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CONVEXITY OF THE RANGE OF CERTAIN INTEGRALS

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ADDENDUM III. CONVEXITY OF THE RANGE OF CERTAIN INTEGRALS

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In this appendix we consider any vector function $f(t) = (f_1, \ldots, f_n)$ whose components are L-integrable in [a, b], and prove that the set function $K(E) = \int_E f(E)dt$ has for range a convex closed set when E describes all measurable subsets of [a, b]. This result, which is proved here rather elementarily through a set of lemmas, is actually a particular case of an analogous one concerning nonatomic vector valued measure functions and due to A. Lyapunov [76].

III 1. SOME PRELIMINARY LEMMAS

If [a, b] is any given interval of length $\ell = b - a$, and α, $0 \leq \alpha \leq 1$, any number, then the point $t = a + \alpha(b - a) = a + \alpha\ell$ divides [a, b] into two parts of measures $\alpha\ell$ and $(1 - \alpha)\ell$. If we divide [a, b] into two equal parts, and we divide each part as above, the corresponding set

$$D_\alpha^2 = [a \leq t < a + \alpha\ell/2] \cup [a + \ell/2 \leq t < a + \ell/2 + \alpha\ell/2]$$

has still measure $\alpha\ell$, and is the union of two disjoint intervals. In general, if we divide [a, b] into 2^k equal parts, and in each part we take corresponding subintervals, then the set

$$D_\alpha^k = \bigcup_{i=1}^{2^k} \left[a + 2^{-k}(i - 1)\ell,\ a + 2^{-k}\ (i - 1 + \alpha)\ell \right] \qquad \text{(III. 1.1)}$$

has measure $\alpha\ell$, and is the union of 2^k disjoint intervals. Also, for

$0 \leq \alpha < \alpha' \leq 1$ and the same k we have $D^k_\alpha \subset D^k_{\alpha'}$ and meas $|D^k_{\alpha'} - D^k_\alpha| = (\alpha' - \alpha)\ell$.

(III 1.i) Given any vector function $f(t) = (f_1, \ldots, f_n)$, $a \leq t \leq b$, whose components are L-integrable in $[a, b]$, and any $\varepsilon > 0$, there is an integer K such that for all $k \geq K$ and α, $0 \leq \alpha \leq 1$, we have

$$\left| \int_{D^k_\alpha} f(t)dt - \alpha \int_a^b f(t)dt \right| \leq \varepsilon.$$

In other words, if $a_o = (a_1, \ldots, a_n)$ denotes the integral of $f(t)$ on $[a, b]$, then the integral on D^k_α, thought of as a function of α, $0 \leq \alpha \leq 1$, is uniformly approximated by the linear function $a_o\alpha$, $0 \leq \alpha \leq 1$.

Proof. It is not restrictive to assume $a = 0$, $b = 1$. We know that there is a continuous vector function $g(t)$, $0 \leq t \leq 1$, such that

$$\int_o^1 |f(t) - g(t)|dt \leq \varepsilon/4.$$

Then $g(t)$ is uniformly continuous in $[0, 1]$, and hence there is $\delta > 0$ such that $t, t' \in [0, 1]$, $|t - t'| \leq \delta$ implies $|g(t) - g(t')| \leq \varepsilon/4$. Let K be the smallest integer with $1/2^K < \delta$. For any $k \geq K$ let $g_k(t)$, $0 \leq t \leq 1$, be the step function defined by $g_k(t) = g(t_{i-1})$ for all $t_{i-1} \leq t \leq t_i$, $i = 1, \ldots, 2^k$, where $t_i = i/2^k$. Then $|g(t) - g_k(t)| \leq \varepsilon/4$ for all $0 \leq t \leq 1$. Thus

$$\Delta = \int_o^1 |f(t) - g_k(t)|dt \leq \int_o^1 |f - g|dt + \int_o^1 |g - g_k|dt \leq \varepsilon/4 + \varepsilon/4 = \varepsilon/2,$$

and

$$\int_{D_\alpha^k} f(t)dt - \alpha\int_0^1 f(t)dt \leq \left|\int_{D_\alpha^k} dt - \int_{D_\alpha^k} g_k dt\right| +$$

$$+ \left|\int_{D_\alpha^k} g_k dt - \alpha\int_0^1 g_k dt\right| + \left|\alpha\int_0^1 g_k dt - \alpha\int_0^1 f(t)dt\right| = s_1 + s_2 + s_3.$$

Here

$$s_2 = \sum_i g_k(t_i)(\alpha/2^k) - \alpha \sum_i g_k(t_i)(1/2^k) = 0,$$

$$|s_1| \leq \int_{D_\alpha^k} |f - g_k| dt \leq \int_0^1 |f - g| dt \leq \varepsilon/2,$$

$$|s_3| \leq \int_0^1 |g_k - f| dt \leq \varepsilon/2,$$

and finally $\Delta \leq \varepsilon/2 + 0 + \varepsilon/2 = \varepsilon$. Statement (III 1.i) is thereby proved.

Statement (III 1.i) has a stronger form which of course is less easy to prove.

(III 1.ii) Given any vector function $f(t) = (f_1,\ldots,f_n)$, $a \leq t \leq b$, whose components are L-integrable in $[a, b]$, then for every α, $0 \leq \alpha \leq 1$, there is a measurable subset E_α of $[a, b]$ such that

$$\int_{E_\alpha} f(t)dt = \alpha\int_a^b f(t)dt, \qquad 0 \leq \alpha \leq 1. \qquad \text{(III. 1.2)}$$

In other words, if $a_0 = (a_1,\ldots,a_n)$ is the integral of f on $[a, b]$, the integral at the first number of (III 1.3) thought of as a function of α, is a linear function of α, say $a_0\alpha$, $0 \leq \alpha \leq 1$.

This statement is a particular case of the following one which we shall prove below.

(III 1.iii) Given any vector function $f(t) = (f_1,\dots,f_n)$, $a \le t \le b$, whose components are L-integrable functions in $[a,b]$, and any measurable subset A of $[a, b]$, then for every α, $0 \le \alpha \le 1$, there is a measurable subset B_α, $B_\alpha \subset A \subset [a, b]$ with

$$\int_{B_\alpha} f(t)dt = \alpha \int_A f(t)dt, \qquad 0 \le \alpha \le 1. \qquad \text{(III. 1.3)}$$

Proof. The proof of (III 1.iii) is made up of parts. (a). Let us prove (iii) for $n = 1$ and f a nonnegative scalar function. If $\varphi(t)$ denotes the characteristic function of A, say $\varphi = 1$ on A and $\varphi = 0$ otherwise, then $f(t)\, \varphi(t)$ is L-integrable in $[a, b]$, and hence $F(t) = \int_a^t f(\tau)\, \varphi(\tau)d\tau$, $a \le t \le b$, is a continuous function taking all values from $F(a) = 0$ to $F(b)$. Thus, there is some c, $a \le c \le b$, with $F(c) = \alpha F(b)$, and, if $B_\alpha = [a,c] \cap A$, also

$$\int_{B_\alpha} f\, dt = \int_0^c f\, \varphi\, dt = F(c) = \alpha F(b) = \alpha \int_a^b f\, \varphi\, dt = \alpha \int_A f\, dt.$$

Thus, (III 1.iii) is proved for $n = 1$ and f scalar nonnegative.

(b) Let us assume that we know how to determine $B_{1/2}$ for every A and a given vector function $f = (f_1,\dots,f_n)$ whose components are nonnegative L-integrable, and let us prove that we can determine all sets B_α, $0 \le \alpha \le 1$, and that we can determine them in such a way that $\alpha < \alpha'$ implies $B_\alpha \subset B_{\alpha'}$.

For the sake of simplicity we shall use the notation

$$\mu(E) = \int_E f\, dt, \quad \mu_i(E) = \int_E f_i dt, \quad i = 1,\dots,n. \qquad \text{(III. 1.4)}$$

First, for $B'_{1/2} = A - B_{1/2}$, we have

$$\mu(B'_{1/2}) = \mu(A) - \mu(B_{1/2}) = \mu(A) - (1/2)\mu(A) = (1/2)\mu(A).$$

Then let us determine sets $B_{1/4} \subset B_{1/2}$, $B'_{3/4} \subset B_{1/2}$ such that

$$\mu(B_{1/4}) = (1/2)\mu(B_{1/2}), \quad \mu(B'_{3/4}) = (1/2)\mu(B'_{1/2})$$

and then, for $B_{3/4} = B_{1/2} \cup B_{3/4}$, also

$$\mu(B_{1/4}) = (1/4)\mu(A), \quad \mu(B_{3/4}) = (3/4)\mu(A),$$

and if $B_0 = \emptyset$, $B_1 = A$, we have $B_0 \subset B_{1/4} \subset B_{1/2} \subset B_{3/4} \subset B_1$.
By repeating this process we obtain sets $B_{i/2^r}$, $i = 0, 1, \dots, 2^r$, $r = 1, 2, \dots,$ so that $\mu(B_{i/2^r}) = i/2^r$, and for $i < j$, $\lambda = i/2^r$, $\lambda' = j/2^r$, also $B_\lambda \subset B_{\lambda'}$.
Now let α be any number $0 < \alpha < 1$, and let $[\lambda_s]$, $[\lambda'_s]$ be sequences of numbers $\lambda_s = i/2^r$, $\lambda'_s = j/2^s$, such that $\lambda_s < \lambda_{s+1} < \alpha < \lambda'_{s+1} < \lambda'_s$, $\lambda_s \to \alpha$, $\lambda'_s \to \alpha$. For

$$B_\alpha = \cup B_{\lambda_s} \qquad B'_\alpha = \cap B_{\lambda'_s}$$

we have $B_\alpha \subset B'_\alpha$ and

$$\lambda_s \int_A f\,dt = \int_{B_{\lambda_s}} f\,dt \leq \int_{B_\alpha} f\,dt \leq \int_{B'_\alpha} f\,dt \leq \int_{B_{\lambda'_s}} dt = \lambda'_s \int_A f\,dt,$$

where $\leq$ means that such a relation holds for each component. As $s \to \infty$ we obtain

$$\alpha = \int_{B_\alpha} f\,dt = \int_{B'_\alpha} f\,dt, \qquad 0 \leq \alpha \leq 1.$$

This proves (b).

(c) Assume that (iii) has been proved for some vector function $f = (f_1,\ldots,f_n)$ whose components are nonnegative L-integrable. Let E, F be any two measurable subsets of $A \subset [a, b]$. Then for every α, $0 \leq \alpha \leq 1$, there is some subset $C(\alpha)$ of $E \cup B$ with $C(0) = E$, $C(1) = F$, such that

$$\int_{C(\alpha)} f\,dt = (1-\alpha)\int_E f\,dt + \alpha\int_F f\,dt, \qquad 0 \leq \alpha \leq 1,$$

$$\left|\int_{C(\alpha)} f_i\,dt - \int_{C(\alpha')} f_i\,dt\right| \leq |\alpha - \alpha'| \left(\int_{E-F} f_i\,dt + \int_{F-E} f_i\,dt\right) \tag{III.1.5}$$

$$0 \leq \alpha,\ \alpha' \leq 1,\ i = 1,\ldots,n.$$

Indeed, let us apply (iii) to the sets E-F and F-E, and the number α. Let $B_\alpha \subset E-F$, $B'_\alpha \subset F-E$ be the corresponding sets and take

$$C(\alpha) = (E \cap F) \cup (E-F-B_\alpha) \cup B'_\alpha.$$

Then

$$\begin{aligned}
\mu(C(\alpha)) &= \mu(E \cap F) + \mu(E - F) - \mu(B_\alpha) + \mu(B'_\alpha) \\
&= \mu(E \cap F) + \mu(E - F) - \alpha\,\mu(E - F) + \alpha\,\mu(F - E) \\
&= \mu(E \cap F) + (1-\alpha)\mu(E - F) + \alpha\,\mu(F - E) \\
&= (1-\alpha)\mu[(E \cap F) \cup (E - F)] + \alpha\,\mu[(E \cap F) \cup (F - E)] \\
&= (1-\alpha)\mu(E) + \alpha\,\mu(F).
\end{aligned}$$

In addition, for each component f_i and $0 \leq \alpha, \alpha' \leq 1$, we have

$$|\mu_i(C(\alpha)) - \mu_i(C(\alpha'))| = |(-\alpha+\alpha')\mu(E - F) + (\alpha - \alpha')\mu(F - E)|$$

$$\leq |\alpha - \alpha'| \ (\mu(E - F) + \mu(F - E)), \ i = 1,\dots,n.$$

Thus (c) is proved.

(d) Statement (iii) has been proved for n = 1 and f scalar nonnegative. Assume that (iii) has been proved for n - 1 and vectors f with nonnegative components, and let us prove it for n. Let $\bar{f}$ be the (n - 1) -vector $\bar{f} = (f_1,\dots,f_{n-2}, \bar{f}_{n-1})$ with $\bar{f}_{n-1} = f_{n-1} + f_n$, and let $\bar{\mu}$, μ_i, $\bar{\mu}_i$ be the set functions defined by (III.1.4) with f replaced by $\bar{f}$, f_i, $\bar{f}_i$. First by (iii) with $\alpha = 1/2$ applied to $\bar{f}$ there is a subdivision of A into two parts E, F, with $E \cap F = \emptyset$, $E \cup F = A$, and

$$\bar{\mu}(E) = \bar{\mu}(F) = (1/2)\bar{\mu}(A). \qquad \text{(III 1.6)}$$

Also, by force of (c), there are sets $C(\alpha) \subset E \cup F = A$, $0 \leq \alpha \leq 1$, with $C(0) = E$, $C(1) = F$, and

$$\bar{\mu}(C(\alpha)) = (1 - \alpha)\bar{\mu}(E) + \alpha\,\bar{\mu}(F) = (1 - \alpha)(1/2)\bar{\mu}(A) + \alpha(1/2)\bar{\mu}(A)$$

$$= (1/2)\bar{\mu}(A), \qquad 0 \leq \alpha \leq 1. \qquad \text{(III 1.7)}$$

Let us prove that $\mu_{n-1}(C(\alpha))$ is a continuous function of α in [0, 1]. Indeed, $\mu_{n-1}(C(\alpha))$ is a scalar, namely the integral of $f_{n-1} \geq 0$ over $C(\alpha)$, and

$$|\mu_{n-1}(C(\alpha)) - \mu_{n-1}(C(\alpha'))| = |\mu_{n-1}[C(\alpha) - C(\alpha')] - \mu_{n-1}[C(\alpha') - C(\alpha)]|$$

$$\leq |\mu_{n-1}[C(\alpha) - C(\alpha')] + \mu_{n-1}[C(\alpha') - C(\alpha)]|$$

$$\leq |\bar{\mu}_{n-1}[C(\alpha) - C(\alpha')] + \bar{\mu}_{n-1}[C(\alpha') - C(\alpha)]|$$

$$\leq |\alpha - \alpha'| \left(\bar{\mu}_{n-1}(E - F) + \bar{\mu}_{n-1}(F - E)\right) \leq 2|\alpha - \alpha'| \int_A (f_{n-1} + f_n)dt.$$

This proves that $\mu_{n-1}(C(\alpha))$ is a continuous function of α for $0 \leq \alpha \leq 1$. On the other hand, $\mu_{n-1}(C(0)) = \mu_{n-1}(E)$, $\mu_{n-1}(C(1)) = \mu_{n-1}(F)$. Since E and F are complementary in A then $\mu_{n-1}(E) \lesseqgtr (1/2)\mu_{n-1}(A)$ according as $\mu_{n-1}(F) \gtreqless (1/2)\mu_{n-1}(A)$. Thus, as α describes $[0, 1]$, $\mu(C(\alpha))$ describes an interval which contains $(1/2)\mu_{n-1}(A)$. We conclude that there is some α, $0 \leq \alpha \leq 1$, such that $\mu_{n-1}(C(\alpha)) = (1/2)\mu_{n-1}(A)$. For this particular value of α, we have from (III 1.7)

$$\int_{C(\alpha)} f_i dt = (1/2) \int_A f_i dt, \quad i = 1,\dots,n-2,$$

$$\int_{C(\alpha)} (f_{n-1} + f_n)dt = (1/2) \int_A (f_{n-1} + f_n)dt,$$

$$\int_{C(\alpha)} f_{n-1}dt = (1/2) \int_A f_{n-1}dt,$$

and hence, by difference, also

$$\int_{C(\alpha)} f_n dt = (1/2) \int_A f_n dt,$$

or

$$\int_{C(\alpha)} f_n dt = (1/2) \int_A f\, dt. \qquad \text{(III 1.8)}$$

We have proved that for the n-vector $f = (f_1, \ldots, f_n)$ we can determine a subset $B_{1/2} = C(\alpha) \subset A$ satisying (III 1.6), where A is any measurable subset of [a, b]. Thus, by (b), we can determine analogous sets B_α for all α, $0 \leq \alpha \leq 1$, and (III 1.iii) is proved for vector valued functions with non-negative components.

(e) We have now to prove (III 1.iii) for vector functions $f = (f_1, \ldots, f_n)$ with L-integrable components of arbitrary signs. For every $i = 1, \ldots, n$, and $j = 1,2$, we consider the sets A_{i1} where $f_i \geq 0$ and A_{i2} where $f_i < 0$. We divide [a, b] into 2^n disjoint measurable sets $A_r = A_{ij_1} \cap A_{2j_2} \cap \ldots \cap A_{nj_n}$, where r denotes any one of the 2^n systems $(j_1, j_2, \ldots, j_n)$ of indices 1 and 2. On each set A_r the components f_i have constant signs, and there are, therefore, sets $B_{r\alpha} \subset A_r$ with $\int_{B_{r\alpha}} f \, dt = \alpha \int_{A_r} f \, dt$, $0 \leq \alpha \leq 1$. The sets $B_\alpha = \cup_r B_{r\alpha}$ then satisfy the requirements of (III 1.iii). Statement (III 1.iii) is thereby proved.

(III 1.iv) Given any vector function $f(t) = (f_1, \ldots, f_n)$, $a \leq t \leq b$, whose components are L-integrable in [a, b], and any two fixed measurable sets $E, F \subset [a, b]$, then for every α, $0 \leq \alpha \leq 1$, there is some set $C(\alpha) \subset E \cup F$, with $C(0) = E$, $C(1) = F$, and

$$\int_{C(\alpha)} f \, dt = (1 - \alpha) \int_E f \, dt + \alpha \int_F f \, dt.$$

This statement is a consequence of parts (b) and (c) of the proof of (III 1.iii).

III 2. THE MAIN STATEMENTS

(III 2.i) Given any vector function $f(t) = (f_1,\ldots,f_n)$, $a \leq t \leq b$, whose components are L-integrable, and any measurable subset A of $[a,b]$, then

$$\mu(E) = \int_E f(t)dt \tag{III 2.1}$$

describes a convex set H as E describes all possible measurable subsets E of A (in other words, the range of $\mu(E)$ is convex).

<u>Proof</u>. If $\mu_1, \mu_2 \in H$, then there are measurable sets E_1, E_2 in A such that $\mu_i = \mu(E_i) = \int_{E_i} f\, dt$, $i = 1,2$. Among all measurable subsets of A there certainly are the sets $C(\alpha)$, $0 \leq \alpha \leq 1$, defined in (III 1.iv). Then

$$\mu(C(\alpha)) = (1 - \alpha) \int_E f\, dt + \alpha \int_F f\, dt = (1 - \alpha)\mu_1 + \alpha\mu_2,$$

that is, all points of the segment $(1 - \alpha)\mu_1 + \alpha\mu_2$, $0 \leq \alpha \leq 1$, belong to H, and H is proved to be convex.

(III 2.ii) Given any two vector functions $f(t) = (f_1,\ldots,f_n)$, $g(t) = (g_1,\ldots,g_n)$, $a \leq t \leq b$, whose components are L-integrable, let E denote any measurable subset of $[a, b]$ and $h_E(t)$, $a \leq t \leq b$, the function $h_E(t) = f(t)$ for $t \in E$, $h_E(t) = g(t)$ for $t \in F = [a, b] - E$. Then

$$\mu(E) = \int_a^b h_E(t)dt \tag{III 2.2}$$

describes a convex subset H of the space E_n as E describes all measurable subsets of $[a, b]$.

Proof. For every E as above and F = [a,b] -E, we have

$$\mu(E) = \int_a^b h_E dt = \int_E f\,dt + \int_F g\,dt = \int_E (f - g)dt + \int_a^b g\,dt.$$

If μ_o is the fixed value of the last integral, and we apply (III 2.i) to the function f-g, we see that the set H of (III 2.ii) is simply a translation of the convex set H of (III 2.i) relative to f-g.

(III 2.iii) The set H of statement (III 2.i) is closed.

Proof of (III 2.iii) for n = 1. We have $\mu(E) = \int_E f\,dt$ where f is a scalar. If $f \geq 0$ the statement is trivial since the values taken by $\mu(E)$ fill the closed segment $[0, \mu(A)]$. Otherwise, let A^+, A^- be the subsets of all $t \in A$ where $f \geq 0$, or $f \leq 0$, and then $A^+ \cap A^- = \emptyset$, $A^+ \cup A^- = A$. For every set $E \subset A$, let $E^+ = E \cap A^+$, $E^- = E \cap A^-$, and then $\mu(E) = \mu(E^+) + \mu(E^-)$. Then $\mu(E)$ takes on its maximum value $\mu^+ = \mu(A^+) \geq 0$ for $E = A^+$, and its minimum value $\mu^- = \mu(A^-) \leq 0$ for $E = A^-$, and the values taken by $\mu(E)$ fill the segment $[\mu_E^-, \mu^+]$. We shall prove (III 2.iii) for $n > 1$ below.

(III 2.iv) If H is the convex set of (III 2.i) and Π: $p \cdot x - c = 0$ any supporting plane for H with $\Pi \cap \text{cl } H \neq 0$, then $\Pi \cap H \neq 0$, that is, there is some $\xi \in H$ with $p \cdot \xi - c = 0$, and some measurable set $E_o \subset A$ with $\xi = \int_{E_o} f\,dt$.

Proof. We may assume $p \cdot x - c \geq 0$ for all $x \in H$, and hence also for all $x \in \text{cl } H$. Thus, $(\text{cl } H) \cap \Pi \neq 0$ implies $c = \text{Inf } (p \cdot x)$ where Inf is taken for all $x \in H$, that is, $c = \text{Inf } p \cdot \int_E f\,dt = \text{Inf} \int_E (p \cdot f)dt$, where Inf is taken for all measurable subsets E of A. On the other hand $\nu(E) = \int_E (p \cdot f)\,dt$ is our usual function μ relative to the scalar function

$g(t) = p \cdot f(t)$, $t \in A$. By (III 2.iii), $\nu(E)$ takes on its maximum and minimum values. Thus, there is some measurable set $E_o \subset A$ with $C = \nu(E_o) = \int_{E_o}(p \cdot f)dt = p \cdot \int_{E_o} f\, dt$, that is, $p \cdot \xi - c = 0$ for $\xi = \int_{E_o} f\, dt$.

Proof of (III 2.iii) for $n > 1$. We have proved (III 2.iii) for $n = 1$. Let us assume that (III 2.iii) has been proved for $1, 2, \ldots, n-1$, and let us prove it for n. Let $f(t) = (f_1, \ldots, f_n)$, $t \in A$, A measurable, and let H be the range of the function $\mu(E) = \int_E f\, dt$ as E describes all measurable subset E of A. By (III 2.i) we know that H is convex, and we have to prove that H is closed. Suppose that this is not true, so that $\text{cl}\, H - H \neq \emptyset$, and let ζ be a point $\zeta \in \text{cl}\, H - H$. Then $\zeta \in \text{bd}\, H$, and by (Vol. 1, App. C2) there is a supporting hyperplane Π: $p \cdot x - c = 0$ through ζ, thus $p \cdot x - c \geq 0$ for all $x \in H$, and $p \cdot \zeta = c$. By (III 2.iv) there is a point $\zeta' \in H$ on Π, that is, $p \cdot \zeta' - c = 0$, and since $\zeta' \in H$, there is a measurable subset E_o of A with $\zeta' = \int_{E_o} f\, dt$. Thus $c = p \cdot \zeta' = p \cdot \int_{E_o} f\, dt = \int_{E_o}(p \cdot f)dt$. Then for every measurable set $E \subset A$ we have

$$\nu(E) = p \cdot \mu(E) - c = p \cdot \int_E f\, dt - p \int_{E_o}(p \cdot f)dt$$

$$= \int_{E-E_o} p \cdot f\, dt + \int_{E_o - E}(-p \cdot f)dt.$$

If $g(t)$, $t \in A$ denotes the scalar $g(t) = p \cdot f(t)$ for $t \in A - E_o$, $g(t) = -p \cdot f(t)$ for $t \in E_o$, then

$$\nu(E) = \int_{(E-E_o) + (E_o - E)} g(t)dt \geq 0$$

for every measurable subset $E \subset A$. This implies that $g(t) \geq 0$ almost everywhere in A.

Let A_o $[A_\delta, \delta > 0]$, be the set of all $t \in A$ with $g(t) \leq 0$ $[g(t) \leq \delta]$. Then all sets A, A_δ are measurable, $A_o \subset A_\delta \subset A_o$, $A_\delta - A_o \to 0$, meas $(A_\delta - A_o) \to 0$ as $\delta \to 0 + 0$. Let μ', μ'' be the functions $\mu'(E) = \mu(E - A_o) = \int_{E-A_o} f\,dt$, $\mu''(E) = \mu(E \cap A_o) = \int_{E\cap A_o} f\,dt$, both defined for all measurable subsets E of A, and let H', H" be the ranges of μ' and μ''. Since $p \cdot \mu''(E) = p \cdot \mu(E \cap A_o) = \int_{A\cap A_o} (p \cdot f)dt = 0$, we see that the range H" of μ'' is contained in the hyperplane Π_o: $p \cdot x = 0$. By a change of coordinates we could, therefore, represent H" by means of an $(n - 1)$-vector function, that is, as the range of the values $\int_{E\cap A_o} f(h)dt$, $h = (h^1,\dots,h^{n-1})$, for an L-integrable $(n - 1)$--vector function g. By the induction argument, H" is therefore a convex closed subset.

Let us prove that $(*)\nu(E_s) \to 0$ implies $\mu'(E_s) \to 0$ for any sequence $[E_s]$ of measurable subsets of A. Let $E'_s = E_s - A_o$, and let us assume that this statement is not true. Then there is some $m > 0$ and a sequence, say still $[E_s]$, with $|\mu'(E_s)| \geq m > 0$, and hence also $|\mu(E'_s)| = |\mu(E_s - A_o)| = |\mu'(E_s)| \geq m > 0$. Since f is L-integrable in A, there is some $\sigma > 0$ such that $\int_E |f|dt < m/2$ on every measurable subset E of A of measure $\leq \sigma$. This implies that meas $E'_s \geq \sigma$ for every s, since otherwise $|\mu(E's)| = |\int_{E_s} f\,dt| \leq \int_{E_s} |f|dt < m/2$, a contradiction. Finally, if we take $\delta > 0$ so that meas $(A_\delta - A_o) < \sigma/2$, we see that the set $E''_s = E'_s - A\delta = E'_s - (A_\delta - A_o)$ has measure $\geq \sigma/2$, and hence $p \cdot f \geq \delta$ everywhere in E''_s, and $\nu(E_s) = \nu(E'_s) \geq \nu(E''_s) \geq \delta\,\sigma/2$, while $\nu(E_s) \to 0$, a contradiction. We have proved $(*)$.

Since $\xi \in$ bd H, there is a sequence $[\xi_s]$ with $\xi_s \in H$, $\xi_s \to \xi$, and hence a sequence of sets $B_s \subset A$ with $\mu(B_s) = \xi_s$, $s = 1,2,\ldots.$ Then $\nu(B_s) = p \cdot \mu(B_s) - c = p \cdot \zeta_s - c \to p \cdot \zeta - c = 0$. By force of (*) we have then $\mu'(B_s) \to 0$ as $s \to \infty$, or $\mu(B_s - A_o) \to 0$. On the other hand $\mu''(B_s) = \mu(B_s \cap B_o) = \mu(B_s) - \mu(B_s - A_o) \to \xi$ as $s \to \infty$. This proves that $\xi \in \text{cl } H''$. Since H'' is closed we have $\xi \in H''$, that is, there is some set $E_1 \subset A$ with $\xi = \mu''(E_1) = \mu(E_1 \cap A_o)$. This proves that there is some measurable subset $E = E_1 \cap A_o \subset A_o \subset A$ with $\mu(E) = \xi$, that is, $\xi \in H$. We have proved that H is closed, and thus, by induction argument, (III 2.iii) is proved for every n.

<u>Bibliographical notes</u>. Beside the original paper by A. Lyapunov [76] on the convexity of the range of vector valued nonatomic measure functions, we mention here the work of D. Blackwell [12], of J.F.C. Kingman and A. P. Robertson [64], and W. de Wilde [119]. The paper by P. Halmos [Bull. Am. Math. Soc. 54, 1948, 416-421] contains an error which was corrected in [Bull. Am. Math. Soc. 1949].