THE UNIVERSITY OF MICHIGAN

COLLEGE OF LITERATURE, SCIENCE, AND THE ARTS

Technical Report No. 23

GENERALIZED SOLUTIONS AS LIMITS OF USUAL SOLUTIONS

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ORA Project 024160

submitted for:

UNITED STATES AIR FORCE
AIR FORCE OFFICE OF SCIENTIFIC RESEARCH
GRANT NO. AFOST-69-1662
ARLINGTON, VIRGINIA

administered through:

OFFICE OF RESEARCH ADMINISTRATION ANN ARBOR

May 1971

ADDENDUM IV. GENERALIZED SOLUTIONS AS LIMITS OF USUAL SOLUTIONS

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In the first part of this appendix we show that the generalized solutions as introduced in (1.8-10) can be uniformly approximated by means of usual solutions. In the second part we give an alternate definition of generalized solutions and we show that these solutions coincide with those introduced in (1.8-10).

IV.1 APPROXIMATION OF GENERALIZED SOLUTIONS BY MEANS OF USUAL SOLUTIONS

As we pointed out in (1.8) it is relevant that generalized solutions can be thought of as limits of usual solutions, and that in the same time, the value of the functional computed on any given generalized solution be the limit of the corresponding values taken by the functional on usual solutions. Since the infimum j of the functional on the class of all generalized solutions is certainly less than or equal the infimum i of the functional on the class of usual solutions, we then are able to conclude that actually $j = i$ as pointed out in (1.8).

We shall prove the possibility of approximating generalized solutions (and corresponding values of the functional) for the canonic (Mayer) problem. The analogous result for Lagrange and other problems will then follow as a corollary because of the usual transformations of one type of problems to the other (1.2).

We shall use below the usual notations for Mayer problems. Thus, A,

$U(t,x)$, M, $f(t,x,u) = (f_1,\dots,f_n)$ are given as usual. To simplify matters, we may disregard the usual set B of the $t_1x_1t_2x_2$-space E_{2n+2}, or what is the same, assume for a moment that $B = E_{m+2}$. Actually, we shall keep the first point of our trajectories fixed, say (t_1,x_1) is a fixed point, $(t_1,x_1) \in A$, and we shall also keep the terminal point t_2 fixed, so that we may simply take for B the set $B = (t_1) \times (x_1) \times (t_2) \times E_n$. Then the usual function $g(t_1,x_1,t_2,x_2)$ shall be considered as a given function of x_2 only in E_n. Then the functional has the form $I = g(x(t_2))$.

Note that for Lagrange problems with functional $I = \int_{t_1}^{t_2} f_o dt$, reduced to Mayer problems by the usual additional variable x^o, differential equation dx^o/dt, and initial value $x^o(t_1) = 0$, then we have a new state variable $\tilde{x} = (x^o,x) = (x^o,x^1,\dots,x^n)$, and the functional takes the Mayer form $I = x^o(t_2)$, exactly in the frame of the assumption above.

(IV.1.i) Let A be closed, let $U(t)$ be a closed set independent of x and let $M = [(t,x,u) | (t,x,) \in A,\ u \in U(t)]$ be closed. Let $f(t,x,u)$ be continuous on M and locally Lipschitzian with respect to x in M, and let g be continuous. Let $y(t)$, $p(t)$, $v(t)$, $t_1 \le t \le t_2$, be a generalized admissible system, whose (generalized) control function p is bounded, and whose trajectory y lies in the interior of A. Then there is a sequence $x_k(t)$, $u_k(t)$, $t_1 \le t \le t_2$, $k = 1,2,\dots$, of (usual) admissible pairs with $x_k(t_1) = y(t_1)$, such that $x_k \to y$ as $k \to \infty$ uniformly in $[t_1,t_2]$, and then consequently $g(x_k(t_2)) \to g(y(t_2))$ as $k \to \infty$.

<u>Proof</u>. It is not restrictive to assume $t_1 = 0$, $t_2 = b$. Let $2d$ be a

positive number which is less than or equal to the distance of the graph $G = [t,y(t)),\ 0 \leq t \leq b]$ from the boundary of A. Let A_o be the closed d-neighborhood of G. Then G and A_o are compact, G lies in the interior of A_o and $A_o \subset A$.

Since $v(t) = (u^{(j)}(t),\ j = 1,\ldots,\gamma)$ is bounded, there is some $N > 0$ such that $|u^{(j)}(t)| \leq N,\ 0 \leq t \leq b,\ j = 1,\ldots,\gamma$. Let M_o be the set of all (t,x,u) with $(t,x) \in A_o,\ 0 \leq t \leq b,\ u \in U(t),\ |u| \leq N$. Obviously $M_o \subset M$, and M_o is a compact subset of E_{1+n+m} (as the intersection of M closed with the compact set $[(t,x,u)|(t,x) \in A_o,\ |u| \leq N]$.

By hypothesis $f(t,x,u)$ is continuous on M_o, and locally Lipschitzian with respect to x, hence bounded, say $|f(t,x,u)| \leq N_1$, and there is a constant $L \geq 0$ such that $|f(t,x,u) - f(t,y,u)| \leq L|x-y|$ for all $(t,x,u),\ (t,y,u) \leq M_o$.

Given $\varepsilon > 0$, let $\varepsilon_1 > 0$ be any number such that $\varepsilon_1 \leq \min [d,\varepsilon]$, and take $\sigma > 0$ so that $3\sigma e^{Lb} \leq \varepsilon_1$. Let $\eta_1 = (2b)^{-1}\sigma$ and let $\delta_1 > 0$ be a number such that $|f(t,x,u) - f(t',x',u')| \leq \eta_1$ for all $(t,x,u),\ (t',x',u') \in M_1$ at a distance $\leq \delta_1$.

Let $\eta_2 = (2N)^{-1}\sigma$. Since the γ functions $u^{(j)}(t)$ are measurable in $[o,b]$, there is a closed subset K of $[o,b]$ with meas $K > b - \eta_2$, such that the γ functions $u^{(j)}(t),\ j = 1,\ldots,\gamma$, are continuous on K. Then, K is compact, and the γ functions $u^{(j)}(t),\ j = 1,\ldots,\gamma$, are uniformly continuous on K. Then there is some $\delta_2 > 0$ such that $|u^{(j)}(t) - u^{(j)}(t')| \leq \delta_1$ for all $j = 1,\ldots,\gamma$, and all $t,\ t' \in K$ with $|t-t'| \leq \delta_2$. Also, we can take $\delta_2 \leq \delta_1$ so small that $t,t' \in [o,b],\ |t-t'| \leq \delta_2$, implies $|y(t) - y(t')| \leq \delta_1$.

Let us divide $I = [o,b]$ into k equal consecutive intervals, say I_{ks},

$s = 1,\dots,k$, each of length b/k. For each I_{ks}, let $[\Sigma_{ksj}, j = 1,\dots,\gamma]$ be any subdivision of I_{ks} into γ measurable disjoint subsets $\Sigma_{ksj} \subset I_{ks}$ (for instance, subintervals) such that

$$\text{meas } \Sigma_{ksj} = \int_{I_{ks}} p_j(t)\, dt, \quad j = 1,\dots,\gamma. \tag{IV.1.2}$$

Then

$$\Sigma_j \text{ meas } \Sigma_{ksj} = \Sigma_j \int_{I_{ks}} p_j(t)\, dt = \int_{I_{ks}} dt = \text{meas } I_{ks}.$$

We now take

$$u(t) = u^{(j)}(t) \text{ for all } t \in \Sigma_{ksj}, \quad j = 1,\dots,\gamma, \quad s = 1,\dots,k. \tag{IV.1.3}$$

Then $u(t)$, $t_1 \leq t \leq t_2$, is a measurable function in I with values $u(t) \in U(t)$, $t_1 \leq t \leq t_2$, and $|u(t)| \leq N$. Let us consider the differential system

$$dx/dt = f(t, x(t), u(t)), \; 0 \leq t \leq b, \tag{IV.1.4}$$

with initial value $x(0) = y(0) = x_o$. Since (o,x_o) is an interior point of A_o, the solution $x(t)$ of (III.1.4) exists in a right neighborhood of $t = 0$, say $[0,\bar{t}]$, and $(t,x(t)) \in A_o$.

Let k_2 be the smallest integer such that $b/k_2 \leq \delta_2$. Hence, for $k \geq k_2$, and for any two points $t, t' \in I_{kso} = I_{ks} \cap K$, we have $|t-t'| \leq b/k \leq \delta_2$, and $|u^{(j)}(t) - u^{(j)}(t')| \leq \delta_1$, $j = 1,\dots,\nu$. Let $I_{kso} = I_{ks} \cap K$, $\Sigma_{ksjo} = \Sigma_{ksj} \cap K$, $I'_{ks} = I_{ks} - K$, $\Sigma'_{ksj} = \Sigma_{ksj} - K$, $K' = I - K$. For any triple k,s,j with meas $\Sigma_{ksjo} > 0$ we select one point $\tau = \tau_{ksj} \in \Sigma_{ksjo}$. If meas $\Sigma_{ksjo} = 0$ we do not select any point, and actually we disregard the corresponding term in the lines below. Obviously, meas $K' \leq \eta_2$.

For any interval $[0,t]$, $0 \leq t \leq \bar{t}$, let K_{to}, K'_t be the sets $K_{to} = K \cap [0,t]$, $K'_t = [0,t] - K$. Note that $\int_{I_{ks}} p_j(\zeta)d\zeta$ - meas $\Sigma_{ksj} = 0$, and hence

$$\Sigma_s \Sigma_j \left| \int_{I_{kso}} p_j(\zeta)d\zeta - \text{meas } \Sigma_{ksjo} \right|$$

$$= \Sigma_s \Sigma_j \left| \left(\int_{I_{ks}} p_j(\zeta)d\zeta - \text{meas } \Sigma_{ksj} \right) - \int_{I'_{ks}} p_j(\zeta)d\zeta + \text{meas } \Sigma'_{ksj} \right|$$

$$\leq 0 + \Sigma_s \int_{I'_{ks}} \Sigma_j p_j(\zeta)d\zeta + \Sigma_s \Sigma_j \text{ meas } \Sigma_{ksj} \leq 2 \text{ meas } K' \leq 2\eta_2.$$

Now $x(t)$ and $y(t)$ are AC with $x(0) = y(0) = x_o$, and

$$dy/dt = h(t,y(t),p(t),v(t)) = \Sigma_j p_j(t) \, f(t,y(t),u^{(j)}(t)),$$
$$0 \leq t \leq b,$$
$$dx/dt = f(t,x(t),u(t)), \quad 0 \leq t \leq \bar{t}, \quad x(0) = y(0) = x_o.$$

For any t we have $u(t) \in U(t)$, hence $f(t,y(t),u(t))$ is defined, and

$$y(t) - x(t) = \int_0^t [h(\zeta,y(\zeta),p(\zeta),v(\zeta)) - f(\zeta,x(\zeta),u(\zeta))]d\zeta$$

$$= \int_0^t [f(\zeta,y(\zeta),u(\zeta)) - f(\zeta,x(\zeta),u(\zeta))]d\zeta$$

$$+ \left(\int_{K_t} + \int_{K'_t} \right) [\Sigma_j p_j(\zeta) \, f(\zeta,y(\zeta),u^{(j)}(\zeta)) - f(\zeta,y(\zeta),u(\zeta))]d\zeta$$

$$= \mu_1 + \mu_2 + \mu_3. \qquad \text{(IV.1.6)}$$

Since $|f| \leq N$ in M_o, we have

$$|\mu_3| = |\int_{K_t'}[\Sigma_j p_j(\zeta) f(\zeta,y(\zeta),u^{(j)}(\zeta)) - f(\zeta,y(\zeta),u(\zeta))]d\zeta|$$

$$\leq \int_{K_t'} [\Sigma_j p_j(\zeta)|f(\zeta,y(\zeta),u^{(j)}(\zeta))| + |f(\zeta,y(\zeta),u(\zeta))]d|$$

$$\leq 2N \text{ meas } K_t' \leq 2N\eta_2 = \sigma.$$

Also, we have

$$|\mu_2| = |\int_{K_t} \Sigma_j p_j(\zeta) f(\zeta,y(\zeta),u^{(j)}(\eta)) - f(\zeta,y(\zeta),u(\zeta))]d\zeta|$$

$$= |\Sigma_s \Sigma_j \{\int_{I_{kso}} p_j(\zeta) f(\zeta,y(\zeta),u^{(j)}(\zeta))d\zeta - \int_{\Sigma_{ksj}} f(\zeta,y(\zeta),u^{(j)}(\zeta))d\zeta\}|$$

$$= |\Sigma_s \Sigma_j \{f(\tau,y(\tau),u^{(j)}(\tau)) [\int_{I_{kso}} p_j (\zeta)d\zeta - \text{meas } \Sigma_{ksjo}]|$$

$$+ \Sigma_s \Sigma_j \int_{I_{kso}} p_j(\zeta)[f(\zeta,y(\zeta),u^{(j)}(\zeta)) - f(\tau,y(\tau),u^{(j)}(\tau))]d\zeta$$

$$+ \Sigma_s \Sigma_j \int_{\Sigma_{ksjo}} [f(\zeta,y(\zeta),u^{(j)}(\zeta)) - f(\tau,y(\tau),u^{(j)}(\tau))]d\zeta,$$

where summations and integrations are extended only over those terms and intervals concerning [0,t]. In each bracket of the second and third sums of the last member we have $|\zeta-\tau| \leq b/k \leq \delta_2 \leq \delta_1$, hence $|y(\zeta) - y(\tau)| \leq \delta_1$, $|u^{(j)}(\zeta) - u^{(j)}(\tau)| \leq \delta_1$, and each bracket has absolute value $\leq \eta_1$. By using this remark and (III.1.5) we obtain

$$|\mu_2| \leq N \Sigma_s \Sigma_j |\int_{I_{kso}} p_j(\zeta)d\zeta - \text{meas } \Sigma_{ksjo}|$$

$$+ \eta_1 \Sigma_s \int_{I_{kso}} \Sigma_j p_j(\zeta)d\zeta + \eta_1 \Sigma_s \Sigma_j \int_{\Sigma_{ksjo}} d\zeta$$

$$\leq 2 N\eta_2 + 2\eta_1 b = \sigma + \sigma = 2\sigma.$$

Finally,

$$|\mu_1| \le \int_0^t L|y(\zeta) - x(\zeta)|d\zeta.$$

Thus, (III.1.6) yields

$$|y(t) - x(t)| \le L \int_0^t |y(\zeta) - x(\zeta)|d\zeta + 3\sigma,$$

and by Gronwall's lemma,

$$|y(t) - x(t)| \le 3\sigma\, e^{Lt} \le 3\sigma\, e^{Lb} \le \varepsilon_1 \le \min\,[d,\varepsilon],$$

and $(t,x(t)) \in A_o$ for all $0 \le t \le \bar{t}$. Thus, $x(t)$ is defined in all of $[0,b]$, or $\bar{t} = b$, the entire graph $[(t,x(t)),\ 0 \le t \le b]$ of x lies in A_o, and

$$x(t) = x_o + \int_0^t f(\zeta,x(\zeta),u(\zeta))d\zeta, \quad u(t) \in U(t),$$

$$|x(t) - y(t)| \le \min[d,\varepsilon], \quad 0 \le t \le b.$$

In particular $|x(t_2) - y(t_2)| \le \min\,[d,\varepsilon]$, and $(t_1,x(t_1),t_2,x(t_2)) \in B$. Since $\varepsilon > 0$ is an arbitrary number, statement (III.1.i) is proved by taking $\varepsilon = k^{-1}$, $k = 1,2,\ldots.$

Statement (IV.1.i) has been generalized so as to include control spaces $U(t,x)$ depending on both t and x, provided f and U vary not too mildly as t and x vary. Namely, let us consider the two following hypotheses:

(p) Given $N > 0$ there is another constant $L \ge 0$ such that $|f(t,x,u) - f(t,y,v)| \le L(|x-y| + |u-v|)$ for all $(t,x,u) \in M$, $(t,y,v) \in M$ with $-N \le t \le N$, $|x|$, $|y| \le N$, $|u|$, $|v| \le N$. If U depends only on t, or $U = U(t)$, then we require only $|f(t,x,u) - f(t,y,u)| \le L|x-y|$ for all (t,x,u), $(t,y,u) \in M$ with $-N \le t \le N$, $|x|$, $|y| \le N$, $|u| \le N$.

(q) Given $N > 0$ there is another constant $H \geq 0$ such that for any two points $(t,x) \in A$, $(t,y) \in A$, $-N \leq t \leq N$, $|x|$, $|y| \leq N$, and any $u \in U(t,x)$ with $|u| \leq N$, there is at least another point $v \in U(t,y)$ with $|u-v| \leq H|x-y|$.

(IV.1.ii) Let A be closed, $U(t,x)$ which may depend on both t and x, and let $M = [(t,x,u)|(t,x) \in A, u \in U(t,x)]$ be closed. Let $(f(t,x,u) = (f_1,\dots,f_n)$ be continuous on M, and let g be continuous. Let us assume that conditions (p) and (q) hold. Let $y(t)$, $p(t)$, $v(t)$, $t_1 \leq t \leq t_2$, be a generalized admissible system, whose control function is bounded, and whose trajectory y lies in the interior of A. Then there is a sequence $x_k(t)$, $u_k(t)$, $t_1 \leq t \leq t_2$, $k = 1,2,\dots$, of usual admissible pairs with $x_k(t_1) = y(t_1)$ and such that $x_k \to y$ as $k \to \infty$ uniformly in $[t_1,t_2]$.

Remark 1. We have mentioned above that the so called Gronwall's lemma: (IV.1.iii). If $u(t) \geq 0$, $v(t) \geq 0$, $0 \leq t < +\infty$, are given functions, $u(t)$ continuous, $v(t)$ L-integrable in every finite interval, if for some nonnegative constant C we have

$$u(t) \leq C + \int_0^t u(\alpha)\, v(\alpha) d\alpha, \quad t \geq 0, \qquad \text{(IV.1.7)}$$

then we have also

$$u(t) \leq C \exp \int_0^t v(\alpha) d\alpha, \quad t \geq 0. \qquad \text{(IV.1.8)}$$

Proof. If $C > 0$, by algebraic manipulation of (IV.1.7), we have

$$uv/(C + \int_0^t u\, v\, d\alpha) \leq v,$$

and, by integration,

$$\log\,(C + \int_0^t u\ v\ d\ \alpha) - \log C \ \leq\ \int_0^t v(\alpha)d\alpha$$

or

$$u\ \leq\ C + \int_0^t u\ v\ d\ \alpha\ \leq\ C\ \exp \int_0^t v(\alpha)d\alpha.$$

If $C = 0$, then (IV.1.7) holds for every constant $C_1 > 0$, and then we have $0 \leq u(t) \leq C_1\ \exp \int_0^t v(\alpha)d\alpha$, $t \geq 0$. This relation, as $C_1 \to 0$ implies $u(t) \equiv 0$. Thereby (IV.1.ii) is proved.

Remark 2. In statement (IV.1.i) the hypothesis that the graph G of y be in the interior of A seems to be requested by the proof, since we obviously need a neighborhood of G on which to define the sequence of usual trajectories approaching y. Actually, it is easy to prove by an example that statement (III.1.i) may not be true without the hypothesis that G is in the interior of A. Take $n = 3$, $m = 1$, A_o made up of the three segments joining the points $0 = (0,0)$, $1 = (2,0)$, $2 = (1,1)$ in the xy-plane and then take $A = E_1 \times A_o \times E_1$. Let U be the fixed set made up of the two points $u = 1$ and $u = -1$, let xyz be the state variables, and take differential system $x' = 1$, $y' = u$, $z' = y$, fixed initial point P_1: $t_1 = 0$, $x_1 = 0$, $y_1 = 0$, $z_1 = 0$, fixed terminal time $t_2 = 2$, $B = (P_1) \times (2) \times E_3$, functional $I = z(t_2)$. Then necessarily $x(t) = t$, $0 \leq t \leq 2$, and $x(t_2) = 2$. This problem has only one admissible usual strategy u, with $u(t) = 1$ for $0 \leq t \leq 1$, $u(t) = -1$ for $1 \leq t \leq 2$, the corresponding trajectory in the xy-plane is the polygonal line 021, and $I = z(t_2) = 1$. The problem has only one admissible generalized (not usual) strategy v, with $u^{(1)}(t) = 1$, $u^{(2)}(t) = -1$, $p_1(t) = p_2(t) = 1/2$, $0 \leq t \leq 2$, the xy-trajectory is the segment 01, and $I = z(t_2) = 0$. Clearly the only generalized (not usual) solution cannot be approached by means of usual solutions. Here A has no interior points.

Remark 3. Unbounded control functions $v(t) = (u^{(1)},\dots,u^{(\gamma)})$, $t_1 \leq t \leq t_2$, are allowed in statements (IV.1.i) and (IV.1.ii) under additional dypotheses, which are analogous to those used for a similar purpose in the proof of Pontryagin's necessary conditions with unbounded controls [see vol.1, App. A]. Namely, statements (IV.1.i) and (IV.1.ii) still hold for v(t) unbounded, under the additional assumption

(δ) There is some number $\delta > 0$ and an L-integrable function $S(t)$, $t_1 \leq t \leq t_2$, such that $t \in [t_1,t_2]$, $|x' - x(t)|$, $|x'' - x(t)| \leq \delta$, $j = 1,\dots,\gamma$, implies

$$|f(t,x',u^{(j)}(t)) - f(t,x'',u^{(j)}(y)| \quad \leq \quad |x' - x''|S(t).$$

As mentioned, the statements above (IV.1.i) and (IV.1.ii) for Mayer problems can be immediately worded for Lagrange problems. For the sake of simplicity we limit ourselves to statement (IV.1.i). Here A, U(t), M are as above, $f_o(t,x,u)$, $f(t,x,u) = (f_1,\dots,f_n)$ are defined on M. Admissible pairs $x(t)$, $u(t)$, $t_1 \leq t \leq t_2$, are defined in (2.9, first paragraph), and the functional is now

$$I[x,u] \quad = \quad \int_{t_1}^{t_2} f_o(t,x(t),u(t))dt.$$

(IV.1.iii) Let A be closed, let U(t) be closed sets independent of x, and let M be closed. Let $f_o(t,x,u)$, $f(t,x,u) = (f_1,\dots,f_n)$ be continuous on M and locally Lipschitzian with respect to x on M. Let y(t), p(t), v(t), $t_1 \leq t \leq t_2$, be a generalized admissible system, whose (generalized) control function v is bounded, and whose trajectory y lies in the interior of A. Then there is a sequence of (usual) admissible pairs $x_k(t)$, $u_k(t)$, $t_1 \leq t \leq t_2$, $k = 1,2,\dots$, such that $x_k \to y$ as $k \to \infty$ uniformly in $[t_1,t_2]$ and such that $I[x_k,u_k] \to I[x,p,v]$.

By introducing the additional variable x^o, the differential equation $dx^o/dt = f_o$, the initial condition $x^o(t_1) = 0$, and new space variable $\tilde{x} = (x^o,x) = (x^o,x^1,\dots,x^n)$ and function $\bar{f}(t,x,u) = (f_o,f) = (f_o,f_1,\dots,f_n)$, the problem is reduced to a Mayer problem to which (IV.1.i) applies.

IV.2 AN ALTERNATE DEFINITION OF GENERALIZED SOLUTIONS

(a) Weak Solutions

We shall denote by U any <u>fixed</u> control space, that is, any arbitrary fixed set U of elements u, and we shall assume that U has a topology, so that U is a topological space. We shall then denote by $\{\phi\}$ the set of all real-valued continuous scalar functions $\phi(u)$, $u \in U$, which are continuous on U. We shall denote by $M(\phi)$, $\phi \in \{\phi\}$, any real valued functional such that (m1) M is linear, that is, ϕ_1, $\phi_2 \in \{\phi\}$, α, β real, implies $M(\alpha\phi_1 + \beta\phi_2) = \alpha M(\phi_1) + \beta M(\phi_2)$; (m2) M is nonnegative, that is $\phi \in \{\phi\}$, $\phi \geq 0$, implies $M(\phi) \geq 0$; (m3) $M(1) = 1$, that is, if ϕ denotes the function $\phi(u) = 1$ in U, then $M(\phi) = \phi = 1$. The following further properties are consequences of these assumptions: (m4) $\phi_1, \phi_2 \in \{\phi\}$, $\phi_1 \leq \phi_2$ implies $M(\phi_1) \leq M(\phi_2)$; (m5) $|M(\varphi)| \leq M(|\phi|)$; (m6) $|M(\phi)| \leq \text{Sup } |\phi|$ where Sup is taken in U, (m7) $\phi_k, \phi \in \{\phi\}$, $k = 1,\ldots,$ $\phi_k \to \phi$ as $k \to \infty$ uniformly in U implies $M(\phi_k) \to M(\phi)$. We shall denote by $\mathcal{F}$ the class of all real valued functionals $M(\phi)$ as above.

We are now in a position to define the concept of weak solution. As in all available expositions of the theory we assume that U is a fixed set.

Let A be any closed subset of the tx-space $E_1 \times E_n$, let U be an arbitrary subset of a Banach space E, and let $M = A \times U$. Let $f(t,x,u) = (f_1,\ldots,f_n)$ be a vector function defined in M.

We shall consider systems $\{x(t),\ t_1 \leq t \leq t_2,\ D,\ M(t,\phi)\}$ where (a) $x(t) = (x^1,\ldots,x^n)$ is an AC vector function on $[t_1,t_2]$; (c) D is a measurable subset of $[t_1,t_2]$ with meas $D = t_2 - t_1$; (d) for every $t \in D$, $M(t,\phi)$ is a real-valued linear functional defined on $\{\phi\}$, that is, on the class of all continuous real-valued function $\phi(u)$, $u \in U$, and for every $t \in D$, $M(t,\phi)$ satisfies the properties (m) above; (e) $x'(t)$ exists and is finite at least everywhere on D (hence, a.e. in $[t_1,t_2]$), and

$$dx^i/dt = M(t;f_i(t,x(t),u)), \quad i = 1,\ldots,n, \ t \in D, \qquad \text{(IV.2.1)}$$

or briefly

$$dx/dt = M(t,f(t,x(t),u)), \quad t \in D. \qquad \text{(IV.2.2)}$$

Any such system $\{x(t),\ D,\ M(t,\phi)\}$ is said to be a weak solution, and $x(t)$ a weak trajectory. It is not necessary to indicate the interval of definition of $x(t)$ since $[t_1,t_2] = \text{cl } D$.

It is important to show that every usual admissible pair $x(t)$, $u(t)$, $t_1 \leq t \leq t_2$, is a weak solution. Indeed, if we take $M(t,\phi) = \phi(u(t))$, then (IV.2.2) reduces to $dx^i/dt = f_i(t,x(t),u(t))$. Analogously, any generalized system $(x(t),p(t),v(t))$ with $p(t) = (p_1,\ldots,p_\nu)$, $v(t) = (u^{(1)},\ldots,u^{(\nu)})$, $\nu \geq n+1$, is a weak solution. Indeed, if we take $M(t,\phi) = \sum_j p_j(t)\phi(u^{(j)}(t))$, then (IV.2.2) reduces to $dx^i/dt = \sum_j p_j(t) f_i(t,x(t),u^{(j)}(t))$. Thus a weak solution appears to be an extension of both usual and generalized solution. Actually, under mild hypotheses, every weak solution is a generalized solution, and is the weak limit of a sequence of usual solutions.

(b) Weak Solutions as Quasi-solutions and Generalized Solutions

We shall denote as in Chapter 1 by R and S the sets $R(t,x) = \text{co } Q(t,x)$, $S(t,x) = \text{cl co } Q(t,x)$.

Equations (IV.2.1), or (IV.2.2), can be thought of as defining a director field, namely

$$dx/dt \in Q^*(t,x),$$

where Q^* is the subset of E_n defined by

$$Q^*(t,x) = \{M(t,f(t,x,u)) \mid M \in \mathcal{M}\} \subset E_n,$$

that is, the subset of all vectors

$$M(t,f(t,x,u)) = \{M(t,f_1),\ M(t,f_2),\ldots,M(t,f_n)\},$$

when M describes the family $\mathfrak{F}$ of all possible real-valued linear functionals M satisfying properties (m).

(IV.2.i) For any space U, we have co $Q(t,x) \subset Q^*(t,x)$, that is $R(t,x) \subset Q^*(t,x)$.

<u>Proof</u>. For every $\bar{u} \in U$, let $M(\phi) = \phi(\bar{u})$, in other words, let M be the Dirac operator $\delta_{\bar{u}}$ at $\bar{u}$, which gives for every $\phi \in \{\phi\}$ the values of ϕ at $\bar{u}$. Then $M(f(t,x,u)) = f(t,x,\bar{u}) \in Q(t,x)$, and as $\bar{u}$ describes U, then $f(t,x,\bar{u})$ describes $Q(t,x)$. Thus $Q \subset Q^*$. Now $\mathfrak{F}$ has a linear structure, namely, if $M_1, M_2 \in \mathfrak{F}$ and $0 \leq \alpha \leq 1$, then $M = \alpha M_1 + (1-\alpha)M_2 \in \mathfrak{F}$. Thus $M(\phi) = \alpha M_1(\phi) + (1-\alpha)M_2(\phi)$, and we conclude that Q^* is convex, and hence Q^* contains the convex hull of Q.

(IV.2.ii) If U is any metric space, then

$$\text{co } Q(t,x) \subset Q^*(t,x) \subset \text{cl co } Q(t,x)$$

<u>Proof</u>. For each point $u \in U$ let $G = G(u)$ be a neighborhood of u where $f(t,x,u)$ as a function of u above has an oscillation $< \varepsilon$, say $|f(u') - f(u'')| \leq \varepsilon$, for all $u', u' \in G = G(u)$. Then the collection $\{G\}$ of all these neighborhoods is a covering of U, that is, the union of all sets G is U, say $\cup G = U$. Then we know (see Remark below for references) that U possesses a <u>partition of unity</u>, that is, U possesses the following important property: Given $\{G\}$, there is another covering $\{G_i', i \in I\}$, I an index set, and certain scalar functions $\alpha_i(u)$, $u \in U$, $i \in I$, with $\alpha_i(u) \in \{\phi\}$, such that (a) $\cup G_i = U$ as

above; (b) each open set G_i is completely contained in at least one set $G \in \{G\}$; (c) $\{G_i, i \in I\}$ is locally finite, that is, for every $u \in U$ there is a neighborhood V of u in U such that $V \cap G \neq \emptyset$ for at most finitely many i; (e) $0 \leq \alpha_i(u) \leq 1$ for all $u \in U$ and $i \in I$; (f) $\sum_{i\in I} \alpha_i(u) = 1$ for all $u \in U$. Note that for each given $u \in U$ the sum in (f) is actually a finite sum: indeed, if V is the relative neighborhood as in (c), then $V \cap G_i \neq \emptyset$ for at most finitely many i, and thus $\alpha_i(u) = 0$ for all remaining $i \in I$, and the sum $\sum_{i\in I} \alpha_i(u)$ possesses at most finitely many terms different from zero (at any given u). Let u_i denote any point $u_i \in G_i$, $i \in I$, and let g denote the function $g(t,x,u) = \sum_{i\in I} f(t,x,u_i)\, \alpha_1(u)$. Then, if $u \in U$ and $\alpha_i(u) \neq 0$ for some $i \in I$, then $u \in G_i$, $u_i \in G_i \subset G$ for some $G \in \{G\}$, and $f(t,x,u_i) = f(t,x,u) + \theta_i$, $|\theta_i| < \varepsilon$. Also

$$\begin{aligned}|g(t,x,u) - f(t,x,u)| &= |\textstyle\sum_i f(t,x,u_i)\, \alpha_i(u) - f(t,x,u)| \\ &= |\textstyle\sum_i \theta_i \alpha_i(u)| \leq \varepsilon \textstyle\sum_i \alpha_i(u) = \varepsilon,\end{aligned}$$

or $|g-f| \leq \varepsilon$ for all $u \in U$. Let $p_i = M(\alpha_i(u))$, $i \in I$. Then, by properties (m) we have $0 \leq p_i \leq 1$. Also we have

$$1 = M(1) = M(\textstyle\sum_i \alpha_i(u)) = \textstyle\sum_i M(\alpha_i(u)) = \textstyle\sum_i p_i .$$

On the other hand

$$|M(g(t,x,u)) - M(f(t,x,u))| \leq \varepsilon,$$

$$\begin{aligned}M(g(t,x,u)) &= M(\textstyle\sum_i f(t,x,u)\alpha_i(u)) = \textstyle\sum_i f(t,x,u_i) M(\alpha_i(u)) \\ &= \textstyle\sum_i p_i f(t,x,u_i) \in \mathrm{co}\, Q(t,x).\end{aligned}$$

Thus, every point $M(f(t,x,u))$, that is, every point of $Q^*(t,x)$, is a point of accumulation of points of co $Q(t,x)$. Thus co $Q(t,x) \subset Q^*(t,x) \subset$ cl co $Q(t,x)$.

(IV.2.iii) If U is any metric space, then whenever $R(t,x)$ is a closed subset of E_n, then $Q^*(t,x) = R(t,x) = S(t,x)$. In particular, this is certainly the case if U is compact.

Indeed R closed implies $R = S$ and hence $Q^* = R = S$ because of (ii). If U is compact, then certainly $Q = f(t,x,U)$ is compact, and then R is compact and hence closed, and the statement applies.

We conclude now with the following theorem:

(IV.2.iv) If U is any metric space, and $\{x(t),D,M\}$ is a weak solution, then $x(t)$, $t_1 \leq t \leq t_2$, is a quasi-trajectory, that is, an AC solution of the orientor field

$$dx/dt \in S(t,x(t)), \quad S(t,x) = \text{cl co } Q(t,x) = \text{cl } R(t,x).$$

If, in addition, $R(t,x(t))$ is known to be closed for almost all $t \in [t_1,t_2]$, in particular, if U is compact since then $R = S$ is also compact, then $x(t)$, $t_1 \leq t \leq t_2$, is a generalized trajectory, that is, there is a probability distribution $p(t) = (p_1,\dots,p_\nu)$, $\nu \geq n+1$, and a vector function $v(t) = (u^{(1)},\dots,u^{(\nu)})$, with $p_j(t)$, $u^{(j)}(t)$ measurable in $[t_1,t_2]$, $u^{(j)}(t) \in U$ a.e. in $[t_1,t_2]$, $0 \leq p_j(t) \leq 1$, $\sum_{j=1}^{\nu} p_j(t) = 1$ for all $t \in [t_1,t_2]$, $dx/dt = \sum_{j=1}^{\nu} p_j(t)\, f(t,x(t),u^{(j)}(t))$ a.e. in $[t_1,t_2]$.

Proof. The first part is obvious since $dx/dt \in Q^*(t,x(t)) \subset S(t,x(t))$ because of (ii). If R $(t,x(t))$ is closed for almost all t, then R = S, and hence

$$dx/dt \in R(t,x(t)) \quad \text{for almost all } t \in [t_1,t_2],$$

where $R(t,x) = \text{co } Q(t,x) \subset \text{co } f(t,x,U)$, and here U is a fixed set which, therefore certainly satisfies the conditions of Theorem (1.6.i) of Chapter 1. If $h(t;x,p) = \sum p_j f(t,x,u^{(i)})$, $p \in \Gamma$, $v \in U^\nu$, as in Chapter V, then we have also $R(t,x) = h(t,x,\Gamma \times U^\nu)$ and again $\Gamma \times U^\nu$ is a fixed set. By the Theorem above, we conclude the proof of our statement.

Remark. Both statements (ii) and (iii) above certainly hold for all topological spaces U for which a partition of unity property holds, thus, in particular, for all paracompact spaces U, more particularly, for all metric spaces U [N. Bourbaki, Topologie Générale, Ch. 9, Sec. 4, no. 3,4,5]. A collection {G} of subsets of a topological space U is said to be an open covering of U if the union of all $G \in \{G\}$ is U, say $\cup G = U$. Another collection {G'} of open subsets of U is said to be a subcovering of {G} if $\cup G' = U$, and if for every $G' \in \{G'\}$ there is at least one $G \in \{G\}$ with $G' \subset G$. A covering {G} of U is said to be locally finite provided, given $u \in U$, there is some neighborhood V of u in U such that $V \cap G \neq 0$ for at most finitely many $G \in \{G\}$. A topological space U is said to be Hausdorff if for any two distinct points $u_1, u_2 \in U$ there are open sets G_1, G_2 in U with $u_1 \in G_1$, $u_2 \in G_2$, $G_1 \cap G_2 = 0$. Also, U is called normal, if for any two closed subsets H_1, H_2 of U with

$H_1 \cap H_2 = 0$, there are open sets G_1, G_2 in U, with $H_1 \subset G_1$, $H_2 \subset G_2$, $G_1 \cap G_2 = 0$. Finally, a topological space U is said to be paracompact provided U is Hausdorff and every open covering of U possesses a locally finite subcovering. Then, every metric space is paracompact [N. Bourbaki, Ibid., Ch. 9, Sec. 4, no. 5, Th. 4]. Every paracompact space is normal [Ibid., Ch. 9, Sec. 4, no. 4]. Every paracompact space possesses the partition of unity property [Ibid., Ch. 9, Sec. 4, no. 4, Cor. To Prop. 4].

(c) Limit of Weak Solutions

Any two weak solutions, or systems $\{x(t), D, M\}$, $\{x(t), D', M'\}$ are said to be equivalent (or identical) provided they have the same trajectory $x(t)$, $t_1 \leq t \leq t_2$, $[t_1, t_2] = \text{cl } D = \text{cl } D'$, and provided, given any vector function $g(t,x,u) = (g_1, \ldots, g_n)$ continuous on A x U, we have

$$x(t) = x(t_1) + \int_{t_1}^{t} M(\tau, g(\tau), x(\tau), u)d\tau = \int_{t_1}^{t} M'(\tau, g(\tau), x(\tau), u))d\tau$$

and $(t, z(t)) \in A$ for all $t_1 \leq t \leq t_2$. In other words, two systems $\{x(t),D,M\}$, $\{x(t),D',M'\}$ are equivalent, or identical, provided the vectors obtainable by integration on every function g along the common trajectory $x(t)$ are the same. This definition establishes an equivalent relation, and we shall denote by C* any equivalent class of such systems $\{x(t),D,M\}$. For the sake of simplicity, we still call C* a weak solution, and any system $\{x(t),D,M\}$ of the equivalent class, a representation of C*.

We shall now introduce in the family {C*} of weak solutions C* a concept of limit. We shall define this concept by using arbitrary representations

$\{x(t),D,M\}$ for every C^*. As we shall see, it is immaterial which of the representations are used, and thus, for the sake of simplicity, we shall word the concept of limit as follows.

We say that a sequence $[C_k]$ of weak solutions, say

$$C_k = [x_k(t),\ t_{1k} \leq t \leq t_{2k},\ D_k,\ M_k(t,\emptyset)],\quad k = 1,2,\ldots,$$

converges to a weak solution

$$C = [x(t),\ t_1 \leq t \leq t_2,\ D,\ M(t,\emptyset)]$$

provided

(a) x_k converges in the metric ρ toward x, or $\rho(x_k, x) \to 0$ as $k \to \infty$, and hence $t_{1k} \to t_1$, $t_{2k} \to t_2$, $x_k(t_{1k}) \to x(t_1)$, $x(t_{2k}) \to x(t_2)$, as $k \to \infty$.

(b) For every vector function $g(t,x,u) = (g_1,\ldots,g_n)$ continuous on $A \times U$ as f, and initial values, say $y_k(t_{1k}) = (y^1_{1k},\ldots,y^n_{1k}) = y_{11}$, such that $y_k(t_{1k}) \to x(t_1)$ as $k \to \infty$, and such that, if

$$y_k(t) = y_{k1} + \int_{t_{1k}}^{t} M_k(t,g(t,x_k(t),u))dt,\quad t_{1k} \leq t \leq t_{2k},\ k = 1,2,\ldots,$$

$$y(t) = x(t_1) + \int_{t_1}^{t} M(t,g(t,x(t),u))dt,\quad t_1 \leq t \leq t_2,$$

with $(t,y_k(t)) \in A$ for $t \in [t_{1k},t_{2k}]$, $(t,y(t)) \in A$ for $t \in [t_1,t_2]$, we also have $\rho(y_k,y) \to 0$ as $k \to \infty$.

With this definition, the family $\{C^*\}$ of weak solutions is an L-space (see Remark below). For $A = E_1 \times E_n$ it has been proved that (C*) is metrizable (E. J. McShane [68 m]).

(d) A Compactness Theorem for Weak Solutions

We shall now prove that, under conditions on A and U, every sequence of usual solutions possesses a subsequence which "converges" toward a weak solution. This statement is included in the following theorem, where we show that every sequence of weak solutions possesses a subsequence which converges in the sense of the previous section, toward a weak solution. For generalized solutions a compactness theorem was proved straightforwardly in (2.7).

(IV.2.v) If A is a compact subset of the tx-space $E_1 \times E_n$, if U is a fixed compact topological space, and $f(t,x,u) = (f_1,\dots,f_n)$ is a continuous vector function on $M = A \times U$, then the set of all weak solutions is compact in the L-topology. We shall precisely prove that, given any sequence of weak solutions $\{x_k(t), t_{1k} \le t \le t_{2k}, D_k, M_k(t,\emptyset)\}$, $k = 1,2,\dots$, there is always a subsequence which converges in the L-topology toward a weak solution $\{x(t), t_1 \le t \le t_2, D, M(t,\emptyset)\}$ and correspondingly the weak trajectories x_k converge in the ρ-metric toward the weak trajectory x.

<u>Proof</u>. The set M is compact and hence there exists a constant M_o such that $(t,x,u) \in M$ implies $-M_o \le t \le M_o$, $|x| \le M_o$, $|u| \le M_o$, and also $|f(t,x,u)| \le M_o$. Then, property (m6) implies $M(t;f(t,x(t),u)) \le M_o$, hence, for every weak trajectory $x(t)$, $t_1 \le t \le t_2$, we have $|dx/dt| \le M_o$ a.e. in $[t_1,t_2]$, and $x(t)$ is Lipschitzian with constant M_o. Also $(t,x(t)) \in A$ for

all $t \in [t_1,t_2]$ where A is compact. Thus, the weak trajectories $x(t)$ are equibounded, equicontinuous, equilipschitzian.

Given the sequence above $\{x_k(t),\ t_{1k} \leq t \leq t_{2k},\ D_k,\ M_k(t,\phi)\}$, then by Ascoli's theorem there is a subsequence of weak trajectories, say still $[x_k]$, which converges in the metric ρ toward a continuous vector function $x_o(t)$, $t_1 \leq t \leq t_2$, and $x_o(t)$ is Lipschitzian with constant M_o, hence AC in $[t_1,t_2]$. Since A is closed, $(t_k,\ x_k(t)) \in A$ implies $(t,x(t)) \in A$ for all $t \in [t_1,t_2]$.

We have $|M_k(t,f(t,x_k(t),u))| \leq M_o$ for all $t \in D_k$, with $D_k \subset [t_{1k},t_{2k}]$, meas $D_k = t_{2k}-t_{1k}$. If CD_k denotes the complement of D_k in $[t_{1k},t_{2k}]$ and we take $D' = (t_1,t_2) - \cup_k CD_k$, then D' is a measurable subset of (t_1,t_2) with meas $D' = t_2 - t_1$. If $\psi_k(t)$, $t_{1k} \leq t \leq t_{2k}$, denotes the function

$$\psi_k(t) = \int_{t_{1k}}^{t} M_k(t,f(t,x_k(t),u))dt, \quad t_{1k} \leq t \leq t_{2k},\ k = 1,2,\ldots,$$

then $\psi_k(t_{1k}) = 0$, and for all $t,t' \in [t_{1k},t_{2k}]$ also

$$|\psi_k(t) - \psi_k(t')| \leq M_o|t-t'|,\ |\psi_k(t)| \leq 2M_o^2.$$

Thus, $\psi_k(t)$ is a sequence of equibounded, equicontinuous, equilipschitzian functions. By Ascoli's theorem there is a further subsequence, which we still denote $[\psi_k]$, and which is convergent in the metric ρ toward a continuous function $\psi_o(t)$, $t_1 \leq t \leq t_2$, and $\psi_o(t)$ is Lipschitzian with constant M_o and hence AC in $[t_1,t_2]$.

We shall now consider the class $\{F\}$ of all scalar functions $F(t,x,u)$ which are continuous on $M = A \times U$. Obviously, there is a sequence F_s,

$s = 1,2,\ldots$, of such functions such that, given any F of the same class and ε, $0 < \varepsilon \leq 1$, there is also some F_s in the sequence with $|F(t,x,u) - F_s(t,x,u| \leq \varepsilon$ for all $(t,x,u) \in M$. For every s we shall consider the scalar functions

$$\psi_{ks}(t) = \int_{t_{1k}}^{t} M_k(t,F_s(t,x_k(t),u))dt, \quad t_{1k} \leq t \leq t_{2k}, \quad k = 1,2,\ldots.$$

Since $|F_s| \leq N_s$ on M for some constant N_s, then for all $t,t' \in [t_{1k},t_{2k}]$ we have

$$\psi_{ks}(t_{1k}) = 0, \quad |\psi_{ks}(t) - \psi_{ks}(t')| \leq N_s|t-t'|, \quad |\psi_{ks}(t)| \leq 2M_oN_s.$$

Thus, by Ascoli's theorem and the usual diagonal process, we can perform the selection of a subsequence, so that ψ_{ks} converges as $k \to \infty$ in the metric ρ toward a continuous function $\psi_{ks}(t)$, $t_1 \leq t \leq t_2$, which is then Lipschitzian with constant N_s in $[t_1,t_2]$ with $\psi_{os}(t_1) = 0$, and this holds for every $s = 1,2,\ldots$.

For every s, the AC function $\psi_{os}(t)$ has finite derivative $\psi'_{os}(t)$ at all points t of a measurable set $D'_s \subset [t_1,t_2]$ with meas $D'_s = t_2-t_1$, and we take as before $D'' = (t_1,t_2) - \bigcup_s C D'_s$, $D = D' \cap D''$, and then $D \subset (t_1,t_2)$, meas $D = t_2-t_1$.

We have now chosen a well determined subsequence, which we still denote by [k], and $t_{1k} \to t_1$, $t_{2k} \to t_2$ as $k \to \infty$ (along the chosen sequence). Thus t_1 and t_2 are also well determined.

Given $F \in \{F\}$ let us take

$$\psi_k(t) = \int_{t_{1k}}^{t} M_k(t,F(t,x_k(t),u))dt, \quad t_{1k} \leq t \leq t_{2k}, \quad k = 1,2,\ldots,$$

and let us prove that ψ_k converges in the metric ρ toward some $\psi(t)$, $t_1 \leq t \leq t_2$, which is AC in $[t_1,t_2]$. Indeed, $|F(t,x,u)| \leq N$ for some constant N and hence

$$\psi_k(t_{1k}) = 0, \quad |\psi_k(t) - \psi_k(t')| \leq N|t-t'|, \quad |\psi_k(t)| \leq 2M_o N, \tag{IV.2.1}$$

for all $t,t' \in [t_{1k},t_{2k}]$, $k = 1,2,\ldots$. On the other hand, given $\varepsilon > 0$, there is some s such that $|F(t,x,u) - F_s(t,x,u)| \leq \varepsilon$ for all $(t,x,u) \in M$, and also there is some k_o such that $k,k' \geq k_o$ implies $\rho(\psi_{ks},\psi'_{ks}) \leq \varepsilon$. Now

$$\begin{aligned} |\psi_k(t) - \psi_{ks}(t)| &= \left|\int_{t_{1k}}^t M_k(t,F(t,x_k(t),u))dt - \int_{t_{1k}}^t M_k(t,F_s(t,x_k(t),u))dt\right| \\ &\leq \int_{t_{1k}}^t |M_k(t,F(t,x_k(t),u)) - M_k(t,F_s(t,x_k(t),u))|dt \\ &= \int_{t_{1k}}^t |M_k[t,F(t,x_k(t),u) - F_s(t,x_k(t),u)]|dt \leq 2M_o\varepsilon, \end{aligned}$$

and an analogous relation holds with k replaced by k'. Thus

$$\begin{aligned} \rho(\psi_k,\psi_{k'}) &\leq \rho(\psi_k,\psi_{ks}) + \rho(\psi_{ks},\psi_{k's}) + \rho(\psi_{k's},\psi_{k'}) \\ &\leq (2M_o+1)\varepsilon, \end{aligned}$$

for all $k,k' \geq k_o$. This proves that $\psi_k(t)$, $k = 1,2,\ldots$, converges in the metric ρ toward a continuous function $\psi(t)$, $t_1 \leq t \leq t_2$. Also (IV.2.1) implies, for every $t,t' \in [t_1,t_2]$,

$$\psi(t_1) = 0, \quad |\psi(t) - \psi(t')| \leq N|t-t'|, \quad |\psi(t)| \leq 2M_o N.$$

Now let us prove that, for any $F \in \{F\}$, the corresponding function $\psi(t)$, $t_1 \leq t \leq t_2$, so determined, has finite derivative $\psi'(t)$ at <u>every</u> point $t \in D$, where D is the subset of (t_1,t_2) determined above with meas $D = t_2-t_1$. Indeed, given $F \in \{F\}$, let us choose F_s as above and note that,

$$F(t,x,u) - F_s(t,x,u) - \varepsilon \leq 0, \quad F(t,x,u) - F_s(t,x,u) + \varepsilon \geq 0$$

for all $(t,x,u) \in M$. Hence,

$$M_k(t,F(t,x_k(t),u)) - M_k(t,F_s(t,x_k(t),u)) - \varepsilon$$

$$\leq \ 0 \ \leq \ M_k(t,F(t,x_k(t),u)) - M_k(t,F_s(t,x_k(t),u)) + \varepsilon$$

a.e. in $[t_{1k},t_{2k}]$, and by integration we conclude that the scalar functions

$$\psi_k(t) - \psi_{ks}(t) - \varepsilon(t-t_{1k}), \qquad \psi_{ks}(t) + \varepsilon(t-t_{1k})$$

are monotone nonincreasing and monotone nondecreasing respectively. Passing to the limit as $k \to \infty$, we obtain two scalar functions

$$\psi(t) - \psi_{os}(t) - \varepsilon(t-t_1), \quad \psi(t) - \psi(t) - \psi_{os}(t) + \varepsilon(t-t_1),$$

which have the same property. If we denote by $\underline{D}$, $\bar{D}$ the usual lower derivative and upper derivative operators, of a scalar function, we obtain

$$\bar{D}\psi(t) - \psi'_{os} - \varepsilon \leq 0, \quad \underline{D}\psi(t) - \psi'_{os}(t) + \varepsilon \geq 0$$

at <u>every</u> point $t \in D$, since $D \subset D_s$ and ψ_{os} has derivative in D_s. Thus,

$$0 \leq \bar{D}\psi(t) - \underline{D}\psi(t) \leq 2\varepsilon, \quad t \in D \subset (t_1,t_2),$$

where D is a fixed set and ε is arbitrary. We conclude that $\psi' = \bar{D}\psi = \underline{D}\psi$ exists and is finite at every $t \in D$. In other words, for every $F \in \{F\}$, the AC function defined above, say $\psi(t) = \psi(t,F)$, $t_1 \leq t \leq t_2$, has finite derivative $\psi'(t)$ at least at every point $t \in D \subset (t_1,t_2)$ with meas $D = t_2-t_1$. We shall define $M(t,\emptyset)$ by taking

$$M(t,F(t,x(t),u)) = (d/dt)\psi(t,F), \quad t \in D, \quad F \in \{F\}.$$

The properties (m1,2,3) for M can be verified immediately. In addition, since $f(t,x,u) = (f_1,\ldots,f_n)$ and each f_i is a scalar function of the class $\{F\}$, we have

$$dx_o^i/dt = (d/dt)\psi(t;f_i) = M(t,f_i(t,x_o(t),u)), \; t \in D, \; i = 1,\ldots,n.$$

We conclude that $\{x_o(t),\ t_1 \leq t \leq t_2,\ D,\ M(t,\emptyset)\}$ is a weak solution.

Finally, assume that $g(t,x,u) = (g_1,\ldots,g_n)$ is any continuous vector function in A x U, that $y_k(t) = (y_k^1,\ldots,y_k^n)$, $t_{1k} \leq t \leq t_{2k}$, $k = 1,2,\ldots,$ are AC vector functions satisfying (a) $(t,y_k(t)) \in A$ for all $t \in [t_{1k},t_{2k}]$, (b)

$$dy_k/dt = M_k(t;g(t,x_k(t),u)) \quad \text{a.e. in } [t_{1k},t_{2k}],$$

and (c) $y_k(t_{1k}) = y_{1k} = (y_{1k}^1,\ldots,y_{1k}^n) \to (y_1^1,\ldots,y_1^n) = y_1 \in E_n$ as $k \to \infty$. Then $t_{1k} \to t_1$, $t_{2k} \to t_2$, and for each $i = 1,\ldots,n$, the function $g_i(t,x,u)$ is a scalar function of the class $\{F\}$, and the differences

$$\psi_k^i(t) = y_k^i(t) - y_{1k}^i, \quad t_{1k} \leq t \leq t_{2k}, \quad k = 1,2,\dots,$$

are functions ψ_k above, and hence converge in the ρ-metric toward an (AC) function $\psi^i(t)$, $t_1 \leq t \leq t_2$. If $y(t) = (y^1,\dots,y^n)$, $y^i(t) = \psi^i(t) + y_1^i$, $i = 1,\dots,n$, then $\rho(y_k,y) \to 0$, as $k \to \infty$,

$$dy/dt = M(t,g(t,x_o(t),u)) \qquad t \in D \subset (t_1,t_2),$$

$$J(x_k,D_k,M_k;g,y_{1k}) = \varphi(t_{1k},y_k(t_{1k}),t_{2k},y_k(t_{2k}))$$

$$\to \varphi(t_1,y(t_1),t_2,y(t_2)) = J(x_o,D,M;g,y_1)$$

Thus, (x_k,D_k,M_k) converges in the L-topology toward (x,D,M). Theorem (IV.2.v) is thereby proved.

Bibliographical notes. Statement (IV.1.i) was observed by R. V. Gamkrelidze [48 a]. The present proof is by Cesari [24 m], who proved also the analogous statement (IV.1.ii) for the case in which U, the control space, depends on both t and x [24 m]. The material of Section (IV.2) is in L. C. Young's [121] and E. J. McShane's papers [77 m,n,o,p].

For a recent discussion of generalized solutions and the question of approximating them by means of usual solutions, see J. Warga [114].