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MOMENTS ABOUT THE AXES OF A SIMPLIFIED FLIGHT TABLE GIMBAL STRUCTURE

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ABSTRACT

The moments about the gimbal axes of a simplified flight table gimbal structure are developed in terms of the Euler angles and their derivatives. From the equations developed, it can be determined how much torque must be produced by the motors turning the flight table gimbal structure about its axes.

The flight table gimbal structure is a physical device to simulate the angular motion of a craft in flight as it maneuvers. The flight table gimbal structure consists of a platform mounted in a gimbaling arrangement to allow complete angular motion about one point which remains fixed. Figure 1 is a schematic drawing of such a gimbaling arrangement.

Referring to figure 1, the outer gimbal axis is fixed in space insofar as the operation of the flight table system is concerned. The middle gimbal axis will always move in the same fixed plane, called the reference plane, perpendicular to the outer gimbal axis. If the coordinates, x_f, y_f, and z_f are fixed with respect to the flight table platform, with z_f along the inner gimbal axis, then the middle gimbal axis will lie along the line of intersection of the x_f - y_f plane and the fixed reference plane.

Rotations about each of the three gimbal axes will generate three independent angles, A_L, A_N, and A_P, where:

A_L = the longitudinal angle, the angle between the x_f-axis and the middle gimbal axis;

A_N = the nodal angle, the angle between the z_f-axis and the outer gimbal axis (Z_o);

A_P = the polar angle, the angle between a line fixed in the reference plane and the middle gimbal axis.

These three angles are the Euler angles (Fig. 3) and the three axes about which they may be generated are called the Euler axes.

In order to simulate the angular motion of a craft, three motors are used to cause appropriate turning of the gimbals about their axes. The motor which turns the inner gimbal is mounted so that the reaction is between the middle gimbal and the flight table. The motor which turns the middle gimbal reacts between the outer and middle gimbals; and the outer gimbal is turned by a motor reacting between the outer gimbal and inertial space. It is desired to find the moment about each of these axes in terms of the Euler angles, angular velocities, and angular accelerations.

FLIGHT SIMULATOR GIMBAL ARRANGEMENT

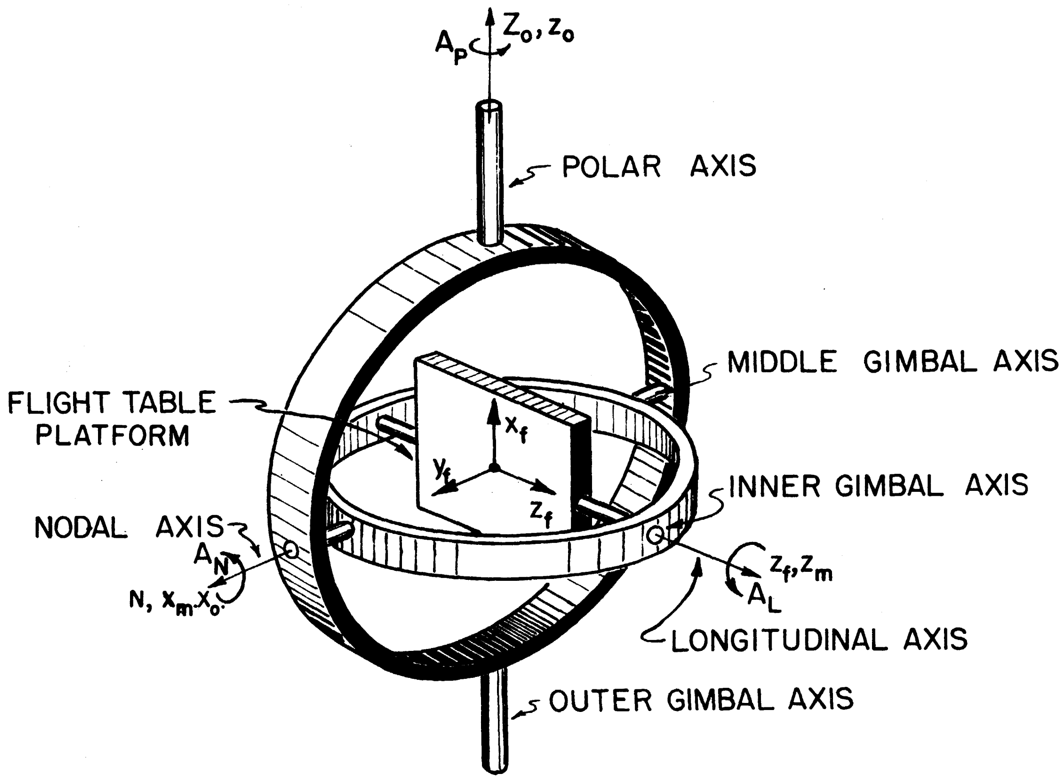

FIG. 1

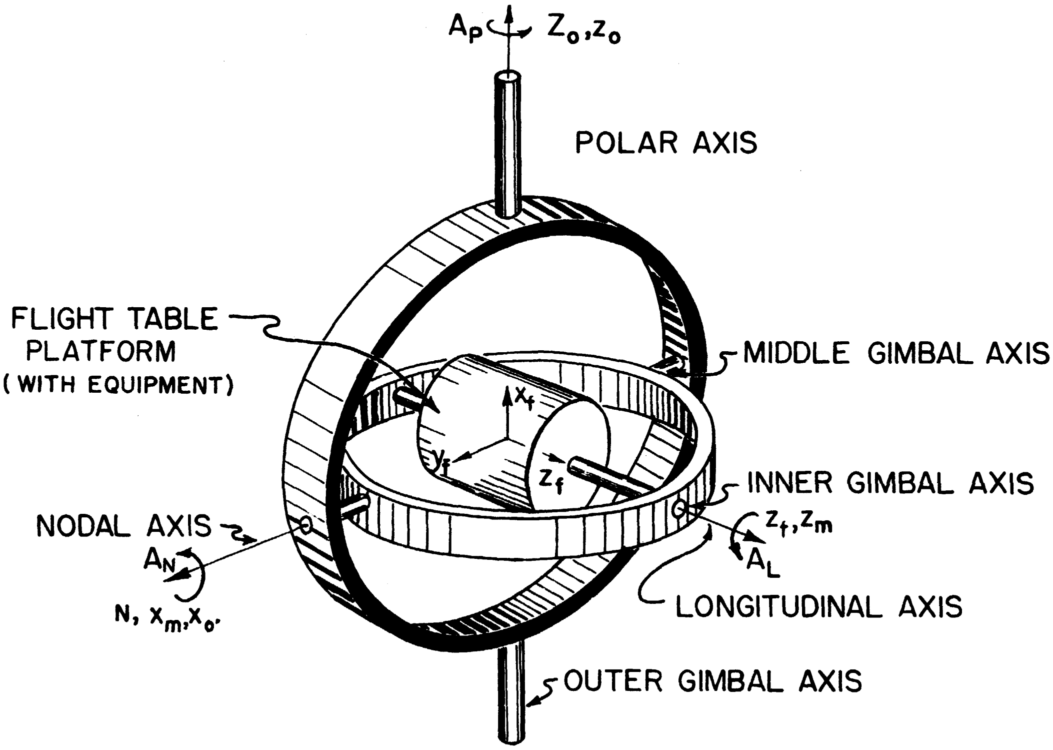

FIG. 2

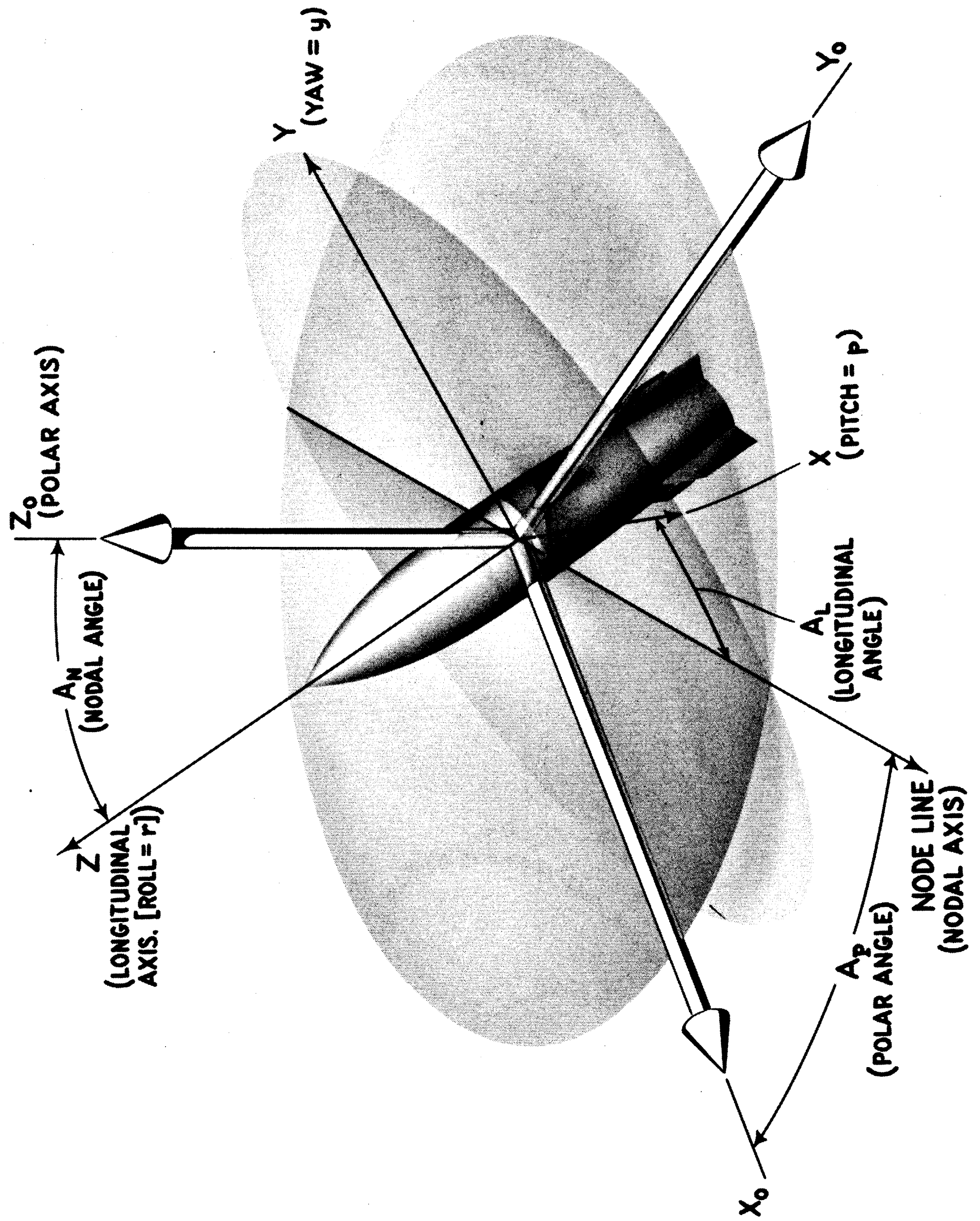

Fig. 3 —GEOMETRIC REPRESENTATION OF MISSILE IN INERTIAL SPACE

The following right-hand orthogonal coordinate systems are used:

Fixed: X_o, Y_o, Z_o with the Z_o-axis along the outer gimbal axis.

Flight Table: x_f, y_f, z_f with the z_f-axis along the inner gimbal axis.

Middle Gimbal: x_m, y_m, z_m with x_m along the middle gimbal axis and z_m along z_f.

Outer Gimbal: x_o, y_o, z_o with x_o along the middle gimbal axis and z_o along Z_o.

Unit triads, such as $\bar{e}_{xf}$, $\bar{e}_{yf}$, $\bar{e}_{zf}$, are placed along each of the coordinate axes systems.

Since the moment of a system is the time rate of change of the angular momentum of the system, the angular momentum of each of the components, inner gimbal, middle gimbal, and outer gimbal, may be found and the time rates of change of these angular moments will give the desired moment.

The coordinate system x_o, y_o, z_o contains two of the Euler axes, viz., x_o (N) and z_o (P); therefore, all vector quantities are referred to this coordinate system.

Angular momentum is defined as:

$$\bar{h} = h_{xo}\bar{e}_{xo} + h_{yo}\bar{e}_{yo} + h_{zo}\bar{e}_{zo} \tag{1}$$

$$\left.\begin{aligned} h_{xo} &= J_{xo}w_{xo} - P_{zo}w_{yo} - P_{yo}w_{zo} \\ h_{yo} &= - P_{zo}w_{xo} + J_{yo}w_{yo} - P_{xo}w_{zo} \\ h_{zo} &= - P_{yo}w_{xo} - P_{xo}w_{yo} + J_{zo}w_{zo} \end{aligned}\right) \tag{2}$$

where: J = moment of inertia

P = product of inertia

$P_{xo} = \sum_i m_i y_o z_o$

$P_{yo} = \sum_i m_i z_o x_o$

$$
\left.\begin{aligned}
P_{zo} &= \sum_i m_i x_o y_o \\
w &= \text{angular velocity}
\end{aligned}\right\} \qquad (3)
$$

Angular Momentum of Flight Table

In order to simplify the resulting equations, it is assumed in the following discussion that the moment of inertia of the inner gimbal about any line in the x_f-y_f plane is the same as the moment of inertia about any other line in the same plane. It will be noted that figure 1 does not indicate this fact. Figure 1 shows only the inner gimbal platform. By proper placing of the craft control system components on the platform, the above assumption can be approximated in actual practice. For purposes of discussion, the inner gimbal, including the platform and its equipment, could be considered as a cylinder whose polar axis is z_f (Fig. 2). Thus x_f, y_f, and z_f become the principal axes of inertia of the inner gimbal.

Since the y_o-z_o plane is a plane of symmetry for the flight table, the products of inertia, P_{fyo} and P_{fzo} are zero. Thus the angular momentum of the flight table becomes:

$$
\left.\begin{aligned}
h_{fxo} &= J_{fxo} w_{fxo} \\
h_{fyo} &= J_{fyo} w_{fyo} - P_{fxo} w_{fzo} \\
h_{fzo} &= J_{fzo} w_{fzo} - P_{fxo} w_{fyo}
\end{aligned}\right\} \quad \text{(cf. table of definitions)} \qquad (4)
$$

In terms of the moments of inertia about the flight table axes:

$$
\left.\begin{aligned}
J_{fxo} &= J_{xf} = J_{yf} \\
J_{fyo} &= J_{yf}\cos^2 A_N + J_{zf}\sin^2 A_N \\
J_{fzo} &= J_{yf}\sin^2 A_N + J_{zf}\cos^2 A_N
\end{aligned}\right\} \qquad (5)
$$

Because of the assumed symmetry of the flight table, the products of inertia with respect to x_f, y_f, and z_f, which would normally appear in the expressions for J_{fyo} and J_{fzo} in equations (5), are zero.

The flight table is considered to have an angular velocity about each of its Euler axes; viz., $\dot{A}_L$, $\dot{A}_N$, and $\dot{A}_P$.

Therefore:

$$\left.\begin{aligned} w_{fxo} &= \dot{A}_N \\ w_{fyo} &= -\dot{A}_L \sin A_N \\ w_{fzo} &= \dot{A}_P + \dot{A}_L \cos A_N \end{aligned}\right) \tag{6}$$

where the dot indicates the time derivative.

If the transformation between the x_o-y_o-z_o and the x_f-y_f-z_f coordinate systems is given by:

$$\left.\begin{array}{c|ccc} & \bar{e}_{xo} & \bar{e}_{yo} & \bar{e}_{zo} \\ \hline \bar{e}_{xf} & m_{11} & m_{12} & m_{13} \\ \bar{e}_{yf} & m_{21} & m_{22} & m_{23} \\ \bar{e}_{zf} & m_{31} & m_{32} & m_{33} \end{array}\right) \tag{7}$$

then the product of inertia, P_{fxo}, is given by:

$$-P_{fxo} = J_{xf} m_{12} m_{13} + J_{yf} m_{22} m_{23} + J_{zf} m_{32} m_{33} \quad \text{(Ref. 1, p. 123)} \tag{8}$$

Substituting the values for the m's from Table A into equation (8):

$$-P_{fxo} = (J_{xf} - J_{zf}) \sin A_N \cos A_N \tag{9}$$

Substituting equations (5), (6), and (9) into equations (4):

$$\left.\begin{aligned} h_{fxo} &= J_{xf} \dot{A}_N \\ h_{fyo} &= (J_{xf} - J_{zf}) \dot{A}_P \sin A_N \cos A_N - J_{zf} \dot{A}_L \sin A_N \\ h_{fzo} &= (J_{yf} \sin^2 A_N + J_{zf} \cos^2 A_N) \dot{A}_P + J_{zf} \dot{A}_L \cos A_N \end{aligned}\right) \tag{10}$$

Angular Momentum of Middle Gimbal

The y_o-z_o plane is a plane of symmetry for the middle gimbal. Therefore, equations (4) apply to the middle gimbal with suitable change of subscript. The angular velocity of the middle gimbal is:

$$\left.\begin{aligned} w_{mxo} &= \dot{A}_N \\ w_{myo} &= 0 \\ w_{mzo} &= \dot{A}_P \end{aligned}\right) \tag{11}$$

In terms of the moments of inertia of the middle gimbal about its coordinate axes:

$$\left.\begin{aligned} J_{mxo} &= J_{xm} \\ J_{myo} &= (J_{ym}\cos^2 A_N + J_{zm}\sin^2 A_N) \\ J_{mzo} &= (J_{ym}\sin^2 A_N + J_{zm}\cos^2 A_N) \\ -P_{mxo} &= (J_{ym} - J_{zm}) \sin A_N \cos A_N \end{aligned}\right) \tag{12}$$

Therefore:

$$\left.\begin{aligned} h_{mxo} &= J_{xm}\dot{A}_N \\ h_{myo} &= (J_{ym} - J_{zm})\, \dot{A}_P \sin A_N \cos A_N \\ h_{mzo} &= (J_{ym} \sin^2 A_N + J_{zm} \cos^2 A_N)\, \dot{A}_P \end{aligned}\right) \tag{13}$$

Angular Momentum of Outer Gimbal

The outer gimbal rotates about a fixed axis. Also it is symmetric to the x_o-y_o and y_o-z_o planes. Therefore, the outer gimbal has an angular momentum of:

$$\bar{h}_o = J_{zo}\, \dot{A}_P\, \bar{e}_{zo} \tag{14}$$

The total angular momentum of the flight simulator is:

$$\left.\begin{aligned} h_{xo} &= (J_{xf} + J_{xm})\, \dot{A}_N \\ h_{yo} &= (J_{yf} + J_{ym} - J_{zf} - J_{zm})\, \dot{A}_P \sin A_N \cos A_N - J_{zf}\dot{A}_L \sin A_N \\ h_{zo} &= [(J_{yf}+J_{ym})\sin^2 A_N+(J_{zf}+J_{zm})\cos^2 A_N+J_{zo}]\dot{A}_P+J_{zf}\dot{A}_L \cos A_N \end{aligned}\right) \tag{15}$$

Moments

The moment of the flight simulator system is given by:

$$\bar{M} = \dot{\bar{h}} = \frac{\delta \bar{h}}{\delta t} + \bar{W} \times \bar{h} \qquad \text{(Ref. 2, p. 345)} \tag{16}$$

Where:

$$\frac{\delta \bar{h}}{\delta t} = \dot{h}_{xo}\bar{e}_{xo} + \dot{h}_{yo}\bar{e}_{yo} + \dot{h}_{zo}\bar{e}_{zo} \tag{17}$$

$$\bar{W} = \dot{A}_P \bar{e}_{zo} \tag{18}$$

therefore:

$$\left.\begin{aligned}
M_{xo} &= (J_{xf} + J_{xm})\ddot{A}_N + J_{zf}\dot{A}_L\dot{A}_P \sin A_N \\
&\quad - (J_{yf} + J_{ym} - J_{zf} - J_{zm})\dot{A}_P^2 \sin A_N \cos A_N \\
M_{yo} &= (J_{yf} + J_{ym} - J_{zf} - J_{zm})\ddot{A}_P \sin A_N \cos A_N - J_{zf}\ddot{A}_L \sin A_N \\
&\quad + [(J_{yf} + J_{ym})\cos^2 A_N + (J_{zf} + J_{zm})\sin^2 A_N]\dot{A}_N\dot{A}_P \\
&\quad - [(J_{yf} + J_{ym})\sin^2 A_N + (J_{zf} + J_{zm})\cos^2 A_N]\dot{A}_N\dot{A}_P \\
&\quad + (J_{xf} + J_{xm})\dot{A}_N\dot{A}_P - J_{zf}\dot{A}_L\dot{A}_N \cos A_N \\
M_{zo} &= [(J_{yf} + J_{ym})\sin^2 A_N + (J_{zf} + J_{zm})\cos^2 A_N + J_{zo}]\ddot{A}_P \\
&\quad + 2(J_{yf} + J_{ym} - J_{zf} - J_{zm})\dot{A}_N\dot{A}_P \sin A_N \cos A_N \\
&\quad + J_{zf}\ddot{A}_L \cos A_N - J_{zf}\dot{A}_L\dot{A}_N \sin A_N
\end{aligned}\right) \tag{19}$$

If M_L, M_N, M_P are the moments along the flight table, middle gimbal, and outer gimbal axes, respectively, then:

$$\left.\begin{aligned}
M_N &= M_{xo} \\
M_P &= M_{zo} \\
M_L &= \bar{M}\cdot\bar{e}_L = \bar{M}\cdot(-\sin A_N\, \bar{e}_{yo} + \cos A_N\, \bar{e}_{zo})
\end{aligned}\right) \tag{20}$$

Therefore:

$$\left.\begin{aligned}
M_L &= J_{zf}\ddot{A}_L + (J_{zf} + J_{zm} + J_{zo})\,\ddot{A}_P \cos A_N \\
&\quad - (J_{xm} - J_{ym} + J_{zf} + J_{zm})\,\dot{A}_N\dot{A}_P \sin A_N \\
M_N &= (J_{xf} + J_{xm})\,\ddot{A}_N + J_{zf}\dot{A}_L\dot{A}_P \sin A_N \\
&\quad - (J_{yf} + J_{ym} - J_{zf} - J_{zm})\,\dot{A}_P^2 \sin A_N \cos A_N \\
M_P &= [(J_{yf} + J_{ym}) \sin^2 A_N + (J_{zf} + J_{zm})\cos^2 A_N + J_{zo}]\ddot{A}_P \\
&\quad + 2(J_{yf} + J_{ym} - J_{zf} - J_{zm})\dot{A}_N\dot{A}_P \sin A_N \cos A_N \\
&\quad + J_{zf}\ddot{A}_L \cos A_N - J_{zf}\dot{A}_L\dot{A}_N \sin A_N
\end{aligned}\right) \qquad (21)$$

Equations (21) give the total moments about the three Euler axes. However, the useful contribution of the motor turning the inner gimbal is that which changes the angular momentum of the inner gimbal. The inner gimbal motor will need to work only against the inertia of the inner gimbal. Therefore, the moment of inertia terms of M_L in equations (21) not containing the letter "f" in the subscript can be dropped to give:

$$M'_L = J_{zf}\ddot{A}_L + J_{zf}\ddot{A}_P \cos A_N - J_{zf}\dot{A}_N\dot{A}_P \sin A_N \qquad (22)$$

which gives the moment or torque the inner gimbal motor must produce.

The motor turning the flight simulator about the middle gimbal axis must not only turn the middle gimbal mass, but also the inner gimbal mass. Therefore, the moment equation will contain moments of inertia for the inner gimbal and also the middle gimbal. Examining the value of M_N in equations (21) it will be found that only the moments of inertia of the inner gimbal and middle gimbal appear; therefore,

$$M'_N = M_N \qquad (23)$$

Likewise:

$$M'_P = M_P \qquad (24)$$

The prime superscripts indicate the value the motors must actually produce to do the turning of the respective gimbals.

TABLE A

Transformation between the inner gimbal and middle gimbal coordinate systems and the outer gimbal coordinate system.

	$\bar{e}_{xo}$	$\bar{e}_{yo}$	$\bar{e}_{zo}$
$\bar{e}_{xf}$	$\cos A_L$	$\sin A_L \cos A_N$	$\sin A_L \sin A_N$
$\bar{e}_{yf}$	$-\sin A_L$	$\cos A_L \cos A_N$	$\cos A_L \sin A_N$
$\bar{e}_{zf}$	0	$-\sin A_N$	$\cos A_N$
$\bar{e}_{xm}$	1	0	0
$\bar{e}_{ym}$	0	$\cos A_N$	$\sin A_N$
$\bar{e}_{zm}$	0	$-\sin A_N$	$\cos A_N$

TABLE OF DEFINITIONS

A_L	= longitudinal angle
A_N	= nodal angle
A_P	= polar angle
$\dot{A}_L$	= angular velocity about L-axis
$\dot{A}_N$	= angular velocity about N-axis
$\dot{A}_P$	= angular velocity about P-axis
$\ddot{A}_L$, $\ddot{A}_N$, $\ddot{A}_P$	= Euler angular acceleration (not the angular acceleration along the Euler axes)
L	= longitudinal axis) Euler axes
N	= nodal axis) Euler axes
P	= polar axis) Euler axes
X_o, Y_o, Z_o	= coordinate axes fixed in space
x_f, y_f, z_f	= coordinate axes fixed in the flight table
x_m, y_m, z_m	= coordinate axes fixed in the middle gimbal
x_o, y_o, z_o	= coordinate axes fixed in the outer gimbal
$\bar{e}_{subscript}$	= unit vector along positive axis as indicated by the subscript
$\bar{h}$	= angular momentum (a vector quantity)

Table of Definitions (cont'd)

h_{xo}, h_{yo}, h_{zo}	= component of total angular momentum of flight table system on the indicated outer gimbal coordinate axis
h_{fxo}, h_{fyo}, h_{fzo}	= component of the angular momentum of the flight table only, on the indicated outer gimbal coordinate axis
h_{mxo}, h_{myo}, h_{mzo}	= component of the angular momentum of the middle gimbal only, on the indicated outer gimbal coordinate axis
J	= moment of inertia
J_{xf}	= moment of inertia of flight table about the x_f-axis
J_{yf}	= moment of inertia of flight table about the y_f-axis
J_{zf}	= moment of inertia of flight table about the z_f-axis
J_{fxo}, J_{fyo}, J_{fzo}	= moment of inertia of flight table only, about the indicated outer gimbal coordinate axis
J_{mxo}, J_{myo}, J_{mzo}	= moment of inertia of middle gimbal only, about the indicated outer gimbal coordinate axis
J_{zo}	= moment of inertia of outer gimbal only, about the z_o-axis

Table of Definitions (cont'd)

m_{ij} = direction cosine (i, j = 1, 2, 3)

$\bar{M}$ = moment (a vector quantity)

M_{xo}, M_{yo}, M_{zo} = component of moment on the indicated outer gimbal coordinate axis

M_L, M_N, M_P = moment along indicated Euler axis

P = product of inertia

P_{fxo} = product of inertia of flight table only, with respect to the x_o-y_o plane and the x_o-z_o plane

P_{mxo} = product of inertia of middle gimbal only, with respect to the x_o-y_o plane and the x_o-z_o plane

$\bar{w}$ = angular velocity (a vector quantity)

w_{fxo}, w_{fyo}, w_{fzo} = angular velocity of flight table only, about the indicated outer gimbal coordinate axis

w_{mxo}, w_{myo}, w_{mzo} = angular velocity of middle gimbal only, about the indicated outer gimbal coordinate axis

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