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A CRITERION FOR THE EXISTENCE IN A STRIP OF PERIODIC SOLUTIONS OF HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

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By
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As in a previous paper[1] we take into consideration a canonical system of hyperbolic partial differential equations (with fixed characteristics) of the form

$$U_{tx} = f(t,x,U,U_t,U_x), \quad -\infty < t < +\infty, \quad -a \leq x \leq a, \tag{0.1}$$

or

$$U_{itx} = f_i(t,x,U,U_t,U_x), \qquad i = 1,\dots,n, \tag{0.2}$$

where $U = (U_1,\dots,U_n)$ is an unknown vector function of the two independent variables t, x, where U_t, U_x, U_{tx} are the vector functions of the partial derivatives with respect to t, to x, and to t,x, where $f(t,x,z,p,q) = (f_1,\dots,f_n)$ denotes a continuous vector function of its arguments for (t,x) in a strip $A = [-\infty < t < +\infty, -a \leq x \leq a]$, and $(z,p,q)\epsilon E_{3n}$, and f is periodic in t of some period T:

$$f(t+T,x,z,p,q) = f(t,x,z,p,q), \quad (t,x)\epsilon A, \quad (z,p,q)\epsilon E_{3n}. \tag{0.3}$$

Let $u(t) = (u_1,\dots,u_n)$, $-\infty < t < +\infty$, be a continuous function of t, periodic in t of period T, and $v(x)$, $-a \leq x \leq a$, a continuous function of x, both with values in E_n, and

$$u(t+T) = u(t), \quad -\infty < t < +\infty, \quad v(o) = 0.$$

In the present paper we give criteria (Section 2) for the existence of a periodic solution $\phi(t,x) = (\phi_1,\dots,\phi_n)$ of period T in x for the Darboux problem

$$
\begin{aligned}
\phi_{tx} &= f(t,x,\phi,\phi_t,\phi_x), \quad (t,x)\epsilon A, \\
\phi(t,o) &= u(t), \quad -\infty < t < +\infty, \\
\phi(t+T,x) &= \phi(t,x), \quad (t,x)\epsilon A, \\
\phi(o,x) &= \phi(T,x) = u(o)+v(x), \quad -a \leq x \leq a .
\end{aligned}
\tag{0.5}
$$

Namely, given u(t), the criteria assure the existence of a number $a > 0$ sufficiently small, of a function v(x), $- a \leq x \leq a$, as above, and of the corresponding solution $\phi(t,x)$ of the Darboux problem (0.5) in A, $\phi(t,x)$ being periodic in t of period T.

To obtain these criteria we make use of previous results in Ref. 1 concerning a modified Darboux problem. Namely, in Ref. 1 we gave existence, uniquenesses, and continous dependence theorems in order that, given u(t), v(x) as above, there exists a pair of vector functions $\phi(t,x) = (\phi_1,\ldots,\phi_n)$, $m(x) = (m_1,\ldots,m_n)$ such that

$$
\begin{aligned}
\phi_{tx} &= f(t,x,\phi,\phi_t,\phi_x)-m(x), \quad (t,x)\epsilon A, \\
\phi(t,o) &= u(t), \quad -\infty < t < +\infty, \\
\phi(t+T,x) &= \phi(t,x), \quad (t,x)\epsilon A, \\
\phi(o,x) &= \phi(T,x) = u(o) + v(x), \quad -a \leq x \leq a.
\end{aligned}
\tag{0.6}
$$

In Section 1 of the present paper we restate and improve the theorems proved in Ref. 1 for the problem (0.6). In Section 2 we prove the criteria for the existence of a solution ϕ to the problem (0.5). These criteria show that we can choose v(x), $- a \leq x \leq a$, in such a way that $m(x) \equiv 0$, $- a \leq x \leq a$. In Section 2 we shall make use of an implicit function theorem of functional analysis we proved in a previous paper (Ref. 2).

SECTION 1. THE MODIFIED DARBOUX PROBLEM

1. WE PROVED IN REF. 1:

THEOREM I (Existence theorem for the modified Darboux problem (0.6).

If a, $T > 0$, and N, N_1, N_2, L, M, b, M_1, M_2, $M_3 \geq 0$ are constants, if A and R are the sets

$$A = [0 \leq t \leq T, \; -a \leq x \leq a],$$

$$R = [o \leq t \leq T, \; -a \leq x \leq a, \; |z| \leq M_1, \; |p| \leq M_2, \; |q| \leq M_3, \; z,p,q \epsilon E_n],$$

if

$$M_1 \geq N + 2^{-1}N_1T + N_2a + LTa, \tag{1.1}$$

$$M_2 \geq N_1 + 2La, \tag{1.2}$$

$$M_3 \geq N_2 + LT, \tag{1.3}$$

if $u(t)$, $o \leq t \leq T$, $v(x)$, $-a \leq x \leq a$, are vector functions which are continuous with $u'(t)$, $v'(x)$, if $f(t,x,z,p,q)$, $(t,x,z,p,q) \epsilon R$, is continuous in R, and

$$u(T) = u(o), \quad |u(0)| \leq N, \quad |u(t_1)-u(t_2)| \leq N_1|t_1-t_2|, \tag{1.4}$$

$$v(o) = 0, \quad |v(x_1) - v(x_2)| \leq N_2|x_1-x_2|, \tag{1.5}$$

$$f(T,x,z,p,q) = f(o,x,z,p,q), \quad |f(t,x,z,p,q)| \leq L, \tag{1.6}$$

$$|f(t,x,z,p_1,q_1) - f(t,x,z,p_2,q_2)| \leq M|p_1-p_2| + b|q_1-q_2| \tag{1.7}$$

$$2Tb < 1, \quad Ma < 1, \tag{1.8}$$

then there exists a vector function $\phi(t,x)$, $(t,x)\epsilon A$, continuous in A together with ϕ_t, ϕ_x, ϕ_{tx}, and a continuous vector function $m(x)$, $-a \leq x \leq a$, such that

$$\phi(t,o) = u(t), \tag{1.9}$$

$$\phi(o,x) = \phi(T,x) = u(o) + v(x), \tag{1.10}$$

$$m(x) = T^{-1}\int_{o}^{T} f(t,x,\phi(t,x),\phi_t(t,x),\phi_t(t,x))dt \tag{1.11}$$

$$\phi_{tx} = f(t,x,\phi(t,x),\phi_t(t,x),\phi_x(t,x)) - m(x) \tag{1.12}$$

for all $o \leq t \leq T$, $-a \leq x \leq a$. Thus, by extending both $\phi(t,x)$ and $f(t,x,z,p,q)$ for all $-\infty < t < +\infty$, $|x| \leq a$, $|z| \leq M_1$, $|p| \leq M_2$, $|q| \leq M_3$, by means of the periodicity of period T in t, Equation (1.12) is satisfied in the strip $-\infty < t < +\infty$, $-a \leq x \leq a$.

2. In the proof of Theorem I (Ref. 1), we denoted by $\omega_1(\alpha)$, $\omega_2(\beta)$, $\omega_3(\gamma)$ continuous monotone functions in $o \leq \alpha < \infty$, $o \leq \beta < \infty$, $o \leq \gamma < \infty$, such that $\omega_1(o) = \omega_2(o) = \omega_3(o)$, and

$$|f(t_1,x,z,p,q) - f(t_2,x,z,p,q)| \leq \omega_1(|t_1-t_2|)$$

$$|f(t,x_1,z,p,q) - f(t,x_2,z,p,q)| \leq \omega_2(|x_1-x_2|)$$

$$|f(t,x,z_1,p,q) - f(t,x,z_2,p,q)| \leq \omega_3(|z_1-z_2|)$$

for all $o \leq t, t_1,t_2 \leq T$, $-a \leq x, x_1,x_2 \leq a$, $|z|,|z_1|,|z_2| \leq M_1$, $|p| \leq M_2$, $|q| \leq M_3$, $z,z_1,z_2,p,q\epsilon E_n$. Analogously, we denote by $\omega_4(\alpha)$, $0 \leq \alpha \leq \infty$, $\omega_5(\beta)$, $o \leq \beta \leq \infty$, continuous monotone functions such that $\omega_4(o) = \omega_5(o) = 0$, and

$$|u'(t_1) - u'(t_2)| \leq \omega_4(|t_1-t_2|), \quad |v'(x_1)-v'(x_2)| \leq \omega_5(|x_1-x_2|),$$

for all $0 \leq t_1,t_2 \leq T$, $-a \leq x_1,x_2 \leq a$. Finally, we take

$$\eta_1(\beta) = (1-2Tb)^{-1}[\omega_5(\beta)+2T\omega_2(\beta)+2T\omega_3(M_3\beta) + 4LMT\beta] , \tag{1.13}$$

$$\eta_2(\alpha) = (1-aM)^{-1}[\omega_4(\alpha)+a\omega_1(\alpha)+a\omega_3(M_2\alpha) + 2Lab\alpha] , \tag{1.14}$$

$$\eta_3(\beta) = \omega_2(\beta) + \omega_3(M_3\beta) + 2LM\beta + b\eta_1(\beta) , \tag{1.15}$$

Both $\eta_1(\alpha)$, $o \leq \alpha < \infty$, $\eta_2(\beta)$, $o \leq \beta < \infty$, are continuous monotone functions with $\eta_1(o) = \eta_2(o) = 0$.

We proved in Ref. 1 that the functions ϕ,m of Theorem I satisfy the following relations

$$|\phi_t(t_1,x) - \phi_t(t_2,x)| \leq \eta_2(|t_1-t_2|) , \tag{1.16}$$

$$|\phi_t(t,x_1) - \phi_t(t,x_2)| \leq 2L|x_1-x_2| , \tag{1.17}$$

$$|\phi_x(t_1,x) - \phi_x(t_2,x)| \leq 2L|t_1-t_2| , \tag{1.18}$$

$$|\phi_x(t,x_1) - \phi_x(t,x_2)| \leq \eta_1(|x_1-x_2|). \tag{1.19}$$

$$|\phi(t_1,x_1) - \phi(t_1,x_2) - \phi(t_2,x_1) + \phi(t_2,x_2)| \leq 2L|t_1-t_2| \, |x_1-x_2| , \tag{1.20}$$

$$|\phi(t,x)| \leq M_1, \ |\phi(t_1,x) - \phi(t_2,x)| \leq M_2|t_1-t_2| , \tag{1.21}$$

$$|\phi(t,x_1) - \phi(t,x_2)| \leq M_3|x_1-x_2|, \tag{1.22}$$

$$|\phi_t(t,x)| \leq M_2, \quad |\phi_x(t,x)| \leq M_3 , \tag{1.23}$$

$$|m(x)| \leq L, \quad |m(x_1) - m(x_2)| \leq \eta_3(|x_1-x_2|) . \tag{1.24}$$

3. Under the conditions of Theorem III, if f,u,u',v,v' are Lipschitz functions in their arguments, then $\phi,\phi_t,\phi_x,\phi_{tx}$,m are all Lipschitzian with constants depending only on the Lipschitz constants of f,u,v,u',v', and the constants listed in Theorem III.

Indeed, if $\omega_1(\alpha) = k_1\alpha$, $\omega_2(\beta) = k_2\beta$, $\omega_3(\gamma) = k_3\gamma$, $\omega_4(\alpha) = k_4\alpha$, $\omega_5(\beta) = k_5\beta$, then

$$\eta_1(\beta) = (1-2Tb)^{-1}(k_5+2Tk_2+2Tk_3M_3+4LMT)\beta = h_1\beta, \tag{1.25}$$

$$\eta_2(\alpha) = (1-aM)^{-1}(k_4+ak_1+aM_2k_3+2Lab)\alpha = h_2\alpha, \tag{1.26}$$

$$\eta_3(\beta) = [k_2+M_3k_3+2LM+b(1-2Tb)^{-1}(k_5+2Tk_2+2Tk_3M_3+4LMT)]\beta = h_3\beta , \tag{1.27}$$

and the other relations of the Remark 1 show that $\phi,\phi_t,\phi_x,\phi_{tx}$,m are all Lipschitzian.

4. THEOREM II (Uniqueness theorem for the modified Darboux problem (0.6))

Under the hypotheses of Theorem I, and $\omega_3(\gamma) = k_3\gamma$ for some constant $k_3 \geq 0$, there is only one vector function $\phi(t,x)$ continuous in A with ϕ_t, ϕ_x, ϕ_{tx} and one vector function m(x) continuous in [-a,a] satisfying (0.6).

THEOREM III (Continuous dependence upon the initial data for the modified Darboux problem (0.6))

Under the conditions of Theorems I and II ($\omega_3(\gamma) = k_3\gamma$), the unique solution $\phi(t,x)$, m(x) of (0.6) depends continuously on u(t), u'(t), v(t), v'(t). In other words, if $(u_1(t), v_1(x))$, $(u_2(t), v_2(x))$ are initial data, and $(\phi_1(t,x), m_1(x))$, $(\phi_2(t,x), m_2(x))$ the corresponding solutions, and

$$\begin{aligned}
\eta &= \max|u_1(t)-u_2(t)| + \max|u_1'(t)-u_2'(t)|, \\
\chi &= \max|v_1(x)-v_2(x)| + \max|v_1'(x)-v_2'(x)|, \\
\epsilon &= \chi+\eta, \\
\alpha &= \max|\phi_1(t,x) - \phi_2(t,x)|, \\
\beta &= \max|\phi_{1t}(t,x) - \phi_{2t}(t,x)|, \\
\gamma &= \max|\phi_{1x}(t,x) - \phi_{2x}(t,x)|, \\
\delta &= \max|m_1(x) - m_2(x)|,
\end{aligned} \tag{1.28}$$

then there is a constant K depending only on the constants $a, T, N, N_1, N_2, L, M, b, M_1, M_2, M_3$ such that

$$\alpha,\beta,\gamma,\delta \leq K\epsilon. \tag{1.29}$$

5. In the proof of Theorem III given in Ref. 1, we proved this theorem for a strip $A_0' = [o \leq t \leq T,\ o \leq x \leq c]$ with $c = a/P$, P integer sufficiently large. The resoning has to be repeated for the remaining strips $A_i'[o \leq t \leq T, (i-1)c \leq x \leq ic]$, $i = 0,1,\ldots,P$. The same reasoning holds for $-a \leq x \leq 0$. Let $\xi_i = ic$, $i = 0,1,\ldots,k$, ($\xi_0 = 0$, $\xi_P = a$). If we take

$$k = (1-2Tb)^{-1}, \quad k' = 1 + 2kTb,$$

and

$$c \leq (4Mk')^{-1}, \quad c \leq (4Tk_3k')^{-1}(1+4k'Ma)^{-1}.$$

We proved in Ref. 1 that, for the strip A_0, we have

$$\alpha_0 \leq 4(1+Tbc+TMk'c(1+2bc))\epsilon = Q_1\epsilon,$$

$$\beta_0 \leq 2(1+2bc)\epsilon + 4k_3k'c\alpha \leq (2(1+2bc) + (4k_3k'c)Q_1)\epsilon = Q_2\epsilon,$$

$$\gamma_0 \leq k\epsilon + 2kTk_3\alpha + 2kTM\beta \leq (k+2kTk_3Q_1+2kTMQ_2)\epsilon = Q_3\epsilon,$$

$$\delta_0 \leq k_3\alpha + M\beta + b\gamma = (k_3+MQ_2+bQ_3)\epsilon = Q_4\epsilon$$

where $\alpha_0,\beta_0,\gamma_0,\delta_0$ are the numers $\alpha,\beta,\gamma,\delta$ relative to the strip A_0 only. Let us denote by Q the largest of the numbers Q_1,Q_2,Q_3,Q_4. In the strip A_1 we can take the same number χ, but $\eta = \eta_0$, $\epsilon = \epsilon_0$ have to be replaced by

$$\eta_1 = \max|\phi_1(t,\xi_1) - \phi_2(t,\xi_1)| + \max|\phi_{1t}(t,\xi_1) - \phi_{2t}(t,\xi_2)| \leq \alpha+\beta,$$

$$\epsilon_1 = \eta_1+\chi \leq \alpha + \beta+\chi.$$

Hence, the corresponding numbers, say α_1, β_1, α_2, β_2 become

$$\alpha_1,\beta_1,\gamma_1,\delta_1 \leq Q\epsilon_1 \leq Q(\alpha+\beta+\chi) \leq Q(2Q(\chi+\eta)+\chi)$$
$$= (Q+2Q^2)\chi + 2Q^2\eta.$$

In the strip A_2 we take the same χ but η_1, ϵ_1 are replaced by

$$\eta_2 \leq \alpha_1 + \beta_1, \quad \epsilon_2 = \eta_2 + \chi \leq \alpha_1 + \beta_1 + \chi,$$

and then we have

$$\alpha_2,\beta_2,\gamma_2,\delta_2 \;\leq\; Q\epsilon_2 \;\leq\; Q(\alpha_1+\beta_1+\chi)$$

$$\leq\; Q[2(Q+2Q^2)\chi + 4Q^2\eta + \chi]$$

$$=\; (Q+2Q+2^2Q^3)\chi + 4Q^3\eta \; .$$

After P of these operations we have

$$\alpha_P,\beta_P,\gamma_P,\delta_P \;=\; (Q+2Q^2+\ldots+2^PQ^{P+1})\chi + 2^PQ^{P+1}\eta .$$

Thus we have only to determine $c = a/P$ with

$$P \;\geq\; (4Mk')a, \quad P \;\geq\; (4Tk_3k')(1+4k'Ma)a \; ,$$

and take

$$K \;=\; Q + 2Q + 2^2Q^3 + \ldots + 2^PQ^{P+1} .$$

Then we have

$$\alpha,\beta,\gamma,\delta \;\leq\; K(\chi+\eta) \;=\; K\epsilon \; . \tag{1.30}$$

This remark completes the proof of Theorem III given in Ref. 1.

6. We shall now improve formula (1.29) concerning the continuous dependence on the initial values u,u',v,v' so that a much sharper estimate can be deduced when u,u' do not vary, that is, then $\eta = 0$. Indeed, if we denote by $\alpha_c,\beta_c,\gamma_c,\delta_c$ the same numbers (1.28) relative to the strip $[o \leq t \leq T, -c \leq x \leq c]$, $(o \leq c \leq a)$, then, for a convenient constant K'', we have

$$\alpha_c,\beta_c \;\leq\; K''(\eta+\chi_c), \quad \beta_c\delta_c \;\leq\; K''(\eta+\chi) \; . \tag{1.31}$$

We shall show that the continuity of v' can be relaxed by allowing an arbitrary set $[\xi]$ of P points of discontinuity of the first type for $v'(x)$ in $[-a,a]$. The number a of Theorems I,II,III and all other constants, in particular K'' in formula (1.31), can be chosen independently of $[\xi]$ and P.

A form of Theorems I,II,III answering the question above reads as follows for discontinuous v' and Lipschitz data.

A Modified Form of Theorems I,II,III:

If a, $T > 0$, and N, N_1, N_2, L, M, b, M_1, M_2, M_3, k_1, k_2, k_3, k_4, k_5, are constants, if A and R are the sets

$$A = [o \leq t \leq T,\ -a \leq x \leq a],$$

$$R = [o \leq t \leq T,\ -a \leq x \leq a,\ |z| \leq M_1, |p| \leq M_2, |q| \leq M_3, z,p,q \epsilon E_n] ,$$

if

$$M_1 \geq N + 2^{-1}N_1T + N_2a + 2LTa + k_5a, \tag{1.32}$$

$$M_2 \geq N_1 + 4La, \tag{1.33}$$

$$M_3 \geq N_2 + LT, \tag{1.34}$$

if $u(t)$, $o \leq t \leq T$, $v(x)$, $-a \leq x \leq a$, $v(o) = 0$, are vector functions, if $u(t)$ is continuous together with $u'(t)$, if $v(t)$ is continuous and $v'(t)$ sectionally continuous, if

$$u(T) = u(o),\ |u(o)| \leq N,\ |u(t_1) - u(t_2)| \leq N_1|t_1-t_2|, \tag{1.35}$$

$$v(o) = 0,\ |v(x_1) - v(x_2)| \leq N_2|x_1-x_2| , \tag{1.36}$$

$$f(T,x,z,p,q) = f(o,x,z,p,q),\ |f(t,x,z,p,q)| \leq L, \tag{1.37}$$

$$|f(t,x,z,p_1,q_1) - f(t,x,z,p_2,q_2)| \leq M|p_1-p_2| + b|q_1-q_2| , \tag{1.38}$$

$$|f(t_1,x_1,z_1,p,q) - f(t_2,x_2,z_2,p,q)| \leq k_1|t_1-t_2| + k_2|x_1-x_2| + k_3|z_1-z_2|, \tag{1.39}$$

$$2Tb < 1,\quad Ma < 1 ,$$

if

$$|u'(t_1)-u'(t_2)| \leq k_4|t_1-t_2| ,\quad 0 \leq t_1,\ t_2 \leq T, \tag{1.40}$$

if $-a < \xi_1 < \xi_2 < \ldots < \xi_Q < a$ are the points of discontinuity of $v(x)$ in $[-a,a]$, and

$$|v'(x_1)-v'(x_2)| \leq k_5|x_1-x_2| , \qquad (1.41)$$

for all $\xi_i < x_1,x_2 < \xi_{i+1}$, $i = 0,1,\ldots,Q$ ($\xi_0 = -a$, $\xi_{Q+1} = a$), then there is a vector function $\phi(t,x)$, $(t,x)\epsilon A$, Lipschitzian in A with partial derivatives $\phi_t(t,x)$ again Lipschitzian in A and partial derivatives $\phi_x(t,x)$, $\phi_{tx}(t,x)$ also Lipschitzian in each strip $A_i = [o \leq t \leq T,\ \xi_i \leq x \leq \xi_{i+1}]$, $i = 0,1,\ldots,Q$, satisfying (1.6)-(1.9) of Theorem I.

The functions ϕ and m are uniquely determined by the initial data $u(t)$, $o \leq t \leq T$, $v(x)$, $-a \leq x \leq a$. If $(u_1(t),v_1(x))$, $(u_2(t),v_2(x))$ are two sets of initial data, and $\eta,\chi,\epsilon,\alpha,\beta,\gamma,\delta$ are defined as in Theorem III (with sup replacing max where needed), then there is a constant K' depending only on the constants $a,t,\ldots,k_5$ listed above such that $\alpha,\beta,\gamma,\delta \leq K'\epsilon$.

Finally, if c is any number $o \leq c \leq a$, and if $\alpha_c,\beta_c,\gamma_c,\delta_c$ are defined as in (1.32) for the strip $o \leq t \leq T$, $o \leq x \leq c$, [or $-c \leq x \leq 0$], then there is a constant $K'' \geq 1$ (depending only on $a,T,\ldots,k_5$ such that

$$\alpha_c,\beta_c \leq K''(\eta+\chi_c), \quad \beta_c,\delta_c \leq K''(\eta+\chi) . \qquad (1.42)$$

7. PROOF OF THE STATEMENT OF NO. 6

As we have mentioned, it is not restrictive to consider only the strip $A' = [o \leq t \leq T,\ o \leq x \leq a]$, the points of discontinuity say $Q < \xi_1 < \ldots < \xi_P < a$, of $v'(t)$ in (o,a), and the strips $A_i' = [o \leq t \leq T,\ \xi_s \leq x \leq \xi_{s+1}]$, $s = 0,1,\ldots,P(\xi_o = 0,\ \xi_{P+1} = a)$. Also, it is not restrictive to assume $v'(x)$ continuous at every $x \neq \xi_1,\ldots,\xi_P$, and verifying $v'(\xi_s-0) \neq v'(\xi_s) = v'(\xi_s+0)$, $s = 1,\ldots,P$, at the points of discontinuity. We shall apply Theorem I to each strip A_s, $s = 0,1,\ldots,P$ in succession, where a is replaced by ξ_1, $\xi_2-\xi_1,\ldots,\xi_P-\xi_{P-1}$, $a-\xi_P$ respectively. The initial conditions for the strip A_o,

$$\phi(t,o) = u(t),\ \phi_t(t,o) = u'(t),\ o \leq t \leq T,$$

is then replaced for the strip A_1 by

$$\phi(t,\xi_1) = u_1(t) = \phi(t,\xi_1-0),\ \phi_t(t,\xi_1) = u_1'(t) = \phi_t(t,\xi_1-0),\ o \leq t \leq T,$$

and, in general

$$\phi(t,\xi_s) = u_s(t) = \phi(t,\xi_s-0), \; \phi_t(t,\xi_s) = u_s'(t) = \phi_t(t,\xi_s-0), \; 0 \le t \le T, \tag{1.43}$$

for the strip A_s, $s = 1,\ldots,P$.

Obviously, the initial condition

$$\phi(o,x) = u(o) + v(x), \quad \phi_x(o,x) = v'(x), \quad o \le x \le \xi_1,$$

for the strip A_o is replaced by

$$\phi(o,x) = u(o) + v(x) = u(o) + v(\xi_s) + [v(x)-v(\xi_s)] = u_s(o)+[v(x)-v(\xi_s)] \tag{1.44}$$

$$\phi_x(o,x) = v'(x), \quad \xi_s \le x \le \xi_{s+1}, \tag{1.45}$$

for the strip A_s, $s = 1,\ldots,P$, since

$$u_s(o) = \phi(o,\xi_s) = \phi(o,\xi_s-0) = u_{s-1}(o) + [v(\xi_s)-v(\xi_{s-1})]$$
$$= u_{s-2}(o) + [v(\xi_s)-v(\xi_{s-2})] = \ldots = u(o) + v(\xi_s) . \tag{1.46}$$

Obviously, we can apply Theorem I to the strip A_o since all relations (1.32-1.42) imply the analogous relations (1.1-1.8).

Assume that we can apply Theorem I to A_{s-1}, and prove that we can apply it to A_s. By (1.46), (1.4), and (1.42) we have

$$u_s(o) = u_{s-1}(o) + \int_{\xi_{s-1}}^{\xi_s} v'(x)dx = u(o) + \int_0^{\xi_s} v'(x)dx ,$$

$$|u_i(o)| \le |u(o)| + \int_0^{\xi_s} |v'(t)|dt \le N + ak_5.$$

By (1.43), (1.20) we have

$$|u_s(t_1) - u_s(t_2)| \leq |u_{s-1}(t_1)-u_{s-1}(t_2)| + 2L|t_1-t_2|(\xi_s-\xi_{s-1})$$

$$\leq |u_{s-2}(t_1)-u_{s-2}(t_2)| + 2L|t_1-t_2|(\xi_s-\xi_{s-2}).$$

By repeating this argument and by force of (1.31) we have

$$|u_s(t_1) - u_s(t_2)| \leq |u(t_1) - u(t_2)| + 2L|t_1-t_2|\xi_s$$

$$\leq (N_1+2La)|t_1-t_2| .$$

Thus, we can replace N and N_1 by $N+aN_2$, N_1+2La respectively in each strip A_i.

Let us prove that we can replace k_4 by

$$k_{4s} = (1-\xi_s M)^{-1}[k_4+\xi_s(k_1+M_2k_3+2Lb)] \tag{1.47}$$

in the strip A_s, $i = 0,1,\ldots,P$. Again this is possible in A_0 since $k_{40} > k_4$. Assume that this is possible in A_{s-1} and let us prove that this is possible in A_s. By force of (1.43), (1.16), 1.26), and (1.47) we have

$$|u_s'(t_1) - u_s'(t_2)| = |\phi_t(t_1,\xi_s-0) - \phi_t(t_2,\xi_s-0)| \leq \eta_2(|t_1-t_2|)$$

$$= (1-(\xi_s-\xi_{s-1})M)^{-1}[k_{4,s-1}+(\xi_s-\xi_{s-1})(k_1+M_2k_3+2Lb)]|t_1-t_2|$$

$$= (1-(\xi_s-\xi_{s-1})M)^{-1}\{(1-\xi_{s-1}M)^{-1}[k_4+\xi_{s-1}(k_1+M_2k_3+2Lb)]$$

$$+(\xi_s-\xi_{s-1})(k_1+M_2k_3+2LB)\}|t_1-t_2| . \tag{1.48}$$

Since, for all $\alpha,\beta > 0$ with $\alpha+\beta < 1$ we have $(1-\alpha)^{-1}(1-\beta)^{-1} < (1-\alpha-\beta)^{-1}$, we have

$$(1-(\xi_s-\xi_{s-1})M)^{-1}(1-\xi_{s-1}M)^{-1} < (1-\xi_s M)^{-1} . \tag{1.49}$$

Since $0 < (\xi_s-\xi_{s-1})M \leq \xi_s M < 1$, we have in succession

$$(1-(\xi_s-\xi_{s-1})M)^{-1} < (1-\xi_s M)^{-1},$$

$$(1-(\xi_s-\xi_{s-1})M)^{-1}(\xi_s-\xi_{s-1}) < (1-\xi_s M)^{-1}(\xi_s-\xi_{s-1}),$$

$$(1-\xi_s M)^{-1}\xi_{s-1}+(1-(\xi_s-\xi_{s-1})M)^{-1}(\xi_s-\xi_{s-1}) < (1-\xi_s M)^{-1}\xi_s. \tag{1.50}$$

By using (1.49) and (1.50), relation (1.48) becomes

$$|u_s'(t_1) - u_s'(t_2)| \leq (1-\xi_s M)^{-1}[k_4+\xi_s(k_1+M_2k_3+2Lb)]|t_1-t_2|$$

$$= k_{4s}|t_1-t_2|,$$

and we have proved the contention that k_4 can be replaced by k_{4s} in A_s. The numbers k_{4s} are of course all smaller than

$$k_4' = (1-aM)^{-1}[k_4 + a(k_1+M_2k_3+2Lb)].$$

Now the relations (1.32), (1.33) of Theorem I, are obviously satisfied in each strip A_s, $s = 0,1,\ldots,P$, since the relations (1.32), (1.33) in the hypotheses of Theorem III are obtained by relations (1.1), (1.2) by replacing N, N_1, k_4 by $N+ak_5$, N_1+2La, k_4' respectively. All other constants and relations are the same.

By induction we conclude that Theorem I can be applied to each strip A_s, $s = 0,1,\ldots,P$, in succession, and thus the existence part of the theorem is proved The uniqueness follows from Theorem II applied to the strips $A_0,A_1,\ldots,A_P$ in succession. The continuous dependence upon the initial data follows from III applied to the strips $A_0,A_1,\ldots,A_P$ in succession and where K' is a constant analogous to K obtained by replacing N, N_1, k_4 by $N+ak_5$, N_2+La, k_4' in the definition of K.

Let us prove now relation (1.42) and, therefore, the continuous dependence of the solution ϕ,m on the initial data u,u',v,v'. First we have

$$\phi_{jtx}(t,x) = f(t,x,\phi_j,\phi_{jt},\phi_{jx}) - m_j(x),$$

$$\phi_j(t,x) = u_j(t) + v_j(x) + \int_0^t\int_0^x [f(t,x,\phi_j,\phi_{jt},\phi_{jx}) - m_j(x)]dt\,dx,$$

$$\phi_{it}(t,x) = u_j'(t) + \int_0^x [f(t,x,\phi_j,\phi_{jt},\phi_{jx}) - m_j(x)]dx, \qquad (1.51)$$

$$\phi_{jx}(t,x) = v_j'(x) + \int_0^t [f(t,x,\phi_j,\phi_{jt},\phi_{jx}) - m_j(x)]dt,$$

$$m_j(x) = T^{-1}\int_0^T f(t,x,\phi_j,\phi_{jt},\phi_{jx})dt,$$

for all $o \leq t \leq T$, $-a \leq x \leq a$, $j = 1,2$, and where the usual conventions hold for $\phi_{jt},\phi_{jx},\phi_{jtx}$ at the lines of discontinuity $[x=\xi_s,\ o \leq t \leq T]$, $s=1,\ldots,P$. We have taken here as set $[\xi_s, s=1,\ldots,P]$ the union of the two sets of points of discontinuity of v_1' and v_2'. Now, for all c, $o < c < a$, and $o \leq x \leq c$, we have

$$v_1(o) = v_2(o) = 0, \quad |v_1'(x)-v_2'(x)| \leq \chi ,$$

$$|v_1(x)-v_2(x)| = \left|\int_o^x [v_1'(x)-v_2'(x)]dx\right| \leq \chi x \leq \chi c,$$

$$|u_1(t)-u_2(t)|,\ |u_1'(t)-u_2'(t)| \leq \eta .$$

By subtracting corresponding relations (1.51) for $j = 1,2$, and standard estimates, we obtain, as in the proof of Theorem III in Ref. 1,

$$\begin{aligned}
|\phi_1(t,x) - \phi_2(t,x)| &\leq \eta + \chi c + Tc(k_3\alpha+M\beta+b\gamma+\delta),\\
|\phi_{1t}(t,x) - \phi_{2t}(t,x)| &\leq \eta + c(k_3\alpha+M\beta+b\gamma+\delta),\\
|\phi_{2x}(t,x) - \phi_{2x}(t,x)| &\leq \chi + T(k_3\alpha+M\beta+b\gamma+\delta),\\
|m_1(x) - m_2(x)| &\leq k_3\alpha + M\beta + b\gamma,
\end{aligned} \qquad (1.52)$$

where $\alpha,\beta,\gamma,\delta$ are computed as in Theorem III for the sole strip $o \leq t \leq T$, $o \leq x \leq c$ (and thus are $\leq$ the numbers $\alpha_c,\beta_c,\gamma_c,\delta_c$ of No. 6). For some t and x, relations (1.52) yield

$$\alpha \leq \eta + \chi c + Tc(k_3\alpha+M\beta+b\gamma+\delta),$$

$$\beta \leq \eta + c(k_3\alpha+M\beta+b\gamma+\delta),$$

$$\gamma \leq \chi + T(k_3\alpha+M\beta+b\gamma+\delta),$$

$$\delta \leq k_3\alpha + M\beta + b\gamma .$$

By elimination of δ we have

$$\alpha \leq \eta + \chi c + 2Tc(k_3\alpha+M\beta+b\gamma),$$

$$\beta \leq \eta + 2c(k_3\alpha+M\beta+b\gamma) ,$$

$$\gamma \le \chi + 2T(k_3\alpha + M\beta + b\gamma) \ .$$

For $k = (1-2Tb)^{-1}$, $k' = 1+2kTb$, we have

$$\gamma \le k\chi + 2kk_3T\alpha + 2kMT\beta \ ,$$

$$\alpha \le \eta + (1+2kTb)\chi c + 2k'k_3T\alpha_c + 2k'MT\beta_c \ ,$$

$$\beta \le \eta + (2kb)\chi c + 2k'k_3\alpha_c + 2k'M\beta_c \ .$$

The same restrictions already used on c, $o < c \le a$, namely

$$2k'Mc \le 1/2 \ , \quad 2k'k_3T(1+4k'Mc)c \le 1/2 \ , \tag{1.53}$$

give

$$\beta \le 2\eta + (4kb)\chi c + (4k'k_3)\alpha_c ,$$

and finally

$$\alpha \le 2(1+4k'MTc)\eta$$

$$+ 2(1+2kTb+8kk'MTbc)\ \chi c = Q_1\eta + Q_1'\chi c \ ,$$

$$\beta \le 2(1+2k'k_3Q_1c)\eta + 4(kb+k'k_3Q_1'c)\chi c = Q_2\eta + Q_2'\chi c \ ,$$

$$\gamma \le 2kT(k_3Q_1+MQ_2)\eta + k(1+2Tc(k_3Q_1'+MQ_2'))\chi$$

$$= Q_3\eta + Q_3'\chi \ ,$$

$$\delta \le (k_3Q_1+MQ_2+bQ_3)\eta + (k_3Q_1'c+MQ_2'c+bQ_3')\chi = Q_4\eta + Q_4'\chi .$$

Let c_o, $o < c_o \le a$, be any number for which relations (1.53) are satisfied. Let Q be the largest of the numbers Q_j, Q_j', $j = 1,2,3,4$, above computed for $c = c_o$. Then, for every $o < c \le c_o$, we have

$$\alpha_c, \beta_c \le Q(\eta + \chi c), \quad \gamma_c, \delta_c \le Q(\eta + \chi) .$$

This proves (1.42) for $o < c \le c_o$, $K'' = Q$.

We shall now take $c = a/P$ with P integer sufficiently large so that $c \leq c_o$. Let us consider the P strips $A_s = [o \leq t \leq T, (s-1)c \leq x \leq sc]$, $s = 1,2,\ldots,P$. We shall denote by $\alpha_s, \beta_s, \gamma_s, \delta_s$ the numbers $\alpha, \beta, \gamma, \delta$ relative to the strip A_s. We have proved above that for the strip A_o we have

$$\alpha_1, \beta_1 \leq Q(\eta + \chi c), \quad \gamma_1, \delta_1 \leq Q(\eta + \chi) .$$

In the strip A_2 we can take the same number $\chi_1 = \chi$, but $\eta = \eta_o$ must be replaced by

$$\eta_1 = \max|\phi_1(t,\xi) - \phi_2(t,\xi)| + \max|\phi_{1t}(t,\xi) - \phi_{2t}(t,\xi)|$$

$$\leq \alpha_1 + \beta_1 \leq 2Q(\eta + \chi c).$$

The reasoning above implies

$$\alpha_2, \beta_2 \leq Q(\eta_1 + \chi c) \leq Q[2Q(\eta + \chi c) + \chi c]$$

$$= 2Q^2\eta + (Q + 2Q^2)\chi c ,$$

$$\gamma_2, \delta_2 \leq Q(\eta + \chi) \leq Q[2Q(\eta + \chi c) + \chi]$$

$$= 2Q^2\eta + (Q + 2Q^2 c)\chi .$$

For the strip A_3 we have

$$\eta_2 \leq \alpha_2 + \beta_2 \leq 4Q^2\eta + (2Q + 4Q^2)\chi c, \qquad \chi_2 = \chi ,$$

and hence

$$\alpha_3, \beta_3 \leq Q(\eta_2 + \chi c) = Q[4Q^2\eta + (2Q + 4Q^2)\chi c + \chi c]$$

$$= 4Q^3\eta + (Q + 2Q^2 + 4Q^3)\chi c$$

$$\gamma_3, \delta_3 \leq Q(\eta_2 + \chi) = Q[4Q^2\eta + (2Q + 4Q^2)\chi c + \chi]$$

$$= 4Q^3\eta + (Q + 2Q^2 c + 4Q^3 c)\chi .$$

By this process we obtain successively $\alpha_s, \beta_s, \gamma_s, \delta_s, s = 1, \ldots, P$. Since $\alpha = \max \alpha_s$, and analogous relations hold for β, γ, δ, we have

$$\alpha, \beta \leq (2^P Q^{P+1})\eta + (Q + 2Q^2 + \ldots + 2^P Q^{P+1})\chi c$$

$$\gamma, \delta \leq (2^P Q^{P+1})\eta + (Q + 2Q^2 c + \ldots + 2^P Q^{P+1} c)\chi \ .$$

Let Q' be the largest of the numbers in parenthesis for $c = a/P \leq c_o$. Then we have

$$\alpha, \beta \leq Q'(\eta + \chi c_o), \qquad \gamma, \delta \leq Q'(\eta + \chi).$$

Now for any c, $c_o \leq c \leq a$, we have $\alpha_c \leq \alpha$, $\beta_c \leq \beta$, $\gamma_c \leq \gamma$, $\delta_c \leq \delta$, and finally

$$\alpha_c, \beta_c \leq Q'(\eta + \chi c), \qquad \gamma_c, \delta_c \leq Q'(\eta + \chi) \ .$$

This proves relations (1.42) for $c_o \leq c \leq a$, $K'' = Q'$. Then relations (1.42) hold for every c by taking $K'' = \max(Q, Q')$.

8. A DIFFERENTIAL RELATION FOR m

We shall assume now that the components of $f(t,x,z,p,q) = (f_1, \ldots, f_n)$ possess continuous partial derivatives f_{iq_j} with respect to the components $q_1, \ldots, q_n$ of q in R. Let $u(t)$, $v(x)$ be given initial data, and let $\phi(t,x)$, $m(x)$ be the corresponding solution. Now let x be any given number, say $o < x < a$. Since we can introduce discontinuities in $v'(x)$ (of the first kind), we may think to change suddenly the value of v' at x, assigning a new value $y = v'(x)$, provided we remain within the limitations listed in No. 6, namely $|y| \leq N_2$. This change does not alter $\phi(t,\xi)$, $m(\xi)$ for $o \leq \xi < x$, but we get a new function $\phi(t,x) = q(t)$ and a new number $m(x) = m$, which are the solutions of the ordinary differential problem

$$dq/dt = F(t,q) - m, \quad o \leq t \leq T, \quad q(o) = y$$

$$m = T^{-1} \int_o^T F(t,q) dt \ , \tag{1.54}$$

where

$$F(t,q) = f(t,x,\phi(t,x),\phi_t(t,x),q),$$

and, for emphasis, we have treated x as a constant. We proved in Ref. 1 that q(t) and m exist, are uniquely determined by y, and depend continuously on y. We already proved that the continuous dependence of q(t) and m upon y is uniform with respect to x and with respect to v,v' for $o \leq x \leq a$, and for v,v' satisfying the limitations listed in Theorems I-III (or No. 6).

We shall now assume that the components f_i of $f(t,x,y,z,p,q) = (f_1,\ldots,f_n)$ possess continuous partial derivatives with respect to the components $q_1,\ldots,q_n$ of q. Then the components F_i of $F = (F_1,\ldots,F_n)$ have continuous partial derivatives with respect to $q_1,\ldots,q_n$

$$F_{iq_j}(t,q) = f_{iq_j}(t,x,\phi(t,x),\phi_t(t,x),q) .$$

Then, as proved in Ref. 1, all components of f(t) and m (here q,m denotes a solution of the problem (1.54)) have continuous partial derivatives with respect to the components y_j of the arbitrary initial value $y = (y_1,\ldots,y_n)$. In particular

$$a_{ij} = \partial m_i/\partial y_j = \left[1\text{-}\exp\left(-\int_o^T F_{iq_j} dt\right)\right] \left[\int_o^T dt \exp\left(-\int_o^T F_{iq_j} dt\right)\right]^{-1} .$$

This relation was actually proved in Ref. 1 for n = 1 (and then there was only one derivative dm/dy). The proof is the same for any n, and we shall not repeat it here, since it is based on usual theorems of continuous dependence and differentiability with respect to parameters and initial values for ordinary differential systems (as in Ref. 4, pp. 155 and 161). In Ref. 1 we assumed F to be of class C' (both in t and q) but the existence and continuity of F_t was never used. Thus (1.55) stands under the condition just stated that the partial derivatives f_{iq_j} exist and are continuous in R. Here we only add that the continuity of the partial derivatives a_{ij} is proved uniformly with respect to x and with respect to v,v' with the usual conventions.

9. THE FUNCTIONAL G(x,y,v')

We shall now assume u(t), $o \leq t \leq T$, as fixed, and study the dependence of m(x) on v(x), v'(x). Obviously, v'(x) determine v(x) since v(o) = 0. Thus, for each x, say $o < x < a$, $m(x) = (m_1,\ldots,m_n)$ depends on the values taken by $v'(\xi)$ for $o \leq \xi < x$. At $\xi = x$ we can always take an arbitrary value y = v'(x) for v'(x) introducing a new point of discontinuity, and the dependence of m(x) on y has been studied in No. 8. Thus m(x) is precisely a vector valued functional depending on $v'(\xi)$, $o \leq \xi < x$, on y = v'(x), and x itself:

$$m(x) = G(x,y;v'(\xi), o \leq \xi < x), \quad o \leq x \leq a.$$

For the sake of brevity we shall write sometimes $G(x,y;v')$. It is understood that all the variables are assumed to satisfy the limitations listed in No. 6. We shall prove for G certain properties G_{1234}.

G1. There is an $M > 0$ such that

$$|G(x,y;v_1'(\xi), o \leq \xi < x) - G(x,y,v_2'(\xi), o \leq \xi < x)| \leq M\chi_x,$$

where $\chi_x = \sup|v_1'(\xi) - v_2'(\xi)|$ for all $o \leq \xi \leq x$.

This is only a different form of the statement $\delta \leq K''\chi$ of No. 6.

G2. There is a constant M such that for any two $x_1, x_2 \epsilon [o,a]$ and y we have

$$|G(x_1,y,v'(\xi), o \leq \xi < x_1) - G(x_2,y,v'(\xi), o \leq \xi < x_2)| \leq M|x_1-x_2| .$$

Indeed we have seen in No. 3 that m(x) satisfies a Lipschitz condition with constant M which depends only on the constants $a, M, \ldots, k_5$. Therefore, if $\overline{v}'(\xi)$ denotes the function which is equal to $v'(\xi)$ for $o \leq \xi < x$, and is constant and equal to y for $x \leq \xi \leq a$, then

$$|G(x_1,y,\overline{v}') - G(x_2,y,\overline{v}')| \leq M|x_1-x_2|,$$

$$|G(x_1,y,v') - G(x_2,y,v')| \leq M|x_1-x_2| ,$$

$$G(x_1,y,\overline{v}') = G(x_1,y,v') .$$

The last relation is trivial since $\overline{v}'(\xi) = v'(\xi)$ for all $o \leq \xi < x_1$. Finally, we have

$$|G(x_1,y,v') - G(x_2,y,v')| \leq 2M|x_1-x_2|.$$

G3. For $x = 0$, G does not depend on v', and $G = (G_1,\ldots,G_n)$ is only a vector valued function of the vector $y = (y_1,\ldots,y_n)$. We shall write $G(o,y,\cdot)$.

G4. For each t, $o \leq t \leq a$, and continuous function $v'(t)$, $o \leq t \leq a$, with values $|v'(t)| \leq N_2$, the components G_i of G are real valued continuous functions of y with continuous first order partial derivatives $a_{ij}(t,y,z) = \partial G_i/\partial y_j$, $i,j = 1,\ldots,n$.

This is only a rewording of statements proved in No. 8. Note that for $t = 0$, all a_{ij} are functions of y only, and we may write $a_{ij}(o,y,\cdot)$. Actually, G_1 can be modified as follows:

G5. There is a constant $M > 0$ and that, if c, $o < c \leq a$, is sufficiently small, then $y, v_1'(\xi), v_2'(\xi)$ as in No. 6 and $o \leq x \leq c$ imply

$$|G(x,y,v_1'(\xi),\ o \leq \xi < x) - G(x,y,v_2'(\xi),\ o \leq \xi < x)| \leq Mc\chi_x .$$

To prove this, let us observe that $\eta = 0$, and that, by No. 6, $\alpha_c, \beta_c \leq Kc\chi_x$, hence, if ϕ_1, m_1, ϕ_2, m_2 are the solutions corresponding to v_1 and v_2, we have

$$|\phi_1(t,x) - \phi_2(t,x)|,\ |\phi_{1t}(t,x) - \phi_{2t}(t,x)| \leq Kc\chi_x .$$

Then we have to solve the two differential problems

$$dq/dt = f(t,x,\phi_i(t,x),\phi_{it}(t,x),q) - m, \quad q(o) = y,$$

$$m = T^{-1} \int_o^T f(t,x,\phi_i(t,x),\phi_{it}(t,x),\ q(t))dt.$$

With the same initial value y. If $q_1(t)$, m_1, $q_2(t)$, m_2 are the two solutions, we have, with obvious notations,

$$|q_1(t)-q_2(t)| = \left|\int_o^t (f_1-f_2)dt + (m_1-m_2)t\right|$$

$$\leq (k_3T+MT)(Kc\chi_x) + |m_1-m_2|T ,$$

$$|m_1-m_2| \leq (k_3+M)(Kc\chi_x)$$

and hence

$$|m_1-m_2| \leq K(M+k_3)c\chi_x , \quad |q_1(t)-q_2(t)| \leq 2K(M+k_3)c\chi_x .$$

The first of these relation obviously proves G5.

SECTION 2. CRITERION FOR THE EXISTENCE OF PERIODIC SOLUTIONS TO THE ORIGINAL PROBLEM

10. AN IMPLICIT FUNCTION THEOREM OF FUNCTIONAL ANALYSIS

In Ref. 2 we have proved an implicit function theorem, a corollary of which will be stated here and applied in No. 11.

Let $H(t,y,z(\xi),\ o \leq \xi < t)$ be a functional, with values in E_n, depending on the real variable t, the real vector $y \epsilon E_n$, and the values in E_n taken by a function $z(\xi)$ in the variable interval $o \leq \xi < t$. We proved, under the hypotheses below, that there is a continuous function $\psi(t)$ with values in E_n such that

$$H(t,\psi(t),\psi(\xi),\ o \leq \xi < t) = 0. \tag{2.1}$$

We state now, with precision both hypotheses and contentions. Let I be the interval $I = [o \leq t \leq a]$ for some $a > 0$, let μ be a point of E_n and Y_o the sphere $Y_o = [y \epsilon E_n|\ |y-\mu| \leq \beta]$ for some $\beta > 0$. Let Z_o be the family of all continuous functions $z(\xi)$, $o \leq \xi \leq a$, with values in Y_o, or $z: I \rightarrow Y_o$. Then $H: I x Y_o x Z_o \rightarrow E_n$. For the sake of simplicity we shall write H in the form $H(t,y,z)$. We shall take in Z_o the topology of the uniform convergence in $[o,a]$.

We shall assume that, for $t = 0$, H does not depend on z, and then we may write $H(o,y,\cdot)$. As in the implicit function theorem we assume $H(o,\mu,\cdot) = 0$. The following statement holds (Ref. 1, Cor. 1, Section 5, No. 2).

(A) If $H(t,y,z) = (H_1,\ldots,H_n)$ is uniformly bounded and continous in t,z, if all H_i have partial derivatives with respect to $y_1,\ldots,y_n$, say $a_{ij}(t,y,z) = \partial H_i/\partial y_j$, $i,j = 1,\ldots,n$, which are bounded and continuous in t,y,z, if $H(t,\mu,\cdot) = 0$, and $\det A_o \neq 0$, with $A_o = [a_{ij}(o,\mu,\cdot)]$, then there is some a_o,β_o, $o < a_o \leq a$, $o < \beta_o \leq \beta$, and a continuous function $\psi(t)$, $o \leq t \leq a_o$, such that $\psi(o) = \mu$, $|\psi(t)-\mu| \leq \beta_o$, and $f(t,\psi(t),\psi) = 0$ for all $o \leq t \leq a_o$.

Also we proved in Ref. 2 that, if there are numbers a',β', $o < a' \leq a$, $o < \beta' \leq \beta$, such that $o \leq t \leq a'$, $|y-\mu| \leq \beta'$, $|z_1(t)-\mu|$, $|z_2(t)-\mu| \leq \beta'$, $o \leq t \leq a'$, imply

$$|G(t,y,z_1) - G(t,y,z_2)| \leq (1/2M) \max_{o \leq \xi \leq t} |z_1(\xi)-z_2(\xi)| \tag{2.2}$$

with $\|A_o^{-1}\| \leq M$, then the function $\psi(t)$, $o \leq t \leq a_o \leq a'$, of statement (A) is unique. Finally, if G depends uniformly and continuously on some parameter α describing a topological space Ω, and the conditions above hold uniformly with respect to α (in particular, $\|A_o^{-1}\| \leq M$ for some constant M independent of α), then the unique function $\psi_\alpha(t)$, $o \leq t \leq a'$, above depends continuously on α in Ω.

11. AN EXISTENCE THEOREM FOR THE ORIGINAL PROBLEM (0.5).

We shall now use again the notations of Section 1.

THEOREM IV

Given constants $a, T, \ldots, k_5$, and functions u and f as in No. 6 (thus, satisfying all relations but (1.36), (1.41)) let us assume that, for some $\mu \epsilon E_n$, $|\mu| < N_2$, we have

$$m(o;\mu) = T^{-1}[q(T)-q(o)] = T^{-1} \int_o^T f(t,o,u(t),u'(t),q(t))dt = 0,$$

where q(t) is the solution of the initial ordinary differential problem

$$dq/dt = f(t,o,u(t),u'(t),q), \quad o \leq t \leq T, \quad q(o) = \mu. \tag{2.3}$$

Assume also that det $A \neq 0$, where $A = [a_{ij}]$ and

$$a_{ij} = \left[1-\exp\left(-\int_o^T f_{iq_j}\, dt\right)\right] \left[\int_o^T dt \exp\left(-\int_o^t f_{iq_j}\, dt\right)\right]^{-1} \tag{2.4}$$

$$f_{iq_j} = f_{iq_j}(t,o,u(t),u'(t),q(t)), \quad o \leq t \leq T.$$

Then there is some a_o, $o < a_o \leq a$, a function $v(x)$, $-a_o \leq x \leq a_o$, continuous in $[-a_o,a_o]$ with $v(o) = 0$, $v'(o) = \mu$, satisfying (1.36), (1.41), and a function $\phi(t,x)$ continuous in the strip $[-\infty < t < +\infty, -a_o \leq x \leq a_o]$ with $\phi_t, \phi_x, \phi_{tx}$, satisfying

$$\phi_{tx}(t,x) = f(t,x,\phi(t,x),\phi_t(t,x),\phi_x(t,x)), \ (t,x)\epsilon A,$$

$$\phi(t,o) = u(t), \ -\infty < t < +\infty,$$

$$\phi(o,x) = \phi(T,x) = v(x), \ -a_o \leq x \leq a_o,$$

$$\phi(t+T,x) \quad = \quad \phi(t,x), \ (t,x)\epsilon A.$$

In addition $\phi(t,x)$ together with $\phi_t(t,x)$, $\phi_x(t,x)$, $\phi_{tx}(t,x)$, μ, $v(x)$, $v'(x)$ are uniquely defined by $u(t)$, and depends continuously on $u(t)$.

Proof of Theorem IV. We have first to prove that the function G of No. 9 verifies the hypotheses of statement (A) of No. 10. Indeed, G12 assure that G is uniformly bounded and continuous in t and z, and G34 assure that G satisfies the remaining hypotheses. Thus, by applying (A) to both intervals $[o,a]$ and $[o,-a]$ we conclude that there is some $a_o > 0$, $\beta_o > 0$, and a continuous function $v'(t)$, $-a_o \le t \le a_o$, with $v'(o) = \mu$, $|v'(t)-\mu| \le \beta_o$, such that

$$G(t,v'(t),v'(\xi),\ o \le \xi < t) \quad = \quad 0, \qquad o \le t \le a_o,$$

$$G(t,v'(t),v'(\xi),\ o \ge \xi > t) \quad = \quad 0, \qquad 0 \ge t \ge -a_o.$$

Since $|\mu| < N_2$, by reducing a_o if needed, we can satisfy $|v'(t)| \le N_2$ in $[-a_o,a_o]$. If $v(t) = \int_0^t v'(\xi)d\xi$, $-a_o \le t \le a_o$, then $v(o) = 0$, and, again by reducing a_o if needed, $v(t)$ satisfies (1.35), (1.41). Finally, if $\phi(t,x)$, $m(x)$ is the solution of problem (0.6) corresponding to $u(t)$, $v(t)$, we have $m(x) \equiv 0$, $-a_o \le x \le a_o$, and $\phi(t,x)$ satisfies (0.6). The uniqueness of the function $v'(x)$ so determined, and therefore of μ,v,ϕ,ϕ_t,ϕ_x are a consequence of the remark following statement (A) and property G5 proved in No. 9. Then $\mu,v,v',\phi,\phi_t,\phi_x$ are continuous functions of u and u', that is, the difference between any two corresponding elements are small when η is small.

12. EXAMPLE OF A DIFFERENTIAL EQUATION (0.1) WITH PERIODIC SOLUTION.

Take $T = 2\pi, n=1$, $f = \sin t+\lambda q$, $\lambda \neq 0$. Then Equation (0.1) becomes

$$u_{tx} \quad = \quad \sin t + \lambda u_x \ . \tag{2.5}$$

Let $u(t)$ be a periodic function of period T, continuous in $(-\infty,+\infty)$ together with $u'(t)$, and let $v(x)$ be a continuous function in some $[-a,a)$ with $v'(x)$, $v(o) = 0$. Then the relations $\phi(o,x) = v(x)$, $\phi(t,o) = u(t)$, $\phi_{tx} = \sin t+\lambda\phi_x$ yield

$$\phi_t(t,x) - \phi_t(t,o) \quad = \quad x \sin t + \lambda\phi(t,x) - \epsilon\phi(t,o)\,,$$

$$\phi_x(t,x) - \lambda\phi(t,x) \quad = \quad x \sin t + u'(t) - \lambda u(t),$$

By integration we obtain

$$\phi(t,x) = e^{\lambda t}\left\{\phi(o,x) + \int_0^t [u'(\tau)-\lambda u(\tau)+ x \sin \tau]e^{-\lambda\tau}\right\} d\tau$$

and, by manipulations, also

$$\phi(t,x) = u(t) + v(x)e^{\lambda x} + (1+\lambda^2)^{-1}x[e^{\lambda x} - \lambda \sin t - \cos t] .$$

Thus, $\phi(t,o) = u(t)$, $\phi(o,x) = u(o) + v(x)$,

$$\phi(2\pi,x) = u(2\pi) + v(x)e^{2\pi\lambda} + (1+\lambda^2)^{-1}x(e^{2\pi\lambda}-1)$$

and hence $\phi(2\pi,x) = \phi(o,x)$ if and only if

$$v(x) = -(1+\lambda^2)^{-1} x.$$

With this function $v(x)$, then $\phi(t,x)$ is periodic in t of period 2π, for all x, $-\infty < x < +\infty$. Also we have $v'(o) = -(1+\lambda^2)^{-1}$.

If we apply Theorem IV to Equation (2.5), we see that we have to consider first the ordinary differential problem

$$dq/dt = \sin t + \lambda q, \quad q(o) = \mu.$$

Hence

$$\begin{aligned} q(t) &= \mu e^{\lambda t} + e^{\lambda t} \int_0^t e^{-\lambda\tau} \sin \tau \, d\tau \\ &= [\mu + (1+\lambda^2)^{-1}]e^{\lambda t} + (1+\lambda^2)^{-1}(\lambda \sin t - \cos t), \end{aligned}$$

and $m(o,\mu) = 0$ if and only if $\mu = -(1+\lambda^2)^{-1}$. Then

$$a = dm/d\mu = [1-e^{-\int_0^T \lambda dt}] \left[\int_0^T e^{-\int_0^t \lambda dt}\right]^{-1} = \lambda \neq 0 .$$

The conditions of Theorem IV are then satisfied, and the existence of a solution $\phi(t,x)$ periodic in t of period 2π in a some strip A can be deduced from Theorem IV, with $\phi(t,o) = u(t)$, $\phi_x(o,o) = \phi_x(2\pi,o) = \mu = -(1+\lambda^2)^{-1}$.

13. EXAMPLE OF A DIFFERENTIAL EQUATION (0.1) WITH PERIOD SOLUTION IN A STRIP

Take $T = 2\pi, n=1, f = \sin t+\lambda q+xL(t,x,z,p,q)$, where L is an arbitrary function, and f satisfies the general hypotheses of No. 6. Then Equation (0.1) becomes

$$u_{tx} = \sin t + \lambda u_x + xL(t,x,u,u_t,u_x). \tag{2.6}$$

For x = 0, both f and f_q are the same as those of No. 12 Hence, the conditions of Theorem IV are satisfied with $\mu = -(1+\lambda^2)^{-1}$, and (2.6) has a solution $\phi(t,x)$ periodic in t of period T, with $\phi(t,o) = u(t)$, for any periodic function u(t) of class C', in the sense of No. 6 and Theorem IV, and therefore in a strip A sufficiently narrow.

14. EXAMPLE OF A DIFFERENTIAL EQUATION (0.1) WHICH HAS A PERIODIC SOLUTION IN A STRIP A BUT NOT IN THE WHOLE PLANE

Let us take $T = 2\pi$, $n=1$, $f = x+(1-x)(\sin t+\lambda q)$. For x = 0, f, f_q reduce to the ones of No. 4, and an application of Theorem IV assures the existence of a periodic solution in some strip A. On the other hand for x = 1, the equation reduces to $u_{tx} = 1$, hence $u_x(2\pi,1) \neq u_x(o,1)$, and the strip A cannot include x = 1.

An equation as (0.1) may have no solution periodic in t in any strip as for example $u_{tx} = 1$, $(t,x)\epsilon E_2$.

15. AN HYPERBOLIC EQUATION REDUCIBLE TO (0.1).

We shall consider the differential equation

$$A(t,x)u_{tt} + u_{tx} = f(t,x,u,u_t,u_x), \tag{2.7}$$

where both A and f are periodic in t of period T.

Here one set of characteristic lines is x = constant. The other set can be obtained by solving the differential equation

$$dt/dx = A(t,x), \quad -a \leq x \leq a, \tag{2.8}$$

and we shall assume $A(t,x) \geq 0$.

Let us assume $A(t,x)$ continuous in (t,x) and Lipschitzian in x in the strip $A = [-\infty < t < +\infty, -a \leq x \leq a]$. For every real τ let $t = X(x,\tau)$ be the solution of (2.7) with $X(a,\tau) = \tau$. Then, for some constant $H > 0$ we have

$$|X(x,\tau) - \tau| \leq H|x|, \quad -a \leq x \leq a,$$

$$X_x(x,\tau) = A(X(x,\tau),x), \quad |X_x(x,\tau)| \leq H.$$

If $\gamma(\tau)$ denotes the line $t = X(x,\tau)$, then $\gamma(\tau)$ cuts the line x = constant at a point (t,x) with $t = X(x,\tau)$. On the other hand, each point $(t,x) \epsilon A$ belongs to one and only one line $\gamma(\tau)$ for some $\tau = \Phi(t,x)$. We have defined, therefore, a change of coordinates $\tau = \Phi(t,x)$, $x = x$, or $t = X(x,\tau)$, $x = x$.

From (2.8) we deduce

$$X_{\tau x}(x,\tau) = (\partial/\partial x)X_\tau(x,\tau) = A_t[X(x,\tau),x]X_\tau(x,\tau),$$

and since $X(o,\tau) = \tau$, we have $X_\tau(o,\tau) = 1$. Hence,

$$X_\tau(x,\tau) = \exp\left[\int_0^x A(X(\xi,\tau),\xi)d\xi\right]$$

$$X_x(x,\tau) = A(X(x,\tau),x) .$$

On the other hand, from $t = X(x,\tau)$, $\tau = \Phi(t,x)$, we deduce $x = X(x,\Phi(t,x))$, hence

$$1 = X_\tau \Phi_\tau, \quad 0 = X_x + X_\tau \Phi_x ,$$

and finally

$$A\Phi_t + \Phi_x = 0,$$

$$\Phi_t = X_\tau^{-1} = \exp\left[-\int_0^x A(X(\xi,\tau),\xi)d\xi\right] ,$$

$$\Phi_x = -X_\tau^{-1}X_x = -A(X(x,\tau),\tau)\exp\left[-\int_0^x A(X(\xi,\tau),\xi)d\xi\right]$$

If $U(\tau,x)$ is obtained from $u(t,x)$ by means of the change of coordinates above, we have

$$U(\tau,x) = u(X(x,\tau),x),\ u(t,x) = U(\phi(t,x),x)$$

and

$$u_t = U_\tau\Phi_t, \quad u_x = U_\tau\Phi_x + U_x ,$$

$$u_{tt} = U_{t\tau}\Phi_t^2 + U_\tau\Phi_{tx}, \quad u_{tx} = U_{\tau\tau}\Phi_t\Phi_x + U_{\tau x}\Phi_t .$$

Equation (2.7) is then changed into

$$U_{\tau x} = g(\tau,x,U,U_x,U_\tau) \tag{2.9}$$

with

$$g = f(X,x,U,U_\tau X_\tau^{-1},-U_\tau X_\tau^{-1}X_x+U_x)X_\tau - U_\tau X_{\tau x}X_\tau ,$$

$$g_q = f_q(X,x,U,U_\tau X_\tau^{-1},-U_\tau X_\tau^{-1}X_x + U_x)X_\tau .$$

For $x = 0$ then $X(o,\tau) = \tau$, $X_\tau(o,\tau) = 1$, $X_x(o,\tau) = A(\tau,o)$, $X_{x\tau}(o,\tau) = A_\tau(\tau,o)$

$$g(\tau,o,U,U_\tau,q) = f(\tau,o,U,U_\tau,-AU_\tau+U_x) - A_\tau U_\tau ,$$

$$g_q(\tau,o,U,U_\tau,q) = f_q(\tau,o,U,U_\tau,-AU_\tau+U_x).$$

Relations (2.3),(2.4) become, with $U(\tau,o) = u(\tau)$,

$$dq/d\tau = f(\tau,o,u,u',-Au'+q) - A_\tau u' , \quad q(o) = \mu \tag{2.10}$$

$$m(o) = q(T)-q(o) = T^{-1}\int_0^T \{f(\tau,o,u,u',-Au'+q) - A_\tau u'\}d\tau = 0 ,$$

$$a = \left[1 - \exp\left(-\int_0^T f_q\, d\tau\right)\right]\left[\int_0^T \exp\left(-\int_0^t f_q\, d\tau\right)\right]^{-1} \neq 0 \tag{2.11}$$

As an example, if we assume $A(t,x) = xB(t,x)$, hence $A(t,o) = 0$, then Equations (2.10), (2.11) are the same as those of Theorem IV. In particular, the equation

$$xB(t,x)\, u_{tt} + u_{tx} = \sin t + \lambda u_x + xL(t,x,u,u_t,u_x)$$

has a periodic solution $\phi(t,x)$ in some strip A with $\phi(t,o) = u(t)$, $u(t)$ arbitrary of class C', and $\phi_x(o,o) = \phi_x(2\pi,o) = -(1+\lambda^2)^{-1}$.

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