COMPARISON OF SEQUENTIAL AND CONTINUOUS INSPECTION STRATEGIES FOR DETERIORATING SYSTEMS

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Comparison of Sequential and Continuous Inspection Strategies for Deteriorating Systems

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Abstract

This paper investigates inspection strategies for a finite state continuous-time Markovian deteriorating system. Two inspection strategies are considered : sequential inspection strategy and continuous inspection strategy. Unlike many previous efforts, the inspection times for the sequential inspection strategy are assumed to be non-negligible. The replacement times and costs for both strategies are non-negligible and state dependent. Our objective here is to minimize the expected long run cost rate. Iterative algorithms are provided to derive the optimal policies for both strategies. The structures of these optimal policies and their corresponding optimal cost rates are discussed and compared.

CONTINUOUS-TIME MARKOVIAN MODEL; EXPECTED LONG RUN COST RATE; SEQUENTIAL INSPECTION STRATEGY; PERIODIC INSPECTION STRATEGY; CONTINUOUS INSPECTION STRATEGY; CONTROL LIMIT TYPE POLICIES

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1 Introduction

Inspection and replacement models for deteriorating systems have been widely studied in the literature. The papers by Barlow and Proschan [1], McCall [3], Pierskalla and Voelker [7], and Valdez-Flores and Feldman [11] are excellent reviews of the models investigated in this area. Besides these surveys, bibliographies for inspection and replacement models are given in Sherif and Smith [10], and Osaki and Nakagawa [6].

In this paper, we consider a finite state continuous-time Markovian model with one absorbing state for deteriorating systems. The absorbing state represents the failure state of the system. The intermediate states reflect the different degrees of deterioration. A similar model for deteriorating systems has been considered before by Luss [2] except he assumed that the transition rates to higher states are all identical, the inspection and replacement are instantaneous, and the costs are state independent. Sengupta [9] considered the same model but he allowed the replacement cost to be an increasing function of the deteriorating states. The model and the sequential inspection strategy considered in this paper are closely related to that studied by Ohnishi et al. [5] except we assume the inspection and replacement times are both non-negligible. Furthermore, the replacement times are also state dependent. Note that the model and the optimal inspection and replacement policy investigated by Mine and Kawai [4] are special cases of our model.

For the sequential inspection strategy, we provide an iterative algorithm to derive the optimal policy which minimizes the expected long run cost rate. We also show that under reasonable assumptions on the time and cost parameters, a control limit type rule holds for replacement, and the optimal time interval between successive inspections is a decreasing function of the state. That is, as the system deteriorates, we would like to inspect more frequently.

In this paper, we also consider a continuous inspection strategy for deteriorating systems. The system is continuously monitored and the current state of the system is always known with certainty. Again, we derive the optimal policy which minimizes the expected long run cost rate and we show that the optimal policy can be computed efficiently under reasonable assumptions on the time and cost parameters.

With the advance in automation and control technology, the sequential and continuous inspection strategies have become widely available. Obviously, there is a cost tradeoff between the two approaches. Usually, continuous monitoring involves a substantial amount of initial investment and the system may be replaced more frequently which may result in a higher total replacement cost. However, the total operating cost of the system can be reduced under continuous monitoring since there is no delay in detecting the system in undesirable states. It is therefore unclear which of the two strategies is better in minimizing the expected long run cost rate. Comparison of the continuous and periodic inspection policies has been considered before by Rosenblatt and Lee [8]. However, they assumed that the system is in one of two states - in control or out of control. In this paper, we present a comparative study of the sequential and continuous inspection strategies for a multi-state model.

This paper is organized as follows. In section 2, we describe a finite state continuous-time Markovian model for deteriorating systems. The sequential and continuous inspection strategies are introduced and they are discussed in detail in Sections 3 and 4, respectively. Finally, in Section 5, we compare the optimal policies of the sequential and continuous inspection strategies.

2 System Description

Consider a system whose deterioration at any point of time can be classified into one of a finite number of states $0, 1, \cdots, n, n+1$. State 0 represents the system before any deterioration takes place. State $n+1$ represents the terminal state (failure) of the deteriorating process. The intermediate states $1, 2, \cdots, n$ are ordered to reflect their relative degree of deterioration (in ascending order). The transition from one state to another is assumed to follow a continuous-time Markovian process with an absorbing state $n+1$. From state i, a direct transition can occur either to state $i+1$ with a transition rate β_i ($\beta_n = 0$) or to state $n+1$ with a transition rate α_i. The total transition rate out of state i is therefore $\lambda_i = \alpha_i + \beta_i$.

Under a sequential inspection strategy, inspection is performed from time to time to identify the current state of the system. The inspection time is assumed to be non-negligible and random with a distribution function $Q(t)$ and a mean value q. Each inspection incurs a fixed cost M. After each inspection, the current state of the system is revealed with certainty. Let E_i represent the time instant when the system is identified to be in state i. At E_i, one of the following decisions is selected: R, replace the system immediately, or $I(t_i)$, $0 \leq t_i \leq \infty$, inspect the system t_i units of time later with $I(\infty)$ interpreted as the limit as $t_i \to \infty$ of $I(t_i)$. $I(\infty)$ is therefore the decision to operate the system without inspection until it fails. If the system fails before time t_i, it can be detected immediately without inspection and it should be replaced. When the time intervals between successive inspections are state independent, i.e., $t_0 = t_1 = \cdots = t_n$, a sequential inspection strategy reduces to a periodic inspection strategy.

Under a continuous inspection strategy, the deteriorating system is monitored continuously and the current state of the system is therefore always known. Once the system enters state i, the decision is either to replace the system or to continue monitoring (CM). For both sequential and continuous inspection strategies, we assume that the replacement costs and times are non-negligible and state dependent. When the system is in state i, the replacement cost is C_i and the replacement time follows a distribution function $R_i(t)$ with a mean value r_i. Furthermore, after the completion of each replacement, the system is renewed (back to state 0). During an inspection or a replacement, it is assumed that the system is neither operating nor deteriorating and this incurs a loss of m per unit time. When the system is operating, the operating cost is a_i per unit time in state i.

The following notations are used throughout this paper.

- S The set of all states of the system, $S = \{0, 1, \cdots, n+1\}$.
- $\mathbb{R}$ The set of all real numbers.
- $P_{ij}(t)$ The probability that the system presently in state i will be in state j after t units of time.
- $F_i(t)$ The failure time distribution of the system starting from state i. Note that $F_i(t) = P_{i,n+1}(t)$ and $\bar{F}_i(t) = 1 - F_i(t) = \sum_{j=i}^{n} P_{ij}(t)$.
- $Q_{ij}(t)$ The expected time that the system spent in state j during $[0, t]$ given that it starts from state i, $Q_{ij}(t) = \int_0^t P_{ij}(u)\, du$.
- $A_i(t)$ The expected operating cost of the system during $[0, t]$ given that it starts from state i, $A_i(t) = \sum_{j=i}^{n} a_j Q_{ij}(t)$.

μ_i The expected time to failure given that the system starts from state i, $\mu_i = \int_0^\infty \bar{F}_i(u)\,du$.

δ A sequence of decisions selected at the time instants E_i, $i = 0, 1, \cdots, n+1$.

$D_\delta(i)$ The decision at the time instant E_i under the policy δ. In particular, we restrict ourselves to $D_\delta(n+1) = R$ for both strategies.

Δ_s The set of all policies, δ, under the sequential inspection strategy with $D_\delta(n+1) = R$.

$X_\delta^s(i)$ The expected time from E_i to the next replacement under the policy δ of the sequential inspection strategy.

$Y_\delta^s(i)$ The expected cost from E_i to the next replacement under the policy δ of the sequential inspection strategy.

$X_\delta^c(i)$ The expected time from E_i to the next replacement under the policy δ of the continuous inspection strategy.

$Y_\delta^c(i)$ The expected cost from E_i to the next replacement under the policy δ of the continuous inspection strategy.

Our objectives here are to derive and to compare the optimal policies of the sequential and the continuous inspection strategies based on minimizing the expected long run cost rate.

3 Sequential Inspection Strategy

In this section, we first formulate the optimization problem under the sequential inspection strategy. A policy improvement algorithm is presented to derive the optimal policy. The properties of the optimal policy are also discussed. Since the model considered here is a direct extension of that considered by Ohnishi et al. [5], the same techniques used in their paper can be applied to derive the results given in this section. The proofs of these results are therefore omitted.

For the sequential inspection strategies, given any $t_i \in [0, \infty]$, $X_\delta^s(i)$ and $Y_\delta^s(i)$ can be calculated respectively by using Equations (3.1) and (3.2) below.

$$X_\delta^s(i) = \begin{cases} \int_0^{t_i} \bar{F}_i(u)\,du + q\bar{F}_i(t_i) + \sum_{j=i}^{n+1} P_{ij}(t_i) X_\delta^s(j) & \text{if } D_\delta(i) = I(t_i) \\ r_i & \text{if } D_\delta(i) = R \end{cases}, \qquad (3.1)$$

and

$$Y_\delta^s(i) \;=\; \begin{cases} A_i(t_i) + (M + mq)\bar{F}_i(t_i) + \sum_{j=i}^{n+1} P_{ij}(t_i) Y_\delta^s(j) & \text{if } D_\delta(i) = I(t_i) \\ R_i & \text{if } D_\delta(i) = R \end{cases}, \tag{3.2}$$

where $R_i = C_i + mr_i$ for all $i \in S$. Since the system is renewed upon the completion of a replacement, the expected long run cost rate of the system under a policy δ is therefore equal to $Y_\delta^s(0)/X_\delta^s(0)$. Here we want to find an optimal inspection and replacement policy $\delta_s^* \in \Delta_s$ such that

$$g_s^* \stackrel{\text{def}}{=} \inf_{\delta \in \Delta_s} \frac{Y_\delta^s(0)}{X_\delta^s(0)} = \frac{Y_{\delta_s^*}^s(0)}{X_{\delta_s^*}^s(0)}$$

where g_s^* is the optimal expected long run cost rate. The following Policy Improvement Algorithm (PIA) can be used to derive δ_s^* and g_s^*.

Step I: [Initial Criteria]

Select a tolerance limit $\epsilon > 0$. Set $k = 0$ and choose an initial value for g_k.

Step II: [Policy Improvement Routine]

Use g_k to construct a policy δ_{k+1} as follows.

Set $V_{\delta_{k+1}}(n+1, g_k) = C_{n+1} + (m - g_k) r_{n+1}$ and $D_{\delta_{k+1}}(n+1) = R$.

For $i = n, n-1, \cdots, 1, 0$,

find $\inf_{t_i \in (0,\infty)} G_{\delta_{k+1}}(i, g_k, t_i)$ where

$$G_{\delta_{k+1}}(i, g_k, t_i) = \frac{1}{1 - P_{ii}(t_i)} \Bigg\{ A_i(t_i) + [M + (m - g_k)q]\, \bar{F}_i(t_i) + \sum_{j=i+1}^{n+1} P_{ij}(t_i) V_{\delta_{k+1}}(j, g_k) - g_k \int_0^{t_i} \bar{F}_i(u)\, du \Bigg\}.$$

Set $V_{\delta_{k+1}}(i, g_k) = \min \left[\inf_{t_i \in (0,\infty)} G_{\delta_{k+1}}(i, g_k, t_i), C_i + (m - g_k) r_i \right]$.

If $V_{\delta_{k+1}}(i, g_k) = G_{\delta_{k+1}}(i, g_k, t_i^*) = \inf_{t_i \in (0,\infty)} G_{\delta_{k+1}}(i, g_k, t_i)$, then $D_{\delta_{k+1}}(i) = I(t_i^*)$.

If $V_{\delta_{k+1}}(i, g_k) = C_i + (m - g_k) r_i$, then $D_{\delta_{k+1}}(i) = R$.

Step III: [Stopping Criterion]

If $| V_{\delta_{k+1}}(0, g_k) | < \epsilon$, then set $\delta_s^* = \delta_{k+1}$ and $g_s^* = g_k$. STOP.

Step IV: [Value Determination Routine]

Set $g_{k+1} = Y_{\delta_{k+1}}^s(0)/X_{\delta_{k+1}}^s(0)$, $k = k + 1$ and GOTO step II.

The following theorem shows that under some assumptions on the time and cost parameters, the optimal policy for the sequential inspection strategy derived from PIA is of control limit type. Also, the optimal time interval between inspections becomes shorter and shorter as the system deteriorates.

Theorem 3.1 $D_{\delta_s^*}(n) \in \{I(\infty), R\}$. *Under the following assumptions,*

(A1) $0 < \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_n < \infty$,

(A2) $0 < \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n < \infty$,

(A3) $0 \leq r_0 \leq r_1 \leq \cdots \leq r_n \leq r_{n+1} - q$,

(A4) $0 < \dfrac{C_0 + M}{r_0 + q} \leq \dfrac{C_1 + M}{r_1 + q} \leq \cdots \leq \dfrac{C_{n+1} + M}{r_{n+1} + q} \leq \dfrac{C_{n+1}}{r_{n+1}}$,

(A5) $\dfrac{a_0}{\lambda_0} - R_0 \leq \dfrac{a_1}{\lambda_1} - R_1 \leq \cdots \leq \dfrac{a_n}{\lambda_n} - R_n$,

there exists a critical state $k_s^* \in S$ *such that*

$$
\begin{aligned}
D_{\delta_s^*}(i) &= R \quad \text{if } k_s^* \leq i \leq n+1, \\
D_{\delta_s^*}(i) &= I(t_i^*) \quad \text{if } 0 \leq i < k_s^*,
\end{aligned}
$$

and $\infty \geq t_0^* \geq t_1^* \geq \cdots \geq t_{k_s^*-1}^* \geq 0$ *where* t_i^*, $i = 0, 1, \cdots, k_s^* - 1$, *are derived from PIA.*

The result $D_{\delta_s^*}(n) \in \{I(\infty), R\}$ tells us that once the system is identified to be in state n, the optimal decision is either to replace the system immediately or to operate the system until it fails. Assumptions **(A1)** and **(A2)** in the theorem above are reasonable since a deteriorating system is more likely to make a transition to a higher state and the terminal state as the system deteriorates. Assumptions **(A3)** and **(A4)** respectively imply that the system becomes more time consuming and expensive for replacement with the deterioration. Assumption **(A5)** indicates that the difference between the expected operation cost and the expected replacement cost in each state is increasing as the system deteriorates. Note that assumption **(A2)** given here is weaker than assumption **(A2)** in [5]. Assumptions **(A3)** to **(A5)** here extend the assumptions **(A3)** to **(A4)** in [5] to take into consideration the non-negligible inspection and replacement times.

4 Continuous Inspection Strategy

In this section, we formulate the optimization problem and discuss the properties of the optimal policy under the continuous inspection strategy. For this strategy, since the current state of the system is always known to us, it is sufficient to consider the set of all policies δ such that $D_\delta(i) = R$ if $k \le i \le n+1$, and $D_\delta(i) = CM$ if $0 \le i < k$ for some $k \in S$. For easier interpretation, let $X_\delta^c(i) = X_k^c(i)$ and $Y_\delta^c(i) = Y_k^c(i)$. These can be calculated recursively using the following equations.

$$X_k^c(i) = \begin{cases} \dfrac{1}{\lambda_i} + \dfrac{\alpha_i}{\lambda_i} r_{n+1} + \dfrac{\beta_i}{\lambda_i} X_k^c(i+1) & \text{if } 0 \le i < k \le n+1 \\ r_k & \text{if } i = k \end{cases}, \tag{4.1}$$

and

$$Y_k^c(i) = \begin{cases} \dfrac{a_i}{\lambda_i} + \dfrac{\alpha_i}{\lambda_i} R_{n+1} + \dfrac{\beta_i}{\lambda_i} Y_k^c(i+1) & \text{if } 0 \le i < k \le n+1 \\ R_k & \text{if } i = k \end{cases}. \tag{4.2}$$

For notation simplicity, define $X_k^c = X_k^c(0)$ and $Y_k^c = Y_k^c(0)$. Obviously, $X_0^c = r_0$ and $Y_0^c = R_0$. Define $g(k) = Y_k^c / X_k^c$. We want to find an optimal critical state $k_c^* \in S$ such that

$$g_c^* \stackrel{\text{def}}{=} \inf_{k \in S} g(k) = \inf_{k \in S} \frac{Y_k^c}{X_k^c} = \frac{Y_{k_c^*}^c}{X_{k_c^*}^c}$$

where g_c^* is the optimal expected long run cost rate for the continuous inspection strategy. Since there is only a finite number of states, it is obvious that both g_c^* and k_c^* exist, and they can be obtained by searching for a $k \in S$ such that $g(k)$ is minimal. Furthermore, we have $g(0) = m + C_0/r_0$ and $g(n+1) = [A_0(\infty) + R_{n+1}] / (\mu_0 + r_{n+1})$, and they are both finite. Therefore, the value of g_c^* is finite and bounded above by the minimum of $g(0)$ and $g(n+1)$.

Let $Z(i)$ be the marginal expected cost rate which is defined as follows. For all $i \in S \setminus \{n+1\}$,

$$Z(i) = \frac{Y_{i+1}^c - Y_i^c}{X_{i+1}^c - X_i^c}.$$

From Equations (4.1) and (4.2), $Z(i)$ can be expressed explicitly as,

$$Z(i) = \frac{Y_{i+1}^c(i) - Y_i^c(i)}{X_{i+1}^c(i) - X_i^c(i)} = \frac{\dfrac{a_i}{\lambda_i} + \dfrac{\alpha_i}{\lambda_i} R_{n+1} + \dfrac{\beta_i}{\lambda_i} R_{i+1} - R_i}{\dfrac{1}{\lambda_i} + \dfrac{\alpha_i}{\lambda_i} r_{n+1} + \dfrac{\beta_i}{\lambda_i} r_{i+1} - r_i}.$$

In the theorem below, we show that when $Z(i)$ and r_i are nonincreasing in i, the optimal critical state k_c^* is the smallest state such that $g(i)$ starts to increase. This result facilitates the search for k_c^*.

Theorem 4.1 *Assume $Z(i)$ is non-decreasing in $i \in S \setminus \{n+1\}$ and r_i is non-decreasing in $i \in S$. If there exists a $k \in S \setminus \{n+1\}$ such that $g(k+1) \geq g(k)$, then $g(i)$ is non-decreasing in $i \in \{k, k+1, \cdots, n+1\}$.*

Proof. If r_i is non-decreasing in $i \in S$, then by repeat substitution

$$\begin{aligned} X_{k+1}^c - X_k^c &= \left(\prod_{i=0}^{k-1} \frac{\beta_i}{\lambda_i}\right)\left(\frac{1}{\lambda_k} + \frac{\alpha_k}{\lambda_k} r_{n+1} + \frac{\beta_k}{\lambda_k} r_{k+1} - r_k\right) \\ &\geq \left(\prod_{i=0}^{k-1} \frac{\beta_i}{\lambda_i}\right)\left(\frac{1}{\lambda_k} + r_{k+1} - r_k\right) > 0. \end{aligned} \tag{4.3}$$

If $g(k+1) \geq g(k)$ for some $k \in S \setminus \{n+1\}$, using Equation (4.3), we have

$$Z(k) - g(k+1) = \frac{Y_{k+1}^c - Y_k^c}{X_{k+1}^c - X_k^c} - \frac{Y_{k+1}^c}{X_{k+1}^c} = \left(\frac{X_k^c}{X_{k+1}^c - X_k^c}\right)[g(k+1) - g(k)] \geq 0. \tag{4.4}$$

Since $Z(i)$ is non-decreasing in $i \in S \setminus \{n+1\}$, Equation (4.4) implies that $Z(k+1) \geq g(k+1)$. By the definition of $g(k+1)$, we have $Y_{k+1}^c - g(k+1)X_{k+1}^c = 0$. Hence,

$$\begin{aligned} [g(k+2) - g(k+1)]X_{k+2}^c &= Y_{k+2}^c - g(k+1)X_{k+2}^c - [Y_{k+1}^c - g(k+1)X_{k+1}^c] \\ &= [Z(k+1) - g(k+1)]\,(X_{k+2}^c - X_{k+1}^c) \geq 0. \end{aligned}$$

Obviously, X_{k+2}^c is strictly positive since all the transition rates of the underlying continuous-time Markov process are assumed to be strictly positive. It follows that $g(k+2) \geq g(k+1)$. The same argument can now be repeated to show that $g(k+3) \geq g(k+2)$, $g(k+4) \geq g(k+3), \cdots,$ $g(n+1) \geq g(n)$. Therefore $g(i)$ is non-decreasing in $i \in \{k, k+1, \cdots, n+1\}$. □

The following two theorems show that under the same assumptions given in Theorem 4.1, if there exists a $k \in S$ such that $Z(k-1) \leq g(k) \leq Z(k)$, then $k = k_c^*$.

Theorem 4.2 *If $Z(i)$ is non-decreasing in $i \in S \setminus \{n+1\}$ and r_i is non-decreasing in $i \in S$, then there exists a $k \in S$ such that $Z(i) \geq g(k)$ for all $i \geq k$, and $Z(i) \leq g(k)$ for all $i < k$.*

Proof. Consider $k = k_c^*$. For all $i \geq k_c^*$,

$$Z(i) - g_c^* \geq Z(k_c^*) - g_c^* = \frac{Y_{k_c^*+1}^c - g_c^* X_{k_c^*+1}^c}{X_{k_c^*+1}^c - X_{k_c^*}^c} \geq 0. \tag{4.5}$$

Equation (4.5) holds by definition of g_c^* and k_c^*, and Equation (4.3) above. Similarly, for $i < k_c^*$,

$$Z(i) - g_c^* \leq Z(k_c^* - 1) - g_c^* = \frac{-Y_{k_c^*-1}^c + g_c^* X_{k_c^*-1}^c}{X_{k_c^*}^c - X_{k_c^*-1}^c} \leq 0.$$

This completes the proof. □

Theorem 4.3 *If there exists a $k \in S$ such that $Z(i) \geq g(k)$ for all $i \geq k$, $Z(i) \leq g(k)$ for all $i < k$, and r_i is non-decreasing in $i \in S$, then $k = k_c^*$.*

Proof. Observe that for all $i \geq k + 1$

$$g(i) = \frac{Y_i^c}{X_i^c} = \frac{\sum_{u=k}^{i-1}(Y_{u+1}^c - Y_u^c) + Y_k^c}{\sum_{u=k}^{i-1}(X_{u+1}^c - X_u^c) + X_k^c}.$$

Since $Z(u) \geq g(k)$ for all $u \geq k$, it is now obvious that $g(i) \geq g(k)$. A similar argument can be used to show that $g(i) \leq g(k)$ for all $i \leq k$. Hence $k = k_c^*$. □

5 Comparison of Inspection Strategies

In this section, we compare the optimal expected long run cost rates and the structures of the optimal policies under the sequential and continuous inspection strategies. We investigate sufficient conditions such that the continuous inspection strategy is preferred to the sequential inspection strategy and show that the optimal critical state for replacement under the periodic inspection strategy is smaller than the optimal critical state under the continuous inspection strategy.

Recall that g_c^* is the optimal expected long run cost rate under the continuous inspection strategy. Intuitively, g_c^* should be less than or equal to g_s^* since there is no delay in detecting or replacing an undesirable system. However, if the inspection cost rate $m + M/q$ is strictly less than

g_c^*, then we have

$$g_s^* \le g_p^* \le \min\left\{m + \frac{M}{q}, m + \frac{C_0}{r_0}, \frac{A_0(\infty) + R_{n+1}}{\mu_0 + r_{n+1}}\right\} < g_c^*$$

where g_p^* is the optimal expected long run cost rate under the periodic inspection strategy. This means that when the inspection cost rate is relatively small, then the sequential inspection strategy is preferred.

In the case when the inspection cost rate is greater than or equal to g_c^*, it is unclear which of the two strategies gives a smaller expected cost rate. From Theorem 3.1, when Assumptions **(A1)** to **(A5)** are satisfied, the optimal policy δ_s^* is of control limit type. Therefore, only the control limit type policies for the sequential inspection strategy are considered in our comparison here. Let Δ_{clt} represent the class of the control limit type policies and define

$$g_{clt}^* \stackrel{\text{def}}{=} \inf_{\delta \in \Delta_{clt}} \frac{Y_\delta^s(0)}{X_\delta^s(0)}.$$

The following theorem provides sufficient conditions such that the continuous inspection strategy is preferred to the sequential inspection strategy. The proof is given in the appendix.

Theorem 5.1 *Under the following sufficient conditions: (1) $Z(i)$ is non-decreasing in $i \in S \setminus \{n+1\}$, (2) r_i is non-decreasing in $i \in S$, and (3) $m + M/q \ge g_c^*$ or $M = q = 0$. Then, $g_{clt}^* \ge g_c^*$.*

Let k_{pclt}^* be the optimal critical state for replacement under the periodic inspection strategy. Theorem 5.2 below shows that under sufficient conditions, k_{pclt}^* is smaller than or equal to the optimal critical state k_c^*. This means that the system is allowed to operate in higher states under the continuous inspection strategy since the condition of the system is always known with certainty.

Theorem 5.2 *Under the following sufficient conditions: (1) $Z(i)$ is non-decreasing in $i \in S \setminus \{n+1\}$, (2) r_i is non-decreasing in $i \in S$, (3) $M = q = 0$. Then, $k_{pclt}^* \le k_c^*$.*

The proof of the theorem is again given in the appendix.

Appendix

Before giving the proofs of Theorems 5.1 and 5.2, let us define the following notations. For all $i, j \in S$ and $t \in [0, \infty]$, define

$$p_{ij}(t) = \frac{P_{ij}(t)}{[1 - P_{ii}(t)]}, \qquad q_{ij}(t) = \frac{\lambda_j Q_{ij}(t)}{[1 - P_{ii}(t)]},$$

$f_i(t) = p_{i,n+1}(t)$, and $\bar{f}_i(t) = \sum_{j=i}^{n} p_{ij}(t)$. Also let

$$S_{ki} = \left\{ \underset{\sim}{\gamma} = (\gamma_0, \gamma_1, \cdots, \gamma_i) : \gamma_0 = 0, \gamma_{j-1} < \gamma_j, \gamma_j \in \{1, 2, \cdots, k\}, j \in \{1, 2, \cdots, i\} \right\}$$

for all $i \leq k$. Given $j, k \in S$, $t_i \in [0, \infty]$ and $i \in S \setminus \{n+1\}$, let

$$\mathcal{Q}_{kj} = \sum_{i=0}^{k} \sum_{\underset{\sim}{\gamma} \in S_{ki}} \left[\prod_{u=1}^{i} p_{\gamma_{u-1}\gamma_u}(t_{\gamma_{u-1}}) \right] q_{\gamma_i j}(t_{\gamma_i}),$$

$$\mathcal{P}_{k,k+1} = \sum_{i=0}^{k} \sum_{\underset{\sim}{\gamma} \in S_{ki}} \left[\prod_{u=1}^{i} p_{\gamma_{u-1}\gamma_u}(t_{\gamma_{u-1}}) \right] p_{\gamma_i, k+1}(t_{\gamma_i}),$$

and

$$\mathcal{F}_k = \sum_{i=0}^{k} \sum_{\underset{\sim}{\gamma} \in S_{ki}} \left[\prod_{u=1}^{i} p_{\gamma_{u-1}\gamma_u}(t_{\gamma_{u-1}}) \right] \bar{f}_{\gamma_i}(t_{\gamma_i}),$$

with $\left[\prod_{u=1}^{0} p_{\gamma_{u-1}\gamma_u}(t_{\gamma_{u-1}}) \right] = 1$. Note that $\mathcal{Q}_{kj} = \mathcal{Q}_{k-1,j}$ whenever $k > j$ and $\mathcal{Q}_{0j} = q_{0j}(t_0)$. Using Kolmogorov's forward and backward equations [5], properties **(P1)** to **(P4)** below can be verified easily.

(P1) For all $i \in S \setminus \{n+1\}$ and $t \in [0, \infty]$, $q_{ii}(t) = 1$.

(P2) For all $0 \leq i \leq j \leq n-1$ and $t \in [0, \infty]$, $\dfrac{\beta_j}{\lambda_j} q_{ij}(t) = q_{i,j+1}(t) + p_{i,j+1}(t)$.

(P3) For all $i \in S \setminus \{n+1\}$ and $t \in [0, \infty]$, $f_i(t) = p_{i,n+1}(t) = \sum_{j=i}^{n} \dfrac{\alpha_j}{\lambda_j} q_{ij}(t)$.

(P4) For each $t \in [0, \infty]$, $q_{ij}(t)$ is totally positive of order 2 (TP_2) in i and j, $i, j \in S \setminus \{n+1\}$.

Properties **(P5)** and **(P6)** can be derived using **(P1)** to **(P3)**.

(P5) Given $i \in S \setminus \{n+1\}$, $j \in S$, $k \in S \setminus \{0\}$, and $t_i \in [0,\infty]$, $\mathcal{Q}_{kj} = \mathcal{P}_{k-1,k} q_{kj}(t_k) + \mathcal{Q}_{k-1,j}$ and $\mathcal{F}_k = \mathcal{P}_{k-1,k}\bar{f}_k(t_k) + \mathcal{F}_{k-1}$.

(P6) For all $k \in S \setminus \{0, n+1\}$, $\prod_{i=0}^{k-1} \frac{\beta_i}{\lambda_i} = \mathcal{Q}_{kk} = \sum_{i=0}^{k} \sum_{\underset{\sim}{\gamma} \in S_{ki}} \left[\prod_{j=1}^{i} p_{\gamma_{j-1}\gamma_j}(t_{\gamma_{j-1}}) \right] q_{\gamma_i k}(t_{\gamma_i})$.

Proof of Theorem 5.1:

Using Kolmogorov's forward equations, Properties **(P1)** to **(P3)**, and recalling that $D_\delta(n+1) = R$, Equations (3.1) and (3.2) can be rewritten as

$$X_\delta^s(i) = \begin{cases} \sum_{j=i+1}^{n} q_{ij}(t_i) \left[\frac{1}{\lambda_j} + \frac{\alpha_j}{\lambda_j} r_{n+1} + \frac{\beta_j}{\lambda_j} X_\delta^s(j+1) - X_\delta^s(j) \right] & \\ \quad + \left[\frac{1}{\lambda_i} + \frac{\alpha_i}{\lambda_i} r_{n+1} + \frac{\beta_i}{\lambda_i} X_\delta^s(i+1) \right] + q\bar{f}_i(t_i) & \text{if } D_\delta(i) = I(t_i) \\ r_i & \text{if } D_\delta(i) = R \end{cases},$$

and

$$Y_\delta^s(i) = \begin{cases} \sum_{j=i+1}^{n} q_{ij}(t_i) \left[\frac{a_j}{\lambda_j} + \frac{\alpha_j}{\lambda_j} R_{n+1} + \frac{\beta_j}{\lambda_j} Y_\delta^s(j+1) - Y_\delta^s(j) \right] & \\ \quad + \left[\frac{a_i}{\lambda_i} + \frac{\alpha_i}{\lambda_i} R_{n+1} + \frac{\beta_i}{\lambda_i} Y_\delta^s(i+1) \right] + (M + mq)\bar{f}_i(t_i) & \text{if } D_\delta(i) = I(t_i) \\ R_i & \text{if } D_\delta(i) = R. \end{cases} \quad \text{(A.1)}$$

For $j \in S \setminus \{n+1\}$, let $Z_j^X = \frac{1}{\lambda_j} + \frac{\alpha_j}{\lambda_j} r_{n+1} + \frac{\beta_j}{\lambda_j} r_{j+1} - r_j$ and $Z_j^Y = \frac{a_j}{\lambda_j} + \frac{\alpha_j}{\lambda_j} R_{n+1} + \frac{\beta_j}{\lambda_j} R_{j+1} - R_j$. By repeat substitution, we have

$$X_k^c = \sum_{j=0}^{k-1} \mathcal{Q}_{jj} Z_j^X + r_0 \qquad \text{and} \qquad Y_k^c = \sum_{j=0}^{k-1} \mathcal{Q}_{jj} Z_j^Y + R_0. \quad \text{(A.2)}$$

Let $\delta \in \Delta_s$ be a control limit type policy with critical state k, i.e., $D_\delta(i) = R$ if $k \le i \le n+1$, and $D_\delta(i) = I(t_i)$, $t_i \in [0,\infty]$, if $0 \le i < k$. For easier interpretation, let $X_\delta^s(i) = X_k^s(i; t_0, t_1, \cdots, t_{k-1})$ and $Y_\delta^s(i) = Y_k^s(i; t_0, t_1, \cdots, t_{k-1})$. Note that $X_k^s(i; t_0, t_1, \cdots, t_{k-1}) = r_i$ and $Y_k^s(i; t_0, t_1, \cdots, t_{k-1}) = R_i$ whenever $i \ge k$. By repeat substitution in Equation (A.1) and using Property **(P2)**, we have for all $k \in S \setminus \{0, n+1\}$,

$$Y_{k+1}^s(0; t_0, t_1, \cdots, t_k) - Y_k^s(0; t_0, t_1, \cdots, t_{k-1}) = \mathcal{P}_{k-1,k} \left[\sum_{j=k}^{n} q_{kj}(t_k) Z_j^Y + (M + mq)\bar{f}_k(t_k) \right]. \quad \text{(A.3)}$$

A similar result holds for $X^s_{k+1}(0;t_0,t_1,\cdots,t_k) - X^s_k(0;t_0,t_1,\cdots,t_{k-1})$. Next observe that for all $k \in S \setminus \{0\}$ and $i \in S$,

$$Y^s_k(0;t_0,t_1,\cdots,t_{k-1}) - Y^c_i = \sum_{j=i}^{n} \mathcal{Q}_{k-1,j} Z^Y_j + \sum_{j=k}^{i-1} (\mathcal{Q}_{k-1,j} - \mathcal{Q}_{jj})\, Z^Y_j + \mathcal{F}_{k-1}(M+mq) \qquad (A.4)$$

where the second term on the right hand side of Equation (A.4) is zero whenever $k \geq i$. Result (A.4) can be readily verified by mathematical induction as follows. Using Equations (A.1) and (A.2), we have for all $i \in S$,

$$\begin{aligned} Y^s_1(0;t_0) - Y^c_i &= [Y^s_1(0;t_0) - R_0] - (Y^c_i - R_0) \\ &= \sum_{j=i}^{n} q_{0j}(t_0) Z^Y_j + \sum_{j=1}^{i-1} [q_{0j}(t_0) - \mathcal{Q}_{jj}]\, Z^Y_j + (M+mq)\bar{f}_0(t_0). \end{aligned} \qquad (A.5)$$

Note that $q_{0j}(t_0) = \mathcal{Q}_{0j}$ and $\bar{f}_0(t_0) = \mathcal{F}_0$, hence Equation (A.5) is the same as Equation (A.4) when $k = 1$. Assume that result (A.4) holds for $k = \tau \in S \setminus \{0, n+1\}$. Now consider

$$\begin{aligned} & Y^s_{\tau+1}(0;t_0,t_1,\cdots,t_\tau) - Y^c_i \\ = \;& [Y^s_{\tau+1}(0;t_0,t_1,\cdots,t_\tau) - Y^s_\tau(0;t_0,t_1,\cdots,t_{\tau-1})] + [Y^s_\tau(0;t_0,t_1,\cdots,t_{\tau-1}) - Y^c_i]. \end{aligned} \qquad (A.6)$$

Applying Equations (A.3) and (A.4), and Properties (**P5**), Equation (A.6) can again be written in the form of (A.4) with $k = \tau + 1$. A similar result holds for $X^s_k(0;t_0,t_1,\cdots,t_{k-1}) - X^c_i$. To show that $g^*_{clt} \geq g^*_c$, we only need to verify that

$$Y^s_k(0;t_0,t_1,\cdots,t_{k-1}) - X^s_k(0;t_0,t_1,\cdots,t_{k-1}) g^*_c \geq 0$$

for all $t_i \in [0,\infty]$, $0 \leq i < k$ and $k \in S$. Obviously,

$$\frac{Y^s_0(0)}{X^s_0(0)} = \frac{R_0}{r_0} = \frac{Y^c_0}{X^c_0} \geq g^*_c.$$

When $k \in S \setminus \{0\}$, we have

$$\begin{aligned} & Y^s_k(0;t_0,t_1,\cdots,t_{k-1}) - X^s_k(0;t_0,t_1,\cdots,t_{k-1}) g^*_c \\ = \;& \left[Y^s_k(0;t_0,t_1,\cdots,t_{k-1}) - Y^c_{k^*_c}\right] - \left[X^s_k(0;t_0,t_1,\cdots,t_{k-1}) - X^c_{k^*_c}\right] g^*_c. \end{aligned} \qquad (A.7)$$

Note that $\mathcal{Q}_{k-1,j} - \mathcal{Q}_{jj} < 0$ for all $j \geq k$ and $Z(j) = Z^Y_j / Z^X_j$ for $j \in S \setminus \{n+1\}$. Under the assumptions given in the theorem, we know from Theorem 4.2 that $Z(i) \geq g^*_c$ for all $i \geq k^*_c$ and

$Z(i) \leq g_c^*$ for all $i < k_c^*$. Using Equations (A.4) and (A.7), and the assumptions stated in the theorem, the result follows. $\square$

Proof of Theorem 5.2:

Given $j, k \in S$ and $t \in [0, \infty]$, define

$$\mathcal{Q}_{kj}^t = \sum_{i=0}^{k} \sum_{\underset{\sim}{\gamma} \in S_{ki}} \left[\prod_{u=1}^{i} p_{\gamma_{u-1}\gamma_u}(t) \right] q_{\gamma_i j}(t), \quad \text{and} \quad \mathcal{F}_k^t = \sum_{i=0}^{k} \sum_{\underset{\sim}{\gamma} \in S_{ki}} \left[\prod_{u=1}^{i} p_{\gamma_{u-1}\gamma_u}(t) \right] \bar{f}_{\gamma_i}(t).$$

Using Property **(P4)**, it is clear that $q_{kj}(t)\mathcal{Q}_{k_c^*-1,u} - q_{ku}(t)\mathcal{Q}_{k_c^*-1,j} \geq 0$ for all $n \geq j > u \geq k \geq k_c^*$ and fixed $t \in [0, \infty]$. Recall that under the assumptions given in the theorem, $Z(i) \geq g(k_c^*)$ for all $i \geq k$ and $Z(i) \leq g(k_c^*)$ if $i < k_c^*$. We can now conclude that for all $t \in [0, \infty]$ and $k \geq k_c^*$,

$$\begin{aligned}
&\left[\sum_{j=k}^{n} q_{kj}(t) Z_j^Y\right] \left(\sum_{j=k_c^*}^{n} \mathcal{Q}_{k_c^*-1,j}^t Z_j^X\right) - \left[\sum_{j=k}^{n} q_{kj}(t) Z_j^X\right] \left(\sum_{j=k_c^*}^{n} \mathcal{Q}_{k_c^*-1,j}^t Z_j^Y\right) \\
= &\sum_{j=k}^{n} \sum_{u=k_c^*}^{k-1} q_{kj}(t) \mathcal{Q}_{k_c^*-1,u}^t Z_j^X Z_u^X [Z(j) - Z(u)] \\
&+ \sum_{j=k+1}^{n} \sum_{u=k}^{j-1} [q_{kj}(t)\mathcal{Q}_{k_c^*-1,u}^t - q_{ku}(t)\mathcal{Q}_{k_c^*-1,j}^t] Z_j^X Z_u^X [Z(j) - Z(u)] \geq 0.
\end{aligned}$$

Let $Y_i^s(0;t) = Y_i^s(0;t,\cdots,t)$ and $X_i^s(0;t) = X_i^s(0;t,\cdots,t)$. Given any $i > k_c^*$, we have from Equations (A.3) and (A.4)

$$\begin{aligned}
&Y_i^s(0;t) X_{k_c^*}^s(0;t) - X_i^s(0;t) Y_{k_c^*}^s(0;t) \\
= &\sum_{k=k_c^*}^{i-1} \left\{ [Y_{k+1}^s(0;t) - Y_k^s(0;t)] X_{k_c^*}^s(0;t) - [X_{k+1}^s(0;t) - X_k^s(0;t)] Y_{k_c^*}^s(0;t) \right\} \\
= &\sum_{k=k_c^*}^{i-1} \mathcal{P}_{k-1,k} \left\{ \left[\sum_{j=k}^{n} q_{kj}(t) Z_j^Y\right] \left(X_{k_c^*}^c + \sum_{j=k_c^*}^{n} \mathcal{Q}_{k_c^*-1,j}^t Z_j^X \right) - \left[\sum_{j=k}^{n} q_{kj}(t) Z_j^X\right] \left(Y_{k_c^*}^c + \sum_{j=k_c^*}^{n} \mathcal{Q}_{k_c^*-1,j}^t Z_j^Y \right) \right\}
\end{aligned}$$

which is obviously nonnegative. That is,

$$\frac{Y_i^s(0;t)}{X_i^s(0;t)} \geq \frac{Y_{k_c^*}^s(0;t)}{X_{k_c^*}^s(0;t)}.$$

for all $i \geq k_c^*$ and $t \in [0, \infty]$. This implies that

$$\inf_{t \in (0,\infty)} \frac{Y_i^s(0;t)}{X_i^s(0;t)} \geq \inf_{u \in (0,\infty)} \frac{Y_{k_c^*}^s(0;u)}{X_{k_c^*}^s(0;u)}$$

and $k_{pclt}^* \leq k_c^*$. $\square$