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PRELIMINARY STUDY OF THE APPLICATION OF ELECTRONIC DIFFERENTIAL ANALYZERS TO AEROELASTIC PROBLEMS

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1. Introduction

In recent years the aero-elastic properties of wing structures have proved to be a major factor in aircraft structural design and analysis [1].* The procedure of replacing time-dependent loads by static loads with a "suitable factor of safety" leads to inefficient structures. This is due to the implied assumption that the stress distribution is the same for static and dynamic problems. Furthermore, the term "suitable factor of safety" is itself nebulous, since the selection of any given factor can only be based on previous experience and therefore is useless when one deals with wings structures of a new type or shape.

It seems more logical to express minimum requirements in terms of the dynamic characteristics of the structure in such a way that the structural weight can be safely reduced to a minimum. One of the more important problems in aeroelasticity is the determination of the response of a wing to gusts of given profile and intensity. This problem is the subject of the report.

2. Summary and Purpose

The purpose of this investigation is to determine whether the gust response problem can be solved conveniently by means of an electronic differential analyzer. Since the greatest difficulty encountered in the problem is a suitable representation of the unsteady aerodynamic forces, it was decided to consider only a simplified problem as far as the elastic representation of the wing is concerned but to investigate thoroughly the representation of all the aerodynamic terms.

The wing was represented by a simple cantilever beam and only flexural oscillations were considered. In the actual case, of course, the wing should be considered as a free-free beam or plate with a large

* Numbers in brackets refer to the bibliography

mass concentrated at the center (representing the fuselage); both bending and twisting should be allowed, together with the rigid-body modes of motion [2]. It is expected, however, that the actual problem will not present any more fundamental difficulties than the simplified problem considered here.

3. Equations of Motion

Consider a tapered cantilever beam of length b/2 which is subjected to a lift load $L(\bar{x}, t)$ per unit length of its span. The equation of motion for this beam is [3]

$$\frac{\partial^2}{\partial \bar{x}^2}\left(EI \frac{\partial^2 \bar{u}}{\partial \bar{x}^2}\right) + \gamma \frac{\partial^2 \bar{u}}{\partial t^2} = L(\bar{x}, t) \tag{1}$$

where

$\bar{u}(x, t)$ = lateral displacement of the beam

$\bar{x}$ = distance along the span

EI = stiffness of wing cross-section

t = time

$b/2$ = semi-span length

$L(\bar{x}, t)$ = lift per unit span of wing

γ = mass per unit length of wing

It will be convenient to put Equation (1) in dimensionless form. Because of the form of the lift functions [1] to be explained later, the new dimensionless time variable is defined as

$$s = \frac{2}{c_o} Ut \tag{2}$$

where

c_o = mean chord of wing

U = airplane velocity.

It is clear from Equation (2) that s is the distance traveled by the wing in unit time expressed in terms of the half-chord. The following additional dimensionless variables are defined:

$$x = \bar{x}/(b/2)$$
$$u = \bar{u}/(b/2)$$

$$c = \text{chord of wing at any spanwise position}$$

$$\phi_c(x) = c/c_o$$

$$\phi_m(x) = \gamma/\gamma_o$$

$$\gamma_o = \text{mass per unit length of wing at the mean chord}$$

$$\phi_f(x) = EI/EI_o$$

$$EI_o = \text{stiffness of wing cross-section at the mean chord.}$$

Substituting the above expressions in Equation (1) gives

$$\frac{EI_o}{(b/2)^3} \frac{\partial^2}{\partial x^2}\left[\phi_f(x) \frac{\partial^2 u}{\partial x^2}\right] + 2\frac{bU^2}{c_o^2} \gamma_o \phi_m(x) \frac{\partial^2 u}{\partial s^2} = L(x, s)$$

or, dividing by $EI_o/(b/2)^3$

$$\frac{\partial^2}{\partial x^2} \phi_f(x) \frac{\partial^2 u}{\partial x^2}\Big] + \lambda^2 \phi_m(x) \frac{\partial^2 u}{\partial s^2} = \frac{(b/2)^3}{EI_o} L(x, s) \tag{3}$$

where

$$\lambda^2 = 4\left[\frac{b}{2}\right]^4 \frac{U^2\gamma_o}{EI_o\, c_o^2} \tag{4}$$

The boundary conditions for a cantilever wing are

$$\left.\begin{aligned} u(0, s) &= \frac{\partial u}{\partial x}(0, s) = 0 \\ \frac{\partial^2 u}{\partial x^2}(1, s) &= \frac{\partial^2 u}{\partial x^3}(1, s) = 0 \end{aligned}\right\} \tag{5}$$

and the initial conditions are

$$\begin{aligned} u(x, 0) &= f(x) \\ \frac{\partial u}{\partial s}(x, 0) &= g(x) \end{aligned} \tag{6}$$

4. The Aerodynamic Loads on the Wing

The aerodynamic loads on the wing can be divided into two parts [1]

$$L(x, s) = L_g(x, s) + L_m(x, s) \tag{7}$$

where $L_g(x, s)$ is the lift on the wing due to a gust of given intensity and profile and $L_m(x, s)$ = lift due to disturbed motion of the wing. The change in lift coefficient due to sharp-edge (i.e., step-function) gust of intensity $\overline{w}$ is [4]

$$\Delta C_{L_g} = 2\pi \frac{\overline{w}}{U} \psi(s) \tag{8}$$

where $\psi(s)$ is the Karman-Sears function which is approximated by

$$\psi(s) = 1 - \frac{1}{2} (e^{-0.130s} + e^{-s}) \tag{9}$$

From Equation (8) it is clear that the lift per unit span due to a sharp-edge gust is

$$L_{1g} = \frac{1}{2} \rho U^2 c \, \Delta C_{L_g}$$

$$= \pi \rho U c \overline{w} \psi(s)$$

For a gust of variable intensity $\overline{w}(s)$ the lift per unit span is therefore

$$L_g = \pi \rho U c \int_0^s \overline{w}'(\sigma) \psi(s-\sigma) \, d\sigma \tag{10}$$

The lift due to disturbed motion L_m is the sum of three effects: the apparent mass effect due to motion of a certain volume of air induced by the wing motion, the so-called quasi-steady effect which is due to the change in apparent angle of attack caused by the wing motion, and the effect of the wake reaching back on the wing. The wake effect is not important unless the frequency of motion is high. The lift due to disturbed motion is [5]

$$L_m = -\pi \rho U^2 \frac{c^2}{c_o^2} \frac{\partial^2 \overline{u}}{\partial s^2} + \pi \rho U^2 c \int_0^s \phi(s-\sigma) \, \alpha'(\sigma) \, d\sigma \tag{11}$$

where

$\alpha(s)$ = angle of attack

$\phi(s)$ = the Wagner function which represents the quasi-steady and wake terms for a step change in angle of attack.

The Wagner function is represented to sufficient accuracy by

$$\phi(s) = 1 - 0.165\, e^{-0.0455s} - 0.335\, e^{-0.300s} \tag{12}$$

The apparent angle of attack due to wing motion is

$$a = -\frac{1}{U}\frac{\partial \bar{u}}{\partial t} = -\frac{2}{c_o}\frac{\partial \bar{u}}{\partial s}$$

hence

$$a'(s) = -\frac{2}{c_o}\frac{\partial^2 \bar{u}}{\partial s^2}$$

and therefore the total lift due to disturbed motion of the wing is

$$L_m = -\pi\rho U^2 c^2 \frac{\partial^2 \bar{u}}{\partial s^2} - 2\pi\rho U^2 c \int_o^s \phi(s-\sigma)\frac{\partial^2 \bar{u}}{\partial \sigma^2}\, d\sigma$$

In terms of the dimensionless variables

$$L_m = -\pi\rho U^2 \left[\frac{b}{2}\right] \phi_c{}^2(x) \frac{\partial^2 u}{\partial s^2} - 2\pi\rho U^2 \left[\frac{b}{2}\right] \phi_c(x) \int_o^s \phi(s-\sigma)\frac{\partial^2 u}{\partial \sigma^2}\, d\sigma \tag{13}$$

The total aerodynamic force acting on the wing is therefore

$$L(x,s) = \pi\rho U^2 c_o\, \phi_c(x) \int_o^s w'(\sigma)\,\psi(s-\sigma)\, d\sigma - \pi\rho U^2 \left[\frac{b}{2}\right] \phi_c{}^2(x) \frac{\partial^2 u}{\partial s^2}$$

$$- 2\pi\rho U^2 \left[\frac{b}{2}\right] \phi_c(x) \int_o^s \phi(s-\sigma)\frac{\partial^2 u}{\partial \sigma^2}\, d\sigma \tag{14}$$

where

$$w(\sigma) = \frac{\bar{w}(\sigma)}{U} = \text{dimensionless gust intensity} \tag{15}$$

The right-hand side of Equation (3) then becomes

$$\frac{(b/2)^3}{EI_o} L(x,s) = \frac{\delta}{R}\, \phi_c(x) \int_o^s \psi(s-\sigma)\, w'(\sigma)\, d\sigma$$

$$- \frac{\delta}{2}\, \phi_c{}^2(x) \frac{\partial^2 u}{\partial s^2} - \delta\phi_c(x) \int_o^s \phi(s-\sigma)\frac{\partial^2 u}{\partial \sigma^2}\, d\sigma$$

Substituting this expression in Equation (3) gives for the equation of motion

$$\frac{\partial^2}{\partial x^2}\left[\phi_f(x)\,\frac{\partial^2 u}{\partial x^2}\right] + \left[\lambda^2\,\phi_m(x) + \frac{\delta}{2}\,\phi_c{}^2(x)\right]\frac{\partial^2 u}{\partial s^2}$$

$$= \frac{\delta}{R}\,\phi_c(x)\,\ell_g(s) + \delta\phi_c(x)\,\ell_m(s) \tag{16}$$

where

$$\ell_g(s) = \int_0^s \psi(s-\sigma)\, w'(\sigma)\, d\sigma, \tag{17}$$

$$\ell_m(s) = -\int_0^s \phi(s-\sigma)\,\frac{\partial^2 u}{\partial \sigma^2}\, d\sigma, \tag{18}$$

$$\delta = \frac{2\pi\rho\, U^2 (b/2)^4}{EI_o} \tag{19}$$

and

$$R = \frac{b}{c_o} = \text{aspect ratio} \tag{19a}$$

The solution of Equation (10) together with the proper boundary and initial conditions (Equations (5) and (6)) will yield the lateral motion of the wing under the unsteady aerodynamic forces.

5. Analytical Solution for Untapered Wing

The solution of Equation (16) is in general impossible to carry out for any case of practical interest such as a tapered wing. In view of the complexity of the wing structure, it is necessary to resort to some approximate method of solution or to make use of automatic computation facilities. However, it will be of some interest to examine and solve the special case of an untapered beam as its solution will bring out an interesting property of Equation (16). For an untapered wing

$$\phi_f(x) = \phi_m(x) = \phi_c(x) \equiv 1$$

and Equation (16) reduces to

$$\frac{\partial^4 u}{\partial x^4} + \left[\lambda^2 + \frac{\delta}{2}\right]\frac{\partial^2 u}{\partial s^2} = \frac{\delta}{R}\int_0^s \psi(s-\sigma)\, w'(\sigma)\, d\sigma - \delta\int_0^s \phi(s-\sigma)\frac{\partial^2 u}{\partial \sigma^2}\, d\sigma \tag{20}$$

Taking the Laplace transform of Equation (20) with respect to the time variable s gives

$$\frac{d^4u^*}{dx^4} + (\lambda^2 + \frac{\delta}{2} + \delta\phi^*)\, p^2 u^* = \frac{\delta}{R}\, p\psi^* w^* + (\lambda^2 + \frac{\delta}{2} + \delta\phi)^* [pf(x) + g(x)] \tag{21}$$

where the starred quantities indicate the Laplace transform of a function which is defined as

$$X^*(p) = L\left\{X(s)\right\} = \int_0^\infty e^{-ps}\, X(s)\, ds \tag{22}$$

If we now consider the problem of a similar wing subjected to the same gust but with the unsteady aerodynamic forces neglected, the equation to be solved is

$$\frac{\partial^4 u_1}{\partial x^4} + \lambda^2 \frac{\partial^2 u_1}{\partial s^2} = \frac{\delta}{R} \int_0^s \psi(s-\sigma)\, w'(\sigma)\, d\sigma\ . \tag{23}$$

and the Laplace transform is

$$\frac{d^4 u_1^*}{dx^4} + \lambda^2 p^2 u_1^* = \frac{\delta}{R}\, p\psi^* w^* \tag{24}$$

Comparison between Equation (21) and (24) reveals that these equations are identical except for the parameter λ^2 which is replaced by $\lambda^2 + \delta/2 + \delta\phi^*$ in Equation (21). Since the quantity $(\lambda^2 + \delta/2 + \delta\phi^*)$ is not a function of x, it simply plays the role of a parameter in Equation (21). Thus, if we know a solution of Equation (21) subject to given boundary conditions, we can obtain the corresponding solution of Equation (24) by simply replacing λ^2 by $(\lambda^2 + \delta/2 + \delta\phi^*)$ in this solution. Then we can obtain the solution including the unsteady aerodynamic terms for the wing by modifying the classical solution for a beam subjected to an arbitrary load. This is done by 1) taking the Laplace transform of the classical solution with respect to time, 2) replacing λ^2 by $(\lambda^2 + \delta/2 + \delta\phi^*)$ in this transform and 3) performing the inverse Laplace transformation.

For the present case, the classical solution would be

$$u_1(x,s) = \sum_{n=1}^{\infty} X_n(x) \Big[D_n \cos \omega_n s + \frac{E_n}{\omega_n} \sin \omega_n s$$

$$+ C_n \int_0^s \cos \omega_n (s-\sigma) \int_0^s \phi(\sigma - \tau)\, w(\tau)\, d\tau\, d\sigma \Big] \tag{25}$$

where

$$X_n(x) = \cosh a_n x - \cos a_n x - \beta_n (\sinh a_n x - \sin a_n x)$$

$$\beta_n = \frac{\sinh a_n - \sin a_n}{\cosh a_n + \cos a_n}$$

$$D_n = 2 \int_0^1 f(\xi)\, X_n(\xi)\, d\xi \tag{26}$$

$$E_n = 2 \int_0^1 g(\xi)\, X_n(\xi)\, d\xi$$

$$C_n = \frac{\delta}{R} \int_0^1 X_n(\xi)\, d\xi$$

$$\omega_n^2 = \frac{a_n^4}{\lambda^2}$$

and a_n are characteristic numbers determined from the equation

$$1 + \cosh a_n \cos a_n = 0$$

The Laplace transform of Equation (25) is

$$u_1^*(x,p) = \sum_{n=1}^{\infty} X_n(x) \left[D_n \frac{p}{p^2 + \omega_n^2} + E_n \frac{1}{p^2 + \omega_n^2} + C_n \frac{p}{p^2 + \omega_n^2} \psi^* w^* \right] \tag{27}$$

Since the parameter λ^2 appears only in $\omega_n^2 = a_n^4/\lambda^2$, we modify this quantity by defining

$$G_n(p) = \frac{a_n^4}{\lambda^2 + \delta/2 + \delta\phi^*} \tag{28}$$

so that the transform of the solution including the unsteady forces is

$$u^*(x,p) = \sum_{n=1}^{\infty} X_n(x) \Big[D_n \frac{p}{p^2 + G_n(p)} + E_n \frac{1}{p^2 + G_n(p)}$$

$$+ C_n \frac{p}{p^2 + G_n(p)} \psi^* w^* \Big]$$

Hence the solution will clearly be of the form

$$u(x, s) = \sum_{n=1}^{\infty} X_n(x) \Big[D_n F_n(s) + E_n H_n(s) + C_n \int_0^s F_n(s-\sigma) \int_0^s \psi(\sigma - \tau) w(\tau) d\tau \, d\sigma \Big] \tag{29}$$

where

$$F_n(s) = L^{-1} \left\{ \frac{p}{p^2 + G_n(p)} \right\}$$
$$H_n(s) = L^{-1} \left\{ \frac{1}{p^2 + G_n(p)} \right\} = \int_0^s F_n(\sigma) \, d\sigma \tag{30}$$

With the function $\phi(s)$ defined by Equation (12) the expression in Equation (30) turns out to be the quotient of two polynomials in p and the inverse Laplace transform of such a function is easily obtained by the use of the Heavyside partial fraction expansion.

It is interesting to note that the unsteady aerodynamic loads, which represent a special type of damping, do not affect the mode shapes of the vibration. This is due to the fact that the aerodynamic loads were calculated from a strip theory which neglects the finite wing aerodynamic effects. Even though this assumption is most likely in error, the results of this analysis constitute an approximate solution which may be of value in obtaining rough estimates of response characteristics for design purposes. For a more accurate analysis it is believed that the only practical method of solution is by the use of some automatic computation method.

6. Representation of the Unsteady Airloads for Solution by the Electronic Differential Analyzer

The solution of classical beam vibration problems on the electronic differential analyzer has been taken up in previous studies [6, 7]. In the present case the flexural beam equations are satisfied in exactly the same manner as in this previous work, namely, by dividing the beam into stations spanwise and by replacing x derivatives by finite differences. The main difficulty in the present problem is to find a suitable method of computing the unsteady airloads. Only the approximate strip theory will be considered here. It is in general difficult to mechanize directly an analog computation of convolution integrals such as those appearing in Equations (10) and (11). For this reason it is convenient to derive differential equations having as their solutions the convolution integrals in Equations (10) and (11). The first term in Equation (11) presents no difficulty since it simply plays the role of an additional inertia term which simply increases the effective mass per unit length of the beam. The second term on the right-hand side of Equation (16), which involves the Wagner function, is defined in Equation (18) and can be computed in the following manner: substituting for ϕ from Equation (12) and taking Laplace transforms with respect to the time variable s gives

$$\ell^*_m = - \left[\frac{1}{p} \cdot \frac{0.165}{p+0.0455} - \frac{0.335}{p+0.30} \right] p^2 u^* \tag{32}$$

where ℓ^*_m is the Laplace transform of ℓ_m. Note that Equation (32) assumes that the initial deflection and velocity of the beam are identically zero. This is permissible since under a steady lift the velocity will actually be zero and this initial deflected position can be chosen as a new origin for deflection.

It should be remembered, however, that when the values for bending moments and stresses are calculated, the constant values due to the steady flight loads must be added to the values resulting from the present analysis.

Equation (32) can easily be put in the form

$$\left[p^3 + 0.3455\, p^2 + 0.01365p \right] \ell^*_m = - \left[\frac{1}{2} p^2 + 0.2808p + 0.01365 \right] p^2 u^*$$

Applying the inverse Laplace transformation now yields

$$\frac{d^3 \ell_m}{ds^3} + 0.3455 \frac{d^2 \ell_m}{ds^2} + 0.01365 \frac{d\ell_m}{ds} = - \left[0.5 \frac{\partial^4 u}{\partial s^4} + 0.2808 \frac{\partial^3 u}{\partial s^3} + 0.01365 \frac{\partial^2 u}{\partial s^2} \right] \tag{33}$$

Equation (33) is an ordinary differential equation with ℓ_m as the unknown and a forcing function depending on quantities related to the beam displacement. This equation can be solved with a differential analyzer circuit having as voltage inputs the displacement quantities available from another analyzer circuit representing the beam equations of motion. It will be necessary to have a separate lift-computing circuit for each station of the beam.

For convenience in establishing the computer circuit, Equation (33) is integrated three times with respect to s with the result

$$\ell_m + 0.3455 \int \ell_m \, ds + 0.01365 \int\int \ell_m \, ds \, ds = -\left[0.5 \frac{\partial u}{\partial s} + 0.2808 \, u + 0.01365 \int u ds\right] \tag{34}$$

Rearranging the terms gives

$$\ell_m = \int \left[- 0.3455 \, \ell_m - 0.01365 \, u - 0.01365 \int \ell_m \, ds \right] ds - 0.5 \frac{\partial u}{\partial s} - 0.2808 \, u \tag{35}$$

The equation is now in convenient form for solution by the electronic differential analyzer.

The first term on the right-hand side of Equation (16), which involves the gust velocity profile and the Karman-Sears function, can be computed in a similar manner. Applying the Laplace transformation and substituting for ψ gives

$$\ell_g^* = \left[\frac{1}{p} - \frac{1}{2} \frac{1}{p + 0.13} - \frac{1}{2} \frac{1}{p + 1}\right] pw^* \tag{36}$$

which can be put in the form

$$(p^3 + 1.13p^2 + 0.13p) \, \ell_g^* = \left[0.565p + 0.13\right] pw^*$$

or

$$(p^2 + 1.13p + 0.13p) \, \ell_g^* = \left[0.565p + 0.13\right] w^* \tag{37}$$

Applying to this last expression the inverse Laplace transformation gives

$$\frac{d^2\ell_g}{ds^2} + 1.13 \frac{d\ell_g}{ds} + 0.13 \, \ell_g = \left[0.565 \frac{dw}{ds} + 0.13 \, w\right] \tag{38}$$

Integrating Equation (38) with respect to the time variable gives

$$0.565\, w - 1.13\, \ell_g - \frac{d\ell_g}{ds} = 0.13 \int \left[\ell_g - w \right] ds \qquad (39)$$

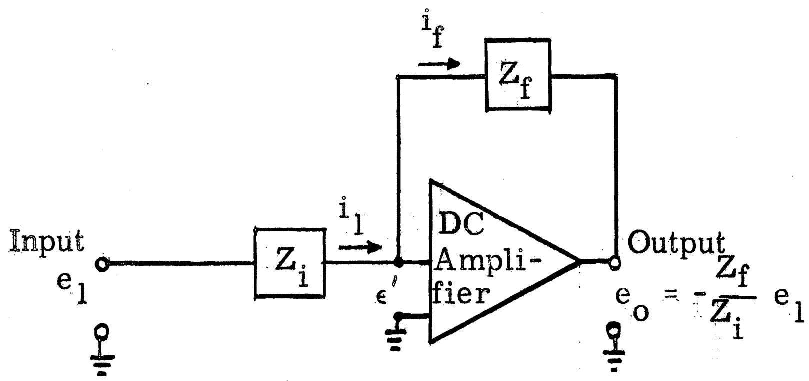
which, again, is now in convenient form for solution by the electronic differential analyzer. Before examining the analyzer circuits for solving the complete gust-response problem, i.e., Equations (16), (35), and 39), let us consider briefly the basic theory of the electronic differential analyzer.

7. Principles of Operation of the Electronic Differential Analyzer

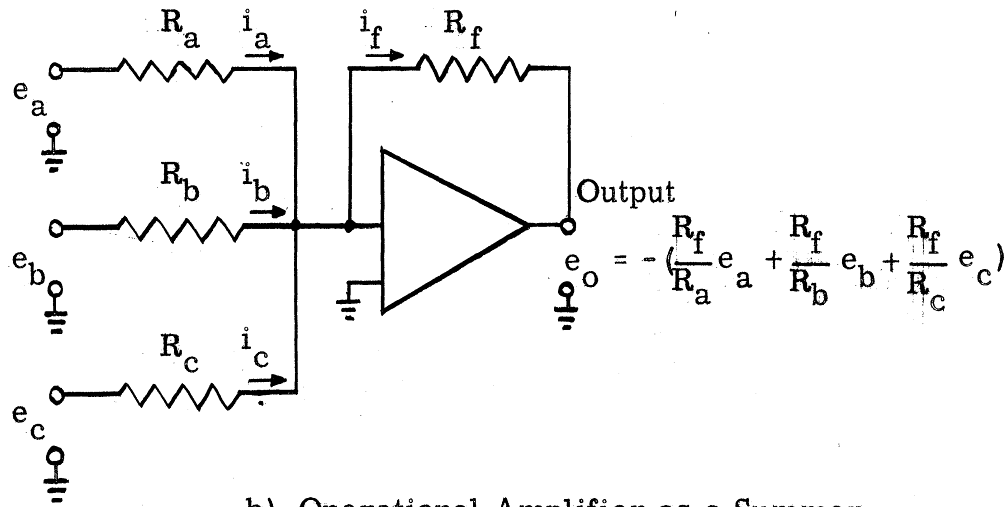
For the reader unfamiliar with the basic operating principles of the electronic differential analyzer we have included here a brief description. Those who wish to become more familiar with this type of computer are directed to other references [8, 9].

The basic unit of the electronic differential analyzer is the operational amplifier, which consists of a high-gain dc amplifier having a feedback impedance Z_f and an input impedance Z_i, as shown in Figure 1a. To a high degree of approximation the output voltage e_o of the operational amplifier is equal to the input voltage multiplied by the ratio of feedback to input impedance, with a reversal of sign. If several input resistors are used, the output voltage is proportional to the sum of the input voltages (Figure 1b). If an input resistor and feedback capacitor are used, the output voltage is proportional to the time integral of the input voltage (Figure 1c).

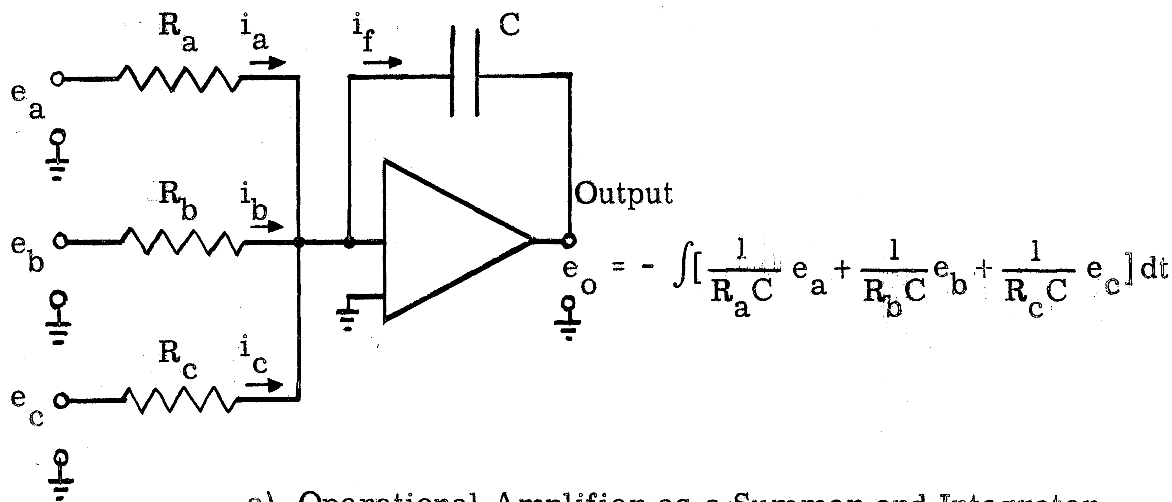
The operational amplifiers shown in Figure 1 can therefore be used to multiple a voltage by a constant factor, invert signs, sum voltages, and integrate a voltage with respect to time. These are the only functions necessary to solve ordinary linear differential equations with constant coefficients. Thus voltages are the physical quantities representing input functions and dependent variables in solving equations with this type of computer, while time represents the independent variable. The way in which operational amplifiers are connected together to actually solve differential equations will become clear from the schematic circuit diagrams in the following sections.



a) Operational Amplifier



b) Operational Amplifier as a Summer



c) Operational Amplifier as a Summer and Integrator

Figure 1. Operational Amplifiers

8. Electronic Differential Analyzer Circuit for Solving the Beam Equation

We have already seen from (16) that the equation representing transverse motion of a beam (and hence, we assume, transverse motion of the aircraft wing) is given by

$$\frac{\partial^2}{\partial x^2}\left[\phi_f(x)\frac{\partial^2 u}{\partial x^2}\right] + \left[\lambda^2\,\phi_m(x) + \frac{\delta}{2}\,\phi_c^{\,2}(x)\right]\frac{\partial^2 u}{\partial s^2} = \frac{\delta}{R}\,\phi_c(x)\ell_g + \delta\phi_c(x)\,\ell_m \tag{40}$$

Here u is the dimensionless transverse displacement of the wing, x is dimensionless distance along the wing, and s is dimensionless time. Equation (40) above is a partial differential equation, and if we are going to solve it with the electronic differential analyzer, we must convert it to one or more ordinary differential equations. This can be done by measuring the wing displacement $u(x, s)$ not at all values of x along the wing but just at certain stations along x, as shown in Figure 2. Let $u_1(s)$ equal the displacement at the first x station, $u_2(s)$ equal the displacement at the second x station, etc. Furthermore, let the separation between stations be a constant Δx. Clearly a good approximation to $\partial u/\partial x$ at the n + 1/2 station can be written as

$$\left.\frac{\partial u}{\partial x}\right|_{n+1/2} = \frac{1}{\Delta x}\left(u_{n+1} - u_n\right) \tag{41}$$

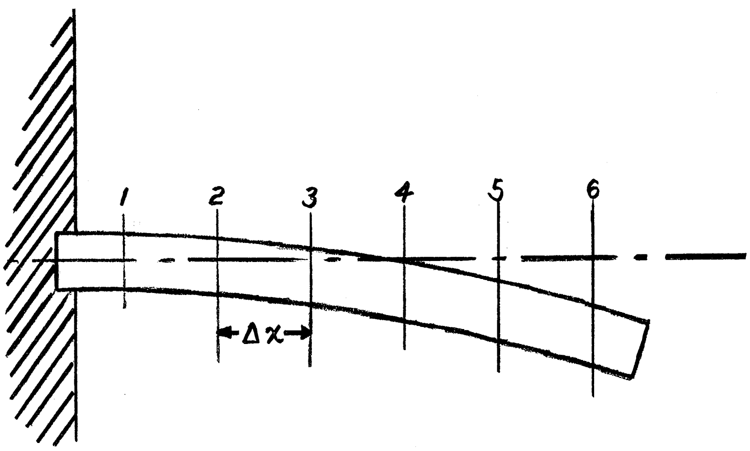


Figure 2. Wing Divided into Stations

In fact, in the limit of infinitely small Δx this is just the definition of $\partial u/\partial x$. Similarly, at the nth station

$$\phi_f \left.\frac{\partial^2 u}{\partial x^2}\right|_n = \frac{1}{\Delta x}\left[\frac{\partial u}{\partial x}_{n+1/2} - \frac{\partial u}{\partial x}_{n-1/2}\right]$$

or

$$\phi_f \left.\frac{\partial^2 u}{\partial x^2}\right|_n = \frac{1}{(\Delta x)^2}\,\phi_{f_n}\left[u_{n+1} - 2u_n + u_{n-1}\right] \tag{42}$$

Using the same difference approximation we can write from Equation (40) the force equilibrium equation at the nth station. Thus

$$\left[\lambda^2 \phi_m + \frac{\delta}{2}\phi_c^2\right]_n \frac{d^2u_n}{ds^2} = \frac{1}{(\Delta x)^2}\left[\phi_f \left.\frac{\partial^2 u}{\partial x^2}\right|_{n+1} - 2\phi_f \left.\frac{\partial^2 u}{\partial x^2}\right|_n\right.$$

$$\left. + \phi_f \left.\frac{\partial^2 u}{\partial x^2}\right|_{n-1}\right] + \frac{\delta}{R}\phi_{c_n}\ell_{g_n}(s) + \delta\phi_{c_n}\ell_{m_n}(s) \tag{43}$$

Equations (42) and (43) are iterated for each station. Note that u_n is a function only of the time variable s. Thus we have converted the original partial differential Equation (40) into a set of simultaneously second-order ordinary differential equations which can be solved by the analyzer. Note that $\Delta x = 1/N$, where N is the number of stations into which the beam is divided.

Equation (43) can be rewritten as

$$\phi_{d_n}\frac{d^2u_n}{ds^2} = -m_{n+1} + 2m_n - m_{n-1} + \frac{\delta}{RN^4}\phi_{c_n}\ell_{g_n} + \frac{\delta}{N^4}\phi_{c_n}\ell_{m_n} \tag{44}$$

where

$$\phi_{d_n} = \frac{1}{N^4}\left[\lambda^2\phi_m + \frac{\delta}{2}\phi_c^2\right]_n$$

Here m_n is proportional to the bending moment at the nth station and is given by

$$m_n = \phi_{f_n}(u_{n+1} - 2u_n + u_{n-1}) \tag{45}$$

The built-in boundary condition at $x = 0$ requires that $u(0, s) = \partial u/\partial x(0, s) = 0$. For the cellular beam this implies that

$u_o = u_1 = 0$ where the actual built-in end occurs at the 1/2 station. Similarly at the free end the bending moment and shear force vanish. For the cellular beam this implies that $m_N = m_{N+1} = 0$, where the actual free end of an N-cell beam occurs at the N + 1/2 station.

For the 4-cell beam shown in Figure (2) and considered in this report, the following equations are used.

$$\phi_{d_2} \frac{d^2u_2}{ds^2} = -m_3 + 2m_2 - m_1 + \frac{\delta}{RN^4} \phi_{c_2} \ell_{g_2} + \frac{\delta}{N^4} \phi_{c_2} \ell_{m_2}$$

$$\phi_{d_3} \frac{d^2u_3}{ds^2} = 2m_3 - m_2 + \frac{\delta}{RN^4} \phi_{c_3} \ell_{g_3} + \frac{\delta}{N^4} \phi_{c_3} \ell_{m_3} \tag{46}$$

$$\phi_{d_4} \frac{d^2u_4}{ds^2} = -m_3 + \frac{\delta}{RN^4} \phi_{c_4} \ell_{g_4} + \frac{\delta}{N^4} \phi_{c_4} \ell_{m_4}$$

where

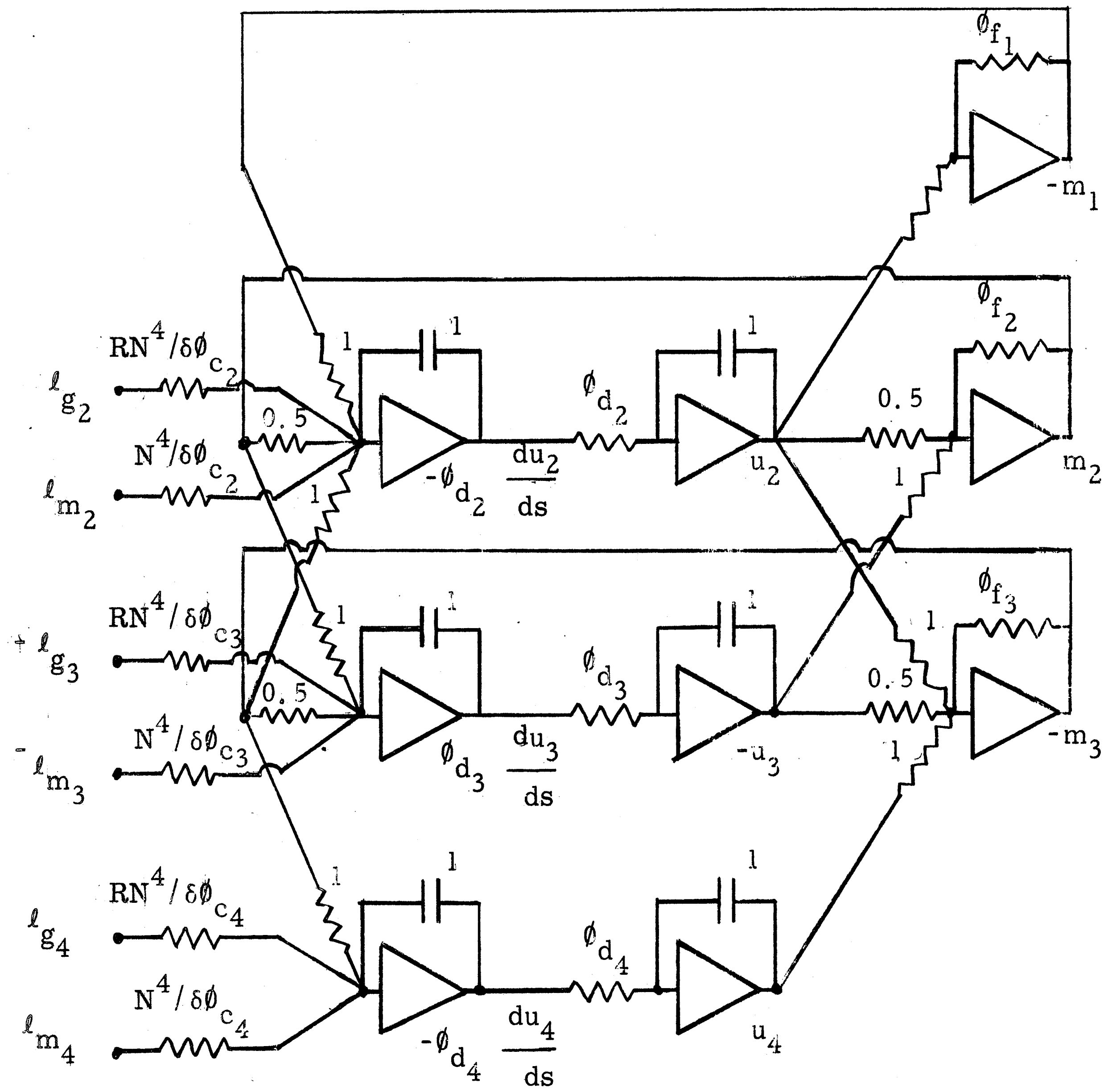
$$m_1 = \phi_{f_1} u_2$$

$$m_2 = \phi_{f_2} (u_3 - 2u_2) \tag{47}$$

$$m_3 = \phi_{f_3} u_4 - 2u_3 + u_2)$$

The electronic differential analyzer circuit for solving these equations is shown in Figure 3. Note that only 9 operational amplifiers are required and that voltage inputs and outputs represent the aerodynamic lifting force and wing displacement u, respectively, at each station.

Although only 4-cells may seem like too crude an approximation to the wing, theoretical solutions for a uniform 4-cell cantilever beam have shown that the mode shapes and frequencies for the first two normal modes of vibration agree within several percent with the exact solutions for a continuous cantilever beam [6, 7]. Note in Figure 3 that there is a one-to-one correspondence between resistor values in the circuit and mass and stiffness coefficients at each station. Thus it is extremely easy to change the wing characteristics on the computer representation. For example, to simulate the effect of a wing-tip tank



Initial-Condition and Ground Connections Omitted for Clarity.

All Resistors are Megohms

All Capacitors are Microfarads

Figure 3. Analyzer Circuit for Solving the 4-Cell Cantilever Beam.

one need only change by the appropriate amount the resistor ϕ_{m4}, which represents the mass at station 4.

9. Electronic Differential Analyzer Circuit for Computing the Unsteady Airloads

Equation (35) gives the lift term ℓ_m due to transverse wing-displacement u. The electronic differential analyzer circuit for solving this equation is shown in Figure 4. Voltage inputs to the circuit are the wing displacement u and velocity $\phi_d \partial u/\partial s$. The voltage output is ℓ_m. This circuit must be repeated at each station for the cellular wing representation.

Equation (39) gives the lift term ℓ_g due to the normalized gust velocity w. The analyzer circuit for solving this equation is shown in Figure 5. Voltage input is the gust velocity w and voltage output is ℓ_g. If we assume that the gust velocity w is independent of the spanwise coordinate x (a reasonable assumption), then the single circuit in Figure 5 can be used for computing ℓ_g at all stations.

It will be remembered that the Karman-Sears function is the change in lift coefficient corresponding to a step function in gust intensity, while the Wagner function represents the change in lift coefficient due to a step change in angle of attack (or in $\partial u/\partial s$).

Therefore the accuracy of the circuit in Figure 4 can be verified by supplying as an input a step function for $\partial u/\partial s$ and a ramp function for u. The output ℓ_m should then be the Wagner function. In the same manner, if a step function in w is imposed at the input of the circuit of Figure 5, the output of the circuit should be the Karman-Sears function. The results of these verifications are shown in Figures 6 and 7. Comparison of these curves with the actual curves of the Karman-Shear and Wagner functions given in Equations (9) and (12) shows excellent agreement.

10. Electronic Differential Analyzer Circuit for the Complete, Gust-Response Problem

The analyzer circuit for the 4-cell cantilever beam representing the aircraft wing is shown in Figure 3. Inputs to this circuit at the nth station are the lift terms ℓ_{g_n} and ℓ_{m_n}. Outputs of the circuit are the wing displacement u_n and velocity du_n/ds. These outputs are used as inputs to the Wagner-function circuit at the nth station (see Figure 4). The output ℓ_{m_n} of the Wagner-function circuit is, in turn, fed back

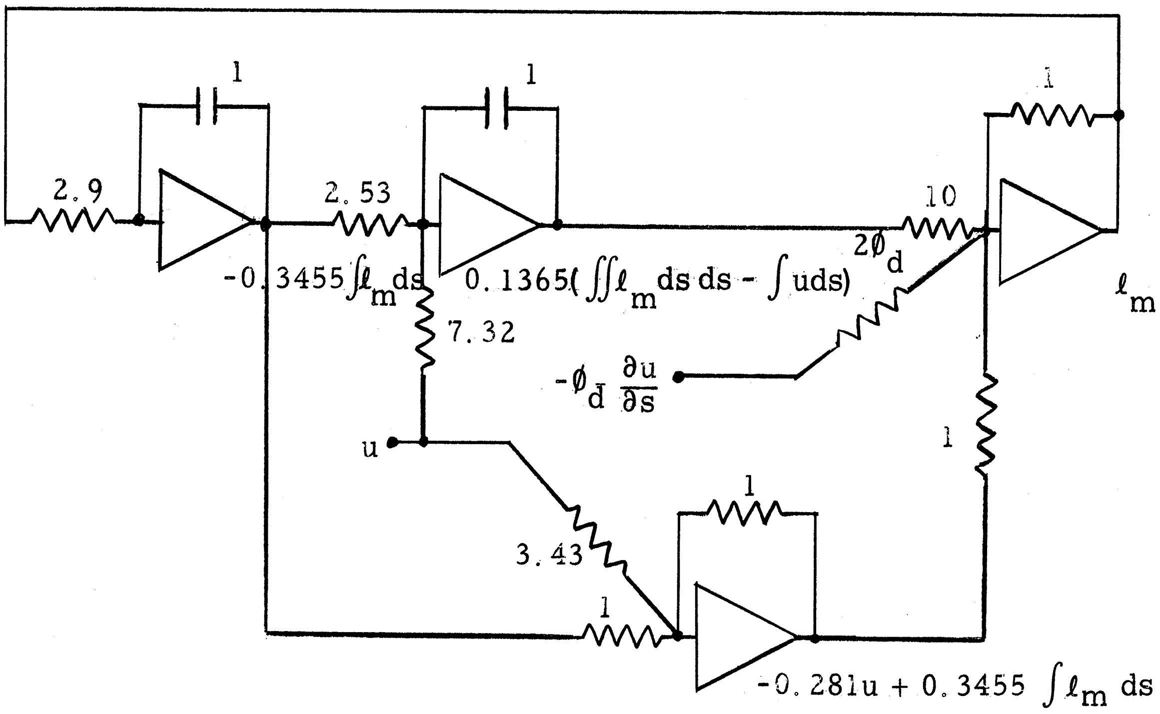


Figure 4. Analyzer Circuit for Computing the Lift ℓ_m as a Function of Wing-Displacement u.

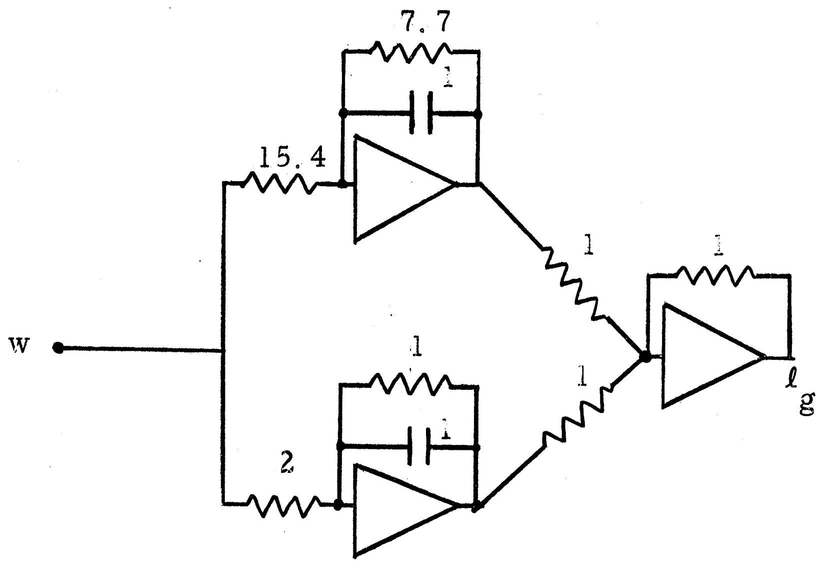


Figure 5. Analyzer Circuit for Computing the Lift ℓ_g as a Function of Gust-Velocity w.

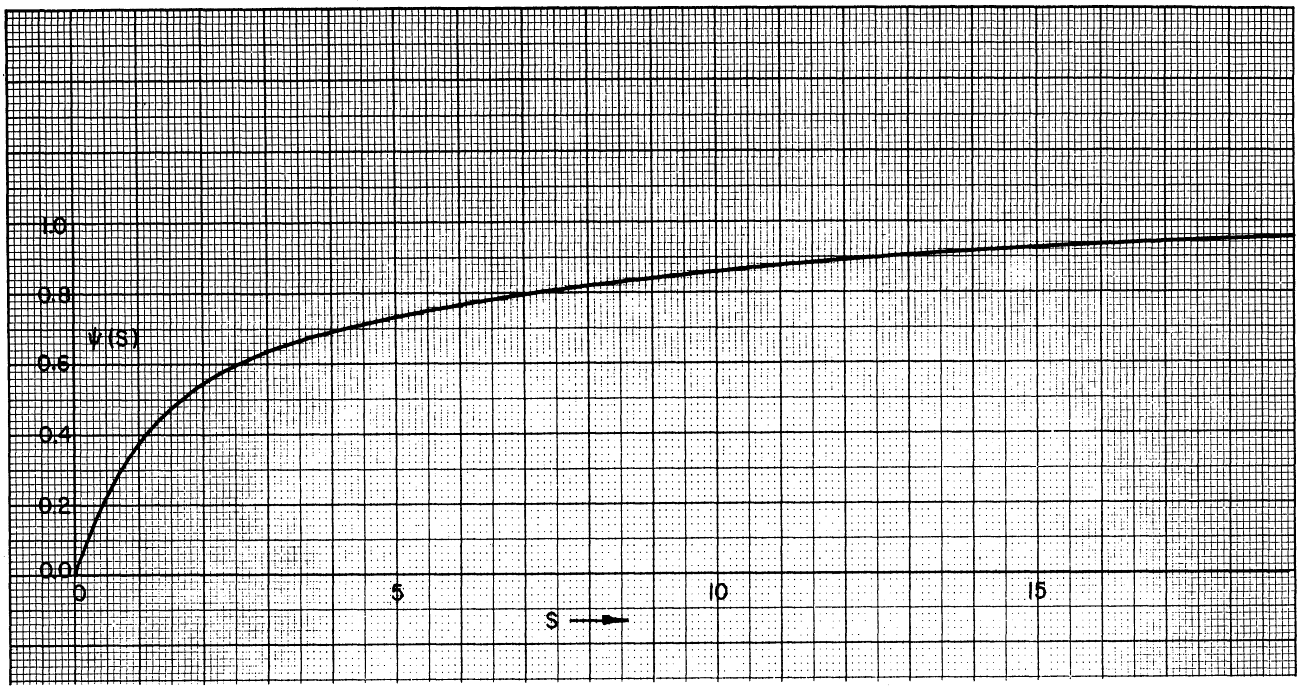


Figure 6. Analyzer Solution for the Karman-Sears Function

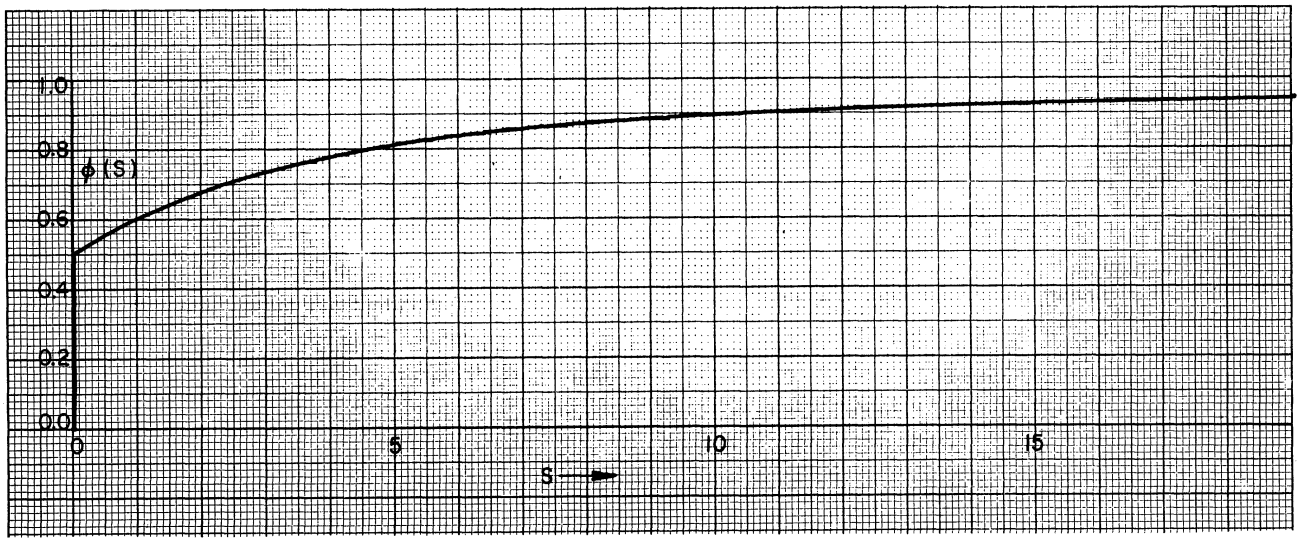


Figure 7. Analyzer Solution for the Wagner Function

to the corresponding input connection on the beam circuit. Also fed into the beam circuit at each station is the lift term ℓ_g computed by the Karmen-Sears function circuit in Figure 5, where w is the gust-velocity input.

The entire representation of the gust-response problem, as defined here, requires 9 operational amplifiers for the 4-cell cantilever-beam representation of the aircraft wing, 3 x 4 or 12 operational amplifiers for the representation of the wake induced lift terms at each station, and 3 operational amplifiers for the lift caused by the gust itself. Thus the total number of amplifiers is 24.

11. Example Solutions of the Gust Response of a Tapered Wing

As an example problem an aircraft wing with a 2:1 taper ratio in chord and thickness and a fundamental bending-mode frequency of 14 cycles per second was set up on the electronic differential analyzer. An aircraft velocity of 500 feet per second and an altitude of sea level were used in the aerodynamic representation. Step functions of gust velocity of 30 feet per second magnitude were applied and the wing response as computed by the electronic differential analyzer was recorded. Figure 8 shows the dimensionless wing displacement near the tip (u_4) as a function of dimensionless time s, along with the velocity $d\dot{u}_2/ds$ near the wing root. Note how the fundamental bending mode is damped by the aerodynamic wake effect in the u_4 recording; note also how the higher modes show up in the wing velocity near the root.

Similar response curves for 30 feet per second step-gust inputs are shown in Figure 9, where a wing tip tank equal to one-half the weight of the half-wing has been added to the circuit. This was accomplished merely by increasing the resistor labeled ϕ_{d4} in Figure 3 by the appropriate amount. The longer period of the response curves is clearly evident.

In Figure 10 is shown the wing response to a unit-sinusoidal gust input of 100 semi-chord lengths in period. Again the response has been computed both with and without the tip tank.

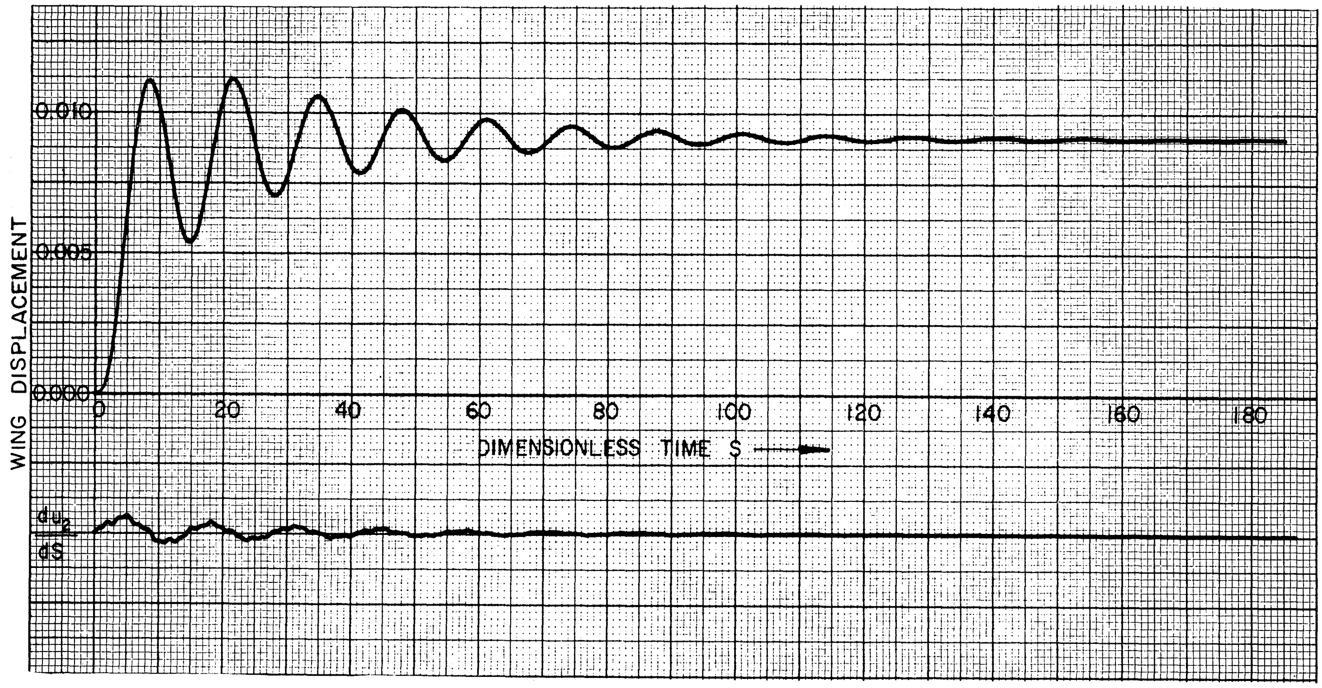


Figure 8. Step-Gust Response of Tapered Wing

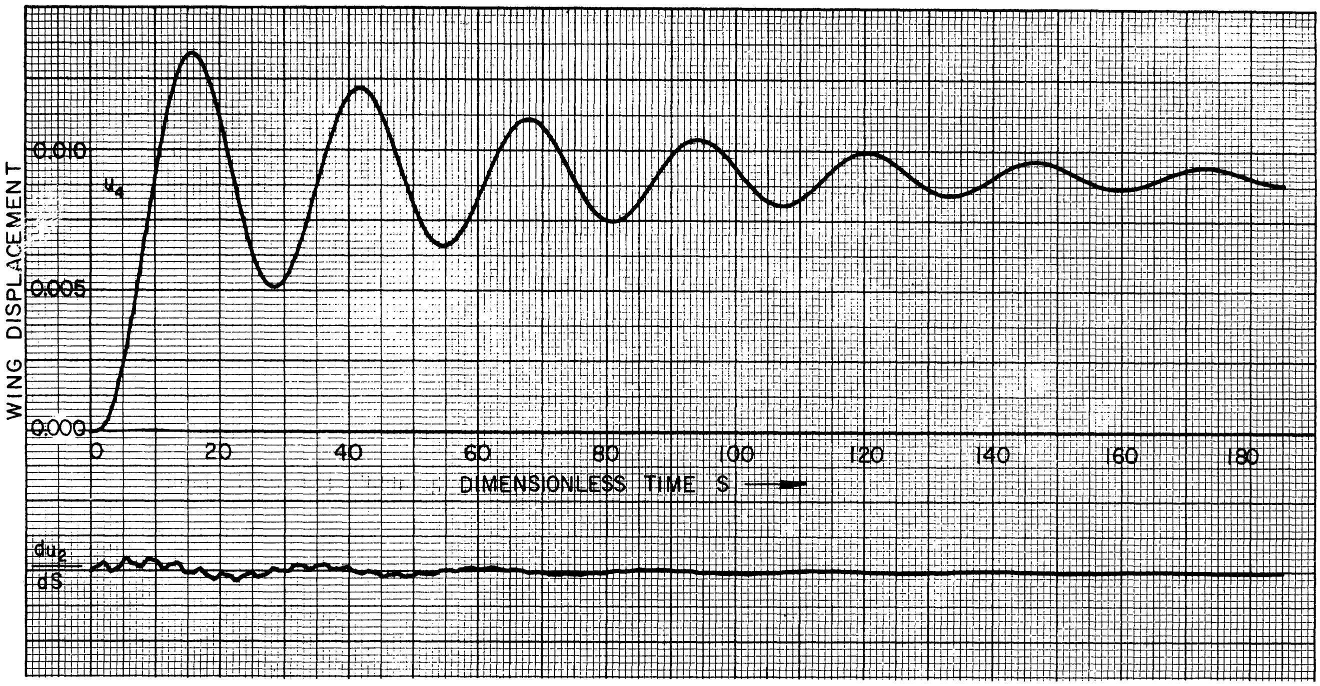


Figure 9. Step-Gust Response of Tapered Wing with Tip-Tank Equal to the Weight of the Half-Wing

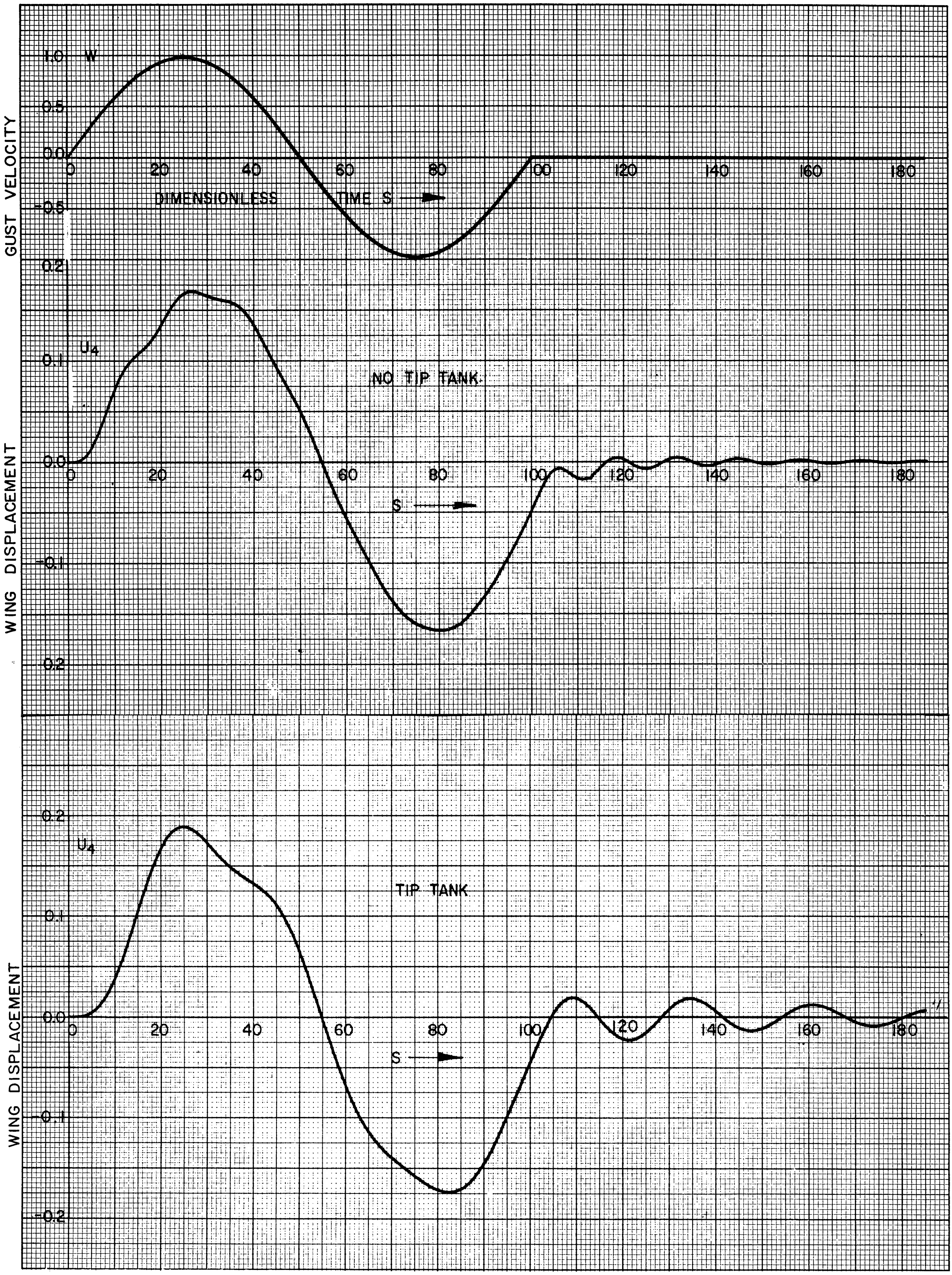


Figure 10. Response of Tapered Wing to a Sinusoidal Gust of Unit Period

Discussion

It is realized that the problem treated in this report does not represent a practical case, principally because of the elastic representation chosen. In the solution of an actual problem it would be necessary to allow bending and twisting of the wing together with rigid body pitching and plunging motions.

In the case of a straight wing this problem would not present additional difficulties and the mechanization of the equations would simply require more amplifiers. It is probable that finite wing aerodynamic corrections can be incorporated into the appropriate circuits.

In the case of swept and delta wings the elastic representation of the wing structure must be modified because of the elastic coupling between bending and torsion present in swept wings and also because delta wings behave more like a plate than a beam.

For the low aspect ratio swept or delta wing it will probably be necessary to use plate representation. The equations of motion for plates of variable thickness are very complex and it will be necessary to resort to approximate representations such as those suggested by Stein [10] or Sechler, Williams and Fung [11].

It has been found that in delta wings of low aspect ratio the chordwise bending deflection is not negligible compared to the other deformations [12]. Hence it will be necessary to include in the aerodynamic terms the effect of this change of camber.

It should be emphasized that the method of solution suggested here by means of an electronic differential analyzer has useful applications in the preliminary design of aircraft wings. Dynamic aeroelastic problems are in general so complex that it does not seem practical to use the available analytical solutions to obtain design estimates. The computer representation suggested here has the advantage that once the circuits are connected up, it is extremely easy to change the mass and stiffness distribution of the wing and to investigate the effect of such changes on the gust response. The differential analyzer used is of a type in common use in the aircraft industry.

Considerable aeroelastic analysis has been carried out on the Caltech analog computer using similar equations [13]. Here the beam and torsional difference equations are represented by passive circuits.

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