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CLOSURE AND LOWER CLOSURE THEOREMS FOR MULTIDIMENSIONAL PROBLEMS

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8. Closure and Lower Closure Theorems for Multidimensional Problems

8.1. Closure Theorems for Orientor Fields

Here G denotes as in (7.1) an open bounded subset of the t-space E_ν, for every $t \in$ cl G we denote by A(t) a given subset of the y-space E_s, and by A the set of all (t,y) with $t \in$ cl G, $y \in A(t)$. We assume A to be closed, and for every $(t,y) \in A$ we denote by Q(t,y) a given subset of the z-space E_r. We consider here pairs $z(t) = (z^1,\dots,z^r)$, $y(t) = (y^1,\dots,y^s)$, $t \in G$, of vector functions $z \in (L_1(G))^r$, $y \in (L_1(G))^s$, such that

$$y(t) \in A(t) \quad , \quad z(t) \in Q(t,y(t)) \quad \text{a.e. in } G. \qquad (8.1.1)$$

We may say that the pair (z,y) is an abstract solution of the orientor field (8.1.1). In applications there are suitable functional relations between z and y, that here are not needed.

(8.1.i) A first closure theorem. Let G, A(t), A, Q(t,y) be as above, G open and bounded, A closed, and let us assume that the sets Q(t,y) satisfy property (Q) in A. Let y(t), $y_k(t)$, $t \in G$, be s-vector functions, $y, y_k \in (L_1(G))^s$, and z(t), $z_k(t)$, $t \in G$, be r-vector functions, $z, z_k \in (L_1(G))^r$, with $y_k(t) \in A(t)$, $z_k(t) \in Q(t,y_k(t))$ a.e. in G, $k = 1,2,\dots$. If $y_k \to y$ strongly in $(L_1(G))^s$, and $z_k \to z$ weakly in $(L_1(G))^r$, then $y(t) \in A(t)$, $z(t) \in Q(t,y(t))$ a.e. in G.

This theorem is a particular case of (8.2.i) below.

8.2 A Closure Theorem for Trajectories With Singular Components

Let G be as above a bounded open subset of the t-space E_ν, and let I_o be an interval containing cl G, $G \subset \text{cl } G \subset I_o \subset E_\nu$. It is not restrictive to assume $I_o = [o,b]$, or $[o,\ldots,o,b,\ldots,b]$ for some $b > o$. For every $t \in I$, $t = (t^1,\ldots,t^\nu)$, let $[o,t]$ denote the interval $[o,\ldots,o,t^1,\ldots,t^\nu]$. Then, given any real-valued L-integrable function $z(t)$, $t \in I_o$, we may consider the function F defined by

$$F(t) = \int_{[o,t]} z(\tau)d\tau = \int_o^t z(\tau)d\tau \quad , \quad t \in I_o, \tag{8.2.1}$$

where the integration in $[o,t]$ will be often denoted simply by $\int_o^t$, and where $d\tau = d\tau^1 \ldots d\tau^\nu$ as usual. Note that $F(t)$ is then a continuous function on I_o with $F(t) = o$ whenever $t = (t^1,\ldots,t^\nu) \in I_o$, $t^1\ldots t^\nu = o$.

We shall consider the countable system $\{t_\rho\}$ of points $t_\rho = (t^1,\ldots,t^\nu)$ with $t^j = \rho_j b$, $j = 1,\ldots,\nu$, and ρ_j rational, and the countable system $\{I\}$ of all intervals $I = [\alpha,\beta] \subset I_o$, $\alpha = (\alpha^1,\ldots,\alpha^\nu)$, $\beta = (\beta^1,\ldots,\beta^\nu)$, $\alpha^i < \beta^i$, $\alpha,\beta \in \{t_\rho\}$. For any given $F(t)$, $t \in I_o$, we may then consider the usual differences $\Delta_I F$ of order ν relative to the 2^ν vertices of I; in other words

$$\Delta_I F = F(\beta) - F(\alpha) \text{ if } \nu = 1,$$

$$\Delta_I F = F(\beta^1,\beta^2) - F(\alpha^1,\beta^2) - F(\beta^1,\alpha^2) + F(\alpha^1,\alpha^2) \text{ if } \nu = 2, \text{ etc.}$$

If F is obtained by integration in $[o,t]$ of a function $z \in L_1(I_o)$ as in (8.2.1), then

$$\Delta_I F = \int_{[\alpha,\beta]} z(\tau)d\tau = \int_\alpha^\beta z(\tau)d\tau, \quad I = [\alpha,\beta] \subset I_o \; , \tag{8.2.2}$$

with the usual conventions.

We recall here that for a function F as in (8.2.2) the following holds: For almost all $t_o \in I_o$ and hypercube of I, $t_o \in I \subset I_o$, of side length $h > o$ we have $|I|^{-1} \Delta_I F \to z(t_o)$ as $h \to o+$.

An arbitrary function $S(t)$, $t \in I_o$, is said to be singular in I_o if for almost all $t_o \in I_o$ we have $|I|^{-1}\Delta_I S \to o$ as $h \to o+$, where as above I denotes any hypercube of side length h with $t_o \in I \subset I_o$.

Let G, A(t), A be as in (8.1.i). For every $(t,y) \in A$ we shall consider subsets $\tilde{\tilde{Q}}(t,y)$ of the $\tilde{z}$-space $E_{r+\sigma}$, $\tilde{z} = (z^1,\dots,z^{r+\sigma})$, $r \geq o$, $\sigma \geq o$, with the following properties:

1. There is an L-integrable function $\psi(t) \geq o$, $t \in G$, such that if $\tilde{z} = (z^1,\dots,z^r, z^{r+1},\dots,z^{r+\sigma}) \in \tilde{\tilde{Q}}(t,y)$, $(t,y) \in A$, then $z^i \geq \psi(t)$, $i = r+1,\dots,r+\sigma$.
2. If $\tilde{\bar{z}} = (\bar{z}^1,\dots,\bar{z}^r,\bar{z}^{r+1},\dots,\bar{z}^{r+\sigma}) \in \tilde{\tilde{Q}}(t,y)$ for some $(t,y) \in A$, then also every point $\tilde{z} = (\bar{z}^1,\dots,\bar{z}^r, z^{r+1},\dots,z^{r+\sigma})$ with $z^i \geq \bar{z}^i$, $i = r+1,\dots,r+\sigma$, belongs to $\tilde{\tilde{Q}}(t,y)$.

(8.2.i) <u>A second closure theorem.</u> Let I_o, G, A(t), A as above, A closed, and for every $(t,y) \in A$ let $\tilde{\tilde{Q}}(t,y)$ be a given subset of the $\tilde{z}$-space $E_{r+\sigma}$ satisfying properties 1 and 2 above, and in addition satisfying property (Q) in A. Let $y(t) = (y^1,\dots,y^s)$, $y_k(t) = (y_k^1,\dots,y_k^s)$, $t \in G$, be elements of $(L_1(G))^s$, let $\tilde{z}(t) = (z^1,\dots,z^{r+\sigma})$, $\tilde{z}_k(t) = (z_k^1,\dots,z_k^{r+\sigma})$, $t \in G$, be elements of $(L_1(G))^{r+\sigma}$, $k = 1,2,\dots$, and take $F^i(t) = \int_o^t z^i(\tau)d\tau$, $F_k^i(t) = \int_o^t z_k^i(\tau)d\tau$ for $t \in I_o$, with

$z^i(t) = z^i_k(t) = o$ for all $t \in I_o - G$, $k = 1,2,\ldots$. Let us assume that $y_k(t) \in A(t)$, $z_k(t) \in \tilde{\tilde{Q}}(t,y_k(t))$ a.e. in G, $k = 1,2,\ldots$, that $y^i_k \to y^i$ strongly in $L_1(G)$, $i = 1,\ldots,s$, that $z^i_k \to z^i$ weakly in $L_1(G)$, $i = 1,\ldots,s$, and that $F^i_k(t) \to F^i(t) + S^i(t)$ pointwise for all $t \in \{t_\rho\}$, $i = r + 1,\ldots,r + \sigma$, as $k \to \infty$, where $S^i(t)$, $t \in I_o$, are nonnegative singular functions on I_o. Then, $y(t) \in A(t)$, $\tilde{z}(t) \in \tilde{\tilde{Q}}(t,y(t))$ a.e. in G.

Proof. We denote by N the sum $N = r + \sigma$, and by $\tilde{z}(t) = (z^1,\ldots,z^{r+\sigma})$, $\tilde{z}_k(t) = (z^1_k,\ldots,z^{r+\sigma}_k)$, $t \in I_o$, the same functions above with $\tilde{z}(t) = \tilde{z}_k(t) = o$ for $t \in I_o$-G, $k = 1,2,\ldots$, and we take also $\psi(t) = o$ for $t \in I_o$-G. Note that $z^i_k(t) \geq - \psi(t)$, $t \in I_o$, $i = r + 1,\ldots,r + \sigma$, and that $\tilde{z}_k(t) \in \tilde{\tilde{Q}}(t,y_k(t))$ a.e. in G, $k = 1,2,\ldots$.

For every $t_o \in G$, $t_o = (t^1_o,\ldots,t^\nu_o)$, let $\delta_o = \delta_o(t_o) > o$ denote the distance of t_o from ∂G, and let $q = q_o = [\bar{t}, \bar{t}+h]$ denote any closed hypercube $g = [\bar{t}^j \leq t^j \leq \bar{t}^j + h$, $j = 1,\ldots,\nu]$, with $\bar{t}$, $\bar{t} + h \in \{t_\rho\}$, $o < h < \delta_o/\nu$, $\bar{t}^j \leq t^j_o \leq \bar{t}^j + h$, $j = 1,\ldots,\nu$. Then $t_o \in q \subset G$, and we denote here, for the sake of simplicity, by h also the ν-vector $(h,\ldots,h)$. Since $y_k \to y$ strongly in Y, there is a subsequence which converges also pointwise a.e. in G. For the sake of simplicity, let us denote this sequence still $[y_k]$. Then, for almost all $t_o \in G$, we have $y_k(t_o) \to y(t_o)$, $(t_o,y_k(t_o)) \in A$, $k = 1,2,\ldots$, where A is a closed set. As $k \to \infty$ we obtain $(t_o,y_o(t_o)) \in A$, or $g_o(t_o) \in A(t_o)$, and this relation holds for almost all $t_o \in G$.

Also, for almost all $t_o \in G$ we have, as $h \to o+$,

$$|q|^{-1} \int_q z^i(t)dt \to z^i(t_o) \quad , \quad i = 1,\ldots,r + \sigma , \tag{8.2.3}$$

$$|q|^{-1} \Delta_q S^i \to o \quad , \quad i = r+1,\dots,r+\sigma . \tag{8.2.4}$$

Because of the pointwise convergence $y_k(t) \to y(t)$ a.e. in G, we have $y_k(t) \to y(t)$ is a subset G_o of G with $|G_o| = |G|$, and we know that there are closed sets C_λ, $\lambda = 1,2,\dots$, with $C_\lambda \subset G_o$, $C_\lambda \subset C_{\lambda+1}$, $|C_\lambda| > |G_o| - \lambda^{-1}$, such that y(t) is continuous on C_λ and $y_k(t) \to y(t)$ uniformly on C_λ as $k \to \infty$ for each $\lambda = 1,2,\dots$. Since G is bounded, and $C_\lambda \subset G_o \subset G$, each set C_λ is compact, and hence y(t) is uniformly continuous on each C_λ.

Let λ be any fixed integer with $\lambda > |G|^{-1}$, hence $|C_\lambda| > o$. Let $\varepsilon > o$ be an arbitrary number. There is some $\delta_o' = \delta_o'(\varepsilon,\lambda) > o$ such that $|t-t'| \le \delta_o$, $t,t' \in C_\lambda$, implies $|y(t)-y(t')| \le \varepsilon/2$. Also, there is some $k(\varepsilon,\lambda) > o$ such that $k \ge k(\varepsilon,\lambda)$, $t \in C_\lambda$ implies $|y_k(t)-y(t)| \le \varepsilon/4$. Then, for $t,t' \in C_\lambda$, $|t-t'| \le \delta_o'(\varepsilon,\lambda)$, $k \ge k(\varepsilon,\lambda)$, we have also

$$|y_k(t)-y_k(t')| \le |y_k(t)-y(t)| + |y(t)-y(t')| + |y(t')-y_k(t')| \le$$
$$\le \varepsilon/4 + \varepsilon/2 + \varepsilon/4 = \varepsilon .$$

Let $X_\lambda(t)$, $X_\lambda^*(t)$, $t \in G$, be the characteristic functions of the sets C_λ and $G-C_\lambda$, so that $X_\lambda + X_\lambda^* = 1$ everywhere in G. All functions $X_\lambda(t)$ and $X_\lambda(t)\, z^j(t)$, $j = 1,\dots$, are of class $L_1(C_1)$, and for every $t_o \in C_\lambda$ we have $X_\lambda(t_o) = 1$, $X_\lambda^*(t_o)\, z^j(t_o) = o$.

Then, for almost all $t_o \in C_\lambda$ we have also, as $h \to o+$,

$$|q|^{-1} \int_q X_\lambda(t)dt \to X_\lambda(t_o) = 1 , \tag{8.2.5}$$

$$|q|^{-1} \int_q X_\lambda^*(t)\, z^i(t)dt \to X_\lambda^*(t_o)\, z^i(t_o) = o, \quad i = 1,\dots,r+\sigma, \tag{8.2.6}$$

$$|q|^{-1} \int_q X_\lambda^*(t)\, \psi(t)dt \to o . \tag{8.2.7}$$

Let C'_λ be the subset of C_λ where this occurs. Then C'_λ is measurable, $C'_\lambda \subset C_\lambda \subset G_o \subset G$, and, if G' denotes the set $\cup_\lambda C'_\lambda$, also

$$|C'_\lambda| = |C_\lambda| \geq |G_o| - \lambda^{-1} > o, \quad |G'| = |\cup_\lambda C'_\lambda| = |G_o| = |G| . \quad (8.1.12)$$

Let H and H^* be the sets

$$H = q \cap C_\lambda \quad , \quad H^* = q - H = q - q \cap C_\lambda = q - C_\lambda .$$

Let $\eta > o$ be any positive number independent of ε. Let t_o be any point of C'_λ , let $y_o = y(t_o)$, $\bar{z}_o = (z^o_o, z_o) = (z^o_o, z^1_o, \ldots, z^r_o) = \bar{z}(t_o)$, and let $M_1 = |\bar{z}(t_o)| + 1$. Let us fix $h > o$ so small that $h < \varepsilon/\nu$, $h < \delta_o/\nu$, $h < \delta'_o/\nu$, where $\delta_o = \delta_o(t_o)$, $\delta'_o = \delta'_o(\varepsilon,\lambda)$, and also so small that

$$|z^i(t_o) - |q|^{-1} \int_q z^i(t)dt| < \min [\eta N^{-1}, 1], \quad i = 1,\ldots,r + \sigma , \quad (8.2.8)$$

$$o \leq 1 - |q|/|H| \leq \min [\eta N^{-1} M_1^{-1}, 1] , \quad (8.2.9)$$

$$||q|^{-1} \int_q X^*_\lambda(t)\, z^i(t)dt | < \eta N^{-1}, \quad i = 1,\ldots,r, \quad (8.2.10)$$

$$||q|^{-1} \Delta_q S^i| < \eta N^{-1}, \quad i = r + 1,\ldots,r + \sigma , \quad (8.2.11)$$

$$|q|^{-1} \int_q X^*_\lambda(t)\, \psi(t)dt < \eta N^{-1} . \quad (8.2.12)$$

This is possible because of relations (8.2.3-7).

For $t \in H$, and $k \geq k(\varepsilon,\lambda)$ we have now

$$|t - t_o| \leq \nu h \leq \min [\varepsilon, \delta_o, \delta'_o],$$

$$|y_k(t) - y_o| = |y_k(t) - y_k(t_o)| + |y_k(t_o) - y(t_o)| \leq \varepsilon + \varepsilon = 2\varepsilon,$$

and hence $(t,y_k(t)) \in N_{3\varepsilon}(t_o,y_o)$ for $t \in H$ and $k \geq k(\varepsilon,\lambda)$. Therefore, we have also

$$\tilde{z}_k(t) \in \tilde{\tilde{Q}}(t_o,y(t_o),3\varepsilon) \text{ for } t \in H \text{ and } k \geq k(\varepsilon,\lambda) \;. \tag{8.2.13}$$

and finally

$$|H|^{-1} \int_H \tilde{z}_k(t)dt \in \text{cl co } \tilde{\tilde{Q}}(t_o,y_o;\, 3\varepsilon) \quad , \quad k \geq k(\varepsilon,\lambda) \;, \tag{8.2.14}$$

since the last set is convex and closed.

Because of the weak convergence of $z_k(t) = (z_k^1,\ldots,z_k)$ to $z(t) = (z^1,\ldots,z^r)$ in $(L_1(G))^r$, we can determine an integer $k' = k'(t_o,\varepsilon,\lambda,\eta,h) \geq k(\varepsilon,\lambda)$ such that, for $k \geq k'(t_o,\varepsilon,\lambda,\eta,h)$, we have

$$\left|\int_H z_k^i(t)dt - \int_H z^i(t)dt\right| \leq \eta N^{-1}|H| \quad , \quad i = 1,\ldots,r. \tag{8.2.15}$$

Now, for $k \geq k'(t_o,\varepsilon,\lambda,\eta,h)$ and $i = 1,\ldots,r$, we have

$$\begin{aligned} &|z^i(t_o) - |H|^{-1} \int_H z_k^i(t)dt| \\ &\quad \leq |z^i(t_o)-|q|^{-1}\int_q z^i(t)dt| + |(|q|^{-1}-|H|^{-1})\int_q z^i(t)dt| \\ &\quad + ||H|^{-1}\int_q(z^i(t)-z_k^i(t))dt| + |(|q|/|H|^{-1})\cdot|q|^{-1}\int_{H*} z^i(t)dt| \\ &\quad = d_1 + d_2 + d_3 + d_4 \;. \end{aligned} \tag{8.2.16}$$

By (8.2.8) we have $d_1 \leq N^{-1}\eta$, by (8.2.15) we have $d_3 \leq N^{-1}\eta$, and by (8.2.9), (8.2.10) we have $d_4 \leq 2N^{-1}\eta$. Also, by (8.2.8),(8.2.9) and the definition of M_1, we have

$$d_2 = |1-|q|\ |H||\cdot||q|^{-1}\int_q z^i(t)dt| \leq \eta N^{-1} M_1^{-1} \cdot M_1 = N^{-1}\eta .$$

Thus, (8.2.16) yields, for $k \geq k'(t_o,\varepsilon,\lambda,\eta,h)$,

$$|z^i(t_o) - |H|^{-1} \int_H z_k^i(t)dt| \leq 5N^{-1}\eta, \quad i = 1,\ldots,r. \tag{8.2.17}$$

We shall now obtain analogous estimates for $i = r + 1,\ldots,r + \sigma$. We know that $z_k^i(t) \geq -\psi(t)$, $t \in G$, $i = r+1,\ldots,r+\sigma$, and hence, by force of (8.2.12) we have for all k

$$|q|^{-1}\int_{H*} z_k^i(t)dt \geq -|q|^{-1} \int_q X*(t)\psi(t)dt \geq - N^{-1}\eta, \quad i = r+1,\ldots,r+\sigma . \tag{8.2.18}$$

Let $F_o^i(t) = F^i(t) + S^i(t)$, $t \in I_o$, $i = r+1,\ldots,r+\sigma$, so that $F_k^i(t) \to F_o^i(t)$ as $k \to \infty$ pointwise at all points $t \in \{t_\rho\}$. By (8.2.2) and the definitions of F_k and F we have now

$$|q|^{-1}\int_q z_k^i(t)dt = |q|^{-1}\Delta_q F_k^i, \quad |q|^{-1}\int_q z^i(t)dt = |q|^{-1}\Delta_q F^i, \quad i=r+1,\ldots,r+\sigma. \tag{8.2.19}$$

On the other hand, by $F_o^i = F^i + S^i$, we have

$$|q|^{-1}\Delta_q F_o^i = |q|^{-1}\Delta_q F^i + |q|^{-1}\Delta_q S^i , \quad i = r + 1,\ldots, r + \sigma . \tag{8.2.20}$$

The 2^ν vertices of q are points of $\{t_\rho\}$, hence $\Delta_q F_k^i \to \Delta_q F_o^i$ as $k \to \infty$, and we can determine the number $k'(t_o,\varepsilon,\lambda,\eta,h)$ above so that, for $k \geq k'(t_o,\varepsilon,\lambda,\eta,h)$ we have also

$$||q|^{-1}\Delta_q F_k^i - |q|^{-1}\Delta_q F_o^i| \leq N^{-1}\eta . \tag{8.2.21}$$

Finally, (8.2.19), (8.2.20), (8.2.21), and (8.2.11) yield

$$\big|\,|q|^{-1}\int_q z_k^i(t)dt - |q|^{-1}\int_q z^i(t)dt\big|$$

$$= \big|\,|q|^{-1}\Delta_q F_k^i - |q|^{-1}\Delta_q F^i\big|$$

$$\leq \big|\,|q|^{-1}\Delta_q F_k^i - |q|^{-1}\Delta_q F_o^i\big| + \big|\,|q|^{-1}\Delta_q S^i\big|$$

$$\leq N^{-1}\eta + N^{-1}\eta = 2N^{-1}\eta\ ,\ i = r+1,\ldots,r+\sigma\ . \tag{8.2.22}$$

We have now, for $i = r+1,\ldots,r+\sigma$ and $k \geq k'(t_o,\varepsilon,\lambda,\eta,h)$,

$$z^i(t_o) - |H|^{-1}\int_H z_k^i(t)dt \geq$$

$$\geq -\big|z^i(t_o) - |q|^{-1}\int_q z^i(t)dt\big|$$

$$-\big|(|q|^{-1} - |H|^{-1})\int_q z^i(t)dt\big|$$

$$-\big|(|q|/|H|)\cdot|q|^{-1}\int_q (z^i(t) - z_k^i(t))dt$$

$$-\big|(|q|/|H|)\cdot|q|^{-1}\int_{H*} z_k^i(t)dt\big|$$

$$= d_5 + d_6 + d_7 + d_8\ . \tag{8.2.23}$$

By (8.2.8) we have $d_5 > -N^{-1}\eta$, by (8.2.9) and (8.2.22) we have $d_7 > -4N^{-1}\eta$, and by (8.2.10), (8.2.18) we have $d_8 > -2N^{-1}\eta$. Finally, by (8.2.8), (8.2.9) and the definition of M_1, also

$$d_6 = -\big|(1-|q|/|H|)\cdot|q|^{-1}\int_q z^i(t)dt\big| \geq -N^{-1}\eta\, M_1^{-1}\cdot M_1 = -N^{-1}\eta\ .$$

Thus, (8.2.23) yields, for $k \geq k'(t_o,\varepsilon,\lambda,\eta,h)$,

$$z^i(t_o) - |H|^{-1} \int_H z_k^i(t)dt \geq - 8N^{-1}\eta \quad , \quad i = r+1,\dots,r+\sigma \ . \qquad (8.2.24)$$

Note that relations (8.2.17) and (8.2.24) can be written in vector form

$$\tilde{z}(t_o) = \tilde{z}_k + \zeta_k + \zeta_k^+ \quad , \quad z_k = |q|^{-1} \int_q z_k(t)dt \quad , \quad k \geq k'(t_o,\varepsilon,\lambda,\eta,h), \qquad (8.2.25)$$

where $|\zeta_k| \leq 8\eta$, $\zeta_k^+ = (o,\dots,o,\zeta_k^{r+1},\dots \zeta_k^{r+\sigma})$, $\zeta_k^{r+1} > o,\dots,\ \zeta_k^{r+\sigma} > o$.
By force of (8.2.14) we have then

$$\tilde{z}_k \in \text{cl co } \tilde{\tilde{Q}}(t_o,y_o,3\varepsilon) \ ,$$

hence, by property 2 of the sets $\tilde{\tilde{Q}}$, also

$$\tilde{z}_k + \zeta_k^+ \in \text{cl co } \tilde{\tilde{Q}}(t_o,y_o,3\varepsilon)$$

and finally

$$\tilde{z}(t_o) = \tilde{z}_k + \zeta_k + \zeta_k^+ \in (\text{cl co } \tilde{\tilde{Q}}(t_o,y_o,3\varepsilon))_{8\eta} \ .$$

Since η is arbitrary and the set in parenthesis is closed, we have

$$\tilde{z}(t_o) \in \text{cl co } \tilde{\tilde{Q}}(t_o,y_o,3\varepsilon)$$

and this relation holds for every $\varepsilon > o$. Then, by property (Q) we have also

$$\tilde{z}(t_o) \in \cap_{\varepsilon > o} \text{ cl co } \tilde{\tilde{Q}}(t_o,y_o,3\varepsilon) = \tilde{\tilde{Q}}(t_o,y_o) \ ,$$

for every point $t_o \in C_\lambda$. Thus, we have $\tilde{z}(t) \in \tilde{\tilde{Q}}(t,y,(t)\ )$ for all $t \in G' = \cup C_\lambda$, that is, almost everywhere in G.

8.3. A Third Closure Theorem for Orientor Fields

We consider here the case where some or all of the r-components z_k^i, $i=1,\dots,r$, of the vectors z_k, or $\tilde{z}_k$, of theorems (8.1.i), (8.2.i), converge strongly to z^i instead of weakly. As we shall see below if all components z_k^i, $i = 1,\dots,r$, converge strongly, then instead of property (Q) we need only require property (U). If, say, ρ components $z_k^1,\dots z_k^\rho$ converge weakly and the remaining $r-\rho$ components $z_k^{\rho+1},\dots,z_k^r$ converge strongly, then the intermediate property (Q_ρ) can be required. We discussed this property (Q_ρ) in Appendix A, and we proved there that property (Q_o) is equivalent to property (U), that property Q_r is equivalent to property (Q), and that property (Q_ρ) implies property $(Q_{\rho-1})$, $1 \le \rho \le r$.

Let us mention here that, if $\tilde{z}_o = (z_o^1,\dots,z_o^{r+\sigma})$ is any point of the $\tilde{z}$-space $E_{r+\sigma}$, $r \ge o$, $\sigma \ge o$, if ρ is any integer, $o \le \rho \le r \le r+\sigma$, and β any positive number, we denote by $N_\beta(\tilde{z}_o,\rho,r)$ and $N(\tilde{z}_o,\rho,r)$ the subsets of $E_{r+\sigma}$ defined by

$$N_\beta(\tilde{z}_o,\rho,r) = [\tilde{z} \in E_{r+\sigma} \,|\, |z^i - z_o^i| \le \beta,\ i = \rho+1,\dots,r] ,$$

$$N(\tilde{z}_o;\rho,r) = N_o(\tilde{z}_o,\rho,r) = [\tilde{z} \in E_{r+\sigma} \,|\, z^i = z_o^i ,\ i = \rho+1,\dots,r].$$

Equivalently, we could take $z_o' = (z_o^{\rho+1},\dots,z_o^r) \in E_{r-\rho}$ and then

$$N_\beta(\tilde{z}_o,\rho,r) = N_\beta(z_o) \times E_{r+\sigma-\rho} \quad , \quad N(\tilde{z}_o,\rho,r) = \{z_o'\} \times E_{r+\sigma-\rho} .$$

Given $G \subset E_\nu$, $A(t) \subset E_s$, $A \subset E_{\nu+s}$, $\tilde{\tilde{Q}}(t,y) \subset E_{r+\sigma}$ as in (8.2) we say that the sets $\tilde{\tilde{Q}}(t,y)$ satisfy property (Q_ρ) at the point $(t_o,y_o) \in A$ provided, for every point $\tilde{z}_o = (z_o^1,\dots,z_o^{r+\sigma}) \in E_{r+\sigma}$ we have

$$\tilde{\tilde{Q}}(t_o,y_o) \cap N(\tilde{z}_o,\rho,r) = \cap_{\delta>o} \cap_{\beta>o} \tilde{\tilde{Q}}(t_o,y_o;\delta) \cap N_\beta(\tilde{z}_o,\rho,r) .$$

(8.3.i) A third closure theorem for orientor fields. The same as (8.2.i) where the sets $\tilde{\tilde{Q}}(t,y) \subset E_{r+\sigma}$, $r \geq o$, $\sigma \geq o$, satisfy property (Q_ρ) for some ρ, $o \leq \rho \leq r \leq r + \sigma$, and where the n-$\rho$ components $z_k^i(t)$, $i = \rho + 1,\dots,r$, of $\tilde{z}_k(t)$, $t \in G$, $k = 1,2,\dots$, converge strongly to the components $z^i(t)$ of $\tilde{z}(t)$, $t \in G$, as $k \to \infty$. Then the same conclusion holds as in (8.2.i).

Proof. The initial steps are the same as for the proof of (8.2.i). Now $y_k \to y$ strongly in $(L_1(G))^s$, $z_k^i \to z^i$ strongly in $L_1(G)$, $i = \rho + 1,\dots,r$; hence, there is a subsequence, say still [k] for the sake of simplicity, such that $y_k \to y$, $z_k^i \to z^i$, $i = \rho + 1,\dots,r$, pointwise a.e. in G, say in a subset G_o of G with $|G_o| = |G|$. There are now closed subsets C_λ, $\lambda = 1,2,\dots$, with $C_\lambda \subset G_o$, $C_\lambda \subset C_{\lambda+1}$, $|C_\lambda| > |G_o| - \lambda^{-1}$, such that all $y(t) = (y^1,\dots,y^s)$, $z^i(t)$, $i = \rho + 1,\dots,r$, are continuous on C_λ, and $y_k(t) \to y(t)$, $z_k^i(t) \to z^i(t)$, $i = \rho + 1,\dots,r$, uniformly on C_λ as $k \to \infty$. As in the proof of (8.2.i) we take $\lambda > |G|^{-1}$, so that $|C_\lambda| > o$, and two numbers $\varepsilon > o$, $\beta > o$ arbitrary. Then there is some $\delta_o' = \delta_o'(\varepsilon,\lambda,\beta) > 0$ so that $|t-t'| \leq \delta_o, t,t' \varepsilon C_\lambda$ implies $(|y(t) - y(t')| \leq \varepsilon/2,) |z^i(t) - z^i(t')| \leq \beta/2$, $i = \rho + 1,\dots,r$. Also, we choose $k(\varepsilon,\lambda,\beta) > o$ such that $k \geq k(\varepsilon,\lambda,\beta)$, $t \in C_\lambda$ implies $|y_k(t)-y(t)| \leq \varepsilon/4$, $|z_k^i(t) - z^i(t)| \leq \beta/4$. Then, for $t, t' \varepsilon C_\lambda$, $|t-t'| \leq \delta(\varepsilon,\lambda,\beta)$, $k \geq k(\varepsilon,\lambda.\beta)$ we have, as in the proof of (5.2.i),

$$|y_k(t)-y_k(t')| \leq \varepsilon, \quad |z_k^i(t) - z_k^i(t')| \leq \beta, \quad i = \rho + 1,\dots,r.$$

We proceed now as for (8.2.i): we define the sets C'_λ, we take any $t_o \in C'_\lambda$, and we fix $h > o$ so that $h < \varepsilon/\nu$, $h < \delta_o/\nu$, $h < \delta'_o/\nu$, where $\delta_o = \delta_o(t_o)$, $\delta'_o = \delta'_o(\varepsilon,\lambda,\beta)$, and so that the corresponding hypercubes q of side length h and the sets $H = q \cap C_\lambda$ satisfy relations (8.2.8-12). Thus, relations (8.2.13) holds now for all $t \in H$ and $k \geq k(\varepsilon,\lambda,\beta)$. On the other hand, for $t \in H = q \cap C_\lambda$, $k \geq k(\varepsilon,\lambda,\beta)$, we certainly have

$$|z_k^i(t) - z^i(t)| \leq \beta, \quad |z^i(t)-z^i(t_o)| \leq \beta \ , \ i = \rho + 1,\dots,r,$$

and hence

$$|z_k^i(t) - z^i(t_o)| \leq 2\beta \quad , \quad i = \rho + 1,\dots,r \ .$$

Thus, relation (8.2.13) can now be rewritten in the stronger form

$$\tilde{z}_k(t) \in \tilde{\tilde{Q}}(t_o,y_o;3\varepsilon) \cap N_{2\beta}(\tilde{z}(t_o),r,\rho),$$

and consequently (8.2.14) becomes

$$|H|^{-1} \int_H \tilde{z}_k(t)dt \in \text{cl co } [\tilde{\tilde{Q}}(t_o,y_o;3\varepsilon) \cap N_{2\beta}(\tilde{z}(t_o),r,\rho)] \qquad (8.2.27)$$

for all $k > k(\varepsilon,\lambda,\beta)$.

Note that strong convergence in $L_1(G)$ implies weak convergence. Thus the arguments of (8.2.i) concerning the components z_k^i, $i = 1,\dots,r + \sigma$, hold without change. In other words, we can determine some number $k'(t_o,\varepsilon,\lambda,\eta,h) \geq k(\varepsilon,\lambda,\beta)$ such that, for $k \geq k'(t_o,\varepsilon,\lambda,\beta,\eta,h)$, we have

$$|z^i(t_o) - |H|^{-1} \int_H z_k^i(t)dt \ | \leq 5N^{-1}\eta \ , \ i = 1,\dots,r \ ,$$

$$z^i(t_o) - |H|^{-1} \int_H z_k^i(t)dt \geq - 8N^{-1}\eta, \; i = r + 1,\ldots,r + \sigma .$$

Thus, the same relation (8.2.25) holds

$$\tilde{z}(t_o) = \tilde{z}_k + \zeta_k + \zeta_k^+ \;, \quad \tilde{z}_k = |H|^{-1} \int_H \tilde{z}_k(t)dt \;,$$

as in the proof of (8.2.i), where $|\zeta_k| > 8\eta$, $\zeta_k^{+i} \geq o$, $i = r + 1,\ldots,r + \sigma$, $k \geq k'(t_o,\varepsilon,\lambda,\beta,\eta,h)$, and where the point z_k is now in the right hand set (8.2.27). Thus, instead of (8.2.26) we have

$$\tilde{z}(t_o) \in (\text{cl co } [\tilde{\tilde{Q}}(t_o,y_o;3\varepsilon) \cap N_{2\beta}(\tilde{z}(t_o),r,\rho)])_{8\eta}$$

where ε,β,η are arbitrary positive numbers. Finally

$$\tilde{z}(t_o) \in \text{cl co } [\tilde{\tilde{Q}}(t_o,y_o,3\varepsilon) \cap N_{2\beta}(\tilde{z}(t_o),r,\rho)]$$

where ε and β are arbitrary. By property (Q_ρ) we have

$$\tilde{z}(t_o) \in \bigcap_{\varepsilon>o} \bigcap_{\beta>o} \text{cl co } [\tilde{\tilde{Q}}(t_o,y_o;3\varepsilon) \cap N_{2\beta}(\tilde{z}(t_o),r,\rho)]$$

$$\tilde{z}(t_o) \in \tilde{\tilde{Q}}(t_o,y_o) \quad \cap \quad N(\tilde{z}(t_o),r,\rho) .$$

Thus, $\tilde{z}(t_o) \in Q(t_o,y(t_o))$ for every $t_o \in C_\lambda{}'$ where λ is arbitrary, and finally $\tilde{z}(t) \in \tilde{\tilde{Q}}(t,y(t))$ a.e. in G.

8.4 Closure Theorems for Control Problems

We shall use here again the notation of (7.1). Thus, G is a bounded open subset of the t-space E_ν, $t = (t^1,\ldots,t^\nu)$, and for every $t \in$ cl co G a subset $A(t)$ of the y-space E_s is assigned. Let A denote the set of all (t,y) with $t \in$ cl G, $y \in A(t)$. For every $(t,y) \in A$ a subset $U(t,y)$ of the u-space

E_m is assigned, and M denotes the set of all (t,y,u) with $(t,y) \in A$, $u \in U(t,y)$. Let $f(t,y,u) = (f_1,\dots,f_r)$ be a given vector function on M. For every $(t,y) \in A$ let $Q(t,y)$ denote the set

$$Q(t,y) = [z \in E_r \mid z = f(t,y,u),\ u \in U(t,y)] \subset E_r .$$

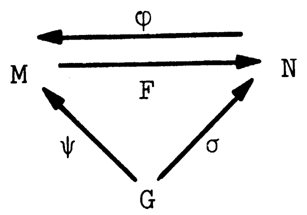
We first state and prove the implicit function theorem for orientor fields which we need below and which is similar to statement (1.6.i).

(8.4.i) An implicit function theorem for orientor fields. Let G, A(t), A, Q(t,y) be as above, G open and bounded, A and M closed, f continuous on M. If $z(t) = (z^1,\dots,z^r)$, $y(t) = (y^1,\dots,y^s)$, $t \in G$, are any two measurable vector functions such that $y(t) \in A(t)$, $z(t) \in Q(t,y(t))$ a. e. in G, then there is some measurable vector function $u(t) = (u^1,\dots,u^m)$, $t \in G$, such that

$$y(t) \in A(t),\ u(t) \in U(t,y(t)),\ z(t) = f(t,y(t),\ u(t)) \text{ a. e. in } G.$$

Proof. The proof is only a modification of the one for (1.6.1) in Section 1. First, M is a (closed) subset of the Euclidean space $E_{\nu+s+m}$; hence, a metric space which is the union of countably many compact subsets. Let $F: M \to N$ be the continuous mapping defined by $(t,y,u) \to (t,y,f(t,y,u))$ so that $N = F(M)$ is the subset of all $(t,y,z) \in E_{\nu+s+r}$ with $(t,y) \in A$, $z = f(t,y,u)$, $u \in U(t,y)$, or $(t,y) \in A$, $z \in Q(t,y)$. Finally, let $\sigma: G \to N$ be the measurable map defined by $t \to (t,y(t),z(t))$. If $\varphi: N \to M$ denotes the partial inverse of F defined in (1.7.iii) such that $F\varphi$ is the identity map on N, then the map $\psi: G \to M$ defined by $\psi = \varphi\sigma$ is a mapping $t \to (t,y(t),u(t))$ such that $z(t) = f(t,y(t),u(t))$ a. e.

in G, $u(t)$, $t \in G$, is measurable, and $u(t) \in U(t,y(t))$ a.e. in G.



Let $Y = (L_1(G))^s$, $Z = (L_1(G))^r$, and let T denote the set of all measurable vector functions $u(t) = (u^1,\dots,u^m)$, $t \in G$.

(8.4.ii) <u>A closure theorem in G</u>. Let G be open and bounded, A and M closed, f continuous on M, and let us assume that the sets $Q(t,y)$ satisfy property (Q) in A. If y_k, z_k, u_k, $k = 1,2,\dots$, is a sequence of elements $y_k \in Y$, $z_k \in Z$, $u_k \in T$, with $y_k(t) \in A(t)$, $u_k(t) \in U(t,y_k(t))$, $z_k(t) = f(t,y_k(t), u_k(t))$ a.e. in G, $k = 1,2,\dots$, and $y_k \to y$ strongly in Y and $z_k \to z$ weakly in Z, then there is at least one element $u \in T$ such that $y(t) \in A(t)$, $u(t) \in U(t,y(t))$, $z(t) = f(t,y(t), u(t))$ a. e. in G.

<u>Proof</u>. First the elements y_k, z_k, $k = 1,2,\dots$, satisfy the orientor field relations $y_k(t) \in A(t)$, $z_k(t) \in Q(t,y_k(t))$ a. e. in G, $k = 1,2,\dots$. Hence, by force of closure theorem (8.1.i) we have also $y(t) \in A(t)$, $z(t) \in Q(t,y(t))$ a. e. in G. Finally, by force of the implicit function theorem (8.4.i) there is a measurable vector function $u: G \to E_m$, or $u \in T$, such that $u(t) \in U(t,y(t))$ and $z(t) = f(t,y(t), u(t))$ a. e. in G.

Closure theorem (8.4.ii) has an analogous version for boundary controls.

Let $\Gamma \subset \partial G$ be a set which can be decomposed into finitely many nonoverlapping parts $\Gamma_1,\dots,\Gamma_N$, each Γ_j being the image under a transformation $\mathcal{T}: I \to \Gamma_i$ of class K of the unit interval I of dimension ν-1, so that a natural area measure function σ is defined on Γ.

As usual we assume that for any $t \in \Gamma$ a subset $B(t)$ of the $\mathring{y}$-space $E_{s'}$ is assigned, and we denote by B the subset of $E_{\nu+s'}$ of all $(t,\mathring{y})$ with $t \in \Gamma$, $\mathring{y} \in B(t)$. For every $(t,\mathring{y}) \in B$ let $V(t,\mathring{y})$ be a given subset of the v-space $E_{m'}$, and let $\mathring{M}$ be the subset of $E_{\nu+s'+m'}$ of all $(t,\mathring{y},v)$ with $(t,\mathring{y}) \in B$, $v \in V(t,\mathring{y})$. Let $g(t,\mathring{y},v) = (g_1,\dots,g_{r'})$ be a given function on $\mathring{M}$, and for every $(t,\mathring{y}) \in B$ let $R(t,\mathring{y})$ denote the set of all $\mathring{z} = (\mathring{z}^1,\dots,\mathring{z}^{r'})$ with $\mathring{z} = g(t,\mathring{y},v)$, $v \in V(t,\mathring{y})$.

Let $\mathring{Y} = (L_1(\Gamma))^{s'}$, $\mathring{Z} = (L_1(\Gamma))^{r'}$, and let $\mathring{T}$ be the set of all σ-measurable vector functions $v(t) = (v^1,\dots,v^{m'})$, $t \in \Gamma$.

(8.4.iii) <u>A closure theorem on Γ.</u> Let Γ be as above, $\Gamma, B, \mathring{M}$ closed, g continuous on $\mathring{M}$, and let us assume that the sets $R(t,y)$ satisfy property (Q) in B. If $\mathring{y}_k$, $\mathring{z}_k$, v_k, $k = 1,2,\dots$, is a sequence of elements $\mathring{y}_k \in \mathring{Y}$, $\mathring{z}_k \in \mathring{Z}$, $v_k \in \mathring{T}$, with $\mathring{y}_k(t) \in B(t)$, $v_k(t) \in V(t, \mathring{y}_k(t))$, $\mathring{z}_k(t) = g(t,\mathring{y}_k(t), v_k(t))$ σ-a.e. in Γ, $k = 1,2,\dots$, and $\mathring{y}_k \to \mathring{y}$ strongly in $\mathring{Y}$ and $\mathring{z}_k \to \mathring{z}$ weakly in $\mathring{Z}$, then there is at least one element $v \in \mathring{T}$ such that $\mathring{y}(t) \in B(t)$, $v(t) \in V(t,\mathring{y}(t))$, $\mathring{z}(t) = g(t,\mathring{y}(t),v(t))$ σ-a.e. in Γ.

<u>Proof</u>. Here Γ is the finite union of sets Γ_i. We can limit ourselves to one set Γ_i. Note that Γ_i is the image of an interval $I \subset E_{\nu-1}$ under a map $\mathcal{T}_i: I \to \Gamma_i$, say $t = t(\tau)$, $\tau \in I$, of class K. Thus, sets of σ-measure zero on Γ_i and sets of measure zero on I correspond. Note that $\mathring{y}_k(t) \to \mathring{y}(t)$ strongly

in $L_1(\Gamma)$ (with respect to the measure σ) if and only if $\mathring{y}_k(t(\tau)) \to \mathring{y}(t(\tau))$ strongly in $L_1(I)$; analogously, $\mathring{z}_k(t) \to \mathring{z}(t)$ weakly in $L_1(\Gamma)$ if and only if $\mathring{z}_k(t(\tau)) \to \mathring{z}(t(\tau))$ weakly in $L_1(I)$. The problem under consideration, which concerns the orientor field

$$\mathring{y}(t) \in B(t),\ \mathring{z}(t) \in R(t,\mathring{y}(t)) \qquad \sigma\text{-a.e. in } \Gamma,$$

where $R(t,\mathring{y}) = g(t,\mathring{y},V(t,\mathring{y}))$, is now transformed into a problem concerning the orientor field

$$\mathring{y}(t(\tau)) \in B(t(\tau)),\ \mathring{z}(t(\tau)) \in R(t(\tau)),\ \mathring{y}(t(\tau)) \qquad \text{a.e. in } I.$$

We have proved in (App. A) that, the sets $R(t,\mathring{y})$ have property (Q) in Γ_i if and only if the sets $R(t(\tau),\mathring{y})$ have property (Q) in I. We can now, therefore, apply theorem (8.4.i), or, equivalently, first closure theorem (8.1.i) for orientor fields, and then implicit function theorem (8.4.i).

8.5. Lower Closure Theorems

Let G, A(t), A, U(t,y), M be as in (8.4), and let $f(t,y,u) = (f_1,\dots,f_r)$, $f_o(t,y,u)$ be continuous functions on M. For every $(t,y) \in A$ let $\tilde{\tilde{Q}}(t,y)$ denote the set

$$\tilde{\tilde{Q}}(t,y) = [(z^o,z) \in E_{r+1} \mid z^o \geq f_o(t,y,u),\ z = f(t,y,u),\ u \in U(t,y)] \subset E_{r+1}.$$

As before let $Y = (L_1(G))^s$, $Z = (L_1(G))^r$, and let T be the set of all measurable vector-functions $u(t) = (u^1,\dots,u^m)$, $t \in G$.

We say that the property of lower closure holds in G provided the following occurs: For every sequence y_k, z_k, u_k, $k = 1,2,\dots,$ of elements $y_k \in Y$, $z_k \in Z$,

$u_k \in T$, with $y_k(t) \in A(t)$, $u_k(t) \in U(t,y_k(t))$, $z_k(t) = f(t,y_k(t),u_k(t))$ a.e. in G, with $z_k^o(t) = f_o(t,y_k(t),u_k(t)) \in L_1(G)$, and $y_k \to y$ strongly in Y, $z_k \to z$ weakly in Z, and $\underline{\lim} \int_G z_k^o(t)dt < +\infty$, then there is at least one element $u \in T$ such that $y(t) \in A(t)$, $u(t) \in U(t,y(t))$, $z(t) = f(t,y(t),u(t))$ a.e. in G, $z^o(t) = f_o(t,y(t), u(t)) \in L_1(G)$, and $\int_G z^o(t)dt \le \underline{\lim} \int_G z_k^o(t)dt$.

(8.5.i) <u>A lower closure theorem in G.</u> Let G be open and bounded, let $A(t)$, A, $U(t,y)$, M as in (8.4), A and M closed, let f, f_o be continuous on M, and let us assume that the sets $\tilde{\tilde{Q}}(t,y)$ have property (Q) in A, and that $f_o(t,y,u) \ge -\psi(t)$ for all $(t,y,u) \in M$ and some $\psi \ge o$, $\psi \in L_1(G)$. Then, the lower closure property holds.

<u>Proof.</u> First, let us consider an interval I_o containing G. It is not restrictive to assume $I_o = [o \le t^j \le b,\ j = 1,\dots,\nu]$, or $I_o = [o,b]$, where o and b denote the ν-vectors $(o,\dots,o)$, $(b,\dots,b)$.

We shall now introduce the auxiliary variables z^o and u^o, the vector $\tilde{u} = (u^o,u^1,\dots,u^m) = (u^o,u)$, the vector function $\tilde{\tilde{f}}(t,y,\tilde{u}) = (\tilde{f}_o,f_1,\dots,f_r)$ with $\tilde{f}_o = u^o$, the control space $\tilde{U}(t,y) = [\tilde{u} = (u^o,u) \mid u^o \ge f_o(t,y,u),\ u \in U(t,y)] \subset E_{m+1}$, and the functional relations

$$z(t) = f(t,y(t), u(t)) \quad , \quad z^o(t) = \tilde{f}_o = u^o(t) \quad \text{a. e. in } G,$$

where $y(t) = (y^1,\dots,y^s)$, $z(t) = (z^1,\dots,z^r)$, $t \in G$,

and we shall take

$$F^o(t) = \int_o^t z^o(\tau)d\tau ,$$

where the integration ranges over the interval $[o,t] \subset I_o$, and $z^o(\tau) = o$ for $\tau \in I_o - G$.

In the new notations the usual sets $Q(t,y) \subset E_r$, $\tilde{\tilde{Q}}(t,y) \subset E_{r+1}$ have now the interpretation

$$\tilde{\tilde{Q}}(t,y) = [\tilde{z} = (z^o,z) \mid z^o \geq f_o(t,y,u),\ z = f(t,y,u),\ u \in U(t,y)]$$

$$= \tilde{\tilde{f}}(t,y,\ \tilde{U}(t,y)) \subset E_{r+1}\ ,$$

$$Q(t,y) = [z \mid z = f(t,y,u),\ u \in U(t,y)] = f(t,y,\ U(t,y)) \subset E_r\ .$$

It is convenient to define f_o and $\tilde{f}_o$ outside G by taking $f_o = \tilde{f}_o = o$ for $t \in I_o - G$

Note that $f_o(t,y,u) \geq -\psi(t)$ for all $(t,y,u) \in M$; hence

$$\int_G z^o(t)dt = \int_G f_o(t,y(t),u(t))dt \geq -\int_G \psi(t)dt = -L,$$

where $L \geq o$ is a fixed number. Hence, if x_k, u_k, $k = 1,2,\dots$, is any sequence of admissible pairs with $x_k \to x$ weakly in S, $\underline{\lim} \int_G z^o(t)dt < +\infty$ as $k \to \infty$, and we denote by i this $\underline{\lim}$, then $-L \leq i < +\infty$, that is, i is finite. We take a subsequence, say still [k] for the sake of simplicity, so that $\int_G z_k^o(t)dt \to i$ as $k \to \infty$, and we can even assume that $\int_G z_k^o(t)dt \leq i + k^{-1} \leq i + 1$ for all k.

We take $z_k^o(t) = \tilde{f}_o = u_k^o(t) = f_o(t,y_k(t),u_k(t))$, $t \in G$, and

$$F_k^o(t) = \int_o^t u_k^o(\tau)d\tau = \int_o^t f_o(\tau,y_k(\tau),\ u_k(\tau))d\tau, \quad t \in I_o,\ k = 1,2,\dots\ .$$

Then $z_k \to z$ weakly in Z, $y_k \to y$ strongly in Y, and

$y_k(t) \in A(t)$, $(z_k^o(t), z_k(t)) \in \tilde{\tilde{Q}}(t,y_k(t)) \subset E_{r+1}$ a.e. in G, $k = 1,2,\dots$.

Let $z_k^{o+}(t) = f_o(t,y_k(t),u_k(t)) + \psi(t)$ for $t \in G$, $z_k^{o+}(t) = o$ for $t \in I_o-G$, and note that $z_k^{o+}(t) \geq o$ in I_o, and $\int_{I_o} z_k^{o+}(t)dt \leq L + i + 1$. Let $F_k^+(t)$, $t \in I_o$, be defined by $F_k^+(t) = \int_o^t z_k^{o+}(\tau)\, d\tau$, $t \in I_o$. Let $z^-(t) = \psi(t)$ for $t \in G$, $z^-(t) = o$ for $t \in I_o-G$, and let $F^-(t)$, $t \in I_o$, be defined by $F^-(t) = \int_o^t z^-(\tau)d\tau$. Then

$$F_k^o(t) = \int_o^t f_o(\tau,y_k(\tau), u_k(\tau))d\tau = F_k^+(t) - F^-(t), \quad t \in I_o .$$

Here $F^-(t)$ is a fixed nonempty continuous function of t in I_o, while the functions $F_k^+(t)$ are nonnegative continuous functions of t in I_o with a common bound

$$o \leq F_k^+(t) \leq F_k^+(b) = \int_{I_o} z_k^{o+}(t)dt \leq L + i + 1 .$$

For every interval $I \subset R$ we denote as usual by $\Delta_I F_k^+$ the differences of order ν of F_k^+ relative to the 2^ν vertices of I; hence

$$\Delta_I F_k^+ = \int z_k^{o+}(t)dt \geq o \quad , \quad \Delta_I F^- = \int \psi(t)dt \geq o .$$

Of course, each of the interval functions $\Delta_I z_k^+$ is also absolutely continuous, but this does not play an essential part below. It interests here to know that the nonnegative interval functions $\Delta_I z_k^+$ are nonnegative, additive, and of uniform bounded variations, the total variations being all $\leq L + i + 1$.

Let us consider the countable lattice point $\{t_\rho\}$, or $t = b\rho \in I_o$, $\rho = (\rho_1,\dots,\rho_\nu)$, where $\rho_1,\dots,\rho_\nu$ are arbitrary rational numbers, $o \leq \rho_j \leq 1$, $j = 1,\dots,\nu$. Let $\{I\}$ be the countable system of intervals $I \subset R$ whose vertices are points $t \in \{t_\rho\}$. We may order the points t_ρ into a sequence. Since the functions $z_k^+(t)$, $t \in R$, are uniformly bounded in R, (and, hence, at each $t = t_\rho$), we can successively select subsequences which are convergent at $t = t_\rho$, and then, by the diagonal process, we can select a unique subsequence, say $[k_s]$, of integers k_s, such that the limits $z_{k_s}^+(t) \to F^+(t)$ as $s \to \infty$ exist for every $t \in \{t_\rho\}$. Then, the interval functions $\Delta_I z_k^+ \to \Delta_I F^+$ also have limits as $s \to \infty$ for every interval I of the countable collection {I}. The limits $\Delta_I z^+$ is a nonnegative additive interval function for $I \in \{I\}$, and $\Delta_I z^+$ has bounded variation, namely a total variation $\leq L + i + l$ (for I describing the collection {I}).

For every $t_o \in I_o$ and interval $I \in \{I\}$ with $t_o \in I$ we may consider the quotients $\Delta_I F^+/\text{meas } I$. We know that for I a hypercube, $I \in \{I\}$, $t_o \in I$, and almost all $t_o \in I_o$, the limit $\Delta_I F^+/\text{meas } I \to z(t_o)$ exists as diam $I \to o$ [App. B.6.i, in conjunction with B.1. Remark 2, case 3]. Moreover, z(t) is finite almost everywhere in I_o, nonnegative, L-integrable in I_o, and zero in I-G. Furthermore, if

$$F(t) = \int_o^t z(\tau)d\tau \quad , \quad t \in I_o \, ,$$

hence

$$\Delta_I F = \int_I z(t)dt \quad , \quad I \subset I_o \, .$$

then $o \leq \Delta_I F \leq \Delta_I F^+$ for every $I \in \{I\}$, and the difference $\Delta_I S = \Delta_I F^+ - \Delta_I F$ is a nonnegative interval function for $I \in \{I\}$, of bounded variation, and singular, that is, $\Delta_I S/|I| \to o$ as diam $I \to o$ for almost all $t_o \in I_o$ (with $t_o \in I \subset I_o$, $I \in \{I\}$, I a hypercube). Equivalently, F^+ possesses a Lebesgue decomposition

$F^+(t) = F(t) + S(t)$, F an integral function, S singular, $S \geq o$, $\Delta_I S \geq o$.
By subtracting the fixed function F^- we have

$$F^o(t) = F^+(t) - F^-(t) = (F^+(t) - F^-(t)) + S(t),$$

or

$$F^o(t) = F(t) + S(t),$$

where $F = F^+ - F^-$ is an integral function, S is singular, ($S \geq o$, $\Delta_I S \geq o$ for $I \in \{I\}$, and

$$F(t) = \int_o^t (z(\tau) - \psi(\tau))d\tau , \quad t \in I_o .$$

We can now apply closure theorem (8.2.i) with $\sigma = 1$, $N = r + 1$. We conclude that

$$y(t) \in A(t), \ (z^o(t), z(t)) \in \tilde{\tilde{Q}}(t,y(t)) \quad \text{a.e. in } G .$$

Then, by implicit function theorem (8.4.i), there is a measurable function $\tilde{u}(t) = (u^o,u) = (u^o,u^1,\ldots,u^m)$, $t \in G$, such that

$$y(t) \in A(t), \quad u(t) \in U(t,y(t)),$$

$$u^o(t) \geq z^o(t) \geq f_o(t,y(t), u(t)) \geq - \psi(t),$$

$$z(t) = f(t,y(t),u(t)).$$

Note that z^o is certainly integrable in G, and so is $f_o(t,y(t),u(t))$ and

$$\int_G z^o(t)dt \leq \int_G f_o(t,y(t),u(t)dt \leq \int_G u^o(t)dt = \int_{I_o} \tilde{f}_o dt$$

$$= F(b) = F^o(b) - S(b) = \lim_{s\to\infty} F^o_{k_s}(b) = S(b) \leq i - S(b) \leq i.$$

Theorem (8.5.i) is thereby proved.

Statement (8.5.i) has an analogous version for boundary controls. Let $\Gamma \subset \partial G$, $B(t)$, B, $V(t,\mathring{y})$, $\mathring{M}$, $\mathring{Y}$, $\mathring{Z}$, $\mathring{T}$ as in (8.4.iii). Let $g(t,\mathring{y},v) = (g_1,\dots,g_{r'})$, $g_o(t,\mathring{y},v)$ be continuous functions on M. For every $(t,\mathring{y}) \in B$ let $\tilde{\tilde{R}}(t,\mathring{y})$ denote the sets

$$\tilde{\tilde{R}}(t,\mathring{y}) = [(z^o,z) \in E_{r'+1} \mid z^o \geq g_o(t,\mathring{y},v),\ z = g(t,\mathring{y},v),\ v \in V(t,\mathring{y})] \subset E_{r'+1}$$

We say that the property of lower closure holds in Γ provided the following occurs: For every sequence y^o_k, z^o_k, v_k, $k = 1,2,\dots$, of elements $y^o_k \in \mathring{Y}$, $z^o_k \in \mathring{Z}$, $v_k \in \mathring{T}$, with $y^o{}_k(t) \in B(t)$, $v_k(t) \in V(t,y^o_k(t))$, $z^o_k(t) = g(t,y^o_k(t), v_k(t))$ σ-a.e. in Γ, with $z^o_k(t) = g_o(t,y^o_k(t), v_k(t)) \in L_1(\Gamma)$, and $\mathring{y}_k \to \mathring{y}$ strongly in $\mathring{Y}$, $z^o_k \to \mathring{z}$ weakly in $\mathring{Z}$, and $\underline{\lim} \int_\Gamma z^o_k(t)\, d\sigma < +\infty$, then there is at least one element $v \in \mathring{T}$ such that $\mathring{y}(t) \in B(t)$, $v(t) \in V(t,\mathring{y}(t))$, $\mathring{z}(t) = g(t,\mathring{y}(t),v(t))$ σ-a.e. in Γ, $z^o(t) = g(t,\mathring{y}(t), v(t)) \in L_1(G)$, and $\int z^o(t)d\sigma \leq \underline{\lim} \int z^o_k(t)d\sigma$.

(8.5ii) <u>A lower closure theorem on ∂G.</u> Let Γ, $B(t)$, B, $V(t,\mathring{y})$, $\mathring{M}$ as in (8.4.iii), B, M closed, let g,g_o be continuous on $\mathring{M}$, and let us assume that the sets $\tilde{\tilde{R}}(t,\mathring{y})$ have property (Q) in Γ, and that the following condition holds (ψ): $g_o(t,\mathring{y},v) \geq -\psi(t)$ for all $(t,\mathring{y},v) \in \mathring{M}$ and some $\psi \geq o$, $\psi \in L_1(\Gamma)$. Then, the lower closure property on Γ holds.

Proof. As in the proof of (8.4.iii) we consider only one set Γ_i as the image of an interval $I \subset E_{\nu+1}$ under a transformation $\mathcal{T}_i$ of class K, or $t = t(\tau)$,

$\tau \in I$. Then

$$\int_\Gamma z^o(t)d\sigma = \int_I z^o(t(\tau))\, J(\tau)d\tau ,$$

where $J(\tau)$ (Jacobian) is a measurable function on I with $0 < K^{-1} \leq J(\tau) \leq K < +\infty$ for some constant $K \geq 1$. Thus, the given problem on Γ can be transformed into an analogous problem on I concerning the integral (7.1.1) and the functional relations

$$\mathring{y}(t(\tau)) \in B(t,\tau),\ z^o(t(\tau))J(\tau) = f(t(\tau),\ \mathring{y}(t(\tau)),\ v(t(\tau))J(\tau) ,$$

$$v(t(\tau)) \in V(t(\tau),\ \mathring{y}(t(\tau)) \quad \text{a.e. in } I .$$

Now, sets of σ-measure zero in Γ_i and sets of measure zero on I correspond. Also, $\mathring{y}_k(t) \to \mathring{y}(t)$ strongly in $L_1(\Gamma)$ (with respect to the measure σ) if and only if $\mathring{y}_k(t(\tau)) \to \mathring{y}(t(\tau))$ strongly in $L_1(I)$; and $z^o_k(t) \to z^o(t)$ weakly in $L_1(\Gamma)$ if and only if $z^o_k(t(\tau))J(\tau) \to z^o(t(\tau))J(\tau)$ weakly on $L_1(I)$. The problem under consideration, which concerns the orientor field

$$\mathring{y}(t) \in B(t), \quad (z^o(t),\ z\ (t)) \in \tilde{\tilde{R}}(t,\mathring{y}(t)) \quad \sigma\text{-a.e. in } \Gamma,$$

where $\tilde{\tilde{R}}(t,\mathring{y}) = [z^o \geq g_o(t,\mathring{y},v),\ z = g(t,\mathring{y},v),\ v \in V(t,\mathring{y})$, is now transformed into a problem concerning the orientor field

$$\mathring{y}(t(\tau)) \in B(t(\tau)) ,\ (z^o(t(\tau))J(\tau),\ z(t(\tau)J(\tau)) \in R^*(\tau,\mathring{y}(t(\tau))),\ \text{a.e. in } I,$$

where $R^*(\tau,\mathring{y}(t(\tau)) = [z^o \geq g(t(\tau),\mathring{y},v)\ J(\tau),\ z = g(t(\tau),\mathring{y},v)J(\tau),\ v \in V(t(\tau),\mathring{y})]$. We have proved in (App. A) that the sets $\tilde{\tilde{R}}(t,\mathring{y})$ have property (Q) in Γ_i if and only if the sets $\tilde{\tilde{R}}(t(\tau),\mathring{y})$ have property (Q) in I, and this in turn if and only if the sets $R^*(\tau,\mathring{y})$ have property (Q) in I. We can now, therefore, apply theorem (8.5.i).

Remark. Under the same hypotheses of (8.3.i) the following is true: If x_k, u_k, $k = 1,2,\dots$, is any sequence of admissible pairs with $x_k \to x$ weakly in S and $\underline{\lim}\ I\,[x_k,u_k] < +\infty$, then there is some $u \in T$ such that the pair x,u is admissible, and for every open set $G_o \subset G$ we have

$$\int_{G_o} f_o(t,y(t),u(t))dt \leq \underline{\lim}_{k\to\infty} \int_{G_o} f_o(t,y_k(t),u_k(t))dt \ .$$

In particular, for $G_o = G$ this relation reduces to $I[x,u] \leq \underline{\lim}\ I[x_k,u_k]$.

To prove this stronger form of (9.3.i) we must first consider the family $\{J\}$ of all cubes $J \subset G$, $J = [b\rho, b\sigma]$, $\rho = (\rho_1,\dots,\rho_\nu)$, $\sigma = (\sigma_1,\dots,\sigma_\nu)$, $\rho_j < \sigma_j$, ρ_j, σ_j rational. This collection is countable. Then, by the diagonal process, we can extract the subsequence, say now $[k_s]$, in the proof of (8.3.i) in such a way that

$$\lim_{s\to\infty} \int_J f_o(t,y_{k_s}(t),u_{k_s}(t))dt = \underline{\lim}_{k\to\infty} \int_J f_o(t,y_k(t),u_k(t))dt \ ,$$

as well as

$$\lim_{s\to\infty} \int_G f_o(t,y_{k_s}(t),u_{k_s}(t))dt = \underline{\lim}_{k\to\infty} \int_G f_o(t,y_k(t),u_k(t))dt \ .$$

We can proceed now as in the proof of (9.3.i), and obtain an element $u \in T$ such that x,u is admissible, and $I[x,u] \leq \lim_{s\to\infty} I[x_{k_s},u_{k_s}] = \underline{\lim}_{k\to\infty} I[x_k,u_k]$. If G_o is any open subset of G, then G_o is the union of countably many disjoint (closed) intervals $J \in \{J\}$. Given $\varepsilon > o$ there is a finite set of these intervals say $J_1,\dots,J_N$, such that

$$\int_{G_o - \cup_s J_s} |f_o(t,y(t),\ u(t))|\,dt < \varepsilon \quad , \qquad \int_{G_o - \cup_s J_s} \psi(t)dt < \varepsilon \ .$$

Hence

$$\int_{G_o} f_o(t,y(t),u(t))dt \leq \Sigma_s \int_{J_s} f_o dt + \varepsilon =$$

$$= \lim_{s\to\infty} \int_{\cup J_s} f_o \ dt + \lim_{s\to\infty} \int_{G_o - \cup_s J_s} (f_o + \psi) \ dt + \varepsilon$$

$$\leq \underline{\lim}_{k\to\infty} \int_{G_o} f_o(t,y_k(t),u_k(t))dt + 2\varepsilon,$$

where $\varepsilon > o$ is arbitrary. This proves the statement.

Remark. Condition (ψ) in statement (9.3.i) is certainly satisfied if, for instance, $f_o(t,y,u) \geq o$ for all $(t,y,u) \in M$, or $f_o(t,y,u) > \nu$ for all $(t,y,u) \in M$, where ν is some real constant. Nevertheless, condition (ψ) in (9.3.i) can be reduced. A generalization of (9.3.i) will be given below (9.3.ii), where we shall use the following weaker assumption.

Condition (ψ_t^*). For every $\bar{t} \in$ cl G there are a neighborhood $N(\bar{t})$ of $\bar{t}$ in cl G, an L-integrable function $\psi(t) \geq o$, and real numbers $b_1,\ldots,b_r$ (all $b_1,\ldots b_r$ and ψ may depend on $\bar{t}$ and $N(\bar{t})$) such that

$$\bar{f}(t,y,u) = f_o(t,y,u) - \Sigma_{j=1}^r b_j \ f_j(t,y,u) \geq -\psi \ (t) \qquad (8.5.2)$$

for all $(t,y,u) \in M$ with $t \in N(\bar{t})$, with exception perhaps of a set of points (t,y,u) whose t-coordinate lies on a set of measure zero in G.

We shall note that, under condition (ψ_t^*), it is natural to consider the sets

$$\tilde{\tilde{Q}}(t,y) = [(z^o,z) | z^o \geq f_o(t,y,u),\ z = f(t,y,u),\ u \in U(t,y)] ,$$

or the analogous sets

$$\tilde{\tilde{Q}}^*(t,y) = [(Z^o,Z) | Z^o \geq \bar{f}_o(t,y,u),\ z = f(t,y,u),\ u \in U(t,y) .$$

It is easy to see that the sets $\tilde{\tilde{Q}}$ are closed, or convex, or satisfy property (Q) if and only if the same occurs for the sets $\tilde{\tilde{Q}}^*$. Indeed, the sets above are transformed into one another by the fixed affine transformation

$$Z^o = z^o - b \cdot z \ , \ Z = z \ .$$

(8.5.ii) The same as (8.3.i) with hypothesis (ψ_t^*) replacing (ψ_t).

Proof. The set $\mathrm{cl}\ G \subset E_\nu$ is closed and bounded, hence compact. For every $\bar{t} \in \mathrm{cl}\ G$ there is a neighborhood $N_s(\bar{t})$ and constants $b = (b_1,\dots,b_r)$ as stated in assumption (ψ_t^*). These neighborhoods form a covering of cl G. Thus, finitely many of the same neighborhoods cover G, say $N_{\delta_\gamma}(t_\gamma)$, $\gamma = 1,\dots,N$. Let $b_{\gamma 1},\dots,b_{\gamma r}$ be the corresponding constants, so that

$$\bar{f}_{o\gamma}(t,y,u) = f_o(t,y,u) = -\sum_{j=1}^{r} b_{\gamma j} f_j(t,y,u) \geq \psi_\gamma(t)$$

for all $(t,y,u) \in M$, $t \in N_{\delta_\gamma}(t_\gamma)$, where $\psi_\gamma(t)$ is L-integrable in $N_{\delta_\gamma}(t_\gamma)$, and $\bigcup_{\gamma=1}^{N} N_{\delta_\gamma}(t_\gamma) \supset \mathrm{cl}\ G$.

The set $H = \{t \mid |t-t_\gamma| = \delta_\gamma, \gamma = 1,\dots,N\}$ is closed and has measure zero in E_ν. Hence, G-H is open, meas (G-H) = meas G, and each component of G-H is also open. If we now take

$$G_1 = (G-H) \cap (\text{int } N_{\delta_1}(t_1))$$

$$G_\gamma = (G-H) \cap (\text{int } N_{\delta_\gamma}(t_\gamma)) - G_1 - \dots - G_{\gamma-1}, \ \gamma = 2,\dots,N ,$$

we see that G-H has been divided into Γ disjoint open sets G_γ, $\gamma = 1,\dots,N$, $G_\gamma \subset N_{\delta_\gamma}(t_\gamma)$, and each component of G-H belongs entirely to one and only one set G_γ. Possibly empty sets G_γ above can be eliminated by suitable reindexing. Since meas H = o we have now

$$\int_G z_k^o(t)dt = \int_G f_o(t,y_k(t))dt = \int_{G-H} f_o dt = \sum_{\gamma=1}^{N} \int_{G_\gamma} f_o dt$$
$$= \sum_{\gamma=1}^{N} \int_{G_\gamma}[f_o(t,y_k(t),u_k(t))dt + \sum_{j=1}^{r} b_{\gamma j} f_j(t,y_k(t),u_k(t))] \, dt \qquad (8.5.3)$$

Let $L = \underline{\lim} \int_G z_k^o(t)dt$ as $k \to \infty$, $L < +\infty$.

On each open set $G_\gamma \subset N_{\delta_\gamma}(t_\gamma)$ we have $\bar{f}_o(t,y,u) \geq -\psi_\gamma(t)$ for all $(t,y,u) \in M$, $t \in G_\gamma$, $\psi_\gamma \in L(G_\gamma)$, and we denote by L_γ the constant $L_\gamma = \int_{G_\gamma} \psi_\gamma \, dt$, $\gamma = 1,\dots,N$. Also, $v_k \to v$ weakly in V as $k \to \infty$. Thus, $v \in L(G)$, and if $X_\gamma(t)$, $t \in G$, denotes the characteristic function of G_γ in G, then

$$\int_G z_k(t)dt \to \int_G z(t)dt \ , \quad \int_G X_\gamma(t)z_k(t)dt \to \int_G X_\gamma(t)z(t)dt \ ,$$

$\gamma = 1,\dots,\Gamma$, as $k \to \infty$, where $z_k(t) = f(t,y_k(t),u_k(t))$ a.e. in G. Hence

$$\int_{G_\gamma} f(t,y_k(t),u_k(t))dt \to \int_{G_\gamma} z(t)dt \ , \ \gamma = 1,\dots,N. \qquad (8.5.4)$$

Again, for every $\gamma = 1,\ldots,N$, we have

$$\int_{G_\gamma} \bar{f}_o(t,y_k(t),u_k(t))dt = \int_{G_\gamma} (f_o - b_\gamma f)dt$$

$$= \int_G f_o\, dt - \Sigma_{s\neq\gamma} \int_{G_s} f_o dt - \Sigma_j\, b_{\gamma j} \int_{G_\gamma} f_j\, dt$$

$$\leq \int_G f_o(t,y_k(t),u_k(t))dt + \Sigma_{s\neq\gamma} \int_{G_s} \psi_s\, dt - \Sigma_j\, b_{\gamma j} \int_{G_\gamma} f_j(t,y_k(t),u_k(t))dt.$$

As $k \to \infty$, we have now

$$L_\gamma \leq \underline{\lim} \int_{G_\gamma} \bar{f}_o(t,y_k(t),u_k(t))dt \leq L + \Sigma_{s\neq\gamma} L_s - \Sigma_j\, b_{\gamma j} \int_{G_\gamma} z_j(t)dt,$$

for every $\gamma = 1,\ldots,N$. Since $L < +\infty$, each of these $\underline{\lim}$ is finite. In addition certainly $L > -\infty$, and thus L is finite.

The sets $\tilde{\tilde{Q}}^*(t,y)$ are closed, convex, and satisfy property (Q) in A_γ, as pointed out in Remark before (8.5.ii), and where $A_\gamma = [(t,y) \mid t \in G_\gamma, (t,y) \in A]$.

We can now apply (8.5.i) to the integrals $\int_{G_\gamma} \bar{f}_o\, dt$, $\gamma = 1,\ldots,N$. By N successive applications of (8.5.i) and successive extractions, we obtain a subsequence which we shall still call [k] for the sake of simplicity, and measurable functions $\bar{u}_\gamma(t)$, $t \in G_\gamma$, such that

$$\bar{u}_\gamma(t) \in U(t,y(t)), \quad z(t) = f(t,y(t),\bar{u}_\gamma(t)), \text{ a.e. in } G_\gamma,$$

$$\int_{G_\gamma} \bar{f}_o(t,y(t),\bar{u}_\gamma(t))dt \leq \underline{\lim} \int_{G_\gamma} \bar{f}_o(t,y(t),u_k(t))dt, \quad \gamma = 1,\ldots,N. \tag{8.5.5}$$

If now $u(t)$, $t \in G$, is defined by taking $u = u_\gamma$ in G_γ, and u arbitrary in $H \cap G$, then

$$u(t) \in U(t,y(t)), \quad z(t) = f(t,y(t),u(t)) \text{ a.e. in } G,$$

$$\int_{G_\gamma} \bar{f}_o(t,y(t),u(t))dt \leq \underline{\lim} \int_{G_\gamma} \bar{f}_o(t,y_k(t),u_k(t))dt, \quad \gamma = 1,\dots,N. \tag{8.5.6}$$

Now relation (8.5.4) becomes

$$\int_{G_\gamma} f(t,y(t),u(t))dt = \underline{\lim} \int_{G_\gamma} f(t,y_k(t),u_k(t))dt, \quad \gamma = 1,\dots,N. \tag{8.5.7}$$

By relations (8.5.3), (8.5.5), and (8.5.7), we have

$$\int_G z^o(t)dt = \int_G f_o(t,y(t),u(t))dt = \Sigma_\gamma \int_{G_\gamma} f_o \, dt$$

$$= \Sigma_\gamma [\int_{G_\gamma} \bar{f}_o(t,y(t),u(t))dt + \Sigma_j b_{\gamma j} \int_{G_\gamma} f_j(t,y(t),u(t))dt]$$

$$\leq \Sigma_\gamma [\underline{\lim} \int_{G_\gamma} \bar{f}_o(t,y_k(t),u_k(t))dt + \Sigma_j b_{\gamma j} \underline{\lim} \int_{G_\gamma} f_j(t,y_k(t),u_k(t))dt]$$

$$\leq \underline{\lim} [\Sigma_\gamma \int_{G_\gamma} \bar{f}_o \, dt + \Sigma_j b_{\gamma j} \int_{G_\gamma} f_j \, dt]$$

$$= \underline{\lim} \int_G f_o(t,y_k(t),u_k(t))dt = \underline{\lim} \int_G z_k^o(t)dt.$$

This proves statement (8.5.ii).