applied magnetic field. The space charge cloud radius was maintained constant and the relative change in transmitted signal amplitude (at resonance) measured as a function of the applied magnetic field. This data is shown in Fig. 6.10 where it is seen that the maximum decrease in transmission occurs at the



3. POLE PIECE
7. FILAMENT LEAD
8. ANODE BAR
9. FILAMENT
10. GLASS ENVELOPE

FIG. 6.9

|← 1" →|

ALL DIMENSIONS UNLESS OTHERWISE SPECIFIED MUST BE HELD TO A TOLERANCE - FRACTIONAL ± 1/64," DECIMAL ± .005," ANGULAR ± 1/2°

ENGINEERING RESEARCH INSTITUTE
UNIVERSITY OF MICHIGAN
ANN ARBOR MICHIGAN

DESIGNED BY JRB	APPROVED BY JRB
DRAWN BY NN	SCALE FULL
CHECKED BY JMD	DATE 3-14-51

PROJECT M-921

TITLE MAGNETRON SPACE CHARGE DIODE

CLASSIFICATION

DWG. NO. B- 11,005

ISSUE | DATE

DWG. NO. B

cyclotron value of magnetic field. This is as predicted by the theory (Eq. III-20) but it is not suggested here that the experimentally observed change in cavity Q necessarily be due to the electron-atom collision mechanism proposed in Chapter III. However as shown in Eqs. III-17 the perturbed velocities pass through a maximum value at $\omega/\omega_c = 1$ so that any mechanism by which the electrons can lose an amount of energy proportional to their velocity would produce the effect shown in Fig. 6.10. About the most that can be claimed for this data, however, is that it does not contradict the theory.

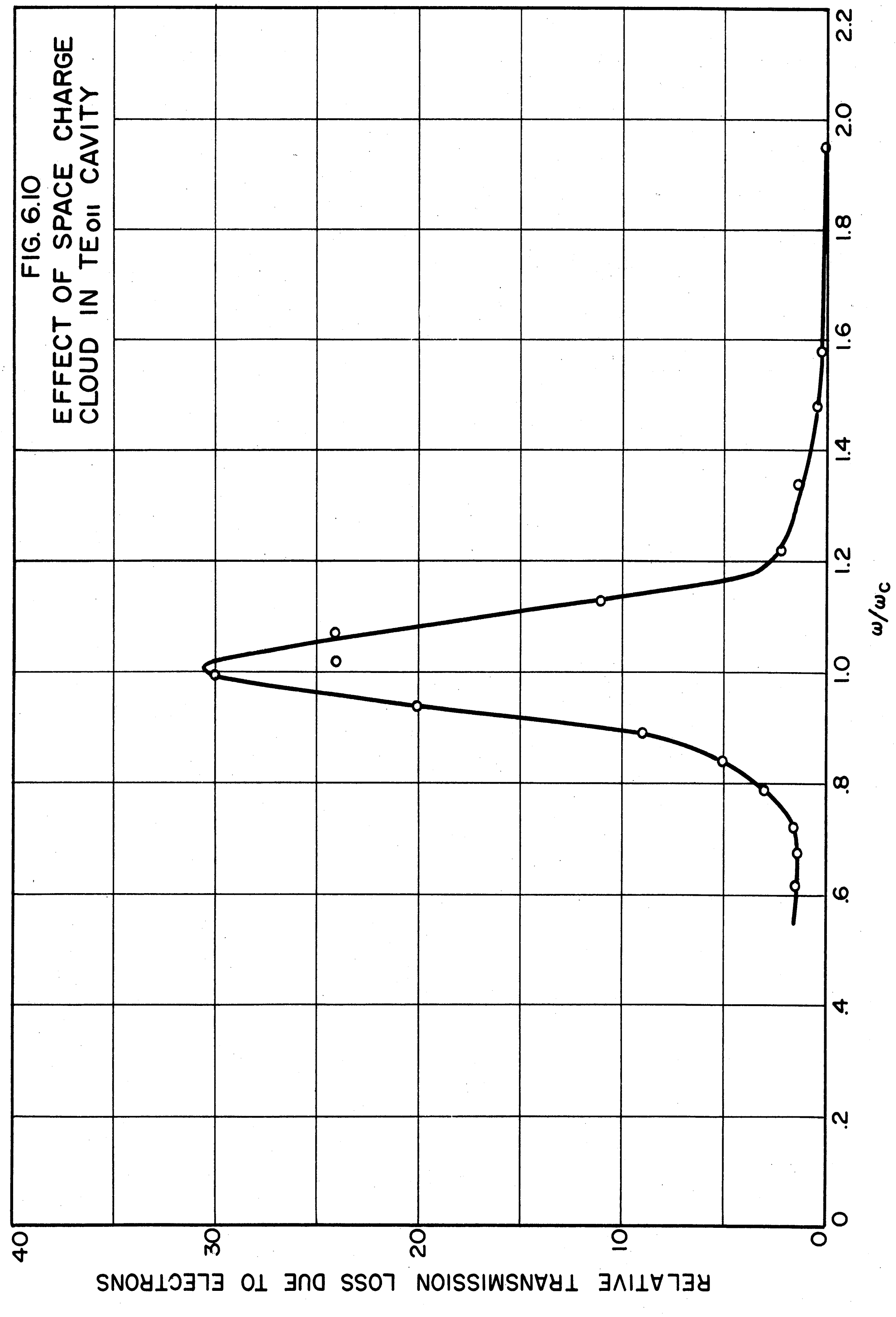


FIG. 6.10
EFFECT OF SPACE CHARGE
CLOUD IN TE_{011} CAVITY

3. Effect of Space Charge on the Resonant Wavelength of a Multicavity Magnetron.

It was mentioned in Section V-2 above that the space charge in a magnetron will change the resonant wavelength (λ_0) of the circuit. Since the electric and magnetic fields in the interaction space of a multicavity magnetron are qualitatively similar to those considered in the analysis of radial wave propagation in Section III-2, it would be expected that the variation in ε_θ vs ω/ω_c be represented by a curve similar to Fig. 3.6. The analysis leading to the curves of ε_θ vs ω/ω_c such as Fig. 3.6, was carried out under the condition $\partial/\partial\theta = 0$. However, while this condition is not satisfied in the multi-anode magnetron, these theoretical results should be at least qualitatively correct for this case, especially in the vicinity of the cathode.

In order to check this, the resonant wavelength of a multicavity magnetron was measured as a function of the applied magnetic field. These measurements were made by determining the Q value of the resonator, maintaining the space charge cloud radius constant by keeping V_a/B_o^2 invariant. The radius of the space charge was adjusted so that no electrons attained synchronism with the rotating wave on the anode, thus avoiding any effects due to beginning of oscillation. Despite this precaution, it is usually somewhat uncertain whether the wavelength shift observed is due to the "bulk" properties of the cloud, analyzed in Chapter III, or to motional effects.[1] This is particularly true in the vicinity of the cyclotron field. For this reason, resonant cavities possessing simpler field configurations (and which cannot oscillate) such as those described in the preceding sections of this

---
1 Lamb and Phillips, Loc. cit.

chapter are more advantageous for the study of the space charge characteristics.

Using Fig. 3.7, an estimate of the variation in ε_e vs ω/ω_c can be found for the value of r_H/r_c used in the test (r_H/r_c = 1.1).

The data obtained from these tests are shown in Figs. 6.11a, 6.11b, 6.11c, and 6.12. The first three of these are typical curves showing the variation in λ_o as the space charge cloud is expanded by increasing the d-c anode voltage at constant magnetic field. G_o is the conductance seen looking into the output coupling system of the tube, its value is determined from measurement of the input standing wave ratio at resonance; I_a is the anode current.

It is seen in Fig. 6.11a that for B = 210 gauss, ω/ω_c = 3, and the resonant wavelength decreases as the space charge is expanded. At this value of magnetic field, the space charge appears as a region with dielectric constant positive and less than unity. Similarly, in Fig. 6.11b, B = 1700 gauss, ω/ω_c = .37 and the expansion of the space charge increases the resonant wavelength. At this value of magnetic field, the space charge is believed to have a negative value of dielectric constant. The sharp rise in λ_o at high voltages seen in Fig. 6.11b is the result of the electrons approaching synchronism with the rotating wave of the magnetron. Fig. 6.11c shows the change in resonant wavelength of a multi-anode magnetron operated with magnetic field below that necessary for oscillation. Under this condition the space charge can be expanded to the anode and since in this region of ω/ω_c, $0 < \varepsilon_e < 1$, a considerable reduction in resonant wavelength is obtained.

Fig. 6.12 shows $\Delta\lambda$ as a function of ω/ω_c (curve A) and can be compared with the $\Delta\lambda$ predicted (curve B) from this theory, and that predicted by the Lamb and Phillips theory (curve C).

The main points to be considered in comparing experimental data of this type with that predicted from the theory, are the values of ω/ω_c at which the $\Delta\lambda$ curve shows a discontinuity. The absolute value of $\Delta\lambda$ for any ω/ω_c region is of less importance, since this cannot be calculated accurately for a multicavity magnetron without an unreasonable amount of mathematical labors.

Examination of Fig. 6.12 reveals the expected sharp rise beginning near $\omega/\omega_c = 1.1$, but the expected decrease in λ_0 near $\omega/\omega_c = .9$ was not observed. It must be remembered that the predicted curve B results from use of Fig. 3.7, which is merely an approximate interpolation based on purely qualitative considerations. If the more exact theory of the plane magnetron (Fig. 3.3) is used, the predicted curve of Fig. 6.12 would be as shown except that there would be no rise between $\omega/\omega_c = .85$ and $\omega/\omega_c = 1.07$. This would not improve the agreement with experiment in the region $.7 < \omega/\omega_c < 1$ however. The theory of Lamb and Phillips indicates a wavelength shift of the form $\frac{1}{\omega_c^2/\omega^2 - 1}$ (curve C) which seems to be in reasonable agreement with the experimental points for $\omega/\omega_c > 1$ but not for $\omega/\omega_c < 1$. Here the direct effect of the cyclotron resonance of the electrons in the space charge tends to obscure the "bulk" effect of the space charge in the resonant circuit. The wavelength variation described in Fig. 6.12 probably represents a combination of these two effects.

It is seen that in general the expected wavelength shifts, predicted from the result of the analysis in Chapter III, are observed except in the neighborhood of $\omega/\omega_c = 1$ where the synchronism effects are predominant.

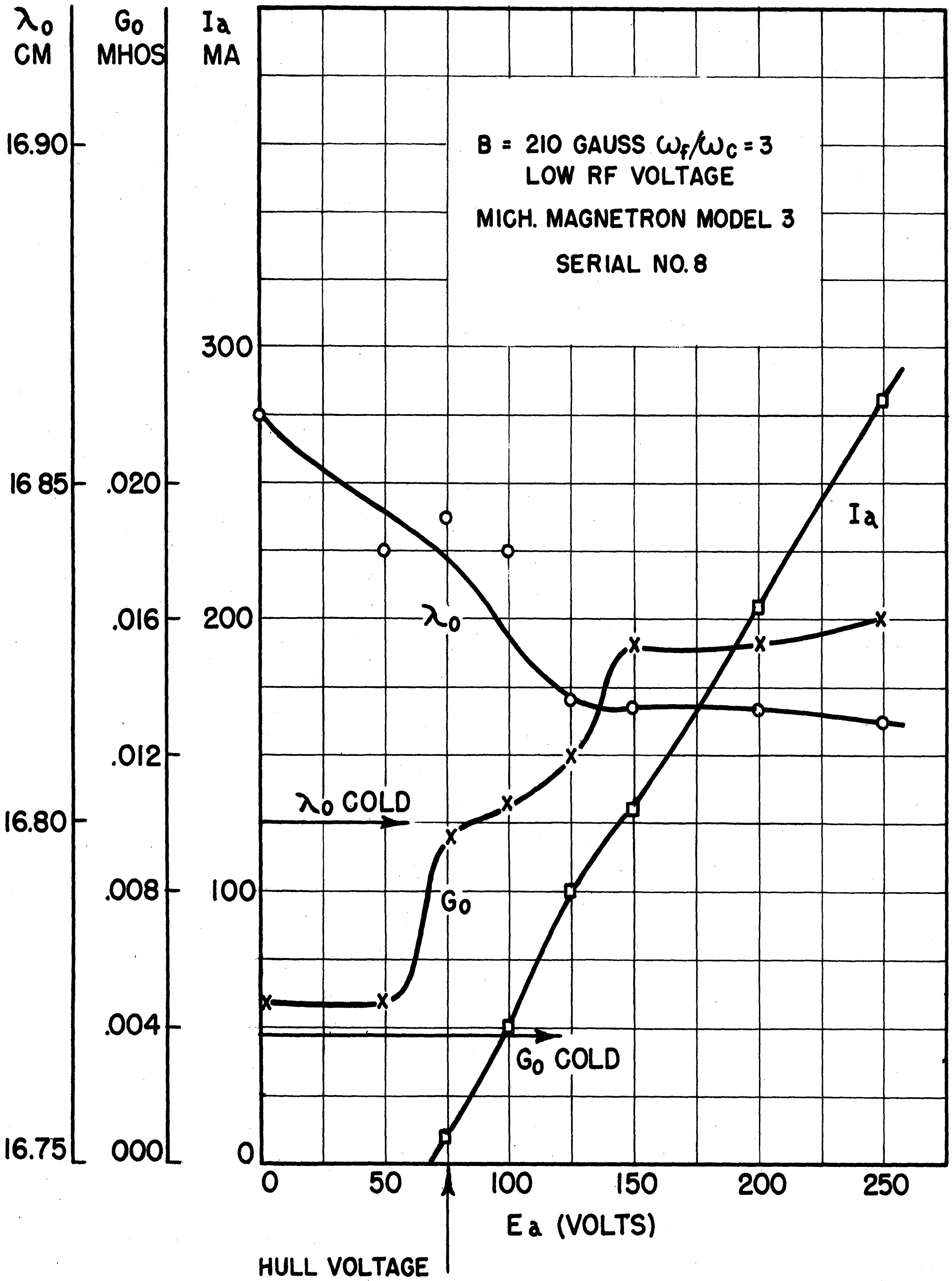


FIG. 6.11a
λ_0 AND G_0 OF HOT MAGNETRON
AS FUNCTION OF PLATE VOLTAGE

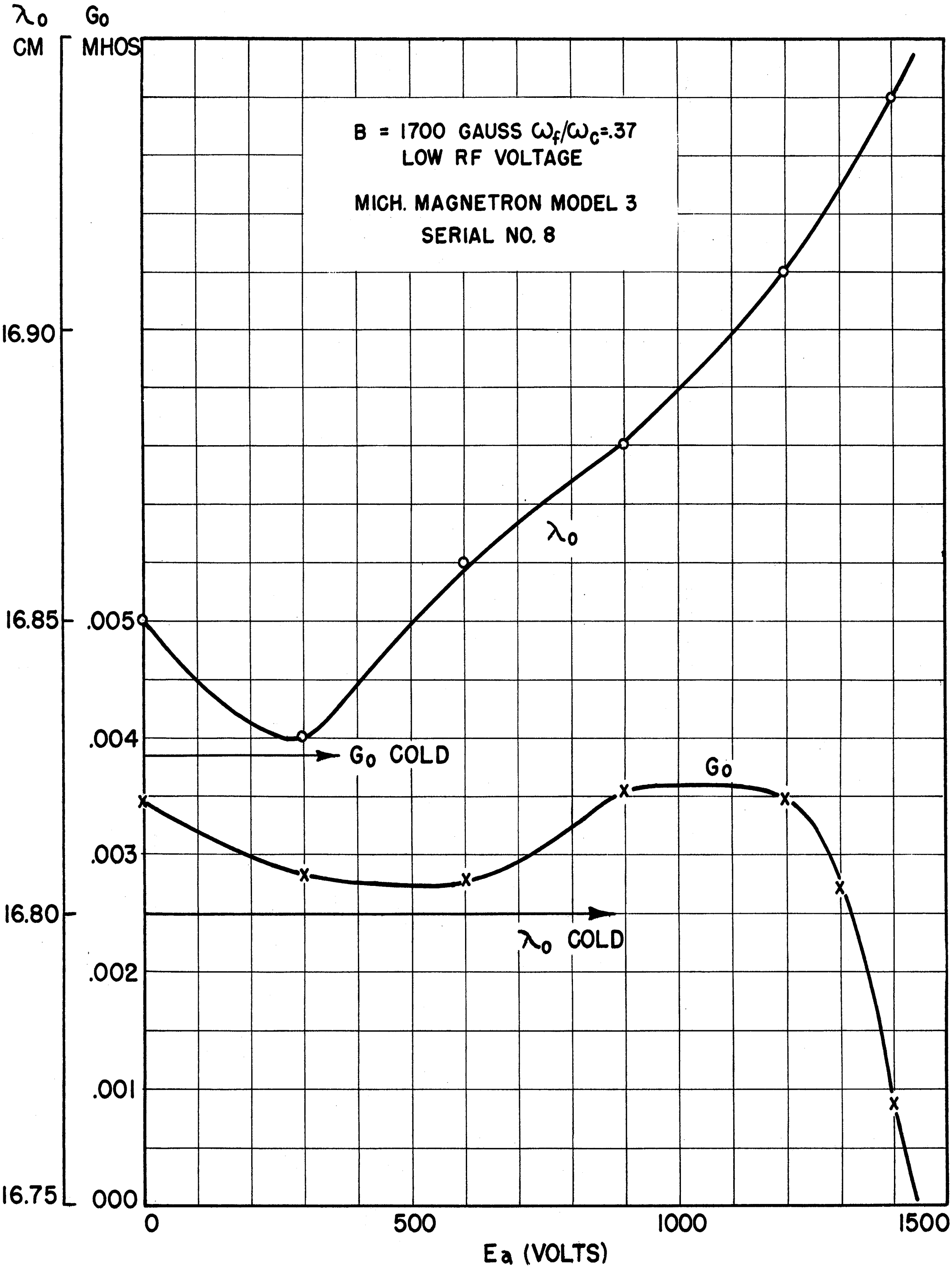


FIG. 6.11b
λ_0 AND G_0 OF HOT MAGNETRON
AS FUNCTION OF PLATE VOLTAGE

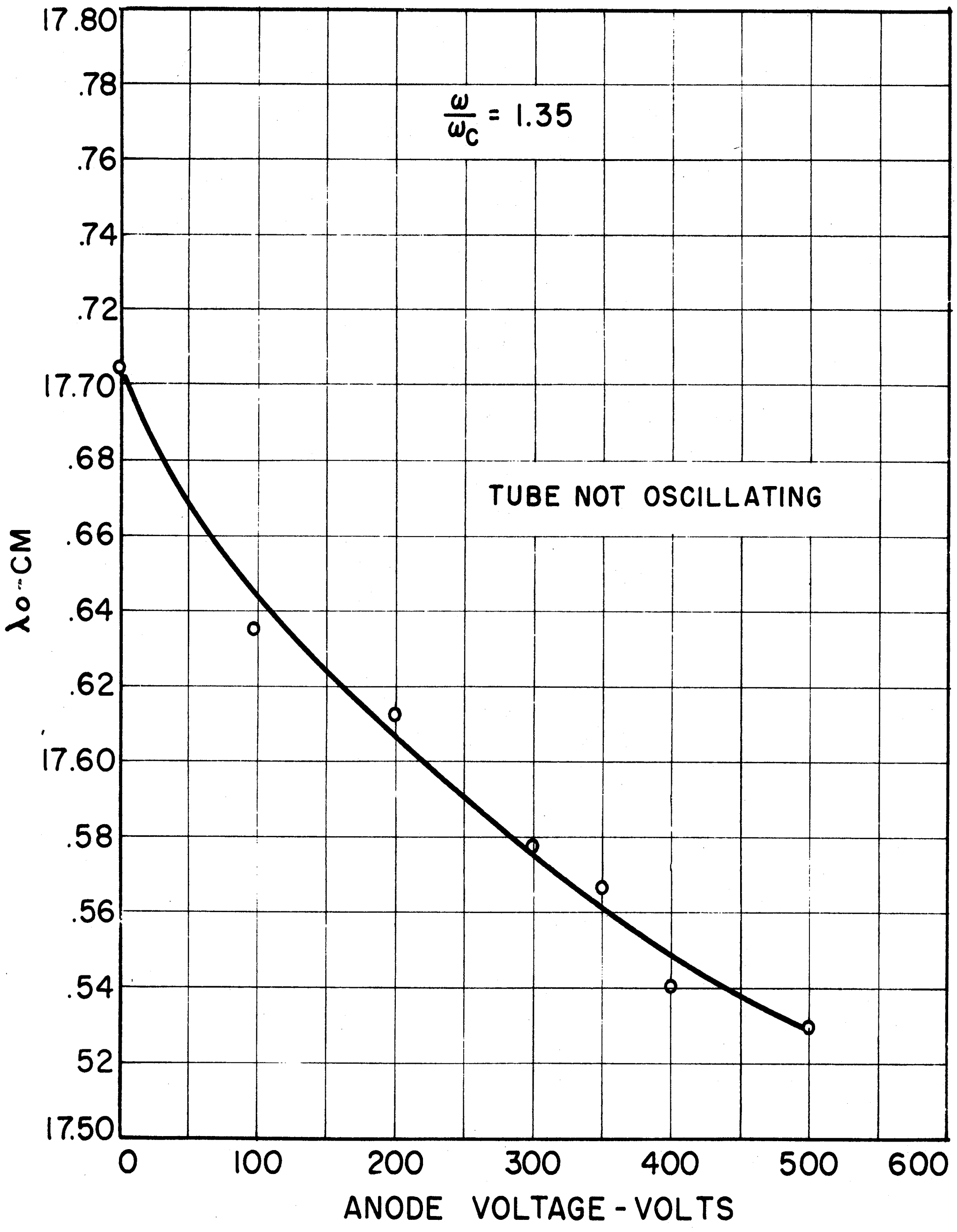


FIG. 6.11c

WAVELENGTH SHIFT IN INTERDIGITAL MAGNETRON DUE TO EXPANDING SPACE CHARGE CLOUD.

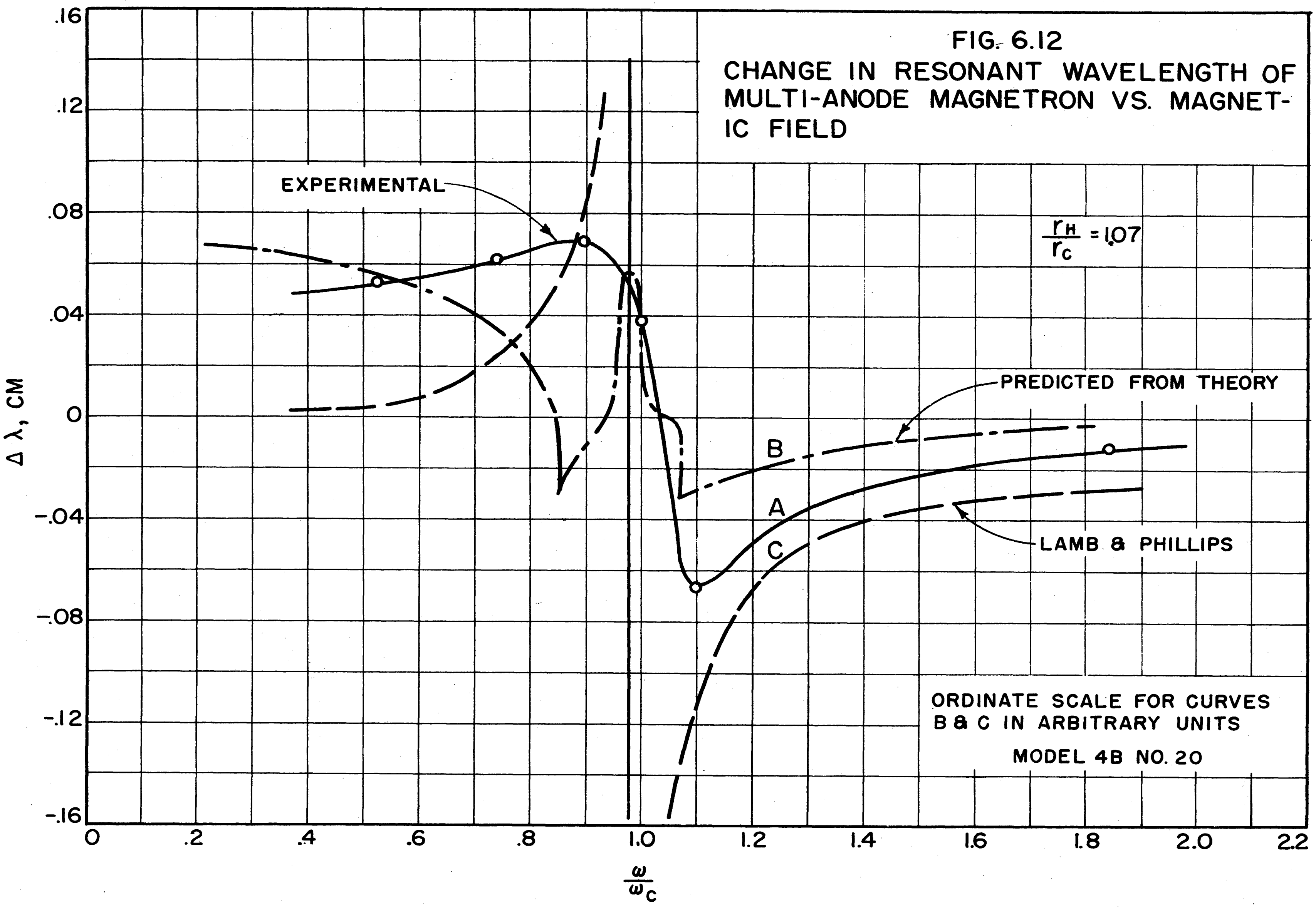


FIG. 6.12
CHANGE IN RESONANT WAVELENGTH OF MULTI-ANODE MAGNETRON VS. MAGNETIC FIELD

VII. CONCLUSIONS

In this chapter the considerations of the magnetron space charge will be concluded by a discussion of the results of the experiments performed in an attempt to check the theory, a resume of the assumptions underlying the analysis, a brief list of the possible applications of a space charge of this type and a few topics suggested for future investigation. Unfortunately this last section contains some rather important points, indicating that this present work is by no means a complete exposition on the subject. It is hoped however that a few of the physical principles have been brought out and that a basis for future investigations has been established.

The magnetron space charge is somewhat unique among electron atmospheres under the influence of a magnetic field (at least the plane magnetron and cylindrical magnetron with propagation along the applied magnetic field) in that no double refraction of the waves was found.

It is believed that this type of space charge cloud could be used with profit more generally for such purposes as frequency modulation.

1. Conclusions - Agreement between Theory and Experiment.

Three experiments have been performed in an effort to determine the validity of the analytical expressions for the dielectric properties of the magnetron space charge, determined in Chapter III.

In order to duplicate experimentally the case of wave propagation in the direction of the magnetic field, a coaxial cavity was constructed so that the travelling waves in the cavity propagate through a cylindrical space charge cloud surrounding a filament serving as part of the center conductor. The results of tests on this tube seem to confirm the theoretical predictions as to values of magnetic field at which certain discontinuities in the properties of the space charge cloud are expected. The magnitude of the wavelength shift observed did not check well with that calculated, but this is believed to be of less importance. In comparing the theoretical singular values of ω/ω_c with those observed, it is remembered that one value should be independent of the potential or space charge distributions, depending only on ω/ω_c. This point was checked quite well by the experiment. The other singular value of ω/ω_c to be sought is dependent on the space charge distribution and therefore cannot be predicted with certainty. The value obtained experimentally appears to correspond to a value of space charge density at the edge of the cloud approximately twice the Hull-Brillouin value. This suggests the existance of a space charge distribution with a peak at the boundary of the cloud. In general, the results of this experiment are considered as satisfactory confirmation of the theory.

The part of the theory concerned with wave propagation in the radial direction was investigated in two experiments. One of these involved measurements on a multianode magnetron in which the results are less conclusive, because the synchronous effects of the electrons at the cyclotron magnetic field tends to obscure the "bulk" effects of the space charge sought here. However it was observed that at values of magnetic field well above the cyclotron field the space charge cloud still exhibited a positive effect on the resonant wavelength of the magnetron, as predicted by the theory. This effect for $B_o \gg B_c$ would definitely be due to the "bulk" properties of the cloud since the synchronous effects would be noticeable only in the neighborhood $B_o = B_c$.

The second of these two latter experiments was performed by inserting a space charge cloud into a cylindrical cavity in the TE_{011} mode. While no effect of the space charge on resonant frequency could be detected; the space charge appeared to abstract net energy from the fields near the cyclotron frequency. This behavior is at least not at variance with the theory.

This analysis of the propagation of electromagnetic waves in the magnetron space charge considered the electron cloud as a medium whose motional behavior can be explained with the aid of the hydrodynamical equation. This treatment is contrasted with that which considers the motion of the individual electrons as obeying Newton's law. The results of the analysis were presented in terms of an equivalent or effective dielectric constant depending only on the ratio of the frequency of the wave to the cyclotron resonance frequency. Since, in the tubes built to test this theory the space charge occupied a region small compared with the wavelength, the space charge cloud was considered as a lumped circuit element and its effect on the associated circuit predicted, by certain qualitative arguments, from the ε_e vs ω/ω_c curves. Agreement between these theoretical predictions and experiments can be interpreted as indicating that the consideration of the space charge as a medium is valid.

Unfortunately only one set of experimental data is considered sufficiently reliable to serve as a check on the validity of the theory and its interpretation; namely the data in Fig. 6.4. It is believed that this data does provide reasonable confirmation of this analytical treatment of the magnetron space charge.

2. Resume of Assumptions.

A brief resume of the *a priori* assumptions, on which this analysis is based, will be given in this section. This will provide a basis for consideration of the validity of the analytical results obtained.

a. It is assumed that the electron space charge can be treated as a gas, the motion of whose particles (although they interact with only inverse square law forces) are mutually interlocked to such an extent that the gas exhibits medium-like behavior. The particles of this gas are considered to possess a random motion which is sufficiently large in comparison with the applied r-f signal voltage so that thermal equilibrium is maintained in the gas. In such a case the equation of motion of the electrons can be derived from the Boltzmann Transport equation. Certain properties (such as thermal conductivity, mean free path, etc.) of such an electronic gas cannot be determined without evaluation of the electron velocity distribution function. This evaluation would entail, in view of the necessity for consideration of both near (binary) and distant encounters, considerable mathematical difficulty. Fortunately the desired dielectric properties of the space charge can be determined without knowledge of the distribution function.

b. In general it is assumed that the amplitude of the perturbed electron velocity (that is, the part of the velocity due to the applied r-f field) is small enough that terms involving its square can be neglected in comparison with terms in the equation involving only the first power of the velocity. That is, the usual small signal theory is used throughout.

c. Certain limitations have been placed on the form of the waves considered to be propagating in the space charge. That is, in general the electromagnetic waves are considered invariant in phase in directions normal to the direction of propagation. In the cases of very simple geometrical r-f structures, this assumption will be fulfilled. The result can be applied to r-f fields of more complex configuration by suitable superposition of waves of the type considered here.

d. The complete solution of the problem in the cases with cylindrical geometry was found to involve lengthy mathematical treatment. Therefore the solutions for the propagation constants in the cylindrical cases were obtained only for the limiting case of no variation with radius of space charge density and angular velocity. This condition would generally be satisfied by use of a vanishingly small cathode. A rigorous solution is obtained therefore only in the two limiting cases of small cathode on the one hand and plane electrodes on the other. It is believed however that the results of the cylindrical analysis provide a useful approximation for $r_H/r_c \gtrsim 3$.

e. In this report, including the considerations of the possible space charge density distributions in Chapter II, the effect of interactions between the discrete electrons is completely neglected. This is not meant to imply that these effects are unimportant, but merely that their consideration is the subject of a separate study.

3. Possible Applications of This Type of Space Charge.

A consideration of the wave propagation characteristics of the magnetron-type space charge suggests possible applications for such a medium other than in an oscillating magnetron. There are, of course, many more possible applications than those listed here, these being merely the most obvious examples.

a. The use of a cylindrical magnetron space charge in a coaxial structure coupled to a resonant oscillator circuit has been suggested before as a means for obtaining frequency modulation. This type of space charge should be capable of producing relatively large changes in reactance with little energy loss by the fields.

b. It has been shown that this space charge can be made to have such characteristics as to prevent wave propagation through this medium. These results were derived on the assumption of a small signal, however it does not appear unreasonable that the behavior, in the presence of a signal of considerable amplitude, will be qualitatively similar. Therefore it is suggested that this space charge can be used in a transmission line as a switch of microwave energy for such purposes as antenna lobe switching, etc.

c. Since the space charge can be made to reflect almost any desired proportion of the incident wave energy, it would seem that a microwave signal could be amplitude modulated without necessity of modulating the generator, by using the space charge as a variable switch in the transmission line.

An engineer seeking to use the magnetron type space charge for one of the above or any similar application naturally requires a knowledge of a number of characteristics of this medium other than its dielectric properties, etc. For example he would be interested in the rate of r-f energy loss in the space charge. In this connection it is believed that this space charge has an advantage over many other forms of electronic modulation, such as the spiral beam, in that its properties are not so critical as to frequency and magnetic field, and a loss (due to an in-phase component of electron current) is not inherent.

For a large signal application this space charge would undoubtedly be more lossy than in the small signal case but from the results of section IV-5 the energy loss would not be expected to be prohibitive.

In any very small signal application such as modulation of a local oscillator the question of noise naturally arises. There is relatively little published, experimental information concerning the noise power output of a non-oscillating magnetron space charge. Rieke[1] presents the results of some measurements on a 10 cm c-w multi-anode magnetron with output power of the order of 100 watts. The noise power output was measured on this tube while in the cutoff condition. The noise power increases as the space charge cloud is expanded; for an anode voltage of about three-fourths of that required for initiation of oscillations, the noise power was observed to be about -60 to -70 db below one watt.

For comparison, the shot noise power from a space charge limited triode in a microwave cavity can be calculated from the relation $P_n \cong 2.5\ kT_c \Delta f$. Using $\Delta f = f/Q = 3 \times 10^9/100$ this power is of the order of -120 db below one watt. Comparing this value to that quoted

---

1 Rieke, F. F., in "Microwave Magnetrons", Radiation Laboratory Series No. 6, McGraw-Hill, page 390, 1948.

above for the magnetron, this latter device appears very unfavorably. However this comparison is not quite fair since the magnetron noise was measured on a device whose power capabilities and therefore size and rotating space charge current are considerably larger than would be used in a modulator structure. If this noise figure of -60 db for the magnetron is scaled down with output power, maintaining signal to noise ratio constant, a noise power of the order of -30 db would be obtained. This latter figure is slightly more favorable in comparison with the triode oscillator tube. From this it appears that the use of a magnetron space charge for low level modulation purposes is not out of the question.

4. Suggested Topics for Future Investigation.

The experiments described in this report, undertaken to obtain some indication of the validity of the theoretical results, by no means exhaust the possibilities of experimental investigations of a magnetron-type space charge. Exploration of a magnetron space charge by means of high frequency waves can have three objectives: (a) by assuming a knowledge of the static space-charge density, experimental data could be used to check the theory developed, (b) to determine purely experimentally the properties of certain configurations of space charge, and (c) assuming the theory to be correct, experimental data could be used to obtain some information on the space charge density. Since the tools with which an experimental study of the static magnetron space charge can be made are extremely limited, this last objective seems to be at least worthy of more consideration.

It is the author's opinion that the major source of error in making measurements of this kind on a space charge lies in the problem of obtaining a space charge of such configuration that it duplicates closely the idealized form considered in the analysis. It is very difficult to obtain an absolutely uniform and laminar magnetic field, so that some leakage of electrons from the ends of the cloud is inevitable unless suitable precautions in the form of shields etc. are taken.

One specific investigation which might increase the understanding and possible applications of this type of space charge involves the application (b) in section 3 above. The measurement of the reflection of microwave energy from a space charge in a coaxial line should provide useful information relative to its use as a microwave switch in a transmission line. This information would be particularly valuable if the signal power used is high (0.1 - 1 kw. or more).

Also it would be desirable to ascertain if the space charge in a plane magnetron could be made to amplify a microwave signal.

Along the line of continued theoretical investigation, very valuable information as far as applications are concerned could be derived from a large signal study of the space charge. If this were done using electric and magnetic fields of simple geometry, the problem may admit of a solution with a reasonable amount of labor and the results might aid in the understanding of the role of the space charge in an oscillating magnetron.

Also a solution for the dielectric properties of the cylindrical space charge for various values of r_H/r_c would yield useful design information.

Finally, continuation of the study of wave propagation parallel to the steady electron velocity and its extension to the cylindrical magnetron should yield results which can be applied profitably to a further understanding of magnetron characteristics.

APPENDICES

Appendix 1 - Derivations from the Boltzmann Equation.

In order to attempt a justification for the treatment of an electronic space charge as a gas obeying the Euler hydrodynamical equation, this latter and other pertinent equations will be derived briefly from the Boltzmann transport equation.[1] This equation considers the gas to be nearly in thermodynamic equilibrium, so that the applied fields cause only a small perturbation on this equilibrium. Therefore it must be assumed a priori that this condition is satisfied.

The Boltzmann equation for the behavior of a gas in an applied vector field of force $\vec{F}$ which is independent of the particle velocity, can be written

$$\frac{\partial f}{\partial t} + \vec{c} \cdot \frac{\partial f}{\partial \vec{r}} + \vec{F} \quad \frac{\partial f}{\partial c} = \left(\frac{\partial f}{\partial t}\right)_{coll} = \mathcal{D} f \qquad \text{A1-1}$$

where f is the (un-normalized) velocity distribution function, $\vec{c}$ the particle velocity[2], and $\vec{r}$ the position vector of a particle with reference to some chosen coordinate system. The term $(\partial f/\partial \hat{c})_{coll}$ denotes the time rate of change of the number density of particles at $\vec{r}$, t with velocities $\vec{c}$, $d\vec{c}$ due to collisions with other particles of the gas, and $\mathcal{D}$ is an operator representing the differential operations on the left side of the Eq. A1-1. It is assumed that the collisions occupy only a small part of the lifetime of a particle.

---

1 Chapman and Cowling - "The Mathematical Theory of Non-Uniform Gases" - Cambridge, 1939, Chapters 3 and 18.

2 $\vec{c}$ is used for particle velocity in this section only, in other sections of this report v will denote particle velocity while c will represent the velocity of light in vacuo.

In the electronic space charge in a magnetron, the particles of the gas are under the influence of the additional (Lorentz) force due to the magnetic field $\vec{B}$. This applied force depends on the particle velocity through the relation $-e\vec{c} \times \vec{B}$. Then Eq. A1-1 becomes:

$$\frac{\partial f}{\partial t} + \vec{c}\cdot\frac{\partial f}{\partial \vec{r}} + \left[\vec{F} - \frac{e}{m}\ \vec{c} \times \vec{B}\right]\cdot\frac{\partial f}{\partial \vec{c}} = \left(\frac{\partial f}{\partial t}\right)_{coll} = \mathcal{D} f \qquad \text{A1-2}$$

To save space in writing equations, let ϕ be any property of the particles, such as their kinetic energy, momentum, etc. Then, multiplying Eq. A1-1 by $\phi d\vec{c}$ and integrating throughout velocity space (assuming the integrals to be convergent), also specifying that ϕf tends to zero as c becomes infinitely large, the result can be written

$$\int \phi \mathcal{D} f\ d\vec{c} = n\Delta\overline{\phi} = \int \phi \left(\frac{\partial f}{\partial t}\right)_{coll} d\vec{c} \qquad \text{A1-3}$$

where $n\Delta\overline{\phi}$ is the change, due to collisions, of the mean value of the sum of the property ϕ over all particles.

If, from all of the electrons represented in velocity space is chosen a number occupying volume $\vec{r}$ to $\vec{r} + d\vec{r}$, whose position is given by the vector $\vec{r}$ at time t and whose velocity is in the range $\vec{c}$ to $\vec{c} + d\vec{c}$, and the sum of the particle property ϕ is taken over this group, the result is written $\Sigma\,\phi$. Then $\phi\left(\frac{\partial f}{\partial t}\right)_{coll} d\vec{c}$ is a measure of the rate of change, due to collisions, of this sum $\Sigma\,\phi$. Therefore $\int \phi \left(\frac{\partial f}{\partial t}\right)_{coll} d\vec{c}$ is the rate of change of $\Sigma\phi$, due to collisions, of all particles in unit volume of real space.

By definition of $\overline{\phi}$:

$$\Sigma\,\phi = n\overline{\phi}$$

where the superscript bar denotes the mean value over the group of particles considered and n is the number density of particles in real space. By the meaning of the mean value

$$n\overline{\phi} = \int \phi f d\vec{c} \qquad \text{A1-4}$$

so that

$$\int \phi \frac{\partial f}{\partial t} d\vec{c} = \frac{\partial}{\partial t}\int \phi f d\vec{c} - \int f \frac{\partial \phi}{\partial t} d\vec{c} = \frac{\partial}{\partial t}(n\overline{\phi}) - n\overline{\frac{\partial \phi}{\partial t}} \qquad \text{A1-5}$$

and similarly

$$\int \phi u \frac{\partial f}{\partial r} d\vec{c} = \frac{\partial}{\partial r}\int \phi u f d\vec{c} - \int f u \frac{\partial \phi}{\partial r} d\vec{c} = \frac{\partial}{\partial x}(n\overline{\phi u}) - n\left(\overline{u\frac{\partial \phi}{\partial x}}\right) \qquad \text{A1-6}$$

where r represents any coordinate direction x y z.

$$\begin{aligned} \int \phi \frac{\partial f}{\partial u} d\vec{c} &= \iiint \frac{\partial}{\partial u}(\phi f)\, du\, dv\, d\omega - \int f \frac{\partial \phi}{\partial u} d\vec{c} \\ &= \iint \Big[\phi f\Big]_{u=-\infty}^{u=\infty} dv\, d\omega - \int f \frac{\partial \phi}{\partial u} d\vec{c} \\ &= -\, n\overline{\frac{\partial \phi}{\partial u}} \end{aligned} \qquad \text{A1-7}$$

where u, v, ω are the components of velocity along the coordinate axes.

Substituting Eq. A1-2 into Eq. A1-3 and using Eqs. A1-5, A1-6 and A1-7:

$$\int \phi \mathcal{D} f d\vec{c} = \frac{\partial n\overline{\phi}}{\partial t} + \nabla \cdot (n\overline{\phi \vec{c}}) - n\left[\overline{\frac{\partial \phi}{\partial t}} + \overline{\vec{c} \cdot \nabla \phi} + (\vec{F} - \frac{e}{m}\vec{c} \times \vec{B}) \cdot \overline{\frac{\partial \phi}{\partial \vec{c}}}\right] \qquad \text{A1-8}$$

If the particles of the gas have a mean or drift velocity $\vec{c}_0$ defined by

$$\vec{c}_0 = \frac{1}{n}\int \vec{c} f d\vec{c}$$

the particle velocities can be referred to a coordinate system moving with velocity $\vec{c}_0$, this "peculiar velocity" or particle velocity referred to the moving system is

$$\vec{C} = \vec{c} - \vec{c}_o$$

Then ϕ can be expressed as a function of $\vec{C}$ instead of $\vec{c}$,

$$\frac{\partial \phi}{\partial \vec{c}} \rightarrow \frac{\partial \phi}{\partial \vec{C}}$$

$$\frac{\partial \phi}{\partial t} \rightarrow \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial \vec{C}} \frac{\partial \vec{C}}{\partial t} = \frac{\partial \phi}{\partial t} - \frac{\partial \phi}{\partial \vec{C}} \frac{\partial \vec{c}_o}{\partial t}$$

$$\frac{\partial \phi}{\partial x} \rightarrow \frac{\partial \phi}{\partial x} - \frac{\partial \phi}{\partial \vec{C}} \frac{\partial \vec{c}_o}{\partial x}$$

$$\vec{c} \cdot \nabla \phi \rightarrow \vec{c} \cdot \nabla \phi - \frac{\partial \phi}{\partial \vec{C}} \cdot (\vec{c} \cdot \nabla) \vec{c}_o$$

Substituting these relations into Eq. A1-8:

$$\int \phi \mathcal{D} f \, d\vec{c} = \frac{\partial n\overline{\phi}}{\partial t} + \nabla \cdot (n\overline{\phi (\vec{c}_o + \vec{C})}) - n\Big[\overline{\frac{\partial \phi}{\partial t}} - \overline{\frac{\partial \phi}{\partial \vec{C}}} \cdot \frac{\partial \vec{c}_o}{\partial t} + \overline{(\vec{c}_o + \vec{C}) \cdot \nabla \phi}$$

$$- \overline{\frac{\partial \phi}{\partial \vec{C}} \cdot \big((\vec{c}_o + \vec{C}) \cdot \nabla\big) \vec{c}_o} + \overline{\Big(\vec{F} - \frac{e}{m} (\vec{c}_o + \vec{C}) \times \vec{B}\Big) \cdot \frac{\partial \phi}{\partial \vec{C}}}\Big] \qquad \text{A1-9}$$

Using the notation $\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{c}_o \cdot \nabla$ Eq. A1-9 reduces to:

$$\int \phi \mathcal{D} f \, d\vec{c} = n\Delta\overline{\phi} = \frac{D}{Dt} n\overline{\phi} + \nabla \cdot n\overline{\phi C} + n\overline{\phi} \nabla \cdot \vec{c}_o$$

$$-n\Big[\overline{\frac{D\phi}{Dt}} + \overline{\vec{C} \cdot \nabla \phi} + \overline{\frac{\partial \phi}{\partial \vec{C}}} \cdot \Big(\vec{F} - \frac{e}{m} \vec{c}_o \times \vec{B} - \frac{D\vec{c}_o}{Dt}\Big) \qquad \text{A1-10}$$

$$- \overline{\frac{\partial \phi}{\partial \vec{C}}} \cdot \Big(\nabla \cdot (\vec{c}_o \overline{\vec{C}}) - \vec{c}_o \cdot (\nabla \cdot \vec{C})\Big) - \overline{\frac{\partial \phi}{\partial \vec{C}} \cdot \frac{e}{m} \vec{C} \times \vec{B}}\Big]$$

Three equations important in gas analysis can be deduced from this equation with no prior knowledge of the form of the distribution function f. This reduction can be accomplished by making use of certain properties of the gas, which are functions of the particle velocity, and whose sum over the particles participating in a collision is conserved, so that $\Delta\overline{\phi} = 0$. These properties are: (a) conservation of number density of particles, (b) conservation of momentum, (c) conservation of energy. These so-called "summational invariants" will be applied to Eq. A1-10 separately.

A. Since the number density of particles is conserved in collisions; letting $\phi = 1$ so that $\overline{\phi} = 1$, it is seen that $\Delta\overline{\phi} = 0$. Then $\overline{\phi\vec{C}} = 0$ (since $\overline{\vec{C}} = 0$ by definition), $\frac{\partial\phi}{\partial C} = \frac{D\phi}{Dt} = \nabla\phi = \Delta\overline{\phi} = 0$ and Eq. A1-10 reduces to

$$n\,\nabla\cdot\vec{c}_0 + \frac{Dn}{Dt} = 0 \qquad \text{A1-11}$$

which is just the equation of continuity of numbers (or density) of particles.

B. Using the law of conservation of momentum; letting $\phi = mU$ (where U is the x component of $\vec{C}$), then $\overline{\phi} = 0$ so that $\Delta\overline{\phi} = 0$. Also

$$\overline{n\phi\vec{C}} = nm\,\overline{U\vec{C}} = p_x\ ^{1}$$

(p is the hydrostatic pressure of the gas),

$$\nabla\phi = \frac{D\phi}{Dt} = 0, \left(\frac{\partial\phi}{\partial C}\right)_x = m, \qquad \overline{\left(\frac{\partial\phi}{\partial C}\right)\cdot\vec{C}} = 0$$

and Eq. A1-10 becomes for the x component of momentum:

---

[1] To verify this, consider that as each particle in $\vec{c}$ to $\vec{c} + d\vec{c}$ crosses a unit surface area it carries with it a quantity ϕ so that the total contribution of this group to the flow of ϕ across the area is, per unit time:

$$\phi C_n\ f\ dC$$

where C_n is the component of C normal to the unit area, $C_n = C\ \vec{n}_1$; $\vec{n}_1$ is a unit normal vector. The total net flux of ϕ is then

$$\int \phi C_n\ f\ d\vec{C} = n\ \overline{C_n\ \phi}$$

In this case $\phi = mU$ and the rate of transport of momentum across the area is by definition the hydrostatic pressure of the gas acting on this surface. Thus

$$p_n = nm\ \overline{(C_n U)}\cdot\vec{n}_1 = nm\ \overline{CU}$$

$$\nabla p_x - nm \left[\vec{F} - \frac{D\vec{u}_o}{Dt} \right] + ne \, \vec{u}_o \times \vec{B} = 0 \qquad \text{A1-12}$$

where $\vec{u}_o$ is the x component of $\vec{c}_o$. In general the equation of momentum is

$$\nabla p - nm \left[\vec{F} - \frac{DC_o}{Dt} \right] + ne \, \vec{c}_o \times \vec{B} - 0 \qquad \text{A1-13}$$

which is the Euler equation of hydrodynamics, Eq. II-1.

C. From the conservation of energy relation, let $\phi = E$, the energy of translation of a particle. Then for the type of gas considered here $\overline{\phi} = 3/2\ kT$ and from considerations similar to that in the footnote to paragraph (B) above, $n\overline{\phi\vec{C}} = n\overline{E\vec{C}} = \vec{q}$ the thermal flux vector (the rate of transfer of heat across unit area). Then $\nabla\phi = \frac{D\phi}{Dt} = 0$

and since

$$\phi = \frac{1}{2} mC^2, \quad \frac{\partial\phi}{\partial C} = m\vec{C}$$

from which it follows that

$$\overline{\frac{\partial\phi}{\partial C}} = 0; \quad n \left(\frac{\partial\phi}{\partial c} \right) \cdot \vec{C} = nm \, \overline{\vec{C}\vec{C}} = p$$

Then Eq. A1-10 becomes:

$$\frac{D}{Dt} \frac{N}{2} nkT + \frac{N}{2} nkT \, \nabla\cdot\vec{c}_o + \nabla\cdot\vec{q} + p\cdot\nabla\cdot\vec{c}_o \qquad \text{A1-14}$$

N = number of degrees of freedom

This reduces to:

$$\frac{DT}{Dt} + \frac{2}{Nkn} \left[p \quad \nabla\cdot\vec{c}_o + \nabla\cdot\vec{q} \right] = 0 \qquad \text{A1-15}$$

which is the equation of thermal energy of the gas.

The Boltzmann Eq. (2) can be solved in general by a series of successive approximations[1] allowing determination of the pressure, thermal flux vector, coefficients of diffusion and thermal conductivity, etc.

---

[1] Chapman and Cowling, 7.3 - 7.5.

In first approximation $q = 0$ and in second approximation

$$q = -\kappa \nabla T \qquad \text{A1-16}$$

where κ is the coefficient of thermal conductivity. Then Eq. A1-15 becomes:

$$\frac{DT}{Dt} + \frac{2}{3kn} \left[p \; \nabla \cdot \vec{c}_0 - \kappa \nabla^2 T \right] = 0 \qquad \text{A1-17}$$

In the case of two constituents of the gas, such as electrons and ions, Eqs. A1-11, A1-13, and A1-17 become:[1]

$$\frac{Dn_1}{Dt} + n_1 \nabla \cdot \vec{c}_0 + \nabla \cdot (n_1 \overline{\vec{C}_1}) = 0$$

A1-18

$$\frac{Dn_2}{Dt} + n_2 \nabla \cdot \vec{c}_0 + \nabla \cdot (n_2 \overline{\vec{C}_2}) = 0$$

$$(n_1 m_1 + n_2 m_2) \frac{D\vec{c}_0}{Dt} = n_1 m_1 \vec{F}_1 + n_2 m_2 \vec{F}_2 + (n_1 e_1 = n_2 e_2) \vec{c}_0 \times \vec{B}$$

A1-19

$$+ (n_1 e_1 \overline{\vec{C}_1} + n_2 e_2 \overline{\vec{C}_2}) \times \vec{B} - \nabla \cdot p$$

$$\frac{N}{3} k(n_1 + n_2) \frac{DT}{Dt} = \frac{N}{2} kT \nabla \cdot (n_1 \overline{\vec{C}_1} + n_2 \overline{\vec{C}_2}) + n_1 m_1 \vec{F}_1 \cdot \overline{\vec{C}_1} + n_2 m_2 \vec{F}_2 \cdot \overline{\vec{C}_2}$$

A1-20

$$+ (n_1 e_1 \overline{\vec{C}_1} + n_2 e_2 \overline{\vec{C}_2}) \cdot (\vec{c}_0 \times \vec{B}) - p. \cdot \nabla \vec{c}_0 + \kappa \nabla^2 T$$

---

1 Chapman and Cowling, page 332.

Appendix 2 - Influence of the Pressure Gradient Term in the Euler Equation.

In this section the effect of the random or thermal motions of the electrons on the propagation of electromagnetic waves in the magnetron space charge will be considered, using an approximate method. In the equation of motion, the effect of these random motions is represented by the pressure gradient term $\frac{\nabla p}{nm}$. A comprehensive treatment of this problem would require a knowledge of the velocity distribution function. This distribution function is unknown and the labor involved in its determination is considered unjustified for the purposes of this treatment. Therefore as a first approximation it will be assumed that the electron gas obeys the ideal gas law so that $p = nkT$. Partly because of the long range forces (Coulomb) existing between particles of the gas, a more exact treatment is difficult.

In order for the thermal motions of the electrons to have an influence on electromagnetic wave propagation, it would seem reasonable to suppose that some perturbation of the thermal velocities must progress with the wave. In his treatment of the effect of Hydrostatic Pressure on the Operation of Travelling Wave Tubes, Parzen[1] considers that both temperature and charge density are perturbed in the same manner as the travelling wave, that is

$$T = T_o + T_1 e^{i\omega t - \gamma s}$$
$$n = n_o + n_1 e^{i\omega t - \gamma s} \qquad \text{A2-1}$$

---

1 Parzen, P. - "The Effect of Hydrostatic Pressure in an Electron Beam on the Operation of Travelling Wave Devices" - Tech. Memo. 391, Federal Telecommunications Lab, March 1950, Recently published in Jour. App. Phys., April 1951.

Using these substitutions in Eqs. A1-13 and A1-17 the propagation in the y direction in a plane magnetron (Section III-1-a) is evaluated. It is found that the effective dielectric constant is given by Eq. III-3-a. That is, the temperature motion has no effect on this wave propagation. However there appears an additional wave, which, if the electron gas is considered isothermal ($T_1 = 0$), is given by

$$\gamma^2 = -\frac{m\,\omega^2}{k\,T_o}\left[1 - \frac{\omega_c^2}{\omega^2}\right] \qquad \text{A2-2}$$

and if the gas is adiabatic ($\kappa = 0$);[1]

$$\gamma^2 = -\frac{m\,\omega^2}{k\,T_o}\left[\frac{\omega_c^2}{\omega^2} - 1\right] \qquad \text{A2-3}$$

This method of accounting for the electron temperature may be useful in the case of the Travelling Wave Tube. However in the magnetron space charge, where the temperature T_o is known to be high, it is not apparent how this temperature can be perturbed in a wave-like fashion, certainly it cannot be considered that the fields of the propagating wave accomplish this since it is fundamental to this analysis that the random energy be large compared with the energy imparted to the electrons by the wave. Therefore it is believed that a more realistic consideration would be that the temperature T_o not exhibit any wave-like properties but the random energy of small volumes of space charge varies with the wave through the variation in space charge density $n_1\, e^{i\omega t - \gamma s}$.

---

[1] One can obtain a rough indication as to whether the electron gas is isothermal or adiabatic by comparing the mean time between collisions of electrons and the period of the propagating wave. Referring to Appendix 3 for notation, if $2\pi/\omega \gg \tau$, a large number of electron-electron collisions take place during a cycle so that energy can be conducted away from any given volume of the gas; this condition will describe an isothermal gas. On the other hand, if $2\pi/\omega \ll \tau$ the gas will be adiabatic. Considering the electron-electron collisions as binary encounters (which condition is not satisfied) Eq. A3-7 can be used to determine the order of magnitude of τ . If the electron diameter is taken as 10^{-12} cm, $1/\tau \simeq 10^{-4}$/sec so that $\tau \gg 2\pi/\omega$ and the electron gas is probably adiabatic.

Then

$$p = (n_o + n_1)\, kT_o \qquad \text{A2-4}$$

so that

$$\frac{\nabla p}{nm} = \frac{kT_o}{n_o m} \nabla n_1 + \frac{kn_1}{n_o m} \nabla T_o \qquad \text{A2-5}$$

From this it is seen that in the case of wave propagation in the z direction, for both plane and cylindrical geometries, since $n_1 = 0$ $\quad p = 0$ and the thermal motions have no effect on the wave propagation.

In the case of propagation in the y direction:

$$\frac{\nabla p}{nm} = \frac{kT_o}{n_o m} \frac{\partial n_1}{\partial y} + \frac{kn_1}{n_o m} \frac{\partial T_o}{\partial y} \qquad \text{A2-6}$$

since n_1 varies only in the direction of wave propagation, and it is improbable that T_o varies with any dimension except distance from the cathode. From the continuity equation

$$n_1 = \frac{\gamma n_o v_y}{i\omega}$$

so that

$$\frac{\nabla p}{nm} = -\frac{kT_o}{n_o m} \frac{\gamma^2 n_o v_y}{i\omega} + \frac{v_y k\, n_o \gamma}{i\omega\, n_o m} \frac{\partial T_o}{\partial y} \qquad \text{A2-7}$$

Then the velocity equations are :

$$v_x = -\frac{e}{m} \frac{E_x}{i\omega}$$

$$v_y = \frac{-\frac{e}{m} E_y + \frac{\omega_c e}{i\omega m} E_x}{i\omega + \frac{k\gamma}{i\omega m}\left(\frac{\partial T_o}{\partial y} - \gamma T_o\right)}$$

let:

$$\psi = \frac{\gamma^2 k\, T_o}{m\, \omega^2}$$

Combining these with the field Eqs. III-13:

$$\left[\frac{\gamma^2}{i\omega\,\mu_0} + \frac{\rho_0\, e/m}{i\omega} - i\omega\,\varepsilon_0\right] E_x + 0 = 0$$

A2-8

$$\left[\frac{\dfrac{-\omega_0\varepsilon}{m\,\omega^2}}{1 + \psi - \dfrac{k\gamma}{m\omega^2}\,\dfrac{\partial T_0}{\partial y}}\right] E_x + \left[\frac{-\dfrac{\rho_0 e}{i\omega m}}{1 + \dfrac{-k\gamma}{m\omega^2}\,\dfrac{\partial T_0}{\partial y}} + i\omega\,\varepsilon_0\right] E_y = 0$$

with solutions

$$\gamma^2 = \mu_0\,\rho_0\,\frac{e}{m} + \omega^2\mu_0\,\varepsilon_0$$

which is the same as Eq. III-15.

Also

$$\frac{-\dfrac{\rho_0\, e}{i\omega m}}{1 + \psi - \dfrac{k\gamma}{m\omega^2}\,\dfrac{\partial T_0}{\partial y}} + i\omega\,\varepsilon_0 = 0$$

so that

$$\gamma = \frac{1}{2T_0}\,\frac{\partial T_0}{\partial y} \pm \sqrt{\left(\frac{1}{2T_0}\,\frac{\partial T_0}{\partial y}\right)^2 - \frac{m}{kT_0}\left(\omega^2 + \frac{\rho_0 e}{m\varepsilon_0}\right)} \qquad \text{A2-10}$$

It is seen that in this case also, the inclusion of the pressure gradient term does not influence the electromagnetic wave propagation but leads to a new wave, which if $\frac{\partial T_0}{\partial y} = 0$ is given by:

$$\gamma^2 = -\frac{m}{kT_0}\,(\omega^2 - \omega_0^{\,2}) \qquad \text{A2-11}$$

which is, of course, the same as Eq. A2-2 found by the other method and is the same as that found by Linder[1] for a plane wave in a simple gas (no magnetic field).

---

1 Linder, E. G., Phys. Rev. 49 - 753, 1936.

Unfortunately, in the case of radial wave propagation the inclusion of the pressure gradient term so complicates the equation corresponding to Eq. III-18 as to make the solution impractical.

These plasma waves (Eq. A2-10) are important in this space charge analysis only by virtue of their being separate from and independent of the propagating electromagnetic waves (this statement is applicable, of course, only in those cases which have been solved). This subject will therefore not be pursued further except to note that these are propagating waves (the group velocity is not zero). Linder[1] has shown the existence of such waves from a wave equation derivable from the Euler equation. For a comprehensive treatment of plasma oscillations see Bohm and Gross.[2, 3]

Eq. A2-9 is seen to be identical to the corresponding expression for the propagation constant obtained in Chapter III, Section 2-a, where the pressure gradient term was not included. While this does not justify rigorously the neglect of the pressure gradient term, it would seem quite reasonable that it have no effect on electromagnetic wave propagation. This follows from the fact the mechanism of the influence of the electron motions on wave propagation is different in the two types of waves. That is, the electromagnetic waves (transverse waves) are influenced principally by the motions of the electrons transverse to the direction of propagation while the plasma waves (longitudinal waves) are influenced by the motions of the electrons parallel to the direction of propagation.

---

1 loc. cit

2 loc. cit

3 Gross, E. P. "Plasma Oscillations in a Static Magnetic Field" - Phys. Rev. V82 - April 15, 1951.

Also the wave motion represented by the pressure gradient term is a longitudinal wave in which the energy is transmitted by physical movement of the particles from one place to another, and is therefore independent of the electromagnetic waves.

Appendix 3 - The Effect of Electron-Ion Collisions.

In this present work the electrons in the magnetron space charge are thought of as moving along certain orbits prescribed by the steady fields. The electric field of the electromagnetic wave propagating in the space charge will periodically perturb the electrons from their steady orbits. During one-half cycle of the propagating wave the electrons will acquire energy from the fields, if there has been no energy lost by the electrons all of this energy will be returned to the wave during the next half cycle and it will propagate undiminished in amplitude. The concept of a wave propagating in a material medium with amplitude undiminished with distance is contrary to our physical experience, so that some mechanism of energy loss by the electron in its perturbed path must be sought. In even the best obtainable vacuum, the space charge will contain of the order of 10^{11} molecules/cm^3 which is of the same order as the electron density. Therefore in their periodic motion the electrons will have a reasonable probability of colliding with gas molecules. These collisions will provide one mechanism for energy loss by the electrons. The inclusion of such an energy loss term avoids the difficulty of equations for the electron amplitude or velocity becoming infinite at certain values of ω/ω_c.

Appleton and Chapman[1] consider an analogy between the velocity equations of electrons vibrating under the influence of a periodic electric field, suffering collisions with gas atoms, and electrons vibrating but experiencing a frictional type force proportional to the velocity. They show that the effect is similar if the frictional coefficient (g)

---
1 Appleton and Chapman - "The Collisional Friction Experienced by Vibrating Electrons in Ionized Air", Proc. Phys. Soc. of London - XLIV, page 246, 1932.

is replaced by the mean inverse time between electron-atom collisions. That is, the term g in Eq. II-3 is $g = \frac{1}{\tau}$ where τ is the mean time between collisions of an electron and a gas atom.

Using a similar line of reasoning to that followed by Appleton and Chapman, this analogy will be extended to include the case in which the electrons are in a magnetic field. It will be shown that the same substitution $g = \frac{1}{\tau}$ is valid in this case also.

For simplicity a stream of electrons with uniform velocity is considered. Then the equations of motion of an electron under the influence of an alternating electric field and a steady magnetic field is given, between collisions, by the following:

$$\ddot{y} = -\frac{e}{m} E_y + \omega_c \dot{x}$$

$$\ddot{x} = -\frac{e}{m} E_x - \omega_c \dot{y} \qquad \text{A3-1}$$

These may be solved to give

$$\dot{x} = \frac{-\frac{e}{m}\frac{1}{i\omega}\left[E_y - \frac{\omega_c}{i\omega} E_x\right]}{1 + \omega_c^2/\omega^2} + C_1$$

$$\text{A3-2}$$

$$\dot{y} = \frac{-\frac{e}{m}\frac{1}{i\omega}\left[E_x + \frac{\omega_c}{i\omega} E_y\right]}{1 - \omega_c^2/\omega^2} + C_2$$

Now suppose that at some time $t = t_1$, out of the large number of electrons a group can be selected which suffer collisions so that those electron's velocities vanish. That is when $t = t_1$, $\dot{x} = \dot{y} = 0$

so that

$$C_1 = \frac{\frac{e}{m}\frac{1}{i\omega}\left[E_y - \frac{\omega_c}{i\omega}E_x\right]}{1 + \omega_c^2/\omega^2}\, e^{j\omega t_1}$$

A3-3

$$C_2 = \frac{\frac{e}{m}\frac{1}{i\omega}\left[E_x + \frac{\omega_c}{i\omega}E_y\right]}{1 - \omega_c^2/\omega^2}\, e^{j\omega t_1}$$

Thus at any time after the collision

$$\dot{x} = \frac{-\frac{e}{m}\frac{1}{i\omega}\left[E_y - \frac{\omega_c}{i\omega}E_x\right]}{1 + \omega_c^2/\omega^2}\left(e^{j\omega t} - e^{j\omega t_1}\right)$$

A3-4

$$\dot{y} = \frac{\frac{e}{m}\frac{1}{i\omega}\left[E_x + \frac{\omega_c}{i\omega}E_y\right]}{1 - \omega_c^2/\omega^2}\left(e^{j\omega t} - e^{j\omega t_1}\right)$$

Let $\theta = t - t_1$. Then if the collisions occur randomly, the number of collisions dN_o which occur in time dt after t_1 is equal to the product of dt and the average number of collisions per unit time, A.

$$dN_o = A\, dt$$

If τ is the average time between collisions and there were N_o collisions at $t = t_1$:

$$A = N_o/\tau$$

Then integrating:

$$\int_{N_o}^{N} \frac{dN_o}{N_o} = \frac{1}{\tau}\int_{t_1}^{t} dt = \frac{\theta}{\tau}$$

so

$$dN = \frac{N_o}{\tau}\, e^{-\theta/\tau}\, d\theta \quad .$$

where dN is the number of electrons colliding in time θ to $\theta + d\theta$

Then to find the mean velocity over all times $t > t_1$:

$$\bar{\dot{x}} = \frac{-\frac{e}{m}\frac{1}{i\omega}\left[E_y - \frac{\omega_c}{i\omega} E_x\right]}{1 + \omega_c^2/\omega^2}\, e^{j\omega t} \int_0^{\infty} \left(e^{-\theta/\tau} - e^{-\theta\left(i\omega + \frac{1}{\tau}\right)} \right) d\theta$$

A3-5

$$= \frac{-\frac{e}{m}\frac{1}{i\omega}\left[E_y - \frac{\omega_c}{i\omega} E_x\right]}{1 + \omega_c^2/\omega^2} \; \frac{1}{i\omega + \frac{1}{\tau}}$$

and similarly for the y velocity component.

Now from Eq. II-7 the equation of motion of an electron subject to a frictional force is:

$$\ddot{y} + g\dot{y} = -\frac{e}{m} E_y + \omega_c \dot{x}$$

$$\ddot{x} + g\dot{x} = -\frac{e}{m} E_x - \omega_c \dot{y}$$

So that the velocities are

$$\dot{y} = \frac{-\frac{e}{m}\left[E_y - \frac{\omega_c}{i\omega + g} E_x\right]}{i\omega + g + \frac{\omega_c^2}{i\omega + g}}$$

A3-6

$$\dot{x} = \frac{-\frac{e}{m}\left[E_x + \frac{\omega_c}{i\omega + g} E_y\right]}{i\omega + g - \frac{\omega_c^2}{i\omega + g}}$$

Comparing these two expressions for the velocities it is seen that, since $g << \omega$, the velocity equations determined from collision considerations and those found using a frictional force are identical. From this one can conclude that the energy lost by each of these processes is equal when the association $g = \frac{1}{\tau}$ is made. While other forms of the electron cloud will change the equations, it is believed that the qualitative reasoning behind the analogy will be valid in any type of electron stream in a magnetic field.

In order to determine a numerical value of the coefficient g, two methods can be used:

1) Appleton and Chapman[1] describe an experiment from which the value of g could be obtained. However, their work was conducted at a gas pressure of about 0.1 mm Hg, considerably above that existing in the type of electron tubes considered here. If it is assumed that the frictional coefficient is proportional to the molecular density and therefore to the pressure, the experimental value of g can be extrapolated to the value at a pressure of 10^{-6} mm Hg. This procedure results in the value $g = 10^4$/sec.

2) The second method for determination of the numerical value of g is through the evaluation of the mean time between collisions, assuming only binary encounters.

In the case where the electrons and gas molecules are considered as smooth, rigid, elastic spheres, affecting each others motion only at a collision, the number of collisions[2] per unit time per unit volume between electrons of mass m and density n, and gas molecules of mass m_g and density n_g is:

$$N_{12} = 2n\, n_g\, \sigma_{12} \left(\frac{2\pi k\, m_0\, T_0}{m\, m_g}\right)^{1/2} \qquad \text{A3-7}$$

where $\sigma_{12} = \frac{\sigma + \sigma_g}{2}$, σ and σ_g being the electron and molecule diameters respectively.

$$m_0 = m + m_g$$

---

1 loc. cit.

2 Chapman and Cowling - page 90.

Since $m << m_g$ and $\sigma << \sigma_g$ this expression reduces to

$$N_{12} = \frac{n\, n_g\, \sigma_g^2}{2} \left(\frac{2\pi\, k\, T_0}{m} \right)^{1/2} \qquad \text{A3-8}$$

The average number of collisions undergone by each electron per unit time is the collision frequency so that $1/\tau = N_{12}/n$

For a typical magnetron space charge, with applied magnetic field of 1000 gauss,

$$n \approx 5 \times 10^{10} \frac{\text{electrons}}{\text{cm}^3} \qquad \sigma_g \approx 4 \times 10^{-8} \text{ cm}$$

$$n_g \approx 10^{11} \frac{\text{molecules}}{\text{cm}^3} \qquad T_0 \approx 10^4 \ {}^\circ\text{K}$$

so that

$$\frac{1}{\tau} = \frac{10^{11} \times 16 \times 10^{-16}}{2} \left[\frac{2\pi \times 1.37 \times 10^{-16} \times 10^4}{9 \times 10^{-28}} \right]^{1/2}$$

and:

$$g = \frac{1}{\tau} = 4 \times 10^4/\text{sec} \qquad \text{A3-9}$$

The close agreement between this result and that obtained by the first method is largely fortuitous, since the assumption of binary encounters is not valid, but at least the order of magnitude of this quantity has been determined.

The assumption that, during the course of an encounter between two particles, their motion is uninfluenced by the other particles in the region is probably valid for molecules interacting with forces varying rapidly with distance (such as an inverse fifth power law). However for the long range inverse square law forces this is not correct. Jeans[1] shows that the effect, on the motion of a given particle, is

---

1 Jeans, J. H. - Astronomy and Cosmogony - Cambridge, 1929, Chapter XII.

greater due to all of the "distant" particles than those few which come very close. These "distant" encounters are not considered by Chapman and Cowling.

It is seen that for electromagnetic waves in the microwave region where $\omega \approx 10^{10}$, the inequality $g \ll \omega$ is valid even if the effect of distant encounters increases g by several orders of magnitude. This inequality is used to simplify the equations developed in Chapter III.

Appendix 4 - The Effects of Electron-Electron Collisions.

In this section the effect of electron-electron collisions in the space charge on the velocity of propagation of electromagnetic waves in this medium will be examined. By an electron-electron collision is meant the process in which two electrons move close enough to each other that their fields of influence overlap and each of their respective orbits is affected by the presence of the other electron.

Using the law of conservation of momentum an attempt will now be made to show that the total dipole moment of the space charge is uninfluenced by collisions between electrons. From this it follows that the dielectric constant and therefore the wave propagation velocity is similarly uninfluenced.

The electrons are thought of as moving along a path under the influence of certain time invariant forces; its path is periodically perturbed by an additional force alternating in time as $e^{i\omega t}$. If the electron velocity is given by

$$v_0 + v_1 e^{i\omega t} \qquad \text{A4-1}$$

the momentum is

$$m\,(v_0 + v_1 e^{i\omega t}) \quad .$$

Using subscripts b and a to denote before and after collision, and superscript numbers to denote one or the other of the two interacting electrons, the conservation of momentum relation states that in a collision

$$m(v_0{}^1 + v_1{}^1 e^{i\omega t})_b + m\,(v_0{}^2 + v_1{}^2 \; e^{i\omega t})_b = \qquad \text{A4-2}$$

$$m(v_0{}^1 + v_1{}^1 \; e^{i\omega t})_a + m(v_0{}^2 + v_1{}^2 \; e^{i\omega t})_a$$

so that

$$v_{o\,b}^{1} + v_{o\,b}^{2} = v_{o\,a}^{1} + v_{o\,a}^{2}$$

$$v_{1\,b}^{1} + v_{1\,b}^{2} = v_{1\,a}^{1} + v_{1\,a}^{2} \quad . \qquad \text{A4-3}$$

These equations show that the sum of the perturbed velocities, and the sum of the unperturbed velocities are conserved in a collision.

The electric dipole moment τ_v , defined as the product of the charge and its displacement from its mean position, can be represented as

$$\tau_v = ea \qquad\qquad a = A\, e^{i\omega t}$$

where A is the maximum amplitude of the electron excursion from its unperturbed position. The velocity of the particle is then

$$v = i\omega a \qquad \text{so that:} \qquad a = \frac{v}{i\omega}$$

and

$$\tau_v = \frac{ev}{i\omega} \quad . \qquad \text{A4-4}$$

Using Eqs. A4-3 it follows from this that

$$\tau_{v\,1b}^{1} + \tau_{v\,1b}^{2} = \tau_{v\,1a}^{1} + \tau_{v\,1a}^{2} \quad . \qquad \text{A4-5}$$

Therefore the sum of the dipole moments of the electrons is conserved in an electron-electron collision. This can be extended to include all electrons in a given region so that the total polarization of the medium remains unchanged as a result of collisions. The polarization P is related to the dielectric constant ε_r by $P = \varepsilon_o E\ (\varepsilon_r - 1)$ so that the dielectric constant and therefore the velocity of wave propagation is unaffected by the collisions. The above reasoning considers, of course, that the collisions cause no periodic variation of electron density, which would effect the wave propagation.

Appendix 5 - Conditions Under Which the Second Order Terms in the Equations of Motion can be Neglected.

The equations of motion of the electrons in the space charge have been linearized in all cases by neglecting the second order term $(v_1 \cdot \nabla)\, v_1$. The complete equation of motion will be examined briefly for a typical case to determine the conditions under which this approximation is valid.

From Eqs. II-7, the equations of motion of the electrons in a plane magnetron space charge under the influence of a wave propagating in the y direction are:

$$i\omega\, v_x - \gamma\, v_x\, v_y = -\frac{e}{m}\, E_x$$

$$i\omega\, v_x - \gamma\, v_y^2 = -\frac{e}{m}\, E_x + \omega_c\, v_x \quad .$$

From the first of these, the x directed velocity is given by:

$$v_x = \frac{-\,e/m\ E_x}{i\omega - \gamma\, v_y}$$

so that:

$$\gamma^2\, v_y^3 + v_y\, \gamma \frac{e}{m} E_y - v_y\, \omega^2 + \frac{e}{m}\,(i\omega\, E_y + \omega_c\, E_x) = 0 \quad .$$

It is seen that the last two terms of this equation represent the y directed velocity relation in the linear approximation. Therefore the remaining two terms represent the non-linear effect and the condition under which these can be neglected will be found as follows. If it is arbitrarily specified that the magnitude of the y directed velocity derived from the linear equation be changed not more than two percent by the inclusion of the non-linear terms, the following relations are valid:

$$(\gamma^2\, v_y^2 + \gamma \frac{e}{m}\, E_y)\, v_y < .02\ \ v_y\, \omega^2$$

This condition will be met if:

$$\gamma^2 v_y^2 < .01\,\omega^2$$

and

$$\gamma \frac{e}{m} E_y < .01\,\omega^2$$

The first of these conditions becomes:

$$\frac{v_y^2}{c^2} < \frac{.01}{\eta^2}$$

where η is the index of refraction of the space charge. This relation is merely a slightly more stringent imposition of the non-relativistic assumption and will be satisfied whenever $v_y^2/c^2 \ll 1$.

The second of these conditions becomes:

$$E < \frac{.01\,\omega c m}{\eta e} =$$

which assumes the value (for $\omega = 6\pi \times 10^9/\text{sec}$, $\eta = 1$):

$$E < 3 \times 10^5 \ \frac{\text{volts}}{\text{m}}$$

which would be satisfied under almost all conditions except for the fields in an oscillating high power magnetron.

Appendix 6 - Calculation of the Shift, Due to Space Charge, of the Resonant Wavelength of a Coaxial Cavity.

The change in resonant frequency due to the insertion of a dielectric in the coaxial cavity shown in Fig. 6.1 is calculated in this section using perturbation methods. The results of this computation can be compared with the experimental data shown in Fig. 6.4.

Following a procedure similar to that used by Bethe and Schwinger[1], let the fields in the unperturbed cavity ($\varepsilon_r = 1$) be represented by the subscript 1 and the fields in the cavity with dielectric be denoted by subscript 2. Then

$$\nabla \times H_1 = i\omega_1 \varepsilon_0 E_1 \qquad \nabla \times E_1 = -i\omega_1 \mu_0 H_1$$

$$\nabla \times H_2 = i\omega_2 \varepsilon_2 \varepsilon_0 E_2 \qquad \nabla \times E_2 = -i\omega_2 \mu_0 H_2 \qquad \text{A6-1}$$

By suitable multiplication and subtraction these become, after integration over the volume of the cavity:

$$i\int_{V_0}\left[(\omega_2\varepsilon_2 - \omega_1)\varepsilon_0 E_1\cdot E_2^* + \mu_0(\omega_2-\omega_1)H_1\cdot H_2^*\right] dV =$$

$$\int_{V_0}\left[E_1\cdot\nabla\times H_2^* - E_2^*\cdot\nabla\times H_1 - H_1\cdot\nabla\times E_2^* + H_2^*\cdot\nabla\times E_1\right] dV \qquad \text{A6-2}$$

The integral on the right becomes, by a well known vector relation

$$= \int_{S_0}\left[E_1 \times H_2^* + E_2^* \times H_1\right] dS \qquad \text{A6-3}$$

---

1 Bethe and Schwinger - "Perturbation Theory for Cavities" - NDRC Report D1-117 PB 18340, March 4, 1943.

where S_c is the boundary wall of the cavity. For a perfectly conducting wall this surface integral vanishes so that the left side above becomes:

$$\int_{V_c}\left[\varepsilon_0(\omega_2\ \varepsilon_2-\omega_1)\ E_1\cdot E_2^{*} - \mu_0(\omega_2-\omega_1)\ H_1\cdot H_2^{*}\right] dV = 0 \qquad \text{A6-4}$$

which can be written

$$(\omega_2-\omega_1)\int_{V_c}\left[\varepsilon_0 E_1\cdot E_2^{*} + \mu_0 H_1\cdot H_2^{*}\right] dV = -(\varepsilon_2-1)\omega_2\varepsilon_0\int E_1\cdot E_2^{*}\ dV \qquad \text{A6-5}$$

If the volume of the inserted dielectric is small compared with the total volume, its presence will cause only small perturbation of the fields so that since the electric displacement ($D = \varepsilon_r\ \varepsilon_0 E$) is continuous across the boundary of the dielectric,[1] in the above expression set $\varepsilon_2\ E_2 = E_1$ so that:

$$\frac{\omega_2-\omega_1}{\omega_2} = \frac{\delta\omega}{\omega_0} = \frac{-\dfrac{(\varepsilon_2 - 1)}{\varepsilon_2}\ \varepsilon_0\displaystyle\int_{V_D}|E_2|^2\ dV}{2\ W} \qquad \text{A6-6}$$

where $W = \frac{1}{2}\int\left[\varepsilon_0\ |E_1|^2 + \mu_0\ |H_1|^2\right] dV$ is the total energy[2] stored in the cavity. The integral in the numerator is taken over the volume of the dielectric.

In this case, using the notation of Fig. A6-1

$$E_r = \frac{I\eta_0}{2\pi r}\ \text{Cos}\ \frac{\pi z}{2(\ell_1+\ell_2)} \qquad \text{where} \qquad \eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

---

1 The cavity is assumed to be in resonance in the TEM mode so that only a radial electric field component exists in the cavity.

2 In the calculation of the stored energy it is assumed that the presence of the dielectric does not materially affect the magnitude of the stored energy, so that in this expression the equality $E1 = E_2$ is made.

$$W = \varepsilon_0 \int_{V_0} |E_r|^2 \, dV = 2\varepsilon_0 \left(\frac{I\eta_0}{2\pi}\right)^2 \left[\int_{r_0}^{r_a} \int_0^{\ell_1} \frac{1}{r^2} \cos^2 \frac{\pi z}{2(\ell_1+\ell_2)} \, 2\pi r \, dr \, dz \; + \right.$$

$$\left. \int_{r_0}^{r_d} \int_{\ell_1}^{\ell_2} \frac{1}{r^2} \cos^2 \frac{\pi z}{2(\ell_1+\ell_2)} \, 2\pi r \, dr \, dz \right] \qquad \text{A6-7}$$

Using the dimensions for this 10 cm cavity

$$W = \left(\frac{I\eta_0}{2\pi}\right)^2 17.1 \, \varepsilon_0 \qquad \text{A6-8}$$

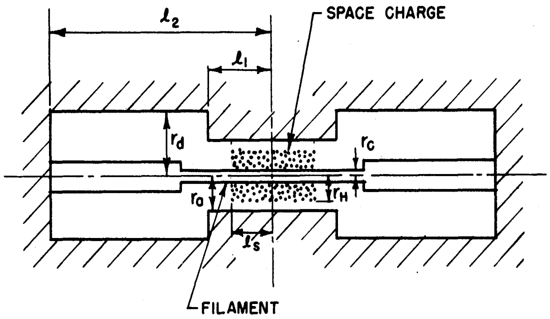


FIG. A6-1
SKETCH SHOWING DIMENSIONS USED IN CALCULATIONS ON COAXIAL CAVITY

For the volume occupied by the dielectric:

$$\int_{V_D} |E_r|^2 \, dV = 2 \left(\frac{I\eta_0}{2\pi}\right)^2 \int_{r_c}^{r_H} \int_0^{\ell_s} \frac{1}{r^2} \cos^2 \frac{\pi z}{2(\ell_1+\ell_2)} \, 2\pi r \, dr \, dz$$

$$= \left(\frac{I\eta_0}{2\pi}\right)^2 \varepsilon_0 \; 1.43 \qquad \text{A6-9}$$

so $$\frac{\delta\omega}{\omega_o} = - \frac{\varepsilon_2 - 1}{\varepsilon_2} \frac{1.43}{2 \times 17.1} = - .042 \frac{\varepsilon_2 - 1}{\varepsilon_2} \qquad \text{A6-10}$$

for example if $\varepsilon_2 = 6$ $(\omega/\omega_2 \simeq 0.6)$ $\delta\lambda = .030$ $\lambda_o = 0.33$ cm

As mentioned before, the magnetron space charge is considered, in certain regions of ω/ω_c, to exhibit properties of a dielectric. From this consideration it would be expected that the resonant wavelength shift of the cavity be the same for the space charge as for a dielectric of the same volume and value of dielectric constant as the space charge.

As seen from Fig. 6.4 the maximum shift obtained experimentally was .08 cm which is less by a factor four than the computed value. This discrepancy is probably due at least in part to improper formation of the space charge cloud, i.e. a cloud which does not approximate sufficiently closely that considered in the analysis, in particular the actual space charge cloud is probably shorter than considered in this computation.

Appendix 7 - Calculation of the Shift in Resonant Wavelength of a TE_{011} Cavity, Due to the Presence of Space Charge.

The change in resonant wavelength of the cylindrical cavity, shown in Fig. 6.7, is calculated in this section for a dielectric inserted along the axis. This calculation is made using the perturbation method outlined in Appendix 6.

The electric field varies as

$$E = 2A\ J_1\left(\frac{r}{a}\ r'_{01}\right)\ \sin\frac{\pi z}{\ell}\ \omega\mu_o\ \frac{a}{r'_{01}} \qquad \text{A7-1}$$

so that the energy stored in the cavity can be found by straightforward methods[1] to be

$$W = A^2\mu_o\ \frac{\omega^2}{c^2}\left(\frac{a}{r'_{01}}\right)^2\ \pi\ell\ a^2\ J_o^2\ (r'_{01}) \qquad \text{A7-2}$$

where a and ℓ are the radius and length of the cavity respectively, and r'_{01} is the first root of the equation $J_o'(r) = 0$. The integral of E_θ^2 in the region occupied by the dielectric is found to be:

$$\varepsilon_o\int_{V_d}|E_\theta|^2\ dV = 8\pi A^2\mu_o\left(\frac{\omega}{c}\right)^2\left(\frac{a}{r'_{01}}\right)^2\frac{\ell}{\pi}\left[\frac{\pi}{2}\ \frac{\ell_s}{\ell} - \frac{1}{4}\sin\frac{2\pi\ell_s}{\ell}\right]\times \int_0^{r_H} r\ J_1^2\left(\frac{r}{a}\ r'_{01}\right)\ dr \qquad \text{A7-3}$$

where ℓ_s is the length and r_H the radius of the space charge, considered as a dielectric.

---

1 See for example: Sarbacher and Edson - "Hyper and Ultra-High Frequency Engineering" - John Wiley and Sons, page 383.

This integral can be evaluated as

$$\int_0^{r_H} r J_1^2\left(\frac{r}{a}\, r'_{01}\right) dr = \frac{r_H^2}{2}\left[J_1'\left(\frac{r_H}{a}\, r'_{01}\right)\right]^2 + \left[1 - \frac{1}{\frac{r_H}{a}\, r'_{01}}\right]\left[J_{01}\left(\frac{r_H}{a}\, r'_{01}\right)\right]^2 \qquad \text{A7-4}$$

With ℓ_s = 1.5 cm r_H = 0.5 cm a = 7.95 cm ℓ = 4.42 cm

λ_o = 10.5 cm; these relations give:

$$W = 45.5 \; A^2 \mu_o \pi \left(\frac{a}{r'_{01}}\right)^2 \left(\frac{2\pi}{\lambda_o}\right)^2 \qquad \text{A7-5}$$

$$\varepsilon_o \int_{V_D} \left|E_\theta\right|^2 dV = .075 \; A^2 \mu_o \pi \left(\frac{a}{r'_{01}}\right)^2 \left(\frac{2\pi}{\lambda_o}\right)^2$$

Since in this case the electric field is everywhere parallel to the surface of the dielectric, the electric field continuity across the boundary yields the relation:

$$\frac{\delta\omega}{\omega} = - \frac{(\varepsilon_r - 1)\,\varepsilon_o \int_{V_D} \left|E_\theta\right|^2 dV}{2\,W} \qquad \text{A7-6}$$

it is seen that

$$\frac{\delta\lambda}{\lambda} = .0157 \; \frac{\varepsilon_r - 1}{\varepsilon_r} \qquad \text{A7-7}$$

So that for ε_r = 8 $\Delta\lambda/\lambda$ = 1.4%

For the values of ω/ω_c such as to yield values of $\varepsilon_{eff} > 1$, this calculation of $\Delta\lambda/\lambda_o$ should be valid for the magnetron type space charge placed on the axis of the cavity.

BIBLIOGRAPHY

REFERENCES WHICH HAVE APPEARED IN THE TEXT

"Space Charge Effects and Frequency Characteristics of C-W Magnetrons", H. W. Welch, Jr., Univ. of Michigan Electron Tube Laboratory Technical Report No. 1, November 15, 1948.

"High Frequency Behavior of a Space Charge Rotating in a Magnetic Field", J. P. Blewett and S. Ramo, Phys. Rev., V57, pages 635-641, April 1940.

"Space Charge Frequency Dependence of a Magnetron Cavity", W. E. Lamb and M. Phillips, J. Appl. Phys., V18, pages 230-238, February, 1947.

"Effects of Space Charge on Frequency Characteristics of Magnetrons", H. W. Welch, Jr., Proc. I.R.E., page 1434, December, 1950.

"The Electrical Conductivity of an Ionized Gas", Cohen, Spitzer and Routley, Phys. Rev. 80, 2, October 15, 1950.

"Excess Energy Electrons and Electron Motion in High Vacuum Tubes", E. G. Linder, Proc. I.R.E. 26, page 346, 1938.

"Effect of High Energy Electron Random Motion upon the Shape of Magnetron Cutoff Curve", E. G. Linder, J. Appl. Phys. 9, page 331, 1938.

"The Theory of Electrons", H. A. Lorentz, B. G. Teubner, Leipzig, 1909, Chapter IV.

"The Refractive Index of an Ionized Medium" II, Chas. Darwin, Proc. Roy. Soc. London V182, page 152, 1944.

"The Effect of a Uniform Magnetic Field on the Motion of Electrons Between Coaxial Cylinders", A. W. Hull, Phys. Rev. V18, page 31, 1921.

"Space Charge in Plane Magnetron", Page and Adams, Phys. Rev. V69, page 492, 1946.

"On the Steady State and Noise Properties of Linear and Cylindrical Magnetrons", R. Q. Twiss, M.I.T. - Ph.D. Thesis, 1950.

"Electronic Theory of the Plane Magnetron", L. Brillouin, Columbia University, AMP Report 129.1R - OSRD 4510, to be published in part in the third volume of "Advances in Electronics".

"Theory of the Magnetron - I", L. Brillouin, Phys. Rev. V60, page 385, 1941.

"Space Charge in Cylindrical Magnetron", Page and Adams, Phys. Rev. V69 page 494, 1946.

"Elektronenbamen und Mechanismus der Schwingungserregung in Schlitzanodenmagnetron", H. G. Moeller, Hochfrequenztechnik und Elak., V47, page 115, 1936.

"The Passage of Steady Current in a Cylindrical Non-slit Magnetron", V. M. Glagolev, Zhur. Tekh. Fiz. USSR - 19, page 943, August 1949. Translated by Naval Research Laboratory, Washington, D. C. NRL Translation No. 318.

"Electronic Orbits in the Cylindrical Magnetron with Static Fields", Radiation Laboratory Special Report 9S, Section V, R.L. Report 122, October, 1941.

"The Influence of Space Charge on Electron Bunching", L. Brillouin, Phys. Rev. V70, page 187, August, 1946.

"Rotating Space Charge in a Magnetron with Solid Anode", I. I. Wasserman, J. Tech. Phys. (USSR), V18, page 785, 1949.

"Measurement of the Circulating Electron Current in a Magnetron", J. Moller, Hochfrequenztechnik und Elak., V47, page 141, July, 1936.

"Electromagnetic Waves, Schelkunoff, D. Van Nostrand and Co., page 406, 1943.

"The Electron Wave Tube - A Novel Method of Generation and Amplification of Microwave Energy", A. V. Haeff, Proc. I.R.E. 37, pages 4-10, 1949.

"High Frequency Amplification by Means of the Interaction Effect between Electron Streams", J. Labus, Arch. Elekt. Ubertragung 4, pages 353-360,

"Wave Propagation in a Slipping Stream of Electrons: Small Amplitude Theory", G. G. Macfarlane and H. G. Hay, Proc. Phys. Soc., Lond. B, LXIII, pages 409-427, 1950.

"Generation and Amplification of Waves in Dense Charged Beams under Crossed Fields", O. Buneman, Nature, V165, page 474, March, 1950.

"Intermediate Differential Equations", E. D. Rainville, John Wiley and Sons, page 90, 1943.

"A Course of Modern Analysis", Whittaker and Watson, Cambridge, page 201, 1946.

"On the Origin of Harmonics in the Ionosphere - At Points where the Dielectric Constant is Zero", K. Forsterling and H. O. Wuster, Comptes Rendus - Acadamie des Sciences - 231 - No. 17, page 831, October 23, 1950.

"On the Propagation of Electromagnetic Waves Through an Atmosphere Containing Free Electrons", H. W. Nichols, and J. C. Schelleng, Bell Syst. Tech. J. - 4, page 215, 1925.

REFERENCES WHICH HAVE APPEARED IN THE TEXT (Cont'd)

"Group and Phase Velocities from the Magneto-Ionic Theory", H. A. Wale and J. P. Stanley, J. Atmos. Terr. Phys. VI - No. 2, page 82, 1950.

"Electromagnetic Theory", J. A. Stratton, McGraw-Hill, page 137, 1941.

"Electrodynamics", Page and Adams, Van Nostrand, page 328, 1940.

"Cosmical Electrodynamics", H. Alfven, Oxford, page 35, 1950.

"Small Signal Theory of Velocity Modulated Electron Beams, W. C. Hahn, G. E. Rev. 42, No. 6, page 258, June, 1939.

"Electron Optical Exploration of Space Charge in a Cutoff Magnetron", D. L. Reverdin, J. Appl. Phys. 22, page 257, March, 1951.

"Microwave Magnetrons", F. F. Rieke, Radiation Laboratory Series No. 6, McGraw-Hill, page 390, 1948.

"The Mathematical Theory of Non-Uniform Gases", - Chapman and Cowling, Cambridge, 1939, Chapters 3 and 18.

"The Effect of Hydrostatic Pressure in an Electron Beam on the Operation of Travelling Wave Devices", P. Parzen, J. Appl. Phys., April, 19 1.

"Plasma Oscillations in a Static Magnetic Field", E. P. Gross, Phys. Rev. V82, April 15, 1951.

"The Collisional Friction Experienced by Vibrating Electrons in Ionized Air", Appleton and Chapman, Proc. Phys. Soc. of London - XLIV, page 246 1932.

"Perturbation Theory for Cavities", Bethe and Schwinger, NDRD Report D1-117 PB 18340, March 4, 1943.

"Hyper and Ultra-High Frequency Engineering", Sarbacher and Edson, John Wiley and Sons, page 383.

ADDITIONAL GENERAL REFERENCES

"Effect of Space Charge in the Magnetron", E. B. Moullin, Camb. Phil. Soc. 36, page 94, 1940.

"Propagation of Electromagnetic Waves in an Atmosphere Containing Free Electrons", Huxley, Phil. Mag. 29, page 313, 1940.

"Propagation of Electromagnetic Waves in an Ionized Medium", Rydbeck, Phil. Mag. 30, page 282, 1940.

"Theory of the Ionosphere", Majumdar, Zeitschrift für Physik 107, page 599, 1937.

"Propagation of Electromagnetic Waves in an Ionized Atmosphere", Huxley, Phil. Mag. 25, page 148, 1938.

"Motions of Electrons in Magnetic Fields and A-C Electric Fields", Huxley, Phil. Mag. 23, page 442, 1937.

"Oscillations in Ionized Gases", Langmuir, Proc. Nat. Acad. Sci. 14, page 627, 1928.

"The Propagation of Electromagnetic Waves in a Refracting Medium in a Magnetic Field", D. R. Hartree, Proc. Camb. Phil. Soc. 27, 143, 1930.

"Wireless Studies of the Ionosphere", E. V. Appleton, Proc. Inst. Elec. Engineers 71, page 642, 1932.

"Collisions Between Electrons and Gas Molecules", Jones and Langmuir, Phys. Rev. 31, page 390, March 1928.

"Physics of the Ionosphere", Mimmo, Rev. Mod. Phys. 9, page 1, January, 1937.

"The Quantitative Study of the Collisions of Electrons with Atoms", Brode, Rev. Mod. Phys. 5, page 257, 1933.

"On the Effect of an External Electromagnetic Field on a Split-Anode Magnetron", S. Ya. Braude, Zh. Tekh. Fig. 13 7/8 431, 1943 (Russian).

"On the Vibrations of the Electronic Plasma", L. Landau, Journ. Phys. USSR 10, 25, 1946.

"Plasma -- Electron Oscillations", E. B. Armstrong, Nature (London) 160, 713, November 1947.

"Frequences de Resonance de la charge d'espace d'un Magnetron", P. Fechner, C. R. Academie des Sciences, page 270, July 24, 1950.

"Mesure des frequences de resonance de la charge d'espace d'un Magnetron a' cavities", P. Fechner, C. R. Academie des Sciences, page 270, July 24, 1950.

ADDITIONAL GENERAL REFERENCES (Cont'd.)

"Uber die Eigenschwingung freier Elektronen in einem Konstanten Magnetfeld" S. Benner - Die Naturwis, 17, 1929, page 120.

"Oscillations in a Plane One-Anode Magnetron" L. Brillouin, OSRD Report No. 5173, AMP Columbia Univ., No. 129.3R, May 1945.

"On the Properties of Tubes in a Constant Magnetic Field" I - O. Doehler, Ann. de Radioelec. - T111, No. 11, January 1948.

"Electro-Magneto - Ionic Optics" - V. A. Bailey, Jour. Roy. Soc. N.S.W. 1948, pages 107-113, V72.

"Group and Phase Velocities from the Magneto-Ionic Theory" - H. A. Whale and J. P. Stanley, Jour. Atmos. Terr. Phys. VI - No. 2, 1950, page 82.

"Space Charge and Field Waves in an Electron Beam", S. Ramo, Phys. Rev. V56, 1939, page 276.

"Energy Conversion in Electronic Devices", D. Gabor, J. Instn Elect. Engrs., Vol. 91, Part III, page 128, 1944.

SYMBOLS USED IN THE TEXT

α	=	attenuation constant of propagating wave
B_o	=	applied magnetic flux density
β	=	phase constant of propagating wave
γ	$= \alpha + j\beta$	propagation constant of propagating wave
c	=	velocity of light (except in Appendix 1)
δ	=	instantaneous magnitude of space charge surface perturbation
D	$= \varepsilon E$	electric displacement vector
E	=	electric field intensity vector
ε_e	$= Re\eta^2$	effective dielectric constant of space charge
ε_o	$= \frac{10^{-9}}{36\pi} \frac{\text{farads}}{\text{m}}$	dielectric constant of free space
ε_r	$= \frac{\varepsilon}{\varepsilon_o}$	relative dielectric constant of a dielectric
e	=	electronic charge
η	$= c/v_p$	index of refraction of space charge
f	=	electron velocity distribution function
g	$= \frac{1}{\tau}$	electron damping coefficient
h	=	anode cathode distance in plane magnetron
Θ	=	angular coordinate in cylindrical magnetron
$\vec{H}$	=	magnetic intensity of propagating wave
κ	=	coefficient of thermal conductivity
ℓ	=	unit of length, also used as defined by Eq. III-27
λ	=	wavelength
λ_o	=	resonant wavelength
μ_o	$= 4\pi \times 10^{-7} \frac{\text{hy}}{\text{m}}$	permeability of free space

SYMBOLS USED IN THE TEXT (Cont'd)

n =	electron number density
n_o =	time invariant electron number density
ξ =	steady electron angular velocity in cylindrical magnetron
r_a =	anode radius of cylindrical magnetron
r_o =	cathode radius of cylindrical magnetron
r_H =	radius of outer edge of cylindrical space charge
ρ =	total electronic charge density
ρ_o =	time invariant electronic charge density
ρ_1 =	a-c component of electronic charge density
σ_e =	effective conductivity of space charge
τ =	mean time between electron-ion collisions
v =	total electron velocity
v_o =	steady (drift) electron motion
v_1 =	a-c component of electron velocity
v_p =	phase velocity of propagating wave
ω =	radian frequency of propagating wave
$\omega_c = \frac{eB_o}{m}$	cyclotron angular velocity
Y_e =	electronic admittance of space charge

ERRATA TO

THE PROPAGATION OF ELECTROMAGNETIC WAVES

IN A

MAGNETRON-TYPE SPACE CHARGE

P. 6 The equation should be written:

$$\frac{\partial f}{\partial t} + c\,\frac{\partial f}{\partial r} + \frac{\vec{F}}{m}\,\frac{\partial f}{\partial c} = \left(\frac{\partial f}{\partial t}\right)_{coll.}$$

P. 7 Eq II-1 Right side should be:

$$= -\frac{e}{m}\left[\vec{E} + \vec{v} \times \vec{B}_o\right] - \frac{1}{nm}\nabla p$$

P. 9 Eq II-2 Right side should be:

$$= -\frac{e}{m}\left[\vec{E} + \vec{v} \times \vec{B}_o\right]$$

P. 9 Equation in center of page should be:

$$\frac{m}{e} F = E + \frac{P}{3\mathcal{E}_o}$$

P. 9 Replace N by n (electronic number density)

P.10 Equation at top of page should be:

$$\frac{e}{m} F = E + \frac{P}{3\mathcal{E}_o}$$

P.10 Eq II-3 Right side should be:

$$= -\frac{e}{m}\left[\vec{E} + \vec{v} \times \vec{B}_o\right]$$

P.11 Fifth term should be: $(\vec{v}_1 \cdot \nabla)\vec{v}_1$

P.15 The dotted curve, referred to in text, which is missing from Fig. 2.3, should be the same as the lower curve except that it will show a peak near $y/h = 1$ and will decrease approximately exponentially for y/h 1.

P.18 The last two sentences on this page should be deleted. This result obtained by Glagolev is at variance with other works of a similar nature and leads to a space-charge density distribution considerably different from the Hull-Brillouin value for small filament. It is believed that there must be an error in his calculation.

ERRATA (Cont'd)

P.37 Fig. 3.5 The lower curve should be deleted for $r_H/r_c \gtrsim 2$.

P.44 Fig. 3.7 The lower three curves should be deleted for $r_H/r_c \gtrsim 2$.

P.99 The scale marked 2" should be changed to 1".

DISTRIBUTION LIST

22 copies - Director, Evans Signal Laboratory
Belmar, New Jersey
FOR - Chief, Thermionics Branch

12 copies - Chief, Bureau of Ships
Navy Department
Washington 25 D. C.
ATTENTION: Code 930A

12 copies - Director, Air Materiel Command
Wright Field
Dayton, Ohio
ATTENTION: Electron Tube Section

4 copies - Chief, Engineering and Technical Service
Office of the Chief Signal Officer
Washington 25, D. C.

2 copies - H. W. Welch, Jr., Research Physicist
Electron Tube Laboratory
Engineering Research Institute
University of Michigan
Ann Arbor, Michigan

1 copy - Engineering Research Institute File
University of Michigan
Ann Arbor, Michigan

W. E. Quinsey, Assistant to the Director
Engineering Research Institute
University of Michigan
Ann Arbor, Michigan

W. G. Dow, Professor
Department of Electrical Engineering
University of Michigan
Ann Arbor, Michigan

Gunnar Hok, Research Engineer
Engineering Research Institute
University of Michigan
Ann Arbor, Michigan

J. R. Black, Research Engineer
Engineering Research Institute
University of Michigan
Ann Arbor, Michigan

G. R. Brewer, Research Associate
Engineering Research Institute
University of Michigan
Ann Arbor, Michigan

J. S. Needle, Instructor
Department of Electrical Engineering
University of Michigan
Ann Arbor, Michigan

Department of Electrical Engineering
University of Minnesota
Minneapolis, Minnesota
ATTENTION: Professor W. G. Shepherd

Westinghouse Engineering Laboratories
Bloomfield, New Jersey
ATTENTION: Dr. J. H. Findlay

Columbia Radiation Laboratory
Columbia University
Department of Physics
New York 27, New York

Electron Tube Laboratory
Department of Electrical Engineering
University of Illinois
Urbana, Illinois

Department of Electrical Engineering
Stanford University
Stanford, California
ATTENTION: Dr. Karl Spangenberg

National Bureau of Standards Library
Room 203, Northwest Building
Washington 25, D. C.

Radio Corporation of America
RCA Laboratories Division
Princeton, New Jersey
ATTENTION: Mr. J. S. Donal, Jr.

Department of Electrical Engineering
The Pennsylvania State College
State College, Pennsylvania
ATTENTION: Professor A. H. Waynick

Document Office - Room 20B-221
Research Laboratory of Electronics
Massachusetts Institute of Technology
Cambridge 39, Massachusetts
ATTENTION: John H. Hewitt

Bell Telephone Laboratories
Murray Hill, New Jersey
ATTENTION: S. Millman

Radio Corporation of America
RCA Victor Division
415 South 5th Street
Harrison, New Jersey
Building 55
ATTENTION: Hans K. Jenny

Magnetron Development Laboratory
Power Tube Division
Raytheon Manufacturing Company
Waltham 54, Massachusetts
ATTENTION: Edward C. Dench

Vacuum Tube Department
Federal Telecommunication Laboratories, Inc.
500 Washinton Avenue
Nutley 10, New Jersey
ATTENTION: A. L. Wing, Jr.

Microwave Research Laboratory
University of California
Berkeley, California
ATTENTION: Professor L. C. Marshall

General Electric Research Laboratory
Schenectady, New York
ATTENTION: P. H. Peters

Cruft Laboratory
Harvard University
Cambridge, Massachusetts
ATTENTION: Professor E. L. Chaffee

Research Laboratory of Electronics
Massachusetts Institute of Technology
Cambridge, Massachusetts
ATTENTION: Professor S. T. Martin

Collins Radio Company
Cedar Rapids, Iowa
ATTENTION: Robert M. Mitchell

Department of Electrical Engineering
University of Kentucky
Lexington, Kentucky
ATTENTION: Professor H. Alexander Romanowitz

Department of Electrical Engineering
Yale University
New Haven, Connecticut
ATTENTION: Dr. H. J. Reich

Department of Physics
Cornell University
Ithaca, New York
ATTENTION: Dr. L. P. Smith

Mrs. Marjorie L. Cox, Librarian
G-16, Littauer Center
Harvard University
Cambridge 38, Massachusetts

Mr. R. E. Harrell, Librarian
West Engineering Library
University of Michigan
Ann Arbor, Michigan

Mr. C. L. Cuccia
RCA Laboratories Division
Radio Corporation of America
Princeton, New Jersey

Dr. O. S. Duffendack, Director
Phillips Laboratories, Inc.
Irvington-on-Hudson, New York

Air Force Cambridge Research Laboratories
Library of Geophysics Directorate
230 Albany Street
Cambridge, Massachusetts
ATTENTION: Dr. E. W. Beth

Raytheon Manufacturing Company
Research Division
Waltham 54, Massachusetts
ATTENTION: W. M. Gottschalk

General Electric Research Laboratory
Schenectady, New York
ATTENTION: Dr. A. W. Hull

Missile and Radar Division
Raytheon Manufacturing Company
Waltham 54, Massachusetts
ATTENTION: Mr. James D. LeVan

Air Force Cambridge Research Laboratories
Library of **Radiophysics Directorate**
230 Albany Street
Cambridge, Massachusetts

Sperry Gyroscope Company
Library Division
Great Neck, Long Island, New York

2 copies - Mr. John Keto
Director, Aircraft Radiation Laboratory
Air Materiel Command
Wright Field
Dayton, Ohio