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CLOSURE, LOWER CLOSURE, AND SEMICONTINUITY THEOREMS

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2.A. THE BASIC CLOSURE THEOREM

2.1. THE SPACE OF CONTINUOUS VECTOR FUNCTIONS OF A SCALAR VARIABLE

We shall consider the collection X of all continuous n-dimensional vector functions defined on finite closed intervals of the real axis: $x(t) = (x^1, \ldots, x^n)$, $a \leq t \leq b$, $x(t) \in E_n$, where a,b are finite but not necessarily the same for all elements of the collection. It is often said that each element of X, or $C: x = x(t)$, $a \leq t \leq b$, is a <u>nonparametric</u> <u>continuous</u> <u>curve</u> of the tx-space $E_1 \times E_n$. The orientation on C is then assumed to be the one corresponding to <u>t</u> increasing. The <u>graph</u> of an element x, or C, of X is the set of all points $(t, x(t)) \in E_1 \times E_n$, $a \leq t \leq b$.

For the sake of simplicity we shall denote by x and y, or $x(t)$, $a \leq t \leq b$, and $y(t)$, $c \leq t \leq d$, any two elements of X. We shall assign to X a distance function $\rho(x,y)$ and by doing so we shall make X a metric space. To define ρ it is convenient to extend first $x(t)$ outside the interval [a,b] by taking $x(t) = x(a)$ for $t \leq a$ and $x(t) = b$ for $t \geq b$. The same we shall do for $y(t)$. Then we take

$$\rho(x,y) = |a - c| + |b - d| + \mathrm{Max}|x(t) - t(t)| ,$$

where Max is taken for $-\infty < t < +\infty$. Because of the way in which $x(t)$, $y(t)$ are defined outside their original finite interval of definition, the maximum above exists. It is left as an exercise for the reader to prove the basic properties: (1) $\rho(x,y) \geq 0$, $x,y \in X$; (2) $\rho(x,y) = 0$ if and only if $a = c$, $b = d$, $x(t) \equiv y(t)$; (3) $\rho(x,y) = \rho(y,x)$, $x,y \in X$; (4) $\rho(x,y) \leq \rho(x,z) + \rho(z,y)$, $x,y,z \in X$.

A sequence $x_k(t)$, $a_k \leq t \leq b_k$, $k = 1,2,\ldots$, of elements of X is said to be convergent toward an element $x(t)$, $a \leq t \leq b$, of X, provided $\rho(x_k,x) \to 0$ as $k \to \infty$. Then $a_k \to a$, $b_k \to b$, and $x_k(t) \to x(t)$ uniformly on $(-\infty,+\infty)$ (after extension of each x_k and x outside their original interval of definition as above). We shall say that the metric ρ defines the uniform topology on X.

As mentioned earlier we shall consider arbitrary classes Ω of elements of X, or continuous curves in the tx-space $E_1 \times E_n$. For our purpose the following simple statement shall be noted.

We shall assume that A is any closed subset of the tx-space E_{1+n}, that B is a closed subset of the $t_1x_1t_2x_2$-space E_{2n+2}, and that $g(t_1,x_1,t_2,x_2)$ is a continuous scalar function on B. We shall denote by Ω the class of continuous vector functions satisfying $(t,x(t)) \in A$ for all $t_1 \leq t \leq t_2$, and $(t_1,x(t_1),t_2,x(t_2)) \in B$. Thus $\Omega \subset X$.

(2.1.i). The class Ω is closed in the uniform topology (metric ρ) and the functional $I[x] = g(t_1,x(t_1),t_2,x(t_2))$ is continuous in Ω.

In other words, if $x_k(t)$, $t_{1k} \leq t \leq t_{2k}$, are elements of X with $(t,x_k(t)) \in A$, $(t_{1k}, x_k(t_{1k}),t_{2k},x_k(t_{2k})) \in B$, and $x(t), t_1 \leq t \leq t_2$, is an element of X such that $\rho(x_k,x) \to 0$ as $k \to \infty$, then $(t,x(t)) \in A$, $(t_1,x(t_1),t_2,x(t_2)) \in B$, and $I[x_k] \to I[x]$ as $k \to \infty$.

Proof of (2.1.i). First $\rho(x_k,x) \to 0$ implies $t_{1k} \to t_1$, $t_{2k} \to t_2$ as $k \to \infty$. Secondly, $x(t)$ is continous in $[t_1,t_2]$, hence its extension is continuous in $(-\infty,+\infty)$; hence, given $\varepsilon > 0$ there is $\delta > 0$ such that $|t - t_1| \leq \delta$ implies $|x(t) - x(t_1)| \leq \varepsilon$, and $|t - t_2| \leq \delta$ implies $|x(t) - x(t_2)| \leq \varepsilon$. Finally, there is $\bar{k}$ such that $k \geq \bar{k}$ implies $|t_{1k} - t_1| \leq \delta$, $|t_{2k} - t_2| \leq \delta$, $\rho(x_k,x) \leq \varepsilon$, and hence

$$|x_k(t_{1k}) - x(t_1)| \leq |x_k(t_{1k}) - x(t_{1k})| + |x(t_{1k}) - x(t_1)| \leq 2\varepsilon \quad ,$$

and analogously $|x_k(t_{2k}) - x(t_2)| \leq 2\varepsilon$ for $k \geq \bar{k}$. This proves that $x_k(t_{1k}) \to x(t_1)$, $x_k(t_{2k}) \to x(t_2)$ as $k \to \infty$. Thus, $(t_{1k},x_k(t_{1k}),t_{2k},x_k(t_{2k})) \in B$ with B closed implies $(t_1,x(t_1),t_2,x(t_2)) \in B$. By the continuity of g then $g(t_{1k}, x_k(t_{1k}),t_{2k},x_k(t_{2k})) \to g(t_1,x(t_1),t_2,x(t_2))$ as $k \to \infty$. Also, $(t_{1k},x_k(t_{1k})) \in A$ with A closed implies $(t_1,x(t_1)) \in A$, and analogously we prove that $(t_2,x(t_2)) \in A$. Finally, for every $t \in (t_1,t_2)$, we have also $t \in (t_{1k},t_{2k})$ for all k sufficiently large, and $(t,x_k(t)) \in A$ then implies $(t,x(t)) \in A$. We have proved that $(t,x(t)) \in A$ for all $t_1 \leq t \leq t_2$, hence x is an element of Ω, and this completes the proof.

Finally, we remind here that Ascoli's theorem in terms of the metric ρ holds in the following form: If $x_k(t)$, $t_{1k} \leq t \leq t_{2k}$, $k = 1,2,\ldots$, are equicontinuous vector functions, and there is a constant M such that $-M \leq t_{1k} \leq t_{2k} \leq M$, $|x_k(t)| \leq M$ for $t \in [t_{1k},t_{2k}]$, $k = 1,2,\ldots$, then there is a subsequence $[x_{k_s}]$ convergent in the metric ρ toward a continuous vector function $x(t)$, $t_1 \leq t \leq t_2$.

2.2. ORIENTOR FIELDS

We assume that a set A is given in the tx-space $E_1 \times E_n$, $x = (x^1,\ldots,x^n)$, and that for every $(t,x) \in A$ a set of "allowable directions" z is assigned, precisely, a nonempty set $Q(t,x)$ of vectors $z = (z^1,\ldots,z^n)$ is given, $Q(t,x) = \{z\}_{tx}$, and this set may depend on $(t,x) \in A$. As mentioned in 1.2 we denote the relation

$$dx/dt \in Q(t,x) \tag{2.2.1}$$

an orientor field. A solution $x(t)$, $t_1 \leq t \leq t_2$, of (2.2.1) is a vector-valued function $x(t) = (x^1,\dots,x^n)$, $t_1 \leq t \leq t_2$, such that (1) $x(t)$ is absolutely continuous (AC) in $[t_1,t_2]$; (2) $(t,x(t)) \in A$ for all $t \in [t_1,t_2]$; (3) $dx/dt \in Q(t,x(t))$ almost everywhere (a.e.) in $[t_1,t_2]$. Thus, for almost all $t \in [t_1,t_2]$ the direction $dx/dt = (x^{1'},\dots,x^{n'})$ of the curve $x = x(t)$ at $(t,x(t))$ is one of the "allowable directions" $z \in Q(t,x(t))$ assigned at $(t,x(t))$.

An orientor field will be said to be autonomous if $Q(t,x)$ depends on x only and not on t. Nevertheless, every orientor field can be written as an autonomous one by a change of coordinates. Indeed, if we add the vector variable x^o satisfying the differential equation $dx^o/dt = 1$ and initial condition $x^o(t_1) = t_1$, and then we use the $(n + 1)$-vector $\tilde{x} = (x^o,x^1,\dots,x^n)$, and direction set $\tilde{Q}(\tilde{x}) = [\tilde{z} = (z^o,z^1,\dots,z^n) = (z^o,z),\ z \in Q(x^o,x),\ z^o = 1]$, then system (2.2.1) becomes

$$d\tilde{x}/dt \in \tilde{Q}(\tilde{x}) \quad .$$

We may use this remark in proofs in order to simplify notations.

Remark 1. For the sake of simplicity, we have assumed the variable x to vary in a Euclidean space E_n. As a careful reader may see, most of the results below are valid even if we allow x to vary in much more general spaces, and attention will be called to this fact when needed.

2.3. MEANING OF CLOSURE THEOREMS

The closure theorems we shall discuss below answer affirmatively a very important question, a question which is relevant even for the particular case of ordinary differential systems. If we have a sequence $[x_k]$ of solutions $x_k(t)$, $t_{1k} \leq t \leq t_{2k}$, of an orientor field (2.2.1), and $[x_k]$ "converges" toward a given function $x_o(t)$, $t_1 \leq t \leq t_2$, then also x_o is a solution of the orientor field (2.2.1). This is essentially true, though under various sets of assumptions on A, on Q, on the mode of convergence, on x_o.

In the closure theorems below we shall always assume that A is a given closed subset of the tx-space $E_1 \times E_n$, and that, for every $(t,x) \in A$, $Q(t,x)$ is a closed convex subset of the z-space E_n. Concerning the mode in which $Q(t,x)$ is allowed to vary as (t,x) describes A, we shall need a very mild property which is usually described as an upper semicontinuity property. Such a property has been introduced by various authors in different ways for different purposes, and we shall introduce some of these definitions as we go along and we need them.

2.4. METRIC UPPER SEMICONTINUITY AND PROPERTY (Q)

Given any set Z in E_n, we shall denote by cl Z, bd Z, co Z the closure of Z, the boundary of Z, and the convex hull of Z, respectively. Thus, cl co Z denotes the closure of the convex hull of Z, or briefly the closed convex hull of Z.

Given a set A of the tx-space $E_1 \times E_n$, a point $(t_o,x_o) \in A$, and a number $\delta > 0$, we shall denote by $N_\delta(t_o,x_o)$, or neighborhood of (t_o,x_o) in A, the set of all $(t,x) \in A$ at a distance $\leq \delta$ from (t_o,x_o). Then $N_\delta(t_o,x_o) \subset A$. Also, given a set $U(t,x)$ of points $z = (z^1,\dots,z^m)$ for each $(t,x) \in A$, a point $(t_o,x_o) \in A$, and a number $\delta > 0$, we shall denote by $U(t_o,x_o;\delta)$ the union of all $U(t,x)$ with $(t,x) \in N_\delta(t_o,x_o)$, in other words

$$U(t_o,x_o,\delta) = \bigcup_{(t,x)\in N_\delta(t_o,x_o)} U(t,x)$$

Thus $U(t_x,x_o) \subset U(t_o,x_o,\delta)$ for all $\delta > 0$.

We say that $U(t,x)$ is <u>metrically upper semicontinuous</u> at a point $(t_o,x_o) \in A$ provided, given $\varepsilon > 0$, there is some $\delta = \delta(t_o,x_o,\varepsilon) > 0$ such that $U(t,x) \subset [U(t_o,x_o)]_\varepsilon$ for all $(t,x) \in N_\delta(t_o,x_o)$ and where $[U]_\varepsilon$ denotes the closed ε-neighborhood of Q. We say that $U(t,x)$ is metrically upper semicontinuous in A if it has this property at every $(t,x) \in A$.

The following analogous property of upper semicontinuity is of interest. We shall say that a variable set $U(t,x)$, $(t,x) \in A \subset E_{1+n}$, $U(t,x) \subset E_m$, has property (U) at a point $(t_o,x_o) \in A$ provided

$$U(t_o,x_o) = \bigcap_{\delta>0} \text{cl}\, U(t_o,x_o;\delta) \quad ,$$

that is,

$$U(t_o,x_o) = \bigcap_{\delta>0} \text{cl}\, \{\bigcup_{(t,x)\in N_\delta(t_o,x_o)} U(t,x)\} \quad .$$

We shall say that $U(t,x)$ has property (U) in A if it has property (U) at every point $(t_o,x_o) \in A_o$.

If M denotes the set of all (t,x,u) with $(t,x) \in A$, $u \in U(t,x)$, then $M \subset E_{1+n+m}$, and M is the <u>graph</u> of $U(t,x)$. The main statement concerning property (U) is the following one:

(2.4.i). If A is closed, then M is closed if and only if $U(t,x)$ has property (U) in A.

We shall prove this statement in [App. A.1.ii] together with others. If $U(t,x)$ has property (U) then $U(t,x)$ is certainly closed as the intersection of closed sets. Property (U) is usually denoted as Kuratowsky's upper semicontinuity.

In the proof of the Closure Theorems below, we shall often require that the sets $U(t,x)$ be closed, convex, and satisfy an analogous property of upper semicontinuity, which we shall denote as property (Q), and which is specific for closed and convex sets.

We shall say that $U(t,x)$, $(t,x) \in A$, has property (Q) at the point $(t_o,x_o) \in A$ provided

$$U(t_o,x_o) = \bigcap_{\delta>0} \text{cl co } U(t_o,x_o,\delta) \quad , \tag{2.4.1}$$

that is,

$$U(t_o,x_o) = \bigcap_{\delta>0} \text{cl co } \{\bigcup_{(t,x)\in N_\delta(t_o,x_o)} U(t,x)\} \quad .$$

We shall say that $U(t,x)$, $(t,x) \in A$, has property (Q) in A if it has property (Q) at every $(t,x) \in A$. If $U(t,x)$ has property (Q) then certainly $U(t,x)$ is closed and convex as the intersection of such sets. For closed convex sets, property (Q) is a way to express the idea of upper semicontinuity in a form which is more general than the metric upper semicontinuity.

The concept of metric upper semicontinuity can be traced very far back (see for instance, F. Hausdorff [1]).

Note that we can always denote (t,x) as a unique variable, say $\tilde{x} \in E_{n+1}$, hence A is a subset of the $\tilde{x}$-space E_{n+1}, and $U(\tilde{x})$ depends on $\tilde{x}$ only. We shall use this remark in the proofs for the sake of simplifying notations.

(2.4.i). If A is closed, if, for every $(t,x) \in A$, $U(t,x)$ is a closed subset of E_m, and $U(t,x)$ is metrically upper semicontinuous in A, then certainly

$U(t,x)$ has property (U) in A. If all sets $U(t,x)$ are closed and contained in a given interval in E_m, then $U(t,x)$ has property (U) in A if and only if $U(t,x)$ is metrically upper semicontinuous in A. If, for every $(t,x) \in A$, the sets $U(t,x)$ are known to be closed and convex, then the same statements above hold for property (Q). The proof is given in App. A (A.1.v; A.1.vii; A.2.iv; A.2.v).

Remark 1. Property (U) for closed sets, and property (Q) for closed convex sets are actually more general than metric upper semicontinuity. This can be shown by the following simple example with $m = 2$, $A = [t | 0 \leq t \leq 1]$, $U(t) = [z = (z^1, z^2) | 0 \leq z^1 < +\infty,\ 0 \leq z^2 \leq tz^1]$. Here $U(t)$ is an angle, and obviously, for $t > t_o$, $U(t)$ is not contained in any $[U(t_o)]_\varepsilon$, no matter how t is close to t_o. $U(t)$ has both properties (U) and (Q) in A, but it is not metrically upper semicontinuous.

More on the comparison properties of upper semicontinuity will be given in App. A.

2.5. THE BASIC CLOSURE THEOREM FOR FUNCTIONS OF A SCALAR VARIABLE

(2.5.i). Closure Theorem 1. Let A be any closed set of the tx-space $E_1 \times E_n$, for every $(t,x) \in A$ let $Q(t,x)$ be a closed convex subset of points $z = (z^1, \dots, z^n)$, and let us assume that $Q(t,x)$ satisfies property (Q) at every point $(t,x) \in A$ with exception perhaps of a set of points whose t coordinate lies on a set H of measure zero on the t-axis. If $x_k(t)$, $t_{1k} \leq t \leq t_{2k}$, $k = 1,2,\dots$, is a sequence of solutions of the orientor field (2.2.1), convergent in the ρ-metric toward an AC function $x(t)$, $t_1 \leq t \leq t_2$, then $x(t)$ is also a solution of the orientor field.

In other words, we know that each $x_k(t)$, $t_{1k} \leq t \leq t_{2k}$, $k = 1,2,\dots$, is AC, that $(t,x_k(t)) \in A$ for every $t \in [t_{1k},t_{2k}]$, and that $dx_k/dt \in Q(t,x_k(t))$ a.e. in $[t_{1k},t_{2k}]$, we know that $\rho(x_k,x) \to 0$, hence $t_{1k} \to t_1$, $t_{2k} \to t_2$, as $k \to \infty$, and that $x(t)$ is AC in $[t_1,t_2]$, and we want to prove that $(t,x(t)) \in A$ for all $t \in [t_1,t_2]$, and that $dx/dt \in Q(t,x(t))$ a.e. in $[t_1,t_2]$.

Proof of (2.5.i). The vector functions $x'(t)$, $t_1 \leq t \leq t_2$, $x_k'(t)$, $t_{1k} \leq t \leq t_{2k}$, are defined a.e. in $[t_1,t_2]$ and $[t_{1k},t_{2k}]$ respectively, $k = 1,2,\ldots$, and are L-integrable in the respective intervals (that is, each component is L-integrable).

Since $\rho(x_k,x) \to 0$, hence $t_{1k} \to t_1$, $t_{2k} \to t_2$ as $k \to \infty$, if $t \in (t_1,t_2)$, or $t_1 < t < t_2$, then $t_{1k} < t < t_{2k}$ for all k sufficiently large, and $(t,x_k(t)) \in A$. Since $x_k(t) \to x(t)$ as $k \to \infty$ and A is closed, we conclude that $(t,x(t)) \in A$ for every $t_1 < t < t_2$. Since $x(t)$ is continuous in $[t_1,t_2]$, and hence continuous at t_1 and t_2, we conclude that $(t,x(t)) \in A$ for every $t_1 \leq t \leq t_2$.

For almost all $t \in [t_1,t_2]$ the derivative $x'(t)$ exists and is finite and $t \in [t_1,t_2] - H$. Let t_o be such a point with $t_1 < t_o < t_2$. Then, there is a $\sigma > 0$ with $t_1 < t_o - \sigma < t_o + \sigma < t_2$, and, for some k_o and all $k \geq k_o$, also $t_{1k} < t_o - \sigma < t_o + \sigma < t_{2k}$. Let $x_o = x(t_o)$. We have $x_k(t) \to x(t)$ uniformly in $[t_o - \sigma, t_o + \sigma]$ and all functions $x(t)$, $x_k(t)$ are continuous in the same interval. Thus, they are equicontinuous in $[t_o - \sigma, t_o + \sigma]$. Given $\varepsilon > 0$, there is $\delta > 0$ such that $t,t' \in [t_o - \sigma, t_o + \sigma]$, $|t - t'| \leq \delta$, $k \geq k_o$, implies

$$|x(t) - x(t')| \leq \varepsilon/2, \quad |x_k(t) - x_k(t')| \leq \varepsilon/2. \qquad (2.5.1)$$

We can assume $0 < \delta < \sigma$, $\delta \leq \varepsilon$. For any h, $0 < h \leq \delta$, let us consider the averages

$$m_h = h^{-1} \int_0^h x'(t_o + s)ds = h^{-1}[x(t_o + h) - x(t_o)] \quad , \qquad (2.5.2)$$

$$m_{hk} = h^{-1} \int_0^h x_k'(t_o + s)ds = h^{-1}[x_k(t_o + h) - x_k(t_o)] \quad . \qquad (2.5.3)$$

Given $\eta > 0$, we can take h so small that

$$|m_h - x'(t_o)| \leq \eta \quad . \qquad (2.5.4)$$

Having so fixed h, let us take $k_1 \geq k_o$ so large that

$$|m_{hk} - m_h| \leq \eta, \quad |x_k(t_o) - x(t_o)| \leq \varepsilon/2 \qquad (2.5.5)$$

for all $k \geq k_1$. This is possible since $x_k(t) \to x(t)$ as $k \to \infty$ both at $t = t_o$ and $t = t_o + h$. Finally, for $0 \leq s \leq h$,

$$|x_k(t_o + s) - x(t_o)| \leq |x_k(t_o + s) - x_k(t_o)| + |x_k(t_o) - x(t_o)|$$

$$\leq \varepsilon/2 + \varepsilon/2 = \varepsilon \quad ,$$

$$|(t_o + s) - t_o| = s \leq h \leq \delta \leq \varepsilon \quad ,$$

$$x_k'(t_o + s) \in Q(t_o + s, x_k(t_o + s)) \quad \text{a.e.}$$

Hence, for almost all s, $0 \leq s \leq h$, $x_k'(t_o + s) \in Q(t_o, x_o, 2\varepsilon)$ and consequently

$$x_k'(t_o + s) \in \text{cl co } Q(t_o, x_o, 2\varepsilon), \quad \text{a.e. in } [0,h].$$

The average m_{hk} as defined by (2.5.3) is then also a point of the same closed and convex set, or

$$m_{hk} \in \text{cl co } Q(t_o, x_o, 2\varepsilon)$$

for the chosen h and every $k \geq k_1$. By relations (2.5.4) and (2.5.5) we deduce

$$|x'(t_o) - m_{hk}| \leq |x'(t_o) - m_h| + |m_h - m_{hk}| \leq 2\eta \quad ,$$

and hence

$$x'(t_o) \in [\text{cl co } Q(t_o, x_o, 2\varepsilon)]_{2\eta} \quad .$$

Here η is an arbitrary number, and the set in brackets is closed, hence

$$x'(t_o) \in \cap_\eta [\text{cl co } Q(t_o, x_o, 2\varepsilon]_{2\eta} \quad = \quad \text{cl co } Q(t_o, x_o, 2\varepsilon) \quad ,$$

for every $\varepsilon > 0$. Thus, by property (Q),

$$x'(t_o) \in \cap_\varepsilon \text{ cl co } Q(t_o, x_o, 2\varepsilon) \quad = \quad Q(t_o, x_o) \quad .$$

We have proved that for almost all $t \in [t_1, t_2]$, we have

$$dx/dt \in Q(t, x(t)) \quad .$$

The Closure Theorem 1 is thereby proved.

The following example illustrates the first closure theorem. Let $n = 1$, $A = E_2$, $Q = Q(t,x) = [z | -1 \leq z \leq 1]$, and $x_k(t)$, $0 \leq t \leq 1$, $k = 1, 2, \ldots$, be defined by $x_k(t) = t - ik^{-1}$ if $ik^{-1} \leq t \leq ik^{-1} + (2k)^{-1}$, $x_k(t) = (i + 1)k^{-1} - t$ if $ik^{-1} + (2k)^{-1} \leq t \leq (i + 1)k^{-1}$ for $i = 0, 1, \ldots, k - 1$. Then $x_k(t) \to x_o(t) = 0$ uniformly in $[0,1]$. On the other hand, $x_k'(t) = \pm 1$ according as t is an interior point of one or the other of the two sets of intervals above, $x_o'(t) = 0$, and $x_k'(t)$, $x_o'(t) \in Q$ for almost all t. Here Q is a closed convex set. If we had taken $Q = Q(t,x) = [z | z = -1 \text{ and } z = +1]$, then obviously $x_k'(t) \in Q$ while $x_o'(t) \notin Q$. Here Q is closed but not convex.

2.6. INTERPRETATION OF CLOSURE THEOREM 1 IN TERMS OF USUAL TRAJECTORIES AND STRATEGIES. CLOSURE AND COMPACTNESS THEOREMS

(a) Let us assume here that A is any closed set of the tx-space E_{1+n}, $x = (x^1,\dots,x^n)$, that, for every $(t,x) \in A$, $U(t,x)$ is a subset of the u-space E_m, $u = (u^1,\dots,u^m)$, that the set M of all (t,x,u) with $(t,x) \in A$, $u \in U(t,x)$, is closed, that the vector function $f(t,x,u) = (f_1,\dots,f_n)$ is continuous on M, that B is a closed subset of the $t_1x_1t_2x_2$-space E_{2n+2}, and that $g(t_1,x_1,t_2,x_2)$ is a continuous scalar function on B. Also we assume that the sets $Q(t,x) = f(t,x, U(t,x)) \subset E_n$ are closed, convex, and satisfy condition (Q) at every point $(\bar{t},\bar{x}) \in A$—with exception perhaps of a set of points whose coordinate $\bar{t}$ lies on a set of measure zero on the t-axis. As stated in (1.1), we say that a pair $x(t)$, $u(t)$, $t_1 \leq t \leq t_2$, is admissible (and that x is a (real) admissible trajectory provided x is AC in $[t_1,t_2]$, u is measurable in $[t_1,t_2]$, $(t,x(t)) \in A$ for all $t \in [t_1,t_2]$, and $u(t) \in U(t,x(t))$, $x' = f(t,x(t),u(t))$ a.e. in $[t_1,t_2]$. For x,u admissible we define as cost functional $I[x,u] = g(t_1,x(t_1),t_2,x(t_2))$. In other words, we have a Mayer problem with state variables $x = (x^1,\dots,x^n)$, and control variables $u = (u^1,\dots,u^m)$.

(2.6.i) (a closure theorem). Under the hypotheses above (in particular the sets $Q(t,x)$ being convex), any AC limit $x(t)$, $t_1 \leq t \leq t_2$, in the ρ-metric for trajectories is a trajectory.

In other words, if $x_k(t)$, $u_k(t)$, $t_{1k} \leq t \leq t_{2k}$, $k = 1,2,\dots$, is any sequence of admissible pairs, if $x(t)$, $t_1 \leq t \leq t_2$, is any AC function, and $\rho(x_k,x) \to 0$ as $k \to \infty$, then x is a trajectory, that is, there is a measurable function $u(t)$, $t_1 \leq t \leq t_2$, such that $x(t)$, $u(t)$, $t_1 \leq t \leq t_2$, is an admissible pair, and $I[x_k,u_k] \to I[x,u]$ as $k \to \infty$. Note that, whenever we wish to disregard boundary conditions, we have only to take $B = E_{2n+2}$, and g and I need not be defined.

Proof. By (2.1.i), $(t,x(t)) \in A$ for $t \in [t_1,t_2]$, and $(t_1,x(t_1),t_2,x(t_2)) \in B$. Also, $x_i'(t) = f(t,x_k(t),u_k(t)) \in Q(t,x_k(t))$ for almost all $t \in [t_{1k},t_{2k}]$, $i = 1,2,\ldots$. By (2.5.i), or closure theorem 1, then $x'(t) \in Q(t,x(t))$ a.e. in $[t_1,t_2]$, and by (1.6.i), or implicit function theorem for orientor fields, there is a measurable function $u(t)$, $t_1 \leq t \leq t_2$, with $u(t) \in U(t,x(t))$, $x'(t) = f(t,x(t),u(t))$ a.e. in $[t_1,t_2]$, that is, x,u is an admissible pair. By (2.1.i) we know that $I[x_k,u_k] \to I[x,u]$.

For Lagrange problems with usual functional $I = \int_{t_1}^{t_2} f_o(t,x(t),u(t))dt$ with $f_o(t,x,u)$ a continuous scalar function on M, the class of admissible pairs is restricted to only those for which $f_o(t,x(t),u(t))$ is L-integrable on $[t_1, t_2]$. By introducing the variable x^o, the additional differential equation $dx^o/dt = f_o(t,x(t),u(t))$, and condition $x^o(t_1) = 0$, we have $I = x^o(t_2)$, and we have again a Mayer problem in the state variable $\tilde{x} = (x^o,x) = (x^o,x^1,\ldots,x^n)$ and control variable $u = (u^1,\ldots,u^m)$. Closure statement (2.6.i) could now be repeated in the new situation, with $\tilde{x}_k,\tilde{x}$ replacing x_k,x, by assuming that the sets $\tilde{Q}(t,x) = \tilde{f}(t,x,U(t,x)) \subset E_{n+1}$ are closed convex, and satisfy property (Q) $\tilde{f} = (f_o,f) = (f_o,f_1,\ldots,f_n)$, and by demanding that $\rho(\tilde{x}_k,\tilde{x}) \to 0$ as $k \to \infty$.

This last requirement is in most cases too demanding in Lagrange problems with unbounded controls, and we shall introduce and discuss the concept of lower closure in (2.9).

(b) Instead of the hypotheses stated at the beginning of (a), let us now assume that A is compact, that M is <u>compact</u>, that $f(t,x,u)$ is continuous on M, that B is closed, and that g is continuous on B. Also, let us assume that the sets $Q(t,x) = f(t,x,U(t,x))$ are convex for all $(t,x) \in A$, with exception perhaps of a set of points whose t coordinate lies on a set of measure zero on the t-axis. We have as before a Mayer problem with cost functional $I[x,u] = g(t_1,x(t_1),t_2,x(t_2))$.

(2.6.ii) (a compactness theorem). Under the hypotheses above (in particular, the sets $Q(t,x)$ being convex), the family of all usual trajectories $x(t)$, $t_1 \leq t \leq t_2$, is sequentially compact in the ρ-metric.

In other words, if $x_k(t)$, $u_k(t)$, $t_{1k} \leq t \leq t_{2k}$, $k = 1,2,\ldots$, is a sequence of admissible pairs, then there is a subsequence $[k_s]$, and an AC function $x(t)$, $t_1 \leq t \leq t_2$, such that $\rho(x_k,u) \to 0$ as $k \to \infty$, and x is a trajectory, that is, there is a measurable function $u(t)$, $t_1 \leq t \leq t_2$, such that $x(t)$, $u(t)$, $t_1 \leq t \leq t_2$, is admissible, and $I[x_k,u_k] \to I[x,u]$ as $k \to \infty$.

Proof. Here A is compact, hence bounded, and A is contained in some set $|t| \leq L$, $|x| \leq L$. Also, f is continuous on M, hence bounded, and we can take L in such a way that $|f| \leq L$ on M. Then $-L \leq t_{1k} \leq t_{2k} \leq L$ for all k, and $|x_k(t)| \leq L$ for all $t \in [t_{1k},t_{2k}]$ and all k. Also, x_k is AC and $|x_k'(t)| = |f(t,x_k(t),u_k(t)| \leq L$ a.e. in $[t_{1k},t_{2k}]$, hence $|x_k(t) - x_k(t')| \leq L|t - t'|$ for all $t,t' \in [t_{1k},t_{2k}]$ and all k. Thus, the trajectories x_k are equibounded and equilipschitzian, hence equicontinuous. By Ascoli's theorem (see (2.1)) there is a subsequence $[k_s]$ and a continuous vector function $x(t)$, $t_1 \leq t \leq t_2$, with $\rho(x_k,x) \to 0$ as $k \to \infty$. Then x is Lipschitzian of the same constant L, hence AC in $[t_1,t_2]$. Now the set M is compact by hypothesis, hence by (A.1.v) the compact sets $U(t,x)$ are metrically upper semicontinuous in A, and by (A.3.i), the sets $Q(t,x)$ are also compact and metrically upper semicontinuous in A. From (A.2.v) we conclude then that the sets $Q(t,x)$ which are convex by hypothesis, have property (Q) in A. Statement (2.6.ii) follows now from (2.6.i).

We may now consider Lagrange problems as in (a). Again we shall assume A and M compact, and we take a scalar $f_o(t,x,u)$ continuous on M. Note that now for $x(t)$, $u(t)$, $t_1 \leq t \leq t_2$, admissible in the usual sense, the function $f_o(t,x(t),u(t))$ is necessarily measurable and bounded, hence L-integrable. Actually, the functions $f_o(t,x(t),u(t))$ are equibounded, and so are the functions $f(t,x(t),u(t)) = (f_o,f_1,\dots,f_n)$. Statement (2.6.ii) holds now also in the new situation, provided we assume that the sets $\tilde{Q}(t,x) = \tilde{f}(t,x,U(t,x))$ are convex, and the conclusion is that the trajectories $\tilde{x}(t)$, $t_1 \leq t \leq t_2$, are now sequentially compact, with $\tilde{x} = (x^o,x) = (x^o,x^1,\dots,x^n)$, and $dx^o/dt = f_o(t,x(t),u(t))$, $x^o(t_1) = 0$. (See (2.9),(2.10) for extensions of these statements.)

2.7. INTERPRETATION OF CLOSURE THEOREM 1 IN TERMS OF GENERALIZED SOLUTIONS CLOSURE AND COMPACTNESS THEOREMS

Here we completely abandon all hypotheses of convexity of the sets Q and $\tilde{Q}$ of (2.6).

(a) Let us assume here that A is any closed set of the tx-space E_{1+n}, $x = (x^1,\dots,x^n)$, that for every $(t,x) \in A$, $U(t,x)$ is a subset of the u-space E_m, $u = (u^1,\dots,u^m)$, that the set N of all $(t,x,p,v) \in E_{1+n+m+m\gamma}$ with $(t,x) \in A$, $p = (p_1,\dots,p_\gamma) \in \Gamma$, $v = (u^{(1)},\dots,u^{(\gamma)}) \in V(t,x) = [U(t,x)]^\gamma$ is closed, that the vector function $f(t,x,u)$ is continuous on the set $M = [t,x,u) | (t,x) \in A, u \in U(t,x)]$, that B is a closed subset of the $t_1x_1t_2x_2$-space E_{2n+2}, and that $g(t_1,x_1,t_2,x_2)$ is a continuous function scalar on B. Also we assume that the sets $R(t,x) = \text{co}\, Q(t,x) = \text{co}\, f(t,x,U(t,x)) \subset E_n$ are closed, convex, and satisfy property (Q) at all points $(t,x) \in A$ with exception perhaps of a set of points (t,x) whose t coordinate lies on a set of measure zero on the t-axis. As stated in

(1.8), we say that $x(t)$, $p(t)$, $v(t)$, $t_1 \leq t \leq t_2$, is an admissible generalized solution (and that x is a generalized trajectory), provided x is AC in $[t_1,t_2]$, $p(t)$, $v(t)$ are measurable in $[t_1,t_2]$, $(t,x(t)) \in A$ for all $t \in [t_1,t_2]$, $p(t) \in \Gamma$, $v(t) \in V(t,x(t))$, $x'(t) = h(t,x(t),p(t),v(t))$ a.e. in $[t_1,t_2]$ (see (1.8)). For x,p,v admissible we define the cost functional $J(x,p,v) = g(t_1,x(t_1),t_2,x(t_2))$. We have here a Mayer-type problem.

(2.7.i) (a closure theorem). Any AC limit $x(t)$, $t_1 \leq t \leq t_2$, in the ρ-metric of generalized trajectories is a generalized trajectory. In particular, any AC limit of (usual) trajectories is certainly a generalized trajectory.

In other words, if $x_k(t)$, $p_k(t)$, $v_k(t)$, $t_{1k} \leq t \leq t_{2k}$, $k = 1,2,\ldots$, is any sequence of admissible generalized systems, if $x(t)$, $t_1 \leq t \leq t_2$, is any AC function, and $\rho(x_k,x) \to 0$ as $k \to \infty$, then x is an admissible generalized system, that is, there are measurable functions $p(t)$, $v(t)$, $t_1 \leq t \leq t_2$, such that $x(t)$, $p(t)$, $v(t)$, $t_1 \leq t \leq t_2$, is an admissible generalized system, and $J[x_k,p_k,v_k] \to J[x,p,v]$ as $k \to \infty$. Note that, whenever we wish to disregard boundary conditions, we have only to take $B = E_{2n+2}$, and g and J need not be defined.

For Lagrange-type problems with usual functional $I = \int_{t_1}^{t_2} f_o(t,x(t),u(t))dt$ replaced by the generalized functional $J = \int_{t_1}^{t_2} h_o(t,x(t),\, p(t).v(t))dt$ (see (1.9)), we consider as in (2.6) the new state variable $\tilde{x} = (x^o,x)$ $= (x^o,x^1,\ldots,x^n)$, the same usual control variable $u = (u^1,\ldots,u^m)$, and generalized control $p = (p_1,\ldots,p_\gamma)$, $v = (u^{(1)},\ldots,u^{(\gamma)})$. Here we assume $f_o(t,x,u)$ continuous on M, we restrict the class of admissible generalized systems to only those for which $h_o(t,x(t),p(t),v(t))$ is L-integrable in $[t_1,t_2]$, we replace in the hypotheses above $\tilde{x}_k,\tilde{x}$ for x_k,x, we assume now that the sets

$\tilde{R}(t,x) = \text{co}\ \tilde{Q}(t,x) = \tilde{h}(t,x,\ \Gamma\ U(t,x)^{\gamma})$ are closed, and satisfy property (Q) with $\tilde{h} = (h_o,h) = (h_o,h_1,\ldots,h_n)$, and we assume that $\rho(\tilde{x}_k,\tilde{x}) \to 0$ as $k \to \infty$. Statement (2.7.1) holds now with no further changes for Lagrange-type problems.

(b) Instead of the hypotheses stated at the beginning of (a), we shall now assume only that A is any compact set of the tx-space E_{1+n}, that the set M is also compact, and that B is closed and g is continuous on B as before. We consider again the Mayer-type problem $J[x,p,v] = g(t_1,x(t_1),t_2,x(t_2))$.

(2.7.ii) (a compactness theorem). The family of all generalized trajectories is sequentially compact in the ρ-metric.

Indeed, here A is compact, M compact, hence N is compact, and the proof proceeds now as for (2.6.ii) with R replaced by Q.

For Lagrange-type problems, with usual functional $I = \int_{t_1}^{t_2} f_o(t,x(t),u(t))dt$ replaced by $J = \int_{t_1}^{t_2} h_o(t,x(t),p(t),v(t))dt$, f_o continuous on M, no other hypothesis is needed. Thus, for A compact, M compact, B closed, statement (2.7.ii) holds for the family of generalized trajectories $\tilde{x}(t) = (x^o,x) = (x^o,x^1,\ldots,x^n)$, $t_1 \leq t \leq t_2$. Note that here the sets $R(t,x)$ are all compact, metrically upper semicontinuous, and since they are convex by definition, they certainly have property (Q), as for the sets Q in (2.6.b).

Part (A) above alone will be used in Chapter 4 (problems with bounded controls). Parts (2.B), (2.C), (2.D) below will be used only in Chapter 5 (problems with unbounded controls).

2.B. CLOSURE THEOREMS FOR FUNCTIONS WITH SINGULAR COMPONENTS

2.8. A CLOSURE THEOREM FOR FUNCTIONS WITH SINGULAR COMPONENTS

We shall need a variant of Closure Theorem 1. We shall assume here that the space E_n is actually a product space $E_s \times E_{n-s}$, hence $x = (y,z)$ with $y \in E_s$, $z \in E_{n-s}$. Analogously, we shall assume that A_o is a subset of the y-space E_s, and we shall take A of the form $A = A_o \times E_{n-s}$, so that we have as usual $A \subset E_n$. We shall finally assume that the orientor field in A has the form

$$dx/dt \in Q(t,y) \quad , \tag{2.8.1}$$

in other words, the set Q depends on t and y only, and not on z. A solution of this orientor field is then an AC n-vector function $x(t) = (y(t),z(t))$, $t_1 \leq t \leq t_2$, with $(t,x(t)) \in A$, that is, $(t,y(t)) \in A_o$ for every $t \in [t_1,t_2]$, and

$$dx/dt \in Q(t,y(t)), \quad \text{a.e. in } [t_1,t_2]$$

that is

$$(y'(t),z'(t)) \in Q(t,y(t)) \tag{2.8.2}$$

(2.8.i) <u>Closure Theorem 2</u>. Let A_o be a closed subset of the ty-space $E_1 \times E_s$, and $A = A_o \times E_{n-s}$, for every $(t,y) \in A_o$ let $Q(t,y)$ denote a closed subset of E_n, and assume that the sets $Q(t,y)$ are convex, closed, and have property (Q) at every point $(t,y) \in A_o$ with exception perhaps of a set of points whose t coordinate lies on a set H of measure zero on the t-axis. Let $x_k(t)$, $t_{1k} \leq t \leq t_{2k}$, $k = 1,2,\ldots$, be a sequence of solutions of the orientor field

(2.8.1), $x_k(t) = (y_k(t), z_k(t))$, for which we assume that the s-vector $y_k(t)$ converges in the ρ-metric toward an AC vector function $y(t)$, $t_1 \leq t \leq t_2$, and that the (n-s)-vector $z_k(t)$ converges pointwise for all t, $t_1 < t < t_2$, toward a vector $z(t)$ which admits of a decomposition $z(t) = Z(t) + S(t)$, where $Z(t)$ is an AC vector function in $[t_1,t_2]$, and $S'(t) = 0$ a.e. in $[t_1,t_2]$, that is, $S(t)$ is a singular function. Then, the AC n-vector $X(t) = [y(t),Z(t)]$, $t_1 \leq t \leq t_2$, is a solution of the orientor field (2.8.1).

In other words, we know that each $x_k(t) = (y_k(t),z_k(t))$, $t_{1k} \leq t \leq t_{2k}$, $k = 1,2,\ldots$, is AC, that $(t,y_k(t)) \in A_o$ for every $t \in [t_{1k},t_{2k}]$ and that $(y_k'(t),z_k'(t)) \in Q(t,y_k(t))$ a.e. in $[t_{1k},t_{2k}]$, we know that $\rho(y_k,y) \to 0$ hence $t_{1k} \to t_1$, $t_{2k} \to t_2$ as $k \to \infty$, that $z_k(t) \to z(t) = Z(t) + S(t)$ for every $t \in (t_1,t_2)$, that $S(t)$ is singular, and $(y(t),Z(t))$ is AC, and we want to prove that $(t,y(t)) \in A_o$ for every $t \in [t_1,t_2]$ and that $(y'(t),Z'(t)) \in Q(t,y(t))$ a.e. in $[t_1,t_2]$. For s = n this Closure Theorem 2 reduces to Theorem 1.

Proof of (2.8.i). The proof that $(t,y(t)) \in A_o$ for every $t \in [t_1,t_2]$ is the same as for Closure Theorem 1.

Let us prove the remaining part of (2.8.i) where we shall need to know only that $z_k(t) \to z(t)$ for almost all $t \in (t_1,t_2)$.

For almost all $t \in [t_1,t_2] - H$ the derivative $X'(t) = [y'(t),Z'(t)]$ exists and is finite, $S'(t)$ exists and $S'(t) = 0$, and $z_k(t) \to z(t)$. Let t_o be such a point with $t_1 < t_o < t_2$. Then, there is a $\sigma > 0$ with $t_1 < t_o - \sigma < t_o + \sigma < t_2$, and, for some k_o and all $k \geq k_o$, also $t_{1k} < t_o$ $t_o + \sigma < t_{2k}$. Let $x_o = X(t_o) = (y_o,Z_o)$, or $y_o = y(t_o)$, $Z_o = Z(t_o)$. Let $z_o = z(t_o)$, $S_o = S(t_o)$. We have $S'(t_o) = 0$, hence $z'(t_o)$ exists and $z'(t_o) = Z'(t_o)$. Also, we know that $z_k(t_o) \to z(t_o)$.

We have $y_k(t) \to y(t)$ uniformly in $[t_o - \sigma, t_o + \sigma]$, and all functions $y(t)$, $y_k(t)$ are continuous in the same interval. Thus, they are equicontinuous in $[t_o - \sigma, t_o + \sigma]$. Given $\varepsilon > 0$, there is $\delta > 0$ such that $t, t' \in [t_o - \sigma, t_o + \sigma]$, $|t - t'| \leq \delta$, $k \geq k_o$ implies

$$|y(t) - y(t')| \leq \varepsilon/2, \quad |y_k(t) - y_k(t')| \leq \varepsilon/2 \quad .$$

We can assume $0 < \delta < \sigma$, $\delta \leq \varepsilon$. For any h, $0 < h \leq \delta$, let us consider the averages

$$m_h = h^{-1} \int_0^h X'(t_o + s)ds = h^{-1}[X(t_o + h) - X(t_o)] \quad , \tag{2.8.3}$$

$$m_{hk} = h^{-1} \int_0^h x_k'(t_o + s)ds = h^{-1}[x_k(t_o + h) - x_k(t_o)] \quad , \tag{2.8.4}$$

where $X = (y, Z)$, $x_k = (y_k, z_k)$.

Given $\eta > 0$ arbitrary, we can fix h, $0 < h \leq \delta < \sigma$, so small that

$$|m_h - X'(t_o)| \leq \eta \quad , \tag{2.8.5}$$

$$|S(t_o + h) - S(t_o)| < \eta\, h/4 \quad . \tag{2.8.6}$$

This is possible since $h^{-1} \int_0^h X'(t_o + s)ds \to X'(t_o)$ and $[S(t_o + h) - S(t_o)]h^{-1} \to 0$ as $h \to 0+$. Also, we can choose h in such a way that $z_k(t_o + h) \to z(t_o)$ as $k \to +\infty$. This is possible since $z_k(t) \to z(t)$ for almost all $t_1 < t < t_2$.

Having so fixed h, let us take $k_1 \geq k_o$ so large that

$$|y_k(t_o) - y(t_o)|, \quad |y_k(t_o + h) - y(t_o + h)| \leq \min[\eta\, h/4, \varepsilon/2],$$

$$|z_k(t_o) - z(t_o)|, \quad |z_k(t_o + h) - z(t_o + h)| \leq \eta\, h/8 \quad .$$

This is possible since $y_k(t) \to Y(t)$, $z_k(t) \to z(t)$ both at $t = t_o$ and $t = t_o + h$. Then, we have

$$|h^{-1}[y_k(t_o + h) - y_k(t_o)] - h^{-1}[y(t_o + h) - y(t_o)]| \leq |h^{-1}[y_k(t_o + h)$$

$$- y(t_o + h)]| + |h^{-1}[y_k(t_o) - y(t_o)]| \leq h^{-1}[\eta\ h/4 + h^{-1}(\eta\ h/4)$$

$$= \eta/2 \quad .$$

Analogously, since $z = Z + S$, we have

$$|h^{-1}[z_k(t_o + h) - z_k(t_o)] - h^{-1}[Z(t_o + h) - Z(t_o)]| = |h^{-1}[z_k(t_o + h) - z_k(t_o)]$$

$$- h^{-1}[z(t_o + h) - z(t_o)] + h^{-1}[S(t_o + h) - S(t_o)]| \leq |h^{-1}[z_k(t_o + h)$$

$$- z(t_o + h)]| + |h^{-1}[z_k(t_o) - z(t_o)]| + |h^{-1}[S(t_o + h) - S(t_o)]|$$

$$\leq h^{-1}(\eta\ h/8) + h^{-1}(\eta\ h/8) + h^{-1}(\eta\ h/4) = \eta/2 \quad .$$

Finally, we have

$$|m_{hk} - m_h| = |h^{-1}[x_k(t_o + h) - x_k(t_o)] - h^{-1}[X(t_o + h) - X(t_o)]| \leq |h^{-1}[y_k(t_o+h)$$

$$- y_k(t_o)] - h^{-1}[y(t_o + h) - y(t_o)]| + |h^{-1}[z_k(t_o + h) - z_k(t_o)]$$

$$- h^{-1}[Z(t_o + h) - Z(t_o)]| \leq \eta/2 + \eta/2 = \eta \quad . \qquad (2.8.7)$$

We conclude that for the chosen value of h, $0 < h \leq \delta < \sigma$, and every $k \geq k_1$ we have

$$|m_h - X'(t_o)| \leq \eta, \quad |m_{hk} - m_h| \leq \eta, \quad |y_k(t_o) - y(t_o)| \leq \varepsilon/2. \qquad (2.8.8)$$

For $0 \leq \tau \leq h$ we have now

$$|y_k(t_o + \tau) - y(t_o)| \leq |y_k(t_o + \tau) - y_k(t_o)| + |y_k(t_o) - y(t_o)| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

$$|(t_o + \tau) - t_o| \leq h \leq \delta \leq \varepsilon , \tag{2.8.9}$$

$$x_k'(t_o + \tau) = (y_k'(t_o + \tau), z_k'(t_o + \tau)) \in Q(t_o + \tau, y_k(t_o + \tau)) \text{ a.e. in } [0,h].$$

Hence, for almost all τ, $0 \leq \tau \leq h$,

$$x_k'(t_o + \tau) = (y_k'(t_o + \tau), z_k'(t_o + \tau)) \in Q(t_o, y_o, 2\varepsilon) ,$$

and consequently

$$x_k'(t_o + \tau) = (y_k'(t_o + \tau), z_k'(t_o + \tau)) \in \text{cl co } Q(t_o, y_o, 2\varepsilon) \text{ a.e. in } [0,h].$$

The average m_{hk} as defined by (2.8.4) is then also a point of the same closed and convex set, or

$$m_{hk} \in \text{cl co } Q(t_o, y_o, 2\varepsilon)$$

for the chosen h and every $k \geq k_1$. By relations (2.8.5) and (2.8.8) we deduce

$$|X'(t_o) - m_{hk}| \leq |X'(t_o) - m_h| + |m_h - m_{hk}| \leq 2\eta ,$$

and hence

$$X'(t_o) \in [\text{cl co } Q(t_o, y_o, 2\varepsilon)]_{2\eta} .$$

Here $\eta > 0$ is an arbitrary number, and the set in brackets is closed, hence

$$X'(t_o) \in \text{cl co } Q(t_o, y_o, 2\varepsilon)$$

for every $\varepsilon > 0$. By property (Q) we have

$$X'(t_o) \in \cap_\varepsilon \text{ cl co } Q(t_o, y_o, 2\varepsilon) \;=\; Q(t_o, y_o) \quad ,$$

where $y_o = y(t_o)$, and $X'(t_o) = (y'(t_o), Z'(t_o))$. We have proved that for almost all $t \in [t_1, t_2]$ we have

$$dX/dt \in Q(t, y(t)) \quad .$$

Closure Theorem 2 is hereby proved.

The following example illustrates Closure Theorem 2. Let $n = 2$, $s = 1$, $n - s = 1$, $A = E_3$, $Q = Q(t,y) = [z = (z^1, z^2) | z^2 \geq 0, -1 \leq z^1 \leq 1]$. If $\varphi(t)$, $0 \leq t \leq 1$, denotes a singular continuous monotone function with $\varphi(0) = 0$, $\varphi(1) = 1$, $\varphi'(t) = 0$ a.e. in $[0,1]$, let us define $\varphi(t)$ in $(-\infty, +\infty)$ by taking $\varphi = 0$ for $t \leq 0$ and $\varphi = 1$ for $t \geq 1$. Let $z_k(t) = k \int_t^{t+k^{-1}} \varphi(\tau) d\tau$, $0 \leq t \leq 1$, $k = 1,2,\ldots$. Here the scalar functions $z_k(t)$ are absolutely continuous, monotone nondecreasing, with $z_k'(t) \geq 0$ and $z_k(t) \to z(t) = \varphi(t)$ uniformly in $[0,1]$ as $k \to \infty$. Let us take $Z(t) = 0$, $y(t) = 0$, $y_k(t) = 0$, $0 \leq t \leq 1$, $k = 1,2,\ldots$, and then $z(t) = Z(t) + \varphi(t)$, $Z(t)$ absolutely continuous, $\varphi(t)$ singular. Here (y_k, z_k) converges uniformly toward (y,z) in $[0,1]$. All pairs (y_k, z_k) are solutions of the orientor field $(y', z') \in Q$, (y, Z) is a solution of the same orientor field, but (y,z) is not.

Remark 1. We could now deduce from (2.8.i) corollaries similar to the closure statements (2.6.i) for usual solutions, and (2.7.i) for generalized solutions. This task can be well left to the reader as an exercise. The essential point is that while for usual solutions we need require explicitly that the sets Q, or $\tilde{Q}$, are convex, for generalized solutions the corresponding sets R are necessarily convex.

2.C. LOWER CLOSURE THEOREMS

2.9. LOWER CLOSURE OF FUNCTIONALS IN INTEGRAL FORM

We shall now introduce the concept of lower closure.

As usual let A be a closed subset of the tx-space $E_1 \times E_n$, for every $(t,x) \in A$ let $U(t,x)$ be a given subset of the u-space E_m, let M be the set of all (t,x,u) with $(t,x) \in A$, $u \in U(t,x)$, and let $\tilde{f}(t,x,u) = (f_o,f_1,\dots,f_n) = (f_o,f)$ be a given continuous vector function on M. Let B be a closed subset of the $t_1x_1t_2x_2$-space E_{2n+2}. As in (1.2) and (2.6, (b)) we consider the functional

$$I[x,u] = \int_{t_1}^{t_2} f_o(t,x(t),u(t))dt \ . \qquad (2.9.1)$$

As in (1.1), (1.2) we shall say that a pair $x(t)$, $u(t)$, $t_1 \le t \le t_2$, is admissible provided $x(t)$ is AC in $[t_1,t_2]$, $u(t)$ is measurable in $[t_1,t_2]$, $(t,x(t)) \in A$ for $t \in [t_1,t_2]$, $u(t) \in U(t,x(t))$ a.e. in $[t_1,t_2]$, $dx/dt = f(t,x(t),u(t))$ a.e. in $[t_1,t_2]$, $f_o(t,x(t),u(t))$ is L-integrable in $[t_1,t_2]$, and $(t_1,x(t_1),t_2,x(t_2)) \in B$. Whenever we wish to disregard boundary conditions, we have only to take $B = E_{2n+2}$.

Let $x(t)$, $t_1 \le t \le t_2$, be any AC vector function (which is the limit in the metric ρ of admissible trajectories). If, for any sequence $x_k(t)$, $u_k(t)$, $t_{1k} \le t \le t_{2k}$, $k = 1,2,\dots$, of admissible pairs with $\rho(x_k,x) \to 0$, $\lim I[x_k,u_k] < +\infty$ as $k \to \infty$, there is some measurable function $u(t)$, $t_1 \le t \le t_2$, such that $x(t)$, $u(t)$, $t_1 \le t \le t_2$, is admissible, and

$$I[x,u] \le \underline{\lim}_{k\to\infty} I[x_k,u_k] \ , \qquad (2.9.2)$$

then we say that $I[x,u]$ has the property of <u>lower closure</u> at the trajectory $x(t)$, $t_1 \leq t \leq t_2$.

Before we prove a sufficient condition for lower closure, the following remarks are needed. First, if x is the limit in the ρ-metric of admissible trajectories as assumed, then by (2.1.i), we know that $(t,x(t)) \in A$ for all $t \in [t_1,t_2]$, and $(t_1,x(t_1), t_2,x(t_2)) \in B$.

Furthermore, if we know that the set M is closed, and that for every $(t,x) \in A$ the sets $Q(t,x) = f(t,x,U(t,x))$ are closed convex subsets of E_n satisfying property (Q) in A, then certainly $x'(t) \in Q(t,x(t))$ a.e. in $[t_1,t_2]$ by force of Closure Theorem 1 (2.5), and then there is some measurable $u(t)$, $t_1 \leq t \leq t_2$, such that

$$u(t) \in U(t,x(t)), \quad x'(t) = f(t,x(t),u(t)) \quad \text{a.e. in } [t_1,t_2], \tag{2.9.3}$$

by force of Implicit Function Theorem (1.6.i) as we have seen in (2.6.i). As usual, we say that any such strategy $u(t)$ generates $x(t)$, $t_1 \leq t \leq t_2$. Obbiously, in the comcept of lower semicontinuity we require more, namely we need a strategy u generating x for which (2.9.2) holds.

It may well occur that x is generated by some strategy $\bar{u}$ for which (2.9.2) does not hold. The following example displays two strategies u and $\bar{u}$, both generating the same trajectory x, such that (2.9.2) holds for u but not for $\bar{u}$.

Indeed, take $m = n = 1$, $t_1 = 0$, $t_2 = 1$, $f_o = 1 + \cos \pi u$, $f = f_1 = \sin \pi u$, $u \in U = [-1 \leq u \leq 1]$, $x(t) = 0$, $0 \leq t \leq 1$, $A = E_2$. Now take $u_k(t) = \pm 2^{-1}$ according as $k^{-1}i \leq t < k^{-1}i + (2k)^{-1}$, or $k^{-1}i + (2k)^{-1} \leq t < (i+1)k^{-1}$, $i = 0,1,\dots,k-1$, $k = 1,2,\dots$, and take $x_k(t) = t - k^{-1}i$, or $x_k(t) = k^{-1}(i+1)-t$, according as t is in one or the other set of intervals above. Then x_k, u_k, $k = 1,2,\ldots$, is a sequence of admissible pairs, $0 \leq x_k(t) \leq (2k)^{-1}$, and $x_k \to x$ as $k \to \infty$ uniformly in $[0,1]$. The trajectory $x(t) = 0$, $0 \leq t \leq 1$, is now generated by both $u(t) = 1$, $0 \leq t \leq 1$, and by $\bar{u}(t) = 0$, $0 \leq t \leq 1$. On the other hand,

$$I[x,u] = 0, \quad I[x,\bar{u}] = 2, \quad I[x_k,u_k] = 1, \quad k = 1,2,\ldots,$$

and thus relation (2.9.2) holds for u but not for $\bar{u}$.

As we shall see in (2.13), the concept of lower closure introduced above contains as a particular case the usual concept of lower semicontinuity, in particular the concept of lower semicontinuity for free problems.

2.10. A SUFFICIENT CONDITION FOR LOWER CLOSURE

Let A, U(t,x), M, B, $f_o(t,x,u)$, $f(t,x,u) = (f_1,\ldots,f_n)$ be defined as in (2.9). For any $(t,x) \in A$ let $\tilde{\tilde{Q}}(t,x)$ be the set of all $\tilde{z} = (z^o,z^1,\ldots,z^n) = (z^o,z)$ with $z^o \geq f_o(t,x,u)$, $z = f(t,x,u)$ for some $u \in U(t,x)$.

(2.10.i). If the sets A, M, B are closed, and $f_o(t,x,u)$, $f(t,x,u) = (f_1,\ldots,f_n)$ are continuous on M, let us assume that the sets $\tilde{\tilde{Q}}(t,x)$ are closed, convex, and satisfy property (Q) at every point $(t,x) \in A$ with exception perhaps of a set of points whose t coordinate lies on a set of measure zero on the t-axis. Let us assume that, for some locally L-integrable scalar function $\psi(t)$ we have $(\psi)f_o(t,x,u) \geq \psi(t)$ for all $(t,x,u) \in M$, with exception perhaps of another set of points whose t coordinate lies on a set of measure zero on the t-axis. Then the integral (2.9.1) has the property of lower closure at every AC vector function $x(t) = (x^1,\ldots,x^n)$, $t_1 \leq t \leq t_2$, which is the limit in the ρ-metric of admissible trajectories. In other words, for every AC vector $x(t) = (x^1,\dots,x^n)$, $t_1 \le t \le t_2$, and sequence $x_k(t)$, $u_k(t)$, $t_{1k} \le t \le t_{2k}$, $k = 1,2,\dots$, of admissible pairs with $\rho(x_k,x) \to 0$, $\underline{\lim}\, I[x_k,u_k] < +\infty$ as $k \to \infty$, there is a measurable function $u(t)$, $t_1 \le t \le t_2$, such that $x(t)$, $u(t)$, $t_1 \le t \le t_2$, is admissible and $I[x,u] \le \underline{\lim}\, I[x_k,u_k]$.

Remark 1. Condition (ψ) in statement (2.10.i) will be drastically reduced in statement (2.10.ii) below. A simple condition under which the sets $\tilde{\tilde{Q}}$ above, if convex, are closed and satisfy property (Q) as requested, and under which also condition (ψ) above is satisfied, will be given in (2.12.i).

Proof of (2.10.i). As usual we introduce auxiliary variables x^o and u^o, vectors $\tilde{x} = (x^o,x^1,\dots,x^n)$, $\tilde{u} = (u^o,u^1,\dots,u^m)$, and vector function $\tilde{f}(t,x,\tilde{u}) = (\tilde{f}_o,f_1,\dots,f_n) = (\tilde{f}_o,f)$ with $\tilde{f}_o = u^o$. Let $\tilde{U}(t,x)$ be the control space $[\tilde{u} = (u^o,u) \mid u^o \ge f_o(t,x,u),\ u \in U(t,x)] \subset E_{m+1}$. Then

$$\begin{aligned} \tilde{\tilde{Q}}(t,x) &= [\tilde{z} = (z^o,z) \mid z^o \ge f_o(t,x,u), \quad z = f(t,x,u), \quad u \in U(t,x)] \\ &= \tilde{\tilde{f}}(t,x,\tilde{U}(t,x)) , \end{aligned} \tag{2.10.2}$$

while

$$Q(t,x) = [z \mid z = f(t,x,u), \quad u \in U(t,x)] = f(t,x,U(t,x)) .$$

We have now an auxiliary canonic problem with

$$d\tilde{x}/dt = \tilde{\tilde{f}}(t,x,\tilde{u}), \quad \tilde{u} \in \tilde{U}(t,x), \quad (t,x) \in A ,$$

$$x^o(t_1) = 0, \quad (t_1,x(t_1),t_2,x(t_2)) \in B ,$$

and functional

$$J = \int_{t_1}^{t_2} u^o(t)dt = x^o(t_2) .$$

Here $\rho(x_k,x) \to 0$ as $k \to \infty$, hence $t_{1k} \to t_1$, $t_{2k} \to t_2$, and thus $t_1 - \ell \leq t_{1k} \leq t_{2k} \leq t_2 + \ell$ for all $k = 1,2,\ldots$, and some constant $\ell > 0$. Finally

$$I[x_k,u_k] = \int_{t_{1k}}^{t_{2k}} f_o(t,x_k(t),u_k(t))dt \geq - \int_{t_1-\ell}^{t_2+\ell} \psi(t)dt = -L ,$$

where $L \geq 0$ is now a fixed number.

Let $i = \varliminf_{k\to\infty} I[x_k,u_k]$, so that $-L \leq i \leq +\infty$. Let us assume $i < +\infty$, and let us consider a subsequence $[k_s]$ such that $I[x_{k_s},u_{k_s}] \to i$ as $s \to \infty$. Let L_1 be a constant such that $I[x_{k_s},u_s] \leq L_1$ for all s.

Now we have

$$x_k^o(t) = \int_{t_{1k}}^{t} f_o(\tau,x_k(\tau),u_k(\tau))d\tau$$

$$= \int_{t_{1k}}^{t} [f_o(\tau,x_k(\tau),u_k(\tau)) + \psi(\tau)]d\tau - \int_{t_{1k}}^{t} \psi(\tau)d\tau , \tag{2.10.3}$$

where $f_o + \psi \geq 0$, and thus if

$$F_k(t) = \int_{t_{1k}}^{t} [f_o(\tau,x_k(\tau),u_k(\tau)) + \psi(\tau)]d\tau, \quad t_{1k} \leq t \leq t_{2k} , \tag{2.10.4}$$

we have

$$0 \leq F_{k_s}(t) \leq L_1 + L$$

for all s, and $F_{k_s}(t)$ is a nondecreasing continuous function in $[t_{1k}, t_{2k}]$. We shall actually extend these functions in the fixed interval $[t_1 - \ell, t_2 + \ell]$ by continuity and constancy outside $[t_{1k}, t_{2k}]$.

By Helly's theorem there exists a subsequence, say still $[k_s]$ for the sake of simplicity, such that $F_{k_s}(t) \to F(t)$ as $s \to \infty$ pointwise in $[t_1 - \ell, t_2 + \ell]$. Then $F(t)$ is a monotone nondecreasing function in $[t_1 - \ell, t_2 + \ell]$, with $0 \leq F(t) \leq L_1 + L$.

Since

$$x^o_{k_s}(t) = F_{k_s}(t) - \int_{t_{1k_s}}^{t} \psi(\tau) d\tau , \qquad (2.10.5)$$

we conclude that $x^o_{k_s}(t) \to x^o(t)$ pointwise in (t_1, t_2), and that

$$x^o(t) = F(t) - \int_{t_1}^{t} \psi(\tau) d\tau, \quad t_1 < t < t_2 . \qquad (2.10.6)$$

Note that the sequence $x^o(t_{2k})$ is bounded, hence we can extract the subsequences (k_s) above, so that the limit $x^o_2 = \underline{\lim}_{s\to\infty} x^o(t_{2k_s})$ exists. Let us define $x^o(t)$ at t_1 and at t_2 by taking $x^o(t_1) = 0$ and $x^o(t_2) = x^o_2$. Let us prove that $x^o(t_1) = 0 \leq x^o(t_1 + 0)$ and $x^o(t_2 - 0) \leq x^o_2 = x^o(t_2)$. Indeed, given $\varepsilon > 0$, we have $F_{k_s}(t_{1k_s}) = 0 \leq F_{k_s}(t_1 + \varepsilon)$, and, as $s \to +\infty$, also $0 \leq F(t_1 + \varepsilon)$. Finally, as $\varepsilon \to 0$, we have $0 \leq F(t_1 + 0)$, and by (2.10.6) also $x^o(t_1) = 0 \leq x^o(t_1 + 0)$. Analogously, we have $t_2 - \varepsilon < t_{2k}$ for k sufficiently large, hence $F_{k_s}(t_2 - \varepsilon) \leq$ $\leq F_{k_s}(t_{2k_s})$, hence, by (2.10.3), and (2.10.4), also

$$x^o_{k_s}(t_2 - \varepsilon) + \int_{t_{1k_s}}^{t_2-\varepsilon} \psi(\tau)d\tau \le x^o_{k_s}(t_{2k_s}) + \int_{t_{1k_s}}^{t_{2k_s}} \psi(\tau)d\tau \ .$$

As $s \to +\infty$ we have

$$x^o(t_2 - \varepsilon) + \int_{t_1}^{t_2-\varepsilon} \psi(\tau)d\tau \le x_2^o + \int_{t_1}^{t_2} \psi(\tau)d\tau \ ,$$

and, as $\varepsilon \to 0+$, also

$$x^o(t_2 - 0) \le x_2^o = x^o(t_2) \ .$$

Now

$$F(t) = x^o(t) + \int_0^t \psi(\tau)d\tau$$

is a monotone, nondecreasing, nonnegative function in $[t_1,t_2]$ with $F(t_1) = 0$, and hence F possesses a unique decomposition $F = F^* + S$ into an AC function F^* and a singular function S, both F^*, S monotone, nondecreasing, nonnegative functions in $[t_1,t_2]$ with $F^*(t_1) = S(t_1) = 0$. If now we set

$$X(t) = F^*(t) - \int_{t_1}^t \psi(\tau)d\tau, \quad t_1 \le t \le t_2 \ ,$$

we have

$$F(t) = x^o(t) + \int_0^t \psi(\tau)d\tau = F^*(t) + S(t) \ ,$$

$$x^o(t) = (F^*(t) - \int_0^t \psi(\tau)d\tau) + S(t) \ ,$$

or

$$x^o(t) = X(t) + S(t) ,$$

where X is AC and $S(t) \geq 0$ is monotone, nondecreasing, and singular. Let $\tilde{x}(t) = (X,x)$, $t_1 \leq t \leq t_2$. Then $dX/dt = dx^o/dt$, $X(t_1) = 0$.

By Closure Theorem 2 we conclude that

$$\tilde{x}'(t) = (X'(t),x'(t)) \in \tilde{\tilde{Q}}(t,x(t)) \quad \text{a.e. in } [t_1,t_2] ,$$

and by Implicit Function Theorem (1.6.i) there is a measurable function $\tilde{u}(t) = (u^o,u)$, $t_1 \leq t \leq t_2$, such that

$$\tilde{u}(t) \in \tilde{U}(t,x(t)), \quad \tilde{x}'(t) = \tilde{\tilde{f}}(t,x(t),\tilde{u}(t)),$$

or

$$u^o(t) \geq f_o(t,x(t),u(t)), \quad u(t) \in U(t,x(t)) ,$$

$$dX/dt = u^o(t), \quad dx/dt = f(t,x(t),u(t))$$

a.e. in $[t_1,t_2]$. On the other hand

$$I[x,u] = \int_{t_1}^{t_2} f_o(t,x(t),u(t))dt \leq \int_{t_1}^{t_2} u^o(t)dt$$

$$= X(t_2) - X(t_1) = X(t_2) = x^o(t_2) - S(t_2)$$

$$\leq x^o(t_2) = x_2^o = \lim_{s\to\infty} x^o_{k_s}(t_{2k_s})$$

$$= \lim_{s\to\infty} I[x_{k_s},u_{k_s}] = i = \underline{\lim}_{k\to\infty} I[x_k,u_k] .$$

Remark 2. In the sufficient condition for lower closure (2.10.i) it is enough to request that the sets $\tilde{\tilde{Q}}(t,x)$ are closed, convex, and have property (Q) at the points $(t,x(t)) \in A$ for almost all $t \in [t_1,t_2]$. In this form, and under suitable regularity hypotheses, the convexity assumption of this sufficient condition for lower closure will be shown to be also necessary [App. A.5].

Remark 3. Condition (ψ) in statement (2.10.i) is certainly satisfied if, for instance, $f_o(t,x,u) \geq 0$ for all $(t,x,u) \in M$, or $f_o(t,x,u) \geq \nu$ for all $(t,x,u) \in M$ where ν is some real constant. Nevertheless, condition(ψ) in (2.10.i) can be reduced. For instance, we may replace it by the following weaker assumption (ψ'): for every compact subset A_o of A there is a locally integrable function $\psi_o(t)$ (which may depend on A_o) such that $f_o(t,x,u) \geq \psi_o(t)$ for all $(t,x) \in A_o$, $u \in U(t,x)$. The proof is the same since we can include all trajectories x and x_k in a unique compact subset A_o of A.

A more drastic generalization of (2.10.i) will be given below (2.10.ii) where we shall use the following much weaker form of condition.

Condition (ψ*). For every $(\bar{t},\bar{x}) \in A$ there are a neighborhood $N(\bar{t},\bar{x})$ of $(\bar{t},\bar{x})$ in A, a locally integrable function $\psi(t)$, and real numbers $b_1,\dots,b_n$ (all $b_1,\dots,b_n$ and ψ may depend on $\bar{t},\bar{x},N$) such that

$$\bar{f}(t,x,u) = f_o(t,x,u) - \sum_{j=1}^{n} b_j f_j(t,x,u) \geq \psi(t)$$

for all $(t,x) \in N(\bar{t},\bar{x})$, $u \in U(t,x)$, with exception perhaps of a set of points (t,x) whose t coordinate lies on a set of measure zero on the t-axis.

Remark 4. We shall note here that, under condition (ψ*), it is natural to consider the sets

$$\tilde{\tilde{Q}}(t,x) = [(z^o,z) | z^o \geq f_o(t,x,u),\ z = f(t,x,u),\ u \in U(t,x)] ,$$

or the analogous sets

$$\tilde{\tilde{Q}}^*(t,x) = [(Z^o,Z) | Z^o \geq \bar{f}_o(t,x,u),\ Z = f(t,x,u),\ u \in U(t,x)] .$$

It is easy to see that the sets $\tilde{\tilde{Q}}$ are closed, or convex, or satisfy property (Q) if and only if the same occurs for the sets $\tilde{\tilde{Q}}^*$. Indeed, the sets above are transformed into one another by the fixed affine transformation

$$Z^o = z^o - b\cdot z, \quad Z = z .$$

(2.10.ii). Let A, B, U(t,x), M, f(t,x,u), $f_o(t,x,u)$ as in (2.11.i), and let us assume that condition (ψ*) holds. With $N(\bar{t},\bar{x})$ as in condition (ψ*), and for every $(t,x) \in N(\bar{t},\bar{x})$, let $\tilde{\tilde{Q}}(t,x)$ denote the set of all $\tilde{z} = (z^o,z^1,\dots,z^n) = (z^o,z)$ with $z^o \geq \bar{f}_o(t,x,u), z = f(t,x,u)$ for $u \in U(t,x)$, and assume that the sets $\tilde{\tilde{Q}}(t,x)$ are closed, convex, and satisfy property (Q) at all points $(t,x) \in N(\bar{t},\bar{x})$, with exception perhaps of a set of points whose t-coordinate lies on a set of measure zero on the t-axis. Then the integral (2.9.1) has the property of lower closure at every AC vector function x(t), $t_1 \leq t \leq t_2$, which is the limit in the ρ-metric of admissible trajectories.

<u>Proof</u>. Let x(t), $t_1 \leq t \leq t_2$, be any AC function as in text, and $x_k(t)$, $u_k(t)$, $t_{1k} \leq t \leq t_{2k}$, k = 1,2,..., be a sequence of admissible pairs with $\rho(x_k,x) \to 0$, $\underline{\lim}\ I[x_k,u_k] < +\infty$. Let A_o be a compact neighborhood (containing

the graph of x and all x_k). By hypothesis, for every $(\bar{t},\bar{x}) \in A_o$ there are numbers $\delta > 0$, $b_1,\ldots,b_n$ real, and a locally integrable function $\psi(t)$, $-\infty < t < +\infty$, such that $\bar{f}_o(t,x,u) \geq \psi(t)$ for all $(t,x) \in N_{2\delta}(\bar{t},\bar{x})$ and $u \in U(t,x)$. We consider the smaller neighborhoods $N_\delta(\bar{t},\bar{x})$ which we consider as open (in A). These too form a cover of the compact set A_o. Thus, finitely many of these N_δ cover A_o, say $N_{\delta_\gamma}(t_\gamma,x_\gamma)$, $\gamma = 1,\ldots,s$. Let $\delta_\gamma > 0$, $b_{\gamma 1},\ldots,$ $b_{\gamma n}$, ψ_γ be the corresponding elements, so that

$$\bar{f}_{os}(t,x,u) = f_o(t,x,u) - \Sigma_{j=1}^{o} b_{\gamma j} f_j(t,x,u) \geq \psi_\gamma(t)$$

for all $(t,x) \in N_{2\delta_\gamma}(t_\gamma,x_\gamma)$, $u \in U(t,x)$, $\gamma = 1,\ldots,s$, and $U_{\gamma=1}^{s} N_{\delta_\gamma}(t_\gamma,x_\gamma) \supset A_o$.

Let $b = \max\,[|b_{\gamma j}|,\ j = 1,\ldots,n,\ \gamma = 1,\ldots,s]$, $\delta_o = \min\,[\delta_\gamma,\ \gamma = 1,\ldots,s]$.

Since $(t,x(t)) \in A_o$ for all $t_1 \leq t \leq t_2$, we can divide the arc C_o: $x = x(t)$, $t_1 \leq t \leq t_2$, into finitely many subarcs, say C_σ, $\sigma = 1,\ldots,N$, each C_σ completely contained in some neighborhood $N_{\delta_\gamma}(t_\gamma,x_\gamma)$. Thus, we have for the arcs C_σ the representations C_σ: $x = x(t)$, $\tau_{\sigma-1} \leq t \leq \tau_\sigma$, with $t_1 = \tau_o < \tau_1 < \ldots < \tau_N = t_2$, and each C_σ lies in a certain $N_{\delta_\gamma}(t_\gamma,x_\gamma)$ which now remains associated to C_σ. Since $\rho(x_k,x) \to 0$ as $k \to \infty$, hence $t_{1k} \to t_1$, $t_{2k} \to t_2$, we see that for all k sufficiently large we have $t_{1k} < \tau_1 < \ldots < \tau_{N-1} < t_{2k}$. Thus for all k sufficiently large, the arc C_k: $x = x_k(t)$, $t_{1k} \leq t \leq t_{2k}$, is divided into the same number N of subarcs, say $C_{k\sigma}$: $x = x_k(t)$, $\tau_{\sigma-1} \leq t \leq \tau_\sigma$, $\sigma = 1,\ldots,N$, where now $\tau_o = t_1$ must be replaced by t_{1k} and $\tau_N = t_2$ must be replaced by t_{2k}. Also, for all k sufficiently large, say for $k \geq k_o$, the arc $C_{k\sigma}$ is completely contained in $N_{2\delta_\gamma}(t_\gamma,x_\gamma)$ for the same γ we have already associated to C_σ. Thus, for $k \geq k_o$, C_σ lies in some $N_{\delta_\gamma}(t_\gamma,x_\gamma)$ and $C_{k\sigma}$ in $N_{2\delta_\gamma}(t_\gamma,x_\gamma)$. Also, $C_{k\sigma} \to C_\sigma$ as $k \to \infty$ in the sense that ρ-distance approaches zero as $k \to \infty$. We shall now consider for each $\sigma = 1,\ldots,N$, the auxiliary functional

$$J = \int_{t'}^{t''} \bar{f}_o(t,x(t),u(t))dt$$

for all admissible pairs x,u with the graph of x lying in $N_{2\delta_\gamma}(t_\gamma,x_\gamma)$. Here by admissible we mean that the conditions a-d of (1.1) are satisfied with A replaced by $N_{2\delta_\gamma}(t_\gamma,x_\gamma)$, and of course $\bar{f}_o(t,x(t),u(t))$ L-integrable as usual.

For each σ we may now apply (2.11.i) to arc C_σ, the sequence $C_{k\sigma}$, k = 1, 2,..., and functional J. We conclude that each C_σ is admissible and that

$$J[C_\sigma] \leq \underline{\lim}_{k\to\infty} J[C_{\sigma k}], \quad \sigma = 1,\ldots,N . \tag{2.10.7}$$

More precisely, for each σ, there is a measurable $u(t)$, $\tau_{\sigma-1} \leq t \leq \tau_\sigma$, such that the pair $x(t)$, $u(t)$, $\tau_{\sigma-1} \leq t \leq \tau_\sigma$ is admissible for the functional J, in particular $u(t) \in U(t,x(t))$, $dx/dt = f(t,x(t),u(t))$, $\tau_{\sigma-1} \leq t \leq \tau_\sigma$ (a.e.), $\sigma = 1,\ldots,N$, and the expression

$$\bar{f}_o(t,x(t),u(t)) = f_o(t,x(t),u(t)) - \Sigma_{j=1}^{n} b_{\gamma j}\, f_j(t,x(t),u(t))$$

is L-integrable in $[\tau_{\sigma-1},\tau_\sigma]$. Since the functions f_j here are certainly L-integrable in the same interval (as derivatives of the AC functions $x^i(t)$ in $[\tau_{\sigma-1},\tau_\sigma]$), we conclude that $f_o(t,x(t),u(t))$ itself is L-integrable in each $[\tau_{\sigma-1},\tau_\sigma]$ and hence in the whole of $[t_1,t_2]$. We have proved that the pair $x(t)$,

$u(t)$, $t_1 \leq t \leq t_2$, is admissible for the original integral I.

Now, given $\varepsilon > 0$, we deduce from (2.10.7) that there is some $\bar{k} \geq k_o$ such that, for $k \geq \bar{k}$, we have

$$J[C_{k\sigma}] - J[C_\sigma] > -\varepsilon/N, \quad \rho(C_{k\sigma}, C_\sigma) < \varepsilon/Nnb, \quad \sigma = 1,\ldots,N\,. \quad (2.10.8)$$

Now we have

$$I[x,u] = \int_{t_1}^{t_2} f_o(t,x(t),u(t))dt = \Sigma_{\sigma=1}^{N} \int_{\tau_{\sigma-1}}^{\tau_\sigma} f_o(t,x(t),u(t))dt$$

$$= \Sigma_{\sigma=1}^{N} \left\{ \left[\int_{\tau_{\sigma-1}}^{\tau_\sigma} \bar{f}_o(t,x(t),u(t))dt + \Sigma_{j=1}^{n} b_{\sigma j}[x^j(\tau_\sigma) - x^j(\tau_{\sigma-1})] \right] \right\}$$

$$I[x_k,u_k] = \int_{t_{1k}}^{t_{2k}} f_o(t,x_k(t),u_k(t))dt = \Sigma_{\sigma=1}^{N} \int_{\tau_{\sigma-1}}^{\tau_\sigma} f_o(t,x_k(t),u_k(t))dt$$

$$= \Sigma_{\sigma=1}^{N} \left\{ \left[\int_{\tau_{\sigma-1}}^{\tau_\sigma} \bar{f}_o(t,x_k(t),u_k(t))dt + \Sigma_{j=1}^{n} b_{\sigma j}\, [x_k^j(\tau_\sigma) - x^j(\tau_{\sigma-1})] \right] \right\}$$

where we have written $b_{\sigma j}$ instead of $b_{\gamma j}$ with γ the index we have associated to σ, and where we have written $\tau_1 = t_1$ and $\tau_N = t_2$ instead of t_{1k} and t_{2k} in the expression for $I[x_k,u_k]$. By difference we have now, remembering relations (2.10.8),

$$I[x_k,u_k] - I[x,u] > -N(\varepsilon/N) +$$

$$+ \Sigma_{\sigma=1}^{N} \Sigma_{j=1}^{n} b_{\sigma j} \left\{ \left[[x_k^j(\tau_\sigma) - x^j(\tau_\sigma)] + [x_k^j(\tau_{\sigma-1}) - x^j(\tau_{\sigma-1})] \right] \right\}$$

where each bracket is now in absolute value $\leq (Nnb)^{-1}\varepsilon$. We conclude that for all $k \geq \bar{k}$ we have

$$I[x_k,u_k] - I[x,u] > -\varepsilon - Nnb[2(Nnb)^{-1}\varepsilon] = -3\varepsilon .$$

Because of the arbitrarity of ε, we have proved the lower closure of I at x.

Remark 5. Statements (2.10.i) and (2.10.ii) have corollaries for generalized solutions. Let A, U(t,x), M, B, $f_o(t,x,u)$, $f(t,x,u) = (f_1,\dots,f_n)$ be defined as in (2.9). Then x(t), p(t), v(t), $t_1 \leq t \leq t_2$, is said to be an admissible generalized system (generalized solution) provided x(t) is AC and p(t), v(t) are measurable in $[t_1,t_2]$; $(t,x(t)) \in A$ for all $t \in [t_1,t_2]$; $p(t) = (p_1,\dots,p_\gamma)$, $p_j(t) \geq 0$, $\Sigma_j p_j(t) = 1$ (that is, $p(t) \in \Gamma$), $v(t) = (u^{(j)}(t),\ j = 1,\dots,\gamma)$, $u^{(j)}(t) \in U(t,x(t))$ a.e. in $[t_1,t_2]$; provided the differential equation

$$dx/dt = \Sigma_{j=1}^{\gamma} p_j(t)\ f(t,x(t),u^{(j)}(t))$$

is satisfied a.e. in $[t_1,t_2]$, the function $\Sigma_{j=1}^{n} p_j(t)\ f_o(t,x(t),u^{(j)}(t))$ is L-integrable in $[t_1,t_2]$, and $(t_1,x(t_1),t_2,x(t_2)) \in B$. Then the corresponding functional is

$$I[x,p,v] = \int_{t_1}^{t_2} \Sigma_{j=1}^{\gamma} p_j(t)\ f_o(t,x(t),u^{(j)}(t))dt .$$

The sets $\tilde{\tilde{R}}(t,x)$ of all points $\tilde{z} = (z^o,z) = (z^o,z^1,\dots,z^n)$ with $z^o \geq \Sigma\, p_j f_o(t,x,u^{(j)})$, $z = \Sigma\, p_j\ f(t,x,u^{(j)})$ for some $(p,v) \in \Gamma \times (U(t,x))^\gamma$ are exactly the sets co $\tilde{\tilde{Q}}(t,x)$ if $\gamma = n + 2$. We take for γ the smallest integer for which this occurs for all $(t,x) \in A$, $1 \leq \gamma \leq n + 2$.

(2.10.iii). (A lower closure theorem for Lagrange problem and generalized solutions). If the sets A, M, B are closed, and f_o, f are continuous on M, let us assume that the convex sets $\tilde{\tilde{R}}(t,x)$ are closed and satisfy property (Q) at every point $(t,x) \in A$ with exception perhaps of a set of points whose t coordinate lies on a set of measure zero on the t-axis. Let us assume that for some locally L-integrable scalar function $\psi(t)$ we have (ψ) $f_o(t,x,u) \geq \psi(t)$ for all $(t,x,u) \in M$, with exception perhaps of another set of points whose t coordinate lies on a set of measure zero on the t-axis. Then the integral (2.10.7) has the property of lower closure at every AC vector function $x(t) = (x^1,\dots,x^n)$, $t_1 \leq t \leq t_2$, which is the limit in the ρ-metric of generalized trajectories. In other words, for every AC function $x(t)$, $t_1 \leq t \leq t_2$, and sequence, $x_k(t)$, $p_k(t)$, $v_k(t)$, $t_{1k} \leq t \leq t_{2k}$, $k = 1,2,\dots$, of generalized systems with $\rho(x_k,x) \to 0$ as $k \to \infty$, $\underline{\lim}\, I[x_k,p_k,v_k] < \infty$ as $k \to \infty$, there are measurable functions $p(t)$, $v(t)$, $t_1 \leq t \leq t_2$, with $p(t) = (p_1,\dots,p_\gamma)$, $p_j(t) \geq 0$, $\sum p_j(t) = 1$, $v(t) = (u^{(j)}(t), j = 1,\dots,\gamma)$, $u^{(j)}(t) \in U(t,x(t))$ a.e. in $[t_1,t_2]$, such that $I[x,p,v] \leq \underline{\lim}\, I[x_k,p_k,v_k]$.

We leave to the reader to state the analogous corollary of (2.10.i) for generalized solutions.

2.11. A VARIANT OF THE LOWER CLOSURE PROPERTY

Statement (2.10.i) holds in a slightly stronger form. To formulate it we need, besides the sets $\tilde{\tilde{Q}}(t,x) \subset E_{n+1}$ of (2.10), also the sets $Q(t,x) = f(t,x, U(t,x)) \subset E_n$. These sets $Q(t,x)$ are the projections on the z-space E_n of the sets $\tilde{\tilde{Q}}(t,x)$ of the $z^o z$-space E_{n+1}. Thus, if the sets $\tilde{\tilde{Q}}(t,x)$ are convex, so are the sets $Q(t,x)$. On the other hand, the sets $\tilde{\tilde{Q}}(t,x)$ may be closed, without the sets $Q(t,x)$ being so. This is shown by the example $n = 2$, $m = 1$, $U = [-\infty < u < +\infty]$, $f_o = (1 + u^2)^{1/2}$, $f = \arctan u$, $-\pi/2 < f < \pi/2$. Then, Q and $\tilde{\tilde{Q}}$ are the fixed sets $Q = [z| \; -\pi/2 < z < \pi/2] \subset E_1$, $\tilde{\tilde{Q}} = [(z^o, z) | \; z^o \geq \sec z$, $-\pi/2 < z < \pi/2] \subset E_2$, and $\tilde{\tilde{Q}}$ is closed, but Q is not. This example shows also that property (Q) for the sets $\tilde{\tilde{Q}}$ does not imply the same property for the sets Q. In the statement below we shall require that both the sets $\tilde{\tilde{Q}}(t,x)$ and the sets $Q(t,x)$ have property (Q).

(2.11.i). If we assume, in addition to the hypotheses of (2.10.i), that both the sets $\tilde{\tilde{Q}}(t,x) \subset E_{n+1}$ and the sets $Q(t,x) \subset E_n$ are closed, convex, and satisfy property (Q) at all points of A with exception perhaps of a set of points whose t coordinate lies on a set of measure zero on the t-axis, then for every sequence $x_k(t)$, $u_k(t)$, $t_{1k} \leq t \leq t_{2k}$, $k = 1,2,\ldots$, of admissible pairs, and any AC vector function $x(t) = (x^1,\ldots,x^n)$, $t_1 \leq t \leq t_2$, with $\rho(x_k,x) \to 0$ as $k \to \infty$, there is a measurable function $u(t)$, $t_1 \leq t \leq t_2$, such that $u(t) \in U(t,x(t))$, $x'(t) = f(t,x(t),u(t))$ a.e. in $[t_1,t_2]$, and

$$I[x,u] \leq \varliminf_{k\to\infty} I[x_k,u_k] \; . \qquad (2.11.1)$$

If $\varliminf I[x_k,u_k] < +\infty$, then certainly the pair x,u is admissible, and (2.11.1) holds.

An analogous variant of theorem (2.11.i) also holds.

<u>Proof</u> <u>of</u> <u>(2.11.i)</u>. First note that $f_o(t,x(t),u(t))$ is measurable in $[t_1,t_2]$ and $\geq \psi(t)$, hence, $I[x,u]$ is finite, or $+\infty$. If the second member of (2.11.1) is finite, then $I[x,u]$ must be finite, hence $f_o(t,x(t),u(t))$ must be L-integrable, and the conclusion of (2.11.i) reduces to the conclusion of (2.10.i) in the case under consideration. If the second member of (2.11.1) is $+\infty$, then (2.11.1) in itself is trivial, but we still have to prove that a measurable

$u(t)$, $t_1 \leq t \leq t_2$, exists with $u(t) \in U(t,x(t))$, $x'(t) = f(t,x(t),u(t))$ a.e. in $[t_1,t_2]$. This, however, is a consequence of closure theorem 1 of (2.5) applied to the AC n-vector function x, the n-vector function f, and the sets $Q(t,x) \subset E_n$. Statement (2.11.2) is thereby proved.

Finally, let us show by an example that an integral $I[x,u]$ may possess the properties of statement (2.10.i), and thus the property of lower closure as defined at the beginning of (2.9), and yet not possess the stronger property of the present section (2.11).

Indeed, take $m = n = 1$, $U = [u| -\infty < u < +\infty]$, $f = \exp(u)$, $f_o = \exp(u^2)$, $A = E_2$, and take $x(t) = 0$, $0 \leq t \leq 1$, $x_k(t) = k^{-1}t$, $0 \leq t \leq 1$, $k = 1,2,\ldots$. Here $x_k \to x$ uniformly in $[0,1]$ as $k \to \infty$ and $I[x_k,u_k] = \exp(\log k)^2 \to +\infty$ as $k \to +\infty$. Obviously, there is no measurable $u(t)$, $0 \leq t \leq 1$, with $-\infty < u(t) < +\infty$, such that $0 = x'(t) = \exp(u(t))$ a.e. in $[0,1]$. The integral I has not the strong property represented by the conclusion of statement (2.11.i). Yet the integral I has the property of lower closure as defined in (2.9) as a consequence of theorem (3.6.i). Indeed, here $\tilde{\tilde{Q}} = [z^o \geq \exp(u^2),\ z = \exp(u),\ u \in E_1]$, or $\tilde{\tilde{Q}} = [z^o \geq \exp(\log z)^2,\ 0 < z < +\infty]$, is a fixed closed convex subset of E_2, and all conditions of (2.10.i) are satisfied. Instead, $Q = [z = \exp(u),\ u \in E_1]$ is the set $Q = [0 < z < +\infty]$, a fixed convex set, and Q is not closed.

2.12. CRITERIA FOR PROPERTY (Q) OF THE SETS $\tilde{\tilde{Q}}(t,x)$

We assume here that the sets A, $U(t,x)$, M, $Q(t,x)$, $\tilde{\tilde{Q}}(t,x)$ are defined as usual, that the sets A and M are closed, and that the functions $f_o(t,x,u)$, $f(t,x,u) = (f_1,\ldots,f_n)$ are continuous on M.

(a) We say that a function $g(t,x,u)$ on M is of <u>slower growth</u> than $f_o(t,x,u)$ as $|u| \to \infty$ in a subset A_o of A if, for every $\varepsilon > 0$ there is some number H, which may depend on ε, f_o and A_o, such that $(t,x) \in A_o$, $|u| \geq H$, $u \in U(t,x)$ implies $|g| \leq \varepsilon f_o$.

(2.12.i). If l and f are of slower growth than f_o as $|u| \to \infty$ in a neighborhood $N_\delta(\bar{t},\bar{x})$ of $(\bar{t},\bar{x})$ in A, and $\tilde{\tilde{Q}}(\bar{t},\bar{x})$ is convex, then the sets $\tilde{\tilde{Q}}(t,x)$ satisfy property (Q) at $(\bar{t},\bar{x})$ (in particular, $\tilde{\tilde{Q}}(\bar{t},\bar{x})$ is closed).

This statement is proved in (App. A.4.i). Note that, if l and f are of slower growth than f_o as $|u| \to \infty$ in A, then not only the sets $\tilde{\tilde{Q}}(t,x)$ of (2.10.i) satisfy property (Q) in A, but also condition (ψ) of (2.10.i) is trivially satisfied with ψ = constant.

<u>Remark 1</u>. As mentioned in (2.10), for generalized solutions of Lagrange problems we consider the convex sets $\tilde{\tilde{R}}(t,x) = \text{co } \tilde{\tilde{Q}}(t,x)$, and we have to verify that these sets $\tilde{\tilde{R}}(t,x)$ are closed and satisfy property (Q). Statement (2.12.i) can be completed as follows:

(2.12.ii). If l and f are of slower growth than f_o as $|u| \to \infty$ in a neighborhood $N_\delta(\bar{t},\bar{x})$ of $(\bar{t},\bar{x})$ in A, then the convex sets $\tilde{\tilde{R}}(t,x) = \text{co } \tilde{\tilde{Q}}(t,x)$ satisfy property (Q) at (t,x) (in particular, $\tilde{\tilde{R}}(\bar{t},\bar{x})$ is closed).

The statement is proved in (App. A.4.). Thus, if l and f are of slower growth than f_o in A, then the convex sets $\tilde{\tilde{R}}(t,x)$ of (2.10.iii) certainly are closed and satisfy property (Q) in A, and even condition (ψ) of (2.10.iii) is trivially satisfied (with ψ = constant).

(b) In the criterion (2.12.iii) below we shall use a different set of hypotheses. At the beginning of (2.11) we noticed that the sets $Q(t,x) \subset E_n$

are the projections of the sets $\tilde{\tilde{Q}}(t,x) \subset E_{n+1}$ on the z-space E_n; hence, the convexity of any set $\tilde{\tilde{Q}}(t,x)$ in E_{n+1} implies the convexity of the corresponding set $Q(t,x)$ in E_n. Nevertheless, as we proved by an example (at the beginning of (2.11)) the sets $\tilde{\tilde{Q}}(t,x)$ may be closed and even satisfy property (Q) at any given point $(\bar{t},\bar{x})$ without this being the case for the sets $Q(t,x)$.

However, the following holds: if the sets $\tilde{\tilde{Q}}(t,x)$ satisfy property (Q) at $(\bar{t},\bar{x})$, then (α) $(z^o,z) \in \cap_\delta \text{ cl co } \tilde{\tilde{Q}}(\bar{t},\bar{x})$ implies $z \in Q(\bar{t},\bar{x})$. Indeed $(z^o,z) \in \cap_\delta \text{ cl co } \tilde{\tilde{Q}}(\bar{t},\bar{x})$ yields $(z^o,z) \in \tilde{\tilde{Q}}(\bar{t},\bar{x})$ by property (Q) at $(\bar{t},\bar{x})$, and then $z \in Q(\bar{t},\bar{x})$.

We shall say that condition (α) holds at the point $(\bar{t},\bar{x}) \in A$ provided:

$$(\alpha) \quad (z^o,z) \in \cap_\delta \text{ cl co } \tilde{\tilde{Q}}(\bar{t},\bar{x}) \text{ implies } z \in Q(\bar{t},\bar{x}) .$$

As mentioned, this condition is necessary for property (Q) of the sets $\tilde{\tilde{Q}}(t,x)$ at $(\bar{t},\bar{x})$. This same condition (α) alone is not sufficient for property (Q) as the following example shows: Take $m = n = 1$, $U = E_1$, $f_o = t^3u^2$, $f = tu$, $0 \leq t \leq 1$. Then $\tilde{\tilde{Q}}(0) = [(z^o,z) | z^o \geq 0,\ z = 0]$, $\tilde{\tilde{Q}}(t) = [(z^o,z) | z^o \geq t^{-1}z^2$, $z \in E_1]$ if $t > 0$, the sets $\tilde{\tilde{Q}}$ do not satisfy condition (Q) at $t = 0$, but condition (α) certainly holds at the same point. Note that condition (α) is trivially satisfied for free problems ($m = n$, $f = u$, $U = E_n$), since $Q = U = E_n$, and all points $z \in E_n$ are in Q.

Now we shall say that condition (X) holds at the point $(\bar{t},\bar{x}) \in A$ provided:

(X) For every $\bar{z} \in Q(\bar{t},\bar{x})$ there is at least one point $\bar{u} \in U(\bar{t},\bar{x})$ with $\bar{z} = f(\bar{t},\bar{x},\bar{u})$ and the following property: given $\varepsilon > 0$ there are numbers $\delta > 0$, and r, $b = (b_1,\dots,b_n)$ real, such that

(X') $f_o(t,x,u) \geq r + \sum_j b_j f_j(t,x,u)$ for all $(t,x) \in N_\delta(\bar{t},\bar{x})$
and $u \in U(t,x)$;

(X") $f_o(\bar{t},\bar{x},\bar{u}) \leq r + \sum_j b_j(\bar{t},\bar{x},\bar{u}) + \varepsilon$.

As we shall see in (App. A. 4), this is a very weak requirement. For free problems, for instance, this condition reduces to a weak form of the well known "seminormal convexity condition" (App. A.6).

(2.12.iii). If conditions (α) and (X) hold at a point $(\bar{t},\bar{x}) \in A$, then the sets $\tilde{\tilde{Q}}(t,x)$ are closed, convex, and satisfy property (Q) at the point $(\bar{t},\bar{x})$.

This statement will be proved in (App. A.4.iv). We shall see there also, that under a slight requirement, the union of (α) and (X) is necessary as well as a sufficient condition for property (Q) of the sets $\tilde{\tilde{Q}}(t,x)$.

(c) _The case of f linear in u._ We shall assume here that A is a given closed subset of the tx-space E_{1+n}, that $U = E_m$, that $f_o(t,x,u)$ and $f(t,x,u) = (f_1,\ldots,f_n)$ are continuous on $M = A \times E_m$, and that f is linear in u, that is,

$$f_i(t,x,u) = \sum_{j=1}^{m} b_{ij}(t,x)u^j + c_i(t,x), \quad i = 1,\ldots,n ,$$

or

$$f(t,x,u) = B(t,x)u + C(t,x) ,$$

where B, C are $n \times m$ and $n \times 1$ matrices with entries continuous in A. For every compact subset A_o of A, the functions b_{ij}, c_i are continuous and bounded on A_o; hence, there are constants G_o, F_o such that $|f(t,x,u)| \leq G_o|u| + F_o$ for all $(t,x) \in A_o$ and $u \in E_n$.

(2.12.iv). If $f_o(t,x,u)$ is convex in u, and f linear in u with $U = E_m$, then the sets $\tilde{\tilde{Q}}(t,x)$ are convex.

Proof. If $\tilde{\zeta} = (\zeta^o, \zeta)$, $\tilde{\eta} = (\eta^o, \eta)$ are any two points of $\tilde{\tilde{Q}}(t,x)$, and $0 \leq \alpha \leq 1$, let $\tilde{z} = (z^o, z) = \alpha\tilde{\zeta} + (1 - \alpha)\tilde{\eta}$. Then there are vectors u, v ϵ E_m such that

$$\zeta^o \geq f_o(t,x,u), \quad \zeta = Bu + C,$$

$$\eta^o \geq f_o(t,x,v), \quad \eta = Bv + C,$$

$$\tilde{z} = \alpha\tilde{\zeta} + (1 - \alpha)\tilde{\eta}, \quad z^o = \alpha\zeta^o + (1 - \alpha)\eta^o, \quad z = \alpha\zeta + (1 - \alpha)\eta \,.$$

If w ϵ E_m denotes the vector $w = \alpha u + (1 - \alpha)v$, we have

$$\begin{aligned} z &= \alpha\zeta + (1 - \alpha)\eta = \alpha(Bu + C) + (1 - \alpha)(Bv + C) \\ &= B(\alpha u + (1 - \alpha)v) + C = Bw + C \\ z^o &= \alpha\zeta^o + (1 - \alpha)\eta^o \geq \alpha f_o(t,x,u) + (1 - \alpha)f_o(t,x,v) \\ &\geq f_o(t,x,\alpha u + (1 - \alpha)v) = f_o(t,x,w) \,. \end{aligned}$$

Thus $\tilde{z} = (z^o, z) \in \tilde{\tilde{Q}}(t,x)$ and $\tilde{\tilde{Q}}(t,x)$ is convex.

(2.12.v). If A is closed, $U = E_m$, $M = A \times E_m$, if $f_o(t,x,u)$ is continuous on M, convex in u, and "seminormal in u at a point $\bar{x} \in A$ (see definition (SN) in (d) below), if $f(t,x,u) = B(t,x)u + C(t,x)$, where the matrices B, C have entries continuous in A, then the sets $\tilde{\tilde{Q}}(t,x) = [(z^o,z) | z^o \geq f_o(t,x,u),$ $z = f(t,x,u), u \in E_m]$ satisfy property (Q) at $(\bar{t},\bar{x})$.

A proof is given in (App. A.4.(v)). This statement for f linear in u, or $f = B(t,x)u + C(t,x)$, is much stronger than the analogous statement (2.12.i). Indeed, we would deduce from (2.12.i) an anologous statement as (2.12.iv) under a growth condition $f_o(t,x,u) \geq \Phi(|u|)$ with $\Phi(\zeta)/\zeta \to +\infty$ as $\zeta \to +\infty$.

Example 1. Take $m = 1$, $n = 2$, $U = E_1$, $f_o = 1$, $f_1 = u$, $f_2 = tu$, $-1 < t \leq 1$. Then the sets Q and $\tilde{\tilde{Q}}$ depend on t, and

$$Q(t) = [z = (z^1,z^2) | z^1 = u,\ z^2 = tu,\ u \in E_1]$$

$$= [z = (z^1,z^2) | z^2 = tz^1,\ z^1 \in E_1] \subset E_2,$$

$$\tilde{\tilde{Q}}(t) = [\tilde{z} = (z^0,z^1,z^2) | z^0 \geq 1,\ z^2 = t\, z^1,\ z^1 \in E_1] \subset E_3 .$$

Each set $Q(t)$ is a straight line in E_2 of slope t, and for each $\delta > 0$ the set $Q(0,\delta)$ contains both lines $z^2 = \pm\delta z^1$, and the convex hull of $Q(0,\delta)$ coincide with the whole plane E_2. Thus $Q(0)$ is the z^1-axis and $\cap_\delta$ cl co $Q(0,\delta)$ is the whole z^1z^2-plane. The set $Q(t)$ does not satisfy property (Q) at $t = 0$, and the same holds for $\tilde{\tilde{Q}}(t)$. Here $f_o = 1$ does not satisfy the weak growth condition requested in (2.12.iv).

Example 2. Take $m = 1$, $n = 2$, $U = E_1$, $f_o = |tu|$, $f_1 = u$, $f_2 = tu$, $-1 \leq t \leq 1$. Then

$$Q(t) = [z = (z^1,z^2) | z^2 = t\, z^1,\ z^1 \in E_1] \subset E_2 ,$$

$$\tilde{\tilde{Q}}(t) = [\tilde{z} = (z^0,z^1,z^2) | z^0 \geq |z^2|,\ z^2 = tz^1,\ z^1 \in E_1] \subset E_3 .$$

As before the set $Q(t)$ does not satisfy property (Q) at $t = 0$. Analogously, for any $\delta > 0$ and $-\delta \leq t \leq \delta$, we see that

$$\tilde{z}' = (z^{0'},z^{1'},z^{2'}) = (1,\delta^{-1},1) \in \tilde{\tilde{Q}}(\delta) ,$$

$$\tilde{z}'' = (z^{0''},z^{1''},z^{2''}) = (1,-\delta^{-1},1) \in \tilde{\tilde{Q}}(-\delta) ,$$

and for $\alpha = 1/2$, also

$$\tilde{z} = \alpha\tilde{z}' + (1-\alpha)\tilde{z}'' = (z^0, z^1, z^2) = (1,0,1) \in \text{co}\, \tilde{\tilde{Q}}(0;\delta) .$$

Hence

$$\tilde{z} = (1,0,1) = \cap_\delta \text{ cl co } \tilde{\tilde{Q}}(0;\delta) \qquad \tilde{z} = (1,0,0) \notin \tilde{\tilde{Q}}(0) ,$$

and $\tilde{\tilde{Q}}(t)$ does not satisfy property (Q) at $t = 0$. Here f_o does not grow at $t = 0$.

Example 3. Take $m = 1$, $n = 2$, $U = E_1$, $f_o = |u|$, $f_2 = tu$, $-1 \le t \le 1$. Then

$$Q(t) = [z = (z^1, z^2) | z^2 = tz^1, z^1 \in E_1] \subset E_2 ,$$

$$\tilde{\tilde{Q}}(t) = [\tilde{z} = (z^0, z^1, z^2) | z^0 \ge |z^1|, z^2 = tz^1, z^1 \in E_1] \subset E_3 .$$

As before $Q(t)$ does not satisfy property (Q), while $\tilde{\tilde{Q}}(t)$ does satisfy property (Q) at every t because of statement (2.12.iv).

Example 4. Take $m = n = 1$, $U = E_1$, $f_o = tu^2$, $f_1 = u$, $0 \le t \le 1$. Then

$$Q(t) = [z | z = u, u \in E_1] \subset E_1 ,$$

$$\tilde{\tilde{Q}}(t) = [\tilde{z} = (z^o, z) | z^o \ge tu^2, z = u, u \in E_1] \subset E_2 .$$

Here $Q(t) = U = E_1$ for every t, $0 \le t \le 1$, and obviously $Q(t)$ satisfies property (Q). On the other hand, $\tilde{\tilde{Q}}(0)$ is the half plane $[z^o \ge 0, z \in E_1]$ while $\tilde{\tilde{Q}}(t)$ for $t > 0$ is the set $\tilde{\tilde{Q}}(t) = [z^o \ge tz^2, z \in E_1]$. Obviously, $\tilde{\tilde{Q}}$ satisfies property (Q) at $t = 0$ (and at every t as well).

Example 5. Take $m = n = 1$, $U = E_1$, $f_o = t^3u^2$, $f_1 = tu$, $0 \le t \le 1$. Then

$$Q(t) = [z | z = tu,\ u \in E_1] \subset E_1 ,$$

$$\tilde{\tilde{Q}}(t) = [\tilde{z} = (z^o, z) | z^o \geq t^3 u^2,\ t = tu,\ u \in E_1] \subset E_2 .$$

Here $Q(0)$ is reduced to the single point $z = 0$, while $Q(t)$ for every $t > 0$ coincides with E_1. Thus $Q(t)$ does not satisfy property (Q) at $t = 0$. Also, $\tilde{\tilde{Q}}(0) = [z^o \geq 0,\ z = 0]$, while $\tilde{\tilde{Q}}(t)$ for $t > 0$ is the set $\tilde{\tilde{Q}}(t) = [z^o \geq t\, z^2,$ $z \in E_1]$ and cl co $\tilde{\tilde{Q}}(0;\delta)$ is the entire half plane $[z^o \geq 0,\ z \in E_1]$. Thus, neither Q nor $\tilde{\tilde{Q}}$ satisfy property (Q) at $t = 0$.

(d) <u>The free problem case</u>: $m = n$, $f = u$, $U = E_n$. Here the sets Q reduce to the fixed, closed, and convex set $Q = U = E_n$. The sets $\tilde{\tilde{Q}}(t,x)$ reduce here to

$$\tilde{\tilde{Q}}(t,x) = [(z^o, u) | z^o \geq f_o(t,x,u),\ u \in E_n] .$$

These sets are closed whenever f_o is continuous, and convex whenever $f_o(t,x,u)$ is convex in u. As mentioned, condition (α) is trivially satisfied. Condition (X) at a point $(\bar{t},\bar{x}) \in A$ reduces to the following simple (and well known) requirement:

(X_f) (= <u>weak seminormality condition</u>) For every $\bar{u} \in E_n$ and $\varepsilon > 0$ there are numbers $\delta > 0$, and r, $b = (b_1,\dots,b_n)$ real such that

(X_f') $f_o(t,x,u) \geq r + b\cdot u$ for all $(t,x) \in N_\delta(\bar{t},\bar{x})$ and $u \in E_n$;

(X_f'') $f_o(\bar{t},\bar{x},\bar{u}) \leq r + b\cdot\bar{u} + \varepsilon$.

Then statement (2.12.ii) yields:

(2.12.vi). For free problems ($m = n$, $f = u$, $U = E_n$), if A is closed, if $f_o(t,x,u)$ is continuous on $M = A \times E_n$ and convex in u, and if f_o is weakly seminormal at a point $(\bar{t},\bar{x}) \in A$, then the sets $\tilde{\tilde{Q}}(t,x)$ satisfy property (Q) at $(\bar{t},\bar{x})$.

Convexity of f_o alone does not imply that the sets $\tilde{\tilde{Q}}(t,x)$ have property (Q) in A. This is shown by the following simple example. Take $n = 1$, $f_o(t,u) = tu$, $0 \leq t \leq 1$, $u \in U = E_1$. Then f_o is continuous and convex in u for every t, but at every t, $0 \leq t \leq 1$, we have $\tilde{\tilde{Q}}(t) = [(z^o,u) | z^o \geq tu,\ u \in E_1]$, a half plane in E_2, while $\cap_\delta$ cl co $\tilde{\tilde{Q}}(t;\delta)$ is the entire plane E_2. Thus the sets $\tilde{\tilde{Q}}$ do not satisfy property (Q) at any t, $0 \leq t \leq 1$.

The usual seminormality condition is a somewhat stronger requirement. By a nonessential modification of the condition used by L. Tonelli and E. J. Mc Shane we shall say that the <u>seminormality condition</u> at a point $(\bar{t},\bar{x}) \in A$ is satisfied provided:

(SN) (seminormality condition) For every $\bar{u} \in E_n$ and $\varepsilon > 0$ there are numbers $\delta > 0$, $\nu > 0$, and r, $b = (b_1,\ldots,b_n)$ real such that

(SN') $f_o(t,x,u) \geq r + b\cdot u + \nu|u - \bar{u}|$ for all $(t,x) \in N_\delta(\bar{t},\bar{x})$ and $u \in E_n$;

(SN") $f_o(\bar{t},\bar{x},\bar{u}) \leq r + b\cdot\bar{u} + \varepsilon$.

Seminormality condition has a very simple and elegant characterization:

(2.12.vii). For free problems ($m = n$, $f = u$, $U = E_n$), if $f_o(t,x,u)$ is convex in u at some $(\bar{t},\bar{x}) \in A$, then f_o is seminormal at $(\bar{t},\bar{x})$ if and only if for no $\bar{u}$, $u_1 \in E_n$, $u_1 \neq 0$, it occurs that $f_o(\bar{t},\bar{x},\bar{u}) = 2^{-1}[f_o(\bar{t},\bar{x},\bar{u} + \lambda u_1) + f_o(\bar{t},\bar{x},\bar{u} - \lambda u_1)]$ for all $\lambda \geq 0$.

A proof of this statement is given in (App. A.6.i). Note that, if we denote by $\tilde{Q}(t,x)$ the set $[(z^o,u) | z^o = f_o(t,x,u), u \in E_n]$, then $\tilde{Q}(t,x)$ is often denoted as the "figurative" of f_o (at the point $(t,x) \in A$). Statement (2.12.vi) then states that f_o is seminormal at $(\bar{t},\bar{x})$ if and only if the figurative contains no straight line. In particular, if say $f_o(\bar{t},\bar{x},u) \to +\infty$ as $|u| \to +\infty$, and $f_o(\bar{t},\bar{x},u)$ is convex in u, then the figurative $\tilde{Q}(\bar{t},\bar{x})$ cannot contain any straight line, f_o is seminormal at $(\bar{t},\bar{x})$, f_o is weakly seminormal, and certainly the sets $\tilde{\tilde{Q}}(t,x)$ satisfy property (Q) at $(\bar{t},\bar{x})$.

2.D. LOWER SEMICONTINUITY THEOREMS

2.13. LOWER SEMICONTINUITY OF FUNCTIONALS IN INTEGRAL FORM

The rather general concept of lower closure defined in (2.9) is a natural extension of the usual concept of lower semicontinuity. Indeed, the definition of lower closure in (2.9) reduces to the usual concept of lower semicontinuity whenever the strategy u is "determined" by the (admissible) trajectory x, and then the functional (2.9.1) can be thought of as depending on the (admissible) trajectory x only:

$$I[x] = \int_{t_1}^{t_2} f_o(t,x(t),u(t))dt . \qquad (2.13.1)$$

This occurs, for instance, for free problems where $u(t) = x'(t)$ (a.e.). Purpose of the present section (2.13) and next one (2.14) is to clarify the concepts, to deduce theorems of lower semicontinuity from our previous theorems of lower closure in (2.10) and (2.11), and to show that the usual theorems of lower semicontinuity for free problems are corollaries of our theorems of lower closure.

(a) As mentioned in (1.5) it may happen that the data A, U(t,x), B, $f_o(t,x,u)$, $f(t,x,u) = (f_1,\dots,f_n)$ are so arranged that, for any admissible pair x(t), u(t), $t_1 \le t \le t_2$, the trajectory x determines uniquely the strategy u (a.e. in $[t_1,t_2]$). Then the functional (2.13.1) can be thought of as being defined for every admissible trajectory x, and we may denote it as I[x]. In (1.5) we referred to these systems as TDS systems. The free problems are in this class. For all these systems the concept of lower closure (2.9) reduces to the one of lower semicontinuity. Let $x(t)$, $t_1 \le t \le t_2$, be any AC vector function which is the limit in the ρ-metric of a sequence of admissible trajectories $x_k(t)$, $t_{1k} \le t \le t_{2k}$, $k = 1,2,\dots$, with $\rho(x_k,x) \to 0$ and $\underline{\lim}\, I[x_k] < +\infty$ as $k \to \infty$ (thus, of course $t_{1k} \to t_1$, $t_{2k} \to t_2$). The functional (2.13.1) is said to be <u>lower</u> semicontinuous at x provided, from any such sequence we can conclude that x is admissible, and that $I[x] \le \underline{\lim}\, I[x_k]$ as $k \to \infty$.

(b) For general TDS systems statements (2.10.i) and (2.11.i) reduce to the following ones.

(2.13.i). For systems TDS, and under the same conditions of (2.10.i) any AC function $x(t)$, $t_1 \le t \le t_2$, with $(t,x(t)) \in A$ for all $t \in [t_1,t_2]$ and which is the uniform limit of admissible trajectories $x_k(t)$, $t_{1k} \le t \le t_{2k}$, $k = 1, 2,\dots$, with $\rho(x_k,x) \to 0$, $\underline{\lim}\, I[x_k] < +\infty$ as $k \to \infty$, then x is admissible and $I[x] \le \underline{\lim}\, I[x_k]$.

(2.13.ii). For systems TDS, and under the conditions of (2.10.i) and additional hypotheses in (2.11.i), any AC function $x(t)$, $t_1 \le t \le t_2$, with $(t,x(t)) \in A$ for all $t \in [t_1,t_2]$ and which is the uniform limit of admissible trajectories $x_k(t)$, $t_{1k} \le t \le t_{2k}$, $k = 1,2,\dots$, that is $\rho(x_k,x) \to 0$ as $k \to \infty$, then x is admissible and $I[x] \le \underline{\lim}\, I[x_k]$.

In the last statement we understand that there is a measurable function $u(t)$, $t_1 \le t \le t_2$, such that $x(t)$, $u(t)$, $t_1 \le t \le t_2$, satisfying all conditions for admissibility but perhaps the L-integrability of $f_o(t,x(t),u(t))$ in $[t_1,t_2]$, and that this condition too is satisfied whenever $\underline{\lim}\, I[x_k] < +\infty$.

Statement (2.10.ii) also has its counterpart here, but we leave its formulation to the reader.

2.14. THEOREMS OF LOWER SEMICONTINUITY FOR FREE PROBLEMS

Let us consider here free problems, that is, systems with $m = n$, $f = u$, $U = E_n$; hence, the strategy $u(t) = x'(t)$ (a.e.) is determined by the trajectory (a.e.). If A and B are as usual closed sets, then $M = A \times E_n$ is also closed, and $f_o(t,x,u)$ is a given continuous scalar function on M. Here a function $x(t) = (x^1,\dots,x^n)$, $t_1 \leq t \leq t_2$, is an <u>admissible trajectory</u> provided x is AC in $[t_1,t_2]$, $(t,x(t)) \in A$ for all $t \in [t_1,t_2]$, $(t_1,x(t_1),t_2,x(t_2)) \in B$, and $f_o(t,x(t), x'(t))$ is L-integrable in $[t_1,t_2]$. Then the cost functional is

$$I[x] \quad = \quad \int_{t_1}^{t_2} f_o(t,x(t),x'(t))dt \;. \tag{2.14.1}$$

The corresponding sets $Q(t,x)$ and $\tilde{\tilde{Q}}(t,x)$ have been already discussed in (2.12) (d), and the concept of weak seminormality has been introduced there.

Our general statement (2.10.ii) in conjunction with (2.12.v) yields:

(2.14.i). (A theorem of lower semicontinuity for free problems). For free problems ($m = n$, $f = u$, $U = E_n$), if A is closed, if $f_o(t,x,u)$ is continuous on $M = A \times E_n$, convex in u, and weakly seminormal with respect to u in A, then the functional (2.14.1) has the property of lower semicontinuity; that is, if $x(t) = (x^1,\dots,x^n)$, $t_1 \leq t \leq t_2$, is an AC function which is the limit in the ρ-metric of admissible trajectories $x_k(t)$, $t_{1k} \leq t \leq t_{2k}$, $k = 1,2,\dots$, with $\rho(x_k,x) \to 0$, and $\underline{\lim}\, I[x_k] < +\infty$ as $k \to +\infty$, then x is admissible, and $I[x] \leq \underline{\lim}\, I[x_k]$.

The condition of weak seminormality is certainly satisfied if $f_o(t,x,u) \to +\infty$ as $|u| \to +\infty$ for every $(t,x) \in A$.

Theorem (2.14.i) is due to L. Tonelli [1] who proved it for f_o of class

C' in u. A proof under the present sole continuity hypotheses was given by L. Turner [1]. The lower semicontinuity theorem (2.14.i) is here a corollary of theorem (2.10.ii) for lower closure of general Lagrange problems.

Statement (2.14.i) without the hypothesis of weak seminormality is not true, as the following simple example shows. Take $n = 2$, $A = E_3$, $f_o = yx' - xy'$, x,y state variables. Then f_o is certainly convex in (x',y'), namely linear. Nevertheless, $I = \int_{t_1}^{t_2} (yx' - xy')dt$ is not lower semicontinuous. Indeed, if we take C: $x = 0$, $y = 0$, $0 \leq t \leq 2\pi$, and C_k: $x = k^{-1}\sin k^2 t$, $y = k^{-1}\cos k^2 t$, $0 \leq t \leq 2\pi$, $k = 1,2,\ldots$, then $C_k \to C$, $I[C_k] = -2\pi$, $k = 1,2,\ldots$, and $I[C] = 0$.

An analogous example for $n = 1$ has been given by L. Tonelli [3]. Nevertheless, Tonelli proved that, for $n = 1$, and f_o continuous in t,x,x' with continuous first order partial derivatives, statement (2.14.i) holds without weak seminormality requirement [1]. Again, the example above shows that this is not the case for $n \geq 2$.

Remark 1. Generalized solutions for free problems form no TDS systems since the strategy (p,v) is not determined by the strategy as mentioned in (1.9). However, our general closure theorems (2.10.ii) and (2.10.ii) apply, and we state below, as an example, a corollary of (2.10.i) for generalized solutions and free problems. Here A is as usual a subset of the tx-space E_{1+n}, $U = E_n$, $M = A \times E_n$, we assume A closed, hence M is closed, and we assume $f_o(t,x,u)$ continuous on M. A generalized solution is as usual (see (1.9)) a system $x(t)$, $p(t)$, $v(t)$, $t_1 \leq t \leq t_2$, with $x(t)$ AC and $p(t)$, $v(t)$ measurable in $[t_1,t_2]$, $(t,x(t)) \in A$ for all $t \in [t_1,t_2]$, $p(t) = (p_1,\ldots,p_\gamma)$, $p_j(t) \geq 0$, $\sum_j p_j(t) = 1$ (that is, $p(t) \in \Gamma$), $v(t) = (u^{(j)}, j = 1,\ldots,\gamma)$, $u^{(j)}(t) \in E_n$ a.e. in

$[t_1,t_2]$, satisfying

$$dx/dt = \Sigma_{j=1}^{\gamma} p_j(t)u^{(j)}(t) \quad \text{a.e. in } [t_1,t_2] ,$$

and such that $\Sigma_j p_j(t)f_o(t,x(t),u^{(j)}(t))$ is L-integrable in $[t_1,t_2]$. Then the corresponding functional is

$$I[x,p,v] = \int_{t_1}^{t_2} \Sigma_{j=1}^{\gamma} p_j(t)f_o(t,x(t),u^{(j)}(t))dt , \qquad (2.14.2)$$

(cfr. (1.9.3) and (1.9.4).

Note that the sets $\tilde{\tilde{R}}(t,x) \subset E_{n+1}$ of all $\tilde{z} = (z^o,z) = (z^o,z^1,\dots,z^n) \in E_{n+1}$ with $z^o \geq \Sigma_j p_j f_o(t,x,u^{(j)})$, $z = \Sigma_j p_j u^{(j)}$ for some $(p,v) \in \Gamma \times E_{n\gamma}$, are exactly the sets co $\tilde{\tilde{Q}}(t,x)$ if $\gamma = n + 2$. We take for γ the minimum integer for which this holds for all $(t,x) \in A$, $1 \leq \gamma \leq n + 2$.

(2.14.ii). (A theorem of lower closure for free problems and generalized solutions). For free problems ($m = n$, $f = u$, $U = E_n$) and $f_o(t,x,u)$ continuous on $M = A \times E_n$, if the convex sets $\tilde{\tilde{R}}(t,x)$ are closed and satisfy property (Q) at every point $(t,x) \in A$ with exception perhaps of a set of points whose t coordinate lies on a set of measure zero on the t-axis, let us assume that (ψ) for some locally integrable scalar function $\psi(t)$ we have $f_o(t,x,u) \geq \psi(t)$ for all $(t,x,u) \in M$, with exception perhaps of another set of points whose t coordinate lies on a set of measure zero on the t-axis. Then the functional (2.14.2) has the property of lower closure; that is, if $x(t) = (x^1,\dots,x^n)$, $t_1 \leq t \leq t_2$, is any AC function which is the limit in the ρ-metric of the trajectories x_k of generalized admissible systems $x_k(t)$, $p_k(t)$, $v_k(t)$, $t_{1k} \leq t \leq t_{2k}$,

$k = 1,2,\ldots$, with $\rho(x_k,x) \to 0$ as $k \to \infty$, and $\underline{\lim}\ I[x_k,p_k,v_k] < +\to$, then x is a generalized admissible trajectory, that is, there are measurable functions $p(t) = (p_1,\ldots,p_\gamma)$, $v(t) = (u^{(j)}, j = 1,\ldots,\gamma)$, $t_1 \le t \le t_2$, with $p(t) \in \Gamma$, $u^{(j)}(t) \in E_n$, such that $x(t)$, $p(t)$, $v(t)$, $t_1 \le t \le t_2$, is a generalized admissible system, and $I[x,p,v] \le \underline{\lim}\ I[x_k,p_k,v_k]$ as $k \to \infty$.

A corollary of (2.10.i). Note that if $|u|$ is of slower growth than $f_o(t,x,u)$ as $|u| \to \infty$ in A (or in some compact neighborhood A_o of the graph of x) then certainly all sets $\tilde{\tilde{R}}(t,x)$ are closed and satisfy condition (Q) in A (or in A_o), and condition ψ is satisfied (with ψ = constant in A_o), as requested in (2.14.ii). This is a consequence of (2.12, Remark 1).

2.15. THEOREMS OF LOWER SEMICONTINUITY FOR PROBLEMS DEPENDING ON HIGHER DERIVATIVES

Let us consider here problems concerning a functional of the form

$$I[y] = \int_{t_1}^{t_2} f_o(t,y(t),y'(t),\ldots,y^{(n)}(t))dt , \qquad (2.15.1)$$

where y denotes a scalar function of t, where f_o is a given function of its n + 2 arguments, where we assume constraints of the form

$$(t,y(t),\ldots,y^{(n-1)}(t)) \in A \subset E_{1+n} ,$$

boundary conditions of the form

$$(t_1,y(t_1),\ldots,y^{(n-1)}(t_1),t_2,y(t_2),\ldots,y^{(n-1)}(t_2)) \in B \subset E_{2n+2} ,$$

and where A and B are given subsets of the indicated spaces. Thus, no constraint on the values of $y^{(n)}(t)$.

By the substitution $x^1 = y$, $x^2 = y', \ldots, x^n = y^{(n-1)}$, $u = y^{(n)}$, and the use of the vector notation $x = (x^1, \ldots, x^n)$, the problem above is reduced to a Lagrange problem for the given $n \geq 1$, with $m = 1$, functional

$$I[x] \quad = \quad \int_{t_1}^{t_2} f_o(t,x(t),u(t))dt \ , \tag{2.15.2}$$

differential system

$$dx^1/dt \quad = \quad x^2, \ldots, \ dx^{n-1}/dt \quad = \quad x^n, \quad dx^n/dt \quad = \quad u \ , \tag{2.15.3}$$

constraints $(t,x(t)) \in A$, and control space $U = E_1$. As mentioned in (1.7) this is a TDS system, that is, the trajectory x determines the strategy u (by means of (2.15.3). This is the reason we have written $I[x]$ instead of the customary $I[x,u]$ in the first member of (2.15.2).

Both statements (2.10.i) and (2.10.ii) yield corollaries for the situation above. We state here a corollary of (2.10.ii) which is rather general.

Note that A is a subset of the tx-space E_{1+n}, $U = E_1$, $M = A \times E_1$, B a subset of the $t_1x_1t_2x_2$-space E_{2n+2}, and $f_o(t,x,u)$ is defined on M. For any $(t,x) \in A$ we denote as usual by $\tilde{\tilde{Q}}(t,x)$ the set of all $\tilde{z} = (z^o, z) \in E_2$ with $z^o \geq f_o(t,x,u)$, $z = u$, $u \in E_1$. We see that these sets $\tilde{\tilde{Q}}(t,\gamma) \subset E_2$ are the same as those we would consider in an analogous "free problem" concerning $f_o(t,x,u)$ (though here $n \geq 1$, $m = 1$). We can speak of the seminormality and weak seminormality of $f_o(t,x,u)$ with respect to u for $u \in E_1$ and $(t,x) \in A$, as we did for free problems.

An admissible trajectory $x(t) = (x^1,\ldots,x^n)$, $t_1 \leq t \leq t_2$, is now any vector function satisfying the following requirements: (a) $x^1,\ldots,x^n$ are AC in $[t_1,t_2]$ and $dx^1/dt = x^2,\ldots,dx^{n-1}/dt = x^n$ for all $t \in [t_1,t_2]$ (thus $u(t) = dx^n/dt$, or equivalently $u(t) = dy^n/dt^n, y = x^1$, is certainly measurable and L-integrable in $[t_1,t_2]$; (b) $(t,x(t)) \in A$ for all $t \in [t_1,t_2]$; (c) $(t_1,x(t_1),t_2,x(t_2)) \in B$; (d) $f_o(t,x(t),u(t))$ is L-integrable in $[t_1,t_2]$. The corresponding functional is now (2.15.2).

Of course, we shall use the ρ-metric on the vector functions $x(t)$, $t_1 \leq t \leq t_2$ (that is, on $y(t)$, $y'(t),\ldots,y^{(n-1)}(t)$ in the old notation).

(2.15.i). (A theorem of lower semicontinuity for problems (2.15.1)). If $A \subset E_{1+n}$ is closed, if $f_o(t,x,u)$ is continuous on $M = A \times E_1$, convex in u, and weakly seminormal with respect to u in A, then the functional (2.15.2) has the property of lower semicontinuity; that is, if $x(t) = (x^1,\ldots,x^n)$, $t_1 \leq t \leq t_2$, is an AC function, satisfying (a) above, is the limit in the ρ-metric of admissible trajectories $x_k(t)$, $t_{1k} \leq t \leq t_{2k}$, $k = 1,2,\ldots$, with $\rho(x_k,x) \to 0$ and $\underline{\lim}\ I[x_k] < +\infty$ as $k \to \infty$, then x is admissible, and $I[x] \leq \underline{\lim}\ I[x_k]$.

The condition of weak seminormality is certainly satisfied if $f_o(t,x,u) \to +\infty$ as $|u| \to +\infty$ for every $(t,x) \in A$.

As a particular case of (2.15.1) we may consider a functional of the form

$$I[y] = \int_{t_1}^{t_2} F(t,y(t),(Ly)(t))dt \ , \tag{2.15.4}$$

where L is a linear differential operator of the form

$$(Ly)(t) = y^{(n)}(t) + a_1(t)y^{(n-1)}(t) + \ldots + a_n(t)y .$$

If we take $x = (x^1,\ldots,x^n)$, $x^1 = y,\ldots,x^n = y^{(n-1)}$, and

$$f_o(t,x,u) = F(t,x^1,u + a_1(t)x^n + \ldots + a_n(t)x^1)$$

we have the same situation as above. Essentially the same theorem (2.15.1) holds where now we may require weak seminormality and convexity of F with respect to its third argument. The condition of weak seminormality is certainly satisfied if F is convex in u, and $F(t,y,u) \to +\infty$ as $|u| \to +\infty$ for all (t,y).

A great many possible extensions of problems (2.15.1) and (2.15.4) can be left as exercises for the reader.