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PERIODIC SOLUTIONS IN A THIN CYLINDER OF WEAKLY NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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1. INTRODUCTION

We consider here partial differential equations

$$L(t,z,u) = 0, \quad z = (z_1,\dots,z_\nu), \qquad u = (u_1,\dots,u_n) \tag{1}$$

which are defined in some cylinder $\Gamma_\sigma = [-\infty < t < +\infty, |z| \le \sigma]$, for which $L(t + T,z,u) = L(t,z,u)$, and for which the Cauchy problem $u(o,z) = u_o(z)$, $|z| \le \sigma$, makes sense. By this we mean that given $u_o(z), |z| \le \sigma$, in a suitable class $\tilde{\mathcal{S}}$, there are numbers s and b, $o < s \le 1$, $b > o$, and a solution $u(t,z)$ to the Cauchy problem in some set $\Gamma' = [o \le t \le b, |z| \le s\,\sigma]$, where s may be rather small. Actually, we limit ourselves, to that large class of Cauchy problems, which, as proved in J. G. Petrovsky ([3], pp. 16-17), can be reduced by suitable substitutions to Cauchy problems for first order partial differential equations.

The problem of the periodic solutions of (1) can then be posed in a form which is similar to the one which is traditional for ordinary differential equations. Indeed, we may first ask (a) whether we can take $b = T$ for a suitable class $\tilde{\mathcal{S}}$ of initial data $u_o(z), |z| \le \sigma$, and then we may ask (b) whether we can choose u_o in $\tilde{\mathcal{S}}$ so that $u(T,z) = u_o(z)$ for all $|z| \le s\,\sigma$, so that the periodicity

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of L guarantees the existence of the solution $u(t,z)$ in the whole thin cylinder $\Gamma_{s\sigma} = [-\infty < t < +\infty, |z| \leq s\sigma]$.

Problem (a) actually belongs to the general theory of partial differential equations. Most methods for the treatment of the Cauchy problem yield criteria under which we can take $b = T$ or b large. Each method concerns of course different classes of partial differential equations, different smoothness hypotheses for the equations, different smoothness requirements for the solutions (usual, or generalized solutions, and so on).

Problem (b) is a problem of functional analysis, since the mapping $u_o(\cdot) \to u(T,\cdot)$, is a mapping from a class $\mathcal{S}$ of functions $u_o(z)$, $|z| \leq \sigma$, into a class $\mathcal{S}'$ of functions $u(T,z)$, $|z| \leq s\sigma$, with the complication that s may be smaller than one, and thus we are dealing with functions, $u_o(\cdot)$ and $u(T,\cdot)$, defined in two different sets.

In the present paper we discuss the argument recently proposed by L. V. Ovcyannikov [2] for proving the convergence — in a suitably reduced domain— of the usual method of successive approximations, when L is assumed to be continuous with respect to t and holomorphic in z, and the solution u is required to be of class C^1 in t and holomorphic in z.

We show first (Nos. 2,3) that the same Ovcyannikov's argument—based on functional analysis—can be applied to nonlinear partial differential equations, and we obtain existence theorems for the Cauchy problem in a nonlinear and very general setting. We then give criteria in order that we can take b large, in particular $b = T$ for L periodic of period T. Finally, we obtain (No. 4) criteria for periodic solutions of partial differential equations by the use of a recent implicit function theorem in functional analysis and we consider a few examples.

2. A NONLINEAR FORM OF OVCYANNIKOV'S THEOREM.

For every s, $o \leq s \leq 1$, let X_s denote a Banach space of elements x with norm $\|x\|_s$ such that

(a) $X_{s'} \supset X_s$ for all $o \leq s' \leq s \leq 1$;

(b) the inclusion operation $j_{s's}: X_s \to X_{s'}$ has norm ≤ 1, where $o \leq s' \leq s \leq 1$.

Thus, $X_o \supset X_s \supset X_1$ for all $o \leq s \leq 1$. Let I_a denote an interval $o \leq t \leq a$ if t is real, or a disc $|t| \leq a$ if t is complex. Let x_o be any given element of X_1. Let f(t), $t \in I_a$, be a bounded continuous function of t, valued in X_1, and , if t complex, we assume also that f is holomorphic in t in the open disc $\overset{o}{I}_a$.

For every $t \in I$ let A(t) be an operator—nonnecessarily linear—with the following properties:

(c) $A(t):X_s \to X_{s'}$, for every $t \in I_a$ and $o < s' < s \leq 1$, and A(t) maps the zero element of X_s into the zero element of $X_{s'}$.

(d) There is a constant $C > o$ such that for all $t \in I_a$, $x_1, x_2 \in X_s$, $o \leq s' < s \leq 1$, we have

$$\| A(t)x_2 - A(t)x_1\|_{s'} \leq C(s-s')^{-1} \| x_2 - x_1\|_s \;;$$

(c) $A(t)$ is a continuous function of t, that is, given $\varepsilon > o$, $o \leq s' < s \leq 1$, there is a $\delta = \delta(\varepsilon, s, s')$ such that for all $t_1, t_2 \in I_a$, $|t_1 - t_2| \leq \delta$, $x \in X_s$ we have $\|A(t_2)x - A(t_1)x\|_{s'} \leq \varepsilon$.

Note that $A(t)$ may not map X_s into itself. We consider the Cauchy problem

$$x'(t) = A(t)\, x(t) + f(t), \quad t \in I_a, \qquad (2)$$
$$x(o) = x_o$$

Note that the second part of requirement (c) is not restrictive since, in the contrary case, we can, write (2) in the form

$$x'(t) = [A(t)x - A(t)(o)] + [A(t)(o) + f(t)] \quad .$$

and now the operator $B(t)$ defined by $B(t)x = A(t)x - A(t)(o)$ has property (c).

<u>Theorem 1.</u> Under the hypotheses (abcde), and for every number b such that $o < b \leq a$, $Ceb < 1$, there exists a continuous function $x(t)$, $t \in I_b$, with values in X_s satisfying (2) for every $o \leq t \leq b$, $o < s < 1 - Ceb$.

<u>Proof.</u> Let us take $x_{-1}(t) = o$, and let $x_k(t)$, $t \in I$, $k = o,1,2,\ldots$ be the sequence defined by

$$x_k(t) = x_o + \int_o^t [A(\tau)x_{k-1}(\tau) + f(\tau)]d\tau \ , \ t \in I_a. \qquad (3)$$

Let us prove that $x_k(t) \in X_s$ for all $o \leq s < 1$, $t \in I_a$, $k = o,1,2,\ldots.$ First,

$$x_o(t) = x_o + \int_o^t f(\tau)d\tau \in X_1 \subset X_s \ , o \leq s \leq 1, \ t \in I_a \ ,$$

by the definitions of x_o and f. Also,

$$x_1(t) = x_o + \int_o^t [A(\tau)x_o(\tau) + f(\tau)]d\tau,$$

hence $A(\tau)x_o(\tau) \in X_s$ for all $o \leq s < 1$, $t \in I_a$, and $x_1(t) \in X_s$ for all $o \leq s < 1$, $t \in I$. From (3) by induction argument we conclude that $x_k(t) \in X_s$ for all $o \leq s < 1$, $t \in I_a$, $k = o,1,2,\ldots$.

Now we have

$$x_o(t) = x_{-1}(t) = x_o + \int_o^t f(\tau)d\tau$$

and we take

$$M = \max \|x_o + \int_o^t f(\tau)d\tau\|_1 \tag{4}$$

where the maximum is taken for $t \in I_b$. Then

$$\|x_o(t)-x_{-1}(t)\|_s \leq \|x_o(t)-x_{-1}(t)\|_1 \leq M, \; o \leq s \leq 1, \; t \in I_a \; . \tag{5}$$

From (3) we obtain

$$x_k(t) - x_{k-1}(t) = \int_o^t [A(\tau)x_{k-1}(\tau)-A(\tau)x_{k-2}(\tau)]d\tau, \; k = 1,2,\ldots \; .$$

and we shall prove that

$$\|x_k(t) - x_{k-1}(t)\|_s \leq M[(1-s)^{-1}Ce|t|]^k \; , \tag{6}$$

$$o \leq s < 1, \quad t \in I_a \; , k = o,1,2,\ldots \quad .$$

As shown by (5) this relation (6) is true for $k = o$. For $k = 1$ we have, by the use of (c),

$$\|x_1(t) - x_o(t)\|_s = \| \int_o^t A(\tau)\ [x_o + \int_o^\tau f(\sigma)d\sigma]d\tau \|_s$$

$$\leq C(1-s)^{-1} \int_o^t M\, d\tau$$

$$= M(1-s)^{-1}\, C|t| < M(1-s)^{-1}\, Ce|t|,$$

and again (6) is proved for $k = 1$.

We assume that (6) is true for k-1 and we prove it for k. Indeed, for all s,η, $o \leq s < s + \eta < 1$, from (6) for k-1 and by the use of (c), we obtain

$$\|x_k(t)-x_{k-}\ (t)\|_s \leq C\eta^{-1} \int_o^t \|x_{k-1}(\tau)-x_{k-2}(\tau)\|_{s+\eta}\, d\tau$$

$$\leq C\eta^{-1} \int_o^t M[(1-s-\eta)^{-1}\, Ce\tau]^{k-1}\, d\tau \tag{7}$$

$$= MC^k t^k e^{k-1}\ \eta^{-1}(1-s-\eta)^{-(k-1)} k^{-1}\ .$$

If we take $\eta = k^{-1}(1-s)$, then $s + \eta < s + (1-s) = 1$ since now $k \geq 2$, and (7) becomes

$$\|x_k(t)-x_{k-1}(t)\|_s = MC^k t^k e^{k-1}(1-s)^{-k}(1-k^{-1})^{-(k-1)}$$

$$= MC^k t^k e^{k-1}(1-s)^{-k}(1+(k-1)^{-1})^{k-1}$$

$$\leq MC^k t^k e^k (1-s)^{-k}\ ,$$

and (6) is proved.

For $o \leq t \leq b$, $o < s < 1-$ Ceb < 1, we have $1-s >$ Ceb, and $(1-s)^{-1}Ce|t| \leq$

$\leq (1-s)^{-1}Ceb < 1$.

We conclude that the series

$$x(t) = x_o + \sum_{k=1}^{\infty} [x_k(t) - x_{k-1}(t)]$$

converges for all $o \leq s < 1 - Ceb$, $t \in I$, that

$$\|x_k(t)\|_s, \|x(t)\|_s \leq M [1-(1-s)^{-1}Cet]^{-1} , \qquad (8)$$

$$\|x(t)-x_k(t)\|_s \leq M[(1-s)^{-1}Cet]^k [1-(1-s)^{-1}Cet]^{-1} , \qquad (9)$$

for all $o \leq s < 1$, $t \in I_b$, $k = 1,2,\ldots$, and thus $x(t) \in X_s$ for all $o \leq s < 1\text{-}Ceb$, $t \in I_b$. The theorem above is thereby proved.

Remark. In the reasoning above the assumption that the norm of A is $\leq C(s-s')^{-1}$ plays an essential role. The argument could not be repeated with the exponent -1 replaced by any other exponent < -1.

Conditions (cde) will turn out to be too restrictive in applications, and we modify them, therefore, as follows.

For a given $\tau > o$ let $X'_s \subset X_s$ be the set $X'_s = [x(x \in X_s, \|x\|_s \leq \tau]$, and let us assume (instead of (cde)):

(c') $A(t) : X'_s \to X_s$ for every $t \in I_a$ and $o \leq s' < s \leq 1$;

(d'),(e') The same as (d), (e) for the elements $x \in X'_s$ only.

Theorem 2. Under hypotheses (abc'd'e'), and constants τ, b, M, C satisfying $o < b \leq a$, $Ceb < 1$, $M < \tau [1\text{-}Ceb]$, where M is defined by (3), there is a continuous function $x(t)$, $t \in I_b$, with values in X'_s satisfying (2) for every $o \leq s < 1 - Ceb$.

The proof is the same as for Theorem 1, where now we note that $\|x_o(t)\|_s =$ $= \|x_o + \int_o^t f(\tau)d\tau\|_s \leq M$, hence $x_o(t) \in X'_s$. Analogously, each successive approximation $x_k(t)$ belongs to X'_s because of (8), and so does $x(t)$.

3. THE CAUCHY PROBLEM FOR PARTIAL DIFFERENTIAL EQUATIONS

The local Cauchy problem (I. G. Petrovsky [3], p. 14) is known to be reducible to an analogous problem for systems of first order partial differential equations (I. G. Petrovsky [3], pp. 14-17).

Let t and $z = (z_1,\ldots,z_\nu)$ the $\nu + 1$ independent variables, t either real or complex, all z_i complex. Let $u(t,z) = (u_1,\ldots,u_n)$ be the n unknown functions, let $u_t = (\partial u_i/\partial t,\ i = 1,\ldots,n)$ denote the system of n first order partial derivatives with respect to t, and let $u_z = (\partial u_i/\partial z_j,\ i = 1,\ldots,n,\ j = 1,\ldots,\nu)$ be the system of $n\nu$ first order partial derivatives with respect to $z_1,\ldots,z_\nu$.

We shall consider a Cauchy problem of the form

$$\partial u_i/\partial t = \sum_{j=1}^{n} \sum_{\ell=1}^{\nu} h_{ij\ell}(t,z,u)\partial u_j/\partial z_\ell + g_i(t,z,u) + f_i(t,z), \qquad (10)$$

$$u_i(o,z) = u_i(z),\ i = 1,\ldots,n,$$

or in vector form

$$u_t = h(t,z,u)u_x + g(t,z,u) + f(t,z),$$
$$u(o,z) = u_o(z),$$

where h is an n x $n\nu$ matrix, and g and f are n-vectors.

For any n-vector $u = (u_1,\ldots,u_n)$ we define $|u|$ by taking $|u| = \max_j |g_j|$. For any matrix $h = (h_{ij})$ we define $|h|$ by taking $|h| = \max_i \sum_j |h_{ij}|$, so that $|hu| \leq |h|\,|u|$.

For the sake of simplicity let us consider only the case where t is real. Let J_a denote a real interval $o \leq t \leq a$, and by B_σ any polydisc $B_\sigma = [z||z_i| \leq \sigma, i,\dots,\nu]$ in the complex z-space C^ν. We shall denote by u and v also any complex n- or $n\nu$-vector variable, and by B_τ, B_h corresponding polydiscs in C^n or $C^{n\nu}$. We shall now consider the following assumptions:

(h_1) Let a, σ, τ be positive numbers. Let us assume that the functions $h(t,z,u)$, $g(t,z,u)$, $f(t,z)$ are continuous in $J_a \times B_\sigma \times B_\tau$ and $J_a \times B_\sigma$ respectively, and that for every $t \in J_a$ they are holomorphic with respect to (z,u) or z in the open set $\overset{o}{B}_\sigma \times \overset{o}{B}_\tau$, or $\overset{o}{B}_\sigma$ respectively.

(h_2) There are constants N_o, N_1, $N_2 \geq o$ such that for all $t \in I_a$, $z \in B_\sigma$, $u, u_1, u_2 \in B_\tau$ we have

$$|h(t,z,u)| \leq N_o \ ,$$

$$|h(t,z,u_1) - h(t,z,u_2)| \leq N_1|u_1-u_2| \ ,$$

$$|g(t,z,u_1) - g(t,z,u_2)| \leq N_2|u_1-u_2| \ .$$

For every s, $o \leq s \leq 1$, we denote by H_s the space of all functions $g(z)$, $z = (z^1,\dots,z^\nu)$, with values in C^1, which are continuous in $B_{s\sigma}$, holomorphic in $\overset{o}{B}_{s\sigma}$, equipped with the maximum norm topology in $B_{s\sigma}$.

<u>Lemma 1</u>. For every $o \leq s' < s \leq 1$, $g(z) \in H_{s\sigma}$, we have $\partial g/\partial z_j \in H_{s'\sigma}$, and for all $z \in B_{s'\sigma}$ we have also

$$|\partial g(z)/\partial z_j| \leq \sigma^{-1}(s-s')^{-1} \max |g(\zeta)| \ , \ j = 1,\dots,\nu, \qquad (11)$$

where max is taken for $\zeta \in B_{s\sigma}$.

Proof. For $z \in B_{s'\sigma}$ we have

$$\partial g(z)/\partial z_1 = (2\pi i)^{-\nu} \int_\Gamma (\zeta_1 - z_1)^{-2}(\zeta_2 - z_2)^{-1}\dots(\zeta_\nu - z_\nu)^{-1} g(\zeta)d\zeta_1\dots d\zeta_\nu$$

where we can take for Γ the oriented manifold $\Gamma = [\zeta \,|\, |\zeta_i - z_i| = (s-s')\pi,$ $i = 1,\dots,\nu]$, and $\Gamma \subset B_{s\sigma}$. Analogous relations hold for the other first order partial derivatives. By taking absolute values we then have (11) for all $z \in B_{s'\sigma}$, $o \le s' < s \le 1$.

Lemma 1 can be interpreted as a property of the linear operators $\partial/\partial z_j$. Namely, for all $o \le s' < s \le 1$ we have

$$\partial/\partial z_j \; : \; H_s \to H_{s'}$$

and the operational norm of $\partial/\partial z_j$ is $\le \sigma^{-1}(s-s')^{-1}$.

We shall denote by H_s^n the space of n-vector functions $u(z) = (u_1,\dots,u_n)$, $z \in B_{s\sigma}$ with $u_i \in H_s$, $i = 1,\dots,n$, and norm $\|u\|_s = \max_i \|u_i\|_s$. We shall denote by $H_s'^n$ the subspace of H_s^n made up of all $u \in H_s^n$ with $\|u\|_s \le \tau$.

Then for every $t \in J_a$ and $u \in H_s'^n$ the following function $U = U(t,z) = A(t)u$ is defined

$$\begin{aligned} U(t,z) &= (A(t)u)(t,z) = \\ &= h(t,z,u)u_z(t,z) + g(t,z,u(t,z)) = g(t,z,o), \end{aligned} \qquad (12)$$

with $t \in J_a$, $z \in B_{s'\sigma}$, $o \le s' < s \le 1$.

Lemma 2. Under the hypotheses (h_1), (h_2), the operator $A(t)$ defined by (12) satisfies

$$A(t) : H_s'^n \to H_s^n \;, \quad \text{and}$$

$$\|A(t)u_1-A(t)u_2\|_{s'} \leq [N_o\sigma^{-1} + N_1(\tau\sigma^{-1} + 1)]\|u_1-u_2\|_s ,$$

for all $u_1, u_2 \in H'_s$, $o \leq t \leq a$, $o \leq s' < s \leq 1$.

The first part is a consequence of Lemma 1. For the second part we have

$$\|A(t)u_1-A(t)u_2\|_{s'} \leq \|[h(t,z,u_1)-h(t,z,u_2)]u_{1z}\|_{s'} + \|h(t,z,u_2)(u_{1z}-u_{2z})\|_{s'} +$$

$$+ \|g(t,z,u_1)-g(t,z,u_2)\|_{s'}$$

$$\leq N_1\|u_1-u_2\|_{s'}\sigma^{-1}(s-s')^{-1}\|u_1\|_s + N_o\sigma^{-1}(s-s')^{-1}\|u_1-u_2\|_s + N_2\|u_1-u_2\|_{s'} \tag{13}$$

$$\leq [N_2 + (N_o+N_1\tau)\sigma^{-1}(s-s')^{-1}] \|u_1-u_2\|_s .$$

If we take

$$C = N_1 + (N_o+N_1\tau)\sigma^{-1} , \tag{14}$$

then (13) becomes

$$\|A(t)u_1-A(t)u_2\|_{s'} \leq C\|u_1-u_2\|_s(s-s')^{-1} ,$$

since

$$N_1 + (N_o+N_1\tau)\sigma^{-1}(s-s')^{-1} \leq [N_2 + (N_o+N_1\tau)\sigma^{-1}](s-s')^{-1} = C(s-s')^{-1} .$$

Let M denote the constant

$$M = \max \|u_o(z)t+\int_o^t [g(\tau,z,o) + f(\tau,z)]d\tau\|_1 \tag{15}$$

where the maximum is taken for $o \leq t \leq a$.

From Theorem 2 we deduce now

Theorem 3 (Cauchy-Kovalevsky). Under hypotheses (h_1),(h_2), and C, M defined by (14) and (15), let us assume that $M < \tau$. Then for all numbers b, s such that $M < \tau$.

$$o < b \leq a \ , \ Ceb < 1 \ , \ o < s < 1 - Ceb \ , \tag{16}$$

$$M[1-(1-s)^{-1}Ceb]^{-1} < \tau \tag{17}$$

there is a solution $u(t,z) = (u_1,\dots,u_n)$ to problem (10) which is continuous in $J_b \times B_{s\sigma}$, and for every $t \in J_b$ holomorphic with respect to $z = (z_1,\dots,z_\nu)$ in the open set $B^o_{s\sigma}$.

Proof. First note that $M < \tau$ by hypothesis, and now we can always choose numbers b, s satisfying relations (16) and (17). Indeed, first we can take $b > o$ as small as we want, and then we can take s as close to zero as we want. The left hand member of relation (17) can now be made as close to M as we want and thus $< \tau$.

Note that with the definition of A(t) in (12), the Cauchy problem (10) takes the form

$$\begin{aligned} u_t &= A(t)u + [g(t,z,o) + f(t,z)] \ , \\ u(o,z) &= u_o(z) \ . \end{aligned}$$

and now we apply Theorem 2.

The method of successive approximations mentioned in No. 2 for the abstract formulation reduces now to the usual method

$$u_{-1}(t,z) = o$$

$$u_o(t,z) = u_o(z) + \int_o^t [g(\tau,z,o) + f(\tau,z)]d\tau ,$$

$$u_k(t,z) = u_o(z) + \int_o^t [h(\tau,z,u_{k-1}(\tau,z)]u_{k-1,z}(\tau,z)+g(\tau,z,u_{k-1}(\tau,z)+f(\tau,z)]d\tau,$$

$k = 1,2,\ldots$.

For problems of the form

$$\partial u_i/\partial t = \sum_{j=1}^{n} \sum_{\ell=1}^{\nu} a_{ij\ell}(t,z)\partial u_j/\partial z_\ell + g_i(t,z,u) + f(t,z) ,$$

$$u_i(o,z) = u_{oi}(z) , \; i = 1,\ldots,n ,$$

we take constants $N_{1ij\ell}$, N_{2i} such that

$$|h_{ij\ell}(t,z)| \leq N_{1ij\ell}$$

$$|g_i(t,z,u_1) - g_i(t,z,u_2)| \leq N_{2i} |u_1-u_2|$$

for all $t \in J_a$, $t \in B_\sigma$, u, u_1, $u_2 \in B_\tau$, and

$$N_o = \max_i \sum_{j=1}^{n} \sum_{\ell=1}^{\nu} N_{1ij\ell}, \quad N_1 = o , \quad N_2 = \max N_{2i} ,$$

$$C = N_2 + N_o\sigma^{-1} ,$$

$$M = \max |u_{oi}(z) + \int_o^t [g_i(t,z,o) + \int_o^t f_i(\tau,z)]d\tau| ,$$

where the last maximum is taken for all $i = 1,\ldots,n$, $t \in I_a$, $z \in B_\sigma$.

Remark. In Theorem 3 we can take $b = a$ whenever $M < \tau$, $Cea < 1$, and then $u(t,z)$ is continuous in $J_a \times B_{s\sigma}$ and for every $t \in J_a$ holomorphic with respect to z in $B^o_{s\sigma}$ for any s such that $o < s < 1 - Cea$, $M[1-(1-s)^{-1}Cea]^{-1} < \tau$.

For linear problems

$$\partial u_i/\partial t = \sum_{j=1}^{n} \sum_{\ell=1}^{\nu} h_{ij\ell}(t,z)\partial u_j/\partial z_\ell + \sum_{j=1}^{n} g_{ij}(t,z)u_j + f_i(t,z) ,$$
$$u(o,z) = u_{oi}(z) , \quad i = 1,\ldots,n , \tag{18}$$

Theorem 3 yields

Corollary. For linear problems (18) where the functions $h_{ij\ell}(t,z)$, $g_{ij}(t,z)$, $f_i(t,z)$ are continuous in $J_a \times B_\sigma$ and for every $t \in J_a$ holomorphic with respect to z in B^o_σ, and the functions $u_{oi}(z)$ are holomorphic in B^o_σ, there is a unique solution u(t,z) which is continuous in $J_a \times B_{s\sigma}$ and for every $t \in J_a$ holomorphic in $B^o_{s\sigma}$ for all s, $o < s < \exp(-Cea)$.

Proof. We have here $|h_{ij\ell}(t,z)| \leq N_{1ij\ell}$, $|g_{ij}(t,z)| \leq N_{2ij}$ for suitable constants $N_{1ij\ell}$, N_{2ij} and all $t \in J_a$, $z \in B_\sigma$. We then take

$$N_o = \max_i \sum_{j=1}^{n} \sum_{\ell=1}^{\nu} N_{1ij\ell} , \quad N_1 = o ,$$

$$N_2 = \max_i \sum_{j=1}^{n} N_{2ij} , \quad C = N_2 + N_o\sigma^{-1} ,$$

$$M = \max \left| u_{oi}(z) + \int_o^t f_i(\tau,z)d\tau \right| ,$$

where the last maximum is taken for all $t \in J_a$, $z \in B_\sigma$, $i = 1,\ldots,n$. Let $\mu \geq 1$ be an integer such that, for $b = a/\mu$ we have $Ceb < 1$, that is, $\mu > Cea$, for $(N_2 + N_o\sigma^{-1})ea < \mu$. Then the solution u(t,z) is continuous in $[o,b] \times B_{s\sigma}$, and for every $o \leq t \leq b$ holomorphic with respect to z in $B^o_{s\sigma}$ for $o < s < 1 - Ceb$, say for all $o < s \leq 1 - Ceb - \eta$, where $o < \eta < 1 - Ceb$ is arbitrary. We can now repeat the argument in the interval $b \leq t \leq 2b$, where now $u_o(z)$ is replaced by $u(b,z)$ and $t = o$ by $t = b$. Then $u(t,z)$ can be continued in $[b,2b] \times B_{s\sigma}$ for all s with $o < s < (1\text{-}Ceb\text{-}\eta)^2$, and so on. By repeating this argument μ times, we conclude that the solution $u(t,z)$ exists and is continuous in $[o,a] \times B_{s\sigma}$ and for every $t \in [o,a]$ is holomorphic with respect to z in $B^o_{s\sigma}$ for all s with $o < s \leq (1\text{-}Ceb\text{-}\eta)^{\mu}$. Since η is arbitrary we see that the statement is true for every $o < s < (1\text{-}Cea\mu^{-1})^{\mu}$ where μ is any integer $\mu > Cea$. As $\mu \to +\infty$ we see that the statement is true for every $o < s < \exp(-Cea)$.

4. APPLICATION TO PERIODIC SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS

We consider now a partial differential system as (10), or (15), where the second members are periodic in t of some period $T = 2\pi/\omega$. The question of the periodic solutions of period T of such systems in a thin cylinder $E_1 \times B_{s\sigma}$ for s, $o < s \leq 1$, sufficiently small, can be reworded as the problem of determining a function $u_o(z)$ holomorphic in B^o_{σ} such that the solution $u(t,z)$ with $u(o,z) = u_o(z)$ can be continued in a cylinder $[o,T] \times B_{s\sigma}$ such that

$$u(T,z) = u(o,z) = u_o(z), \quad z \in B_{s\sigma} . \tag{19}$$

Then, by periodicity, $u(t,z)$ can be extended to the whole of $E_1 \times B_{s\sigma}$.

For linear systems (16) we have seen that the first requirement is always satisfied.

Note that $u(t,z)$ depends functionally on $u_o(z)$, and we shall denote it by $u(t,z;u_o)$. Thus, relation (19) becomes

$$u(T, \cdot ;u_o) = u_o , \tag{20}$$

where u_o in the first member is a holomorphic function in B_σ, or $u_o \in H_1$, while u_o in the second member is only a holomorphic function in $B_{s\sigma}$, or $u_o \in H_s$, $o < s \leq s_o \leq 1$, and suitable s_o.

A situation where problem (20) can be handled easily in the case where system (10), or (18), contains a small parameter, and for $\varepsilon = o$ the reduced system is known to possess periodic solutions of period T.

In two of the examples below we use an implicit function theorem in functional analysis we have proved elsewhere [1b].

Example 1. Let us consider the problem of periodic solutions of period $T = 2\pi$ of the equation

$$u_t = \varepsilon u_z + \sin t , \tag{21}$$

where ε is a small parameter, $\nu = 1$, $n = 1$. If we take an arbitrary function $u_o(z)$, $z \in B_\pi$, $u_o \in H_1$, then the method of successive approximations

$$u_{-1}(t,z) = o \quad , \quad u_k(t,z) = u_o(z) + \int_o^t [\varepsilon u_{k-1,z}(\tau,z) + \sin \tau]d\tau, \ k = 1,2,\ldots,$$

yields

$$u_o(t,z) = u_o(z) + 1 - \cos t ,$$

$$u_1(t,z) = u_o(z) + \varepsilon t u_o'(z) + 1 - \cos t , \qquad (') = d/dt ,$$

$$u_2(t,z) = u_o(z) + \varepsilon t u_o'(z) + 2^{-1} \varepsilon^2 t^2 u''(z) + 1 - \cos t,$$

and hence

$$u_k(t,z) = 1 - \cos t + u_o(z) + \varepsilon t u_o'(z) + (\varepsilon^2 t^2/2!)u''(z) + \ldots + (\varepsilon^k t^k/k!)u^{(k)}/z).$$

By the previous analysis we know that the series

$$u(t,z) = 1\text{-}\cos t + \sum_{k=o}^{\infty} (\varepsilon^k t^k/k!)u^{(k)}(z) ,$$

is uniformly convergent in $[o,2\pi] \times B_{s\sigma}$ for all s, $o \leq s \leq s_o$, and ε, $|\varepsilon| \leq \varepsilon_o$, and suitable $s_o > o$, $\varepsilon_o > o$. The problem of determining $u_o(z)$ in such a way that $u(2\pi,z) = u_o(z)$ in $B_{s\sigma}$ reduces here to

$$\sum_{k=1}^{\infty} (\varepsilon^k t^k/k!)u^{(k)}(z) = o, \quad z \in B_{s\sigma} \quad ,$$

for all ε in absolute value sufficiently small. The only solution is $u_o'(z) = o$, or $u_o(z) = C$, a constant. Thus, all periodic solutions of (21) for ε in absolute value sufficiently small are of the form

$$u(t,z) = C + 1 - \cos t.$$

Example 2. Let us consider the problem of periodic solutions of period $T = 2\pi$ of the equation

$$u_t = \varepsilon(u + u_z) + \sin t,$$

where ε is a small parameter. If we take an arbitrary $u_o(z)$, $z \in B_\sigma$, $u_o \in H_1$, then the method of successive approximations

$$u_{-1}(t,z) = o, \; u_k(t,z) = u_o(z) + \int_o^t [\varepsilon(u_{k-1}(\tau,z)+u_{k-1,z}(\tau,z)) + \sin \tau] \, d\tau,$$

$k = o,1,2,\ldots$, yields

$$u_o(t,z) = u_o(z) + 1 - \cos t$$

$$u_1(t,z) = u_o(z) + \varepsilon t u_o(z) + \varepsilon t u_o'(z) + \varepsilon(t-\sin t) + 1 - \cos t \ ,$$

$$u_2(t,z) = u_o(z) + \varepsilon t u_o(z) + (\varepsilon^2 t^2/2)u_o(z) + \varepsilon t u_o'(z) + \varepsilon^2 t^2 u_o'(z) +$$

$$+ (\varepsilon^2 t^2/2)u_o''(z) + 1 - \cos t + \varepsilon(t - \sin t) \ ,$$

where $(') = d/dz$. At the limit as $k \to o$ we have

$$u(t,z,\varepsilon) = 1 - \cos t + \varepsilon(t - \sin t) + u_o(z) + \varepsilon t(u_o(z) + u_o'(z)) + o(\varepsilon^2)$$

for all $(t,z) \in [o,2\pi] \times B_{s\sigma}$, and all s, ε, $o \leq s \leq s_o < 1$, $|\varepsilon| \leq \varepsilon_o$, and suitable s_o and ε_o positive.

We now discuss the question as to whether $u_o(z)$ exists such that $u(T;z,\varepsilon) = u_o(z)$, $|z| \leq s'\sigma$, at least for all $|z| \leq s'\sigma$, $o < s' \leq s \leq 1$, and s' sufficiently small. If $u_o(o) = C$, an arbitrary constant, we see that a necessary condition is that $u_o'(o) = -C$. We shall now write u in the form $u(t,z,\varepsilon; u_o(\zeta), |\zeta| \leq \sigma)$ to emphasize that u depends functionally on $u_o(z)$. Also, we take $v(z) = u_o'(z)$ as a new unknown function, and we write μ for $v(o) = u_o'(o)$. Then the functional equation $u(2\pi,z,\varepsilon) = u_o(z)$ takes the form

$$W(z,\varepsilon;v(\zeta), |\zeta| \leq \sigma) = \varepsilon^{-1} [u(2\pi,z,\varepsilon,u_o(\zeta), |\zeta| \leq \sigma) - u_o(z)]$$

$$= u(2\pi,z,\varepsilon, c + \int_o^{\zeta} v(\alpha)d\alpha, |\zeta| \leq \sigma) - c - \int_o^{z} v(\alpha)d\alpha = o, \ |z| \leq s\sigma \ ,$$

where W is holomorphic in z, where $W(o,o,\cdot)$ reduces to $c + \mu = o$, and the functional determinant $\partial W(o,o,\cdot)/\partial\mu = 1 \neq o$. By an implicit function theorem similar to the ones in [1b], we know that there are numbers s', ε_o and a function $u_o(z;\varepsilon)$, $|z| \leq \sigma$, $|\varepsilon| \leq \varepsilon_o$, $o \leq s' \leq s \leq 1$, $\varepsilon_o > o$, such that $u(2\pi,z,\varepsilon)-u_o(z,\varepsilon) = o$

for all $|z| \leq s'\sigma$, $|\varepsilon| \leq \varepsilon_o$. Also,

$$u_o(z,\varepsilon) = c - cz + O(\varepsilon) + O(z^2) ,$$

$$u(t,z,\varepsilon) = u_o(z,\varepsilon) + \varepsilon t(u_o(z,\varepsilon) + u_o'(z,\varepsilon)) + 1 - \cos t + \varepsilon(t - \sin t) + O(\varepsilon) .$$

Example 3. Let us consider the problem of periodic solutions of period $T = 2\pi$ of the equation

$$u_{tt} + \omega^2 u = \varepsilon(1-u^2)u_t + \varepsilon p\omega \cos(\omega t + \alpha) + \varepsilon \beta u_z , \tag{22}$$

where $\nu = 1$, $n = 1$, $T = 2\pi/\omega$, $\omega > o$, p, β are real constant, and ε is a small real parameter. For $\beta = o$, equation (22) reduces to the usual van der Pol equation with a forcing term ([19], (8.5.26), p. 133). The usual transformation

$$y_1 = i\omega u + u_t , \quad y_2 = i\omega u - u_t ,$$

yields the system of partial differential equations

$$y_{1t} = i\omega y_1 + \varepsilon f , \quad y_{2t} = -i\omega y_2 - \varepsilon f ,$$

$$f = 2^{-1}[1 + (2\omega)^{-2}(y_1+y_2)^2] (y_1-y_2) + p\omega \cos(\omega t + \alpha)+(2i\omega)^{-1} \beta(y_{1z} + y_{2z}) .$$

Finally, the transformation $Y_1 = e^{-i\omega t}y_1$, $Y_2 = e^{i\omega t}y_2$, yields

$$Y_{1t} = \varepsilon e^{-i\omega t}F, \qquad Y_{2t} = -\varepsilon e^{i\omega t}F,$$

$$F(t,z,Y) = 2^{-1}[1+(2\omega)^{-2}(e^{i\omega t}Y_1 + e^{-i\omega t}Y_2)^2] (e^{i\omega t}Y_1 - e^{-i\omega t}Y_2)$$

$$+ p\omega \cos(\omega t + \alpha) + (2i\omega)^{-1} \beta(e^{i\omega t}Y_1 + e^{-i\omega t}Y_2),$$

where $Y = (Y_1, Y_2)$. If $Y_{10}(z)$, $Y_{20}(z)$, $|z| \leq \sigma$, denote arbitrary functions, the method of successive approximations

$$Y_1^{(-1)} = o, \qquad Y_2^{(-1)} = o,$$

$$Y_1^{(k)}(t,z) = Y_{10}(z) + \varepsilon \int_o^t e^{-i\omega t} F(\tau, z, Y^{(k-1)}(\tau,z))d\tau ,$$

$$Y_2^{(k)}(t,z) = Y_{20}(z) - \varepsilon \int_o^t e^{i\omega t} F(\tau, z, Y^{(k-1)}(\tau,z))d\tau ,$$

$k = o,1,2,\ldots$, yields first

$$Y_1^{(o)}(t,z) = Y_{10}(z) + \varepsilon p\omega \, [2^{-1} t \, e^{i\alpha} - (4i\omega)^{-1}(e^{-2i\omega t - i\alpha} - e^{-i\alpha})] ,$$

$$Y_2^{(o)}(t,z) = Y_{20}(z) - \varepsilon p\omega \, [2^{-1} t \, e^{-i\alpha} + (4i\omega)^{-1}(e^{2i\omega t + i\alpha} - e^{i\alpha})] .$$

Also we have

$$Y_1^{(1)}(t,z) = Y_{10}(z) + \varepsilon \, \{2^{-1}[t \, Y_{10}(z) - (-2i\omega)^{-1} e^{-2i\omega t} Y_{20}(z)]$$

$$+ 2^{-1}(2\omega)^{-2}[(2i\omega)^{-1} e^{2i\omega t} Y_{10}^{3}(z) + tY_{10}^{2}(z)Y_{20}(z) - (-2i\omega)^{-1} e^{2i\omega t} Y_{10}(z)Y_{20}^{2}(z)$$

$$- (-4i\omega)^{-1} \, e^{-4i\omega t} \, Y_{20}^{3}(z) \,]$$

$$+ p\omega \, [2^{-1} t \, e^{i\alpha} - (4i\omega)^{-1} \, (e^{-2i\omega t - i\alpha} - e^{-i\alpha})]$$

$$+ (2i\omega)^{-1} \, [t \, Y_{10z}(z) + (-2i\omega)^{-1} \, e^{-2i\omega t} \, Y_{20z}(z)] + \ldots\} ,$$

$$Y_2^{(1)}(t,z) = Y_{20}(z) - \varepsilon \, \{2^{-1} \, [(2i\omega)^{-1} \, e^{2i\omega t} \, Y_{10}(z) - t \, Y_{20}(z)]$$

$$+ 2^{-1}(2\omega)^{-2} \, [(4i\omega)^{-1} \, e^{4i\omega t} \, Y_{10}^{3}(z) + (2i\omega)^{-1} \, e^{2i\omega t} \, Y_{10}^{2}(z) \, Y_{20}(z)$$

$$- t \, Y_{10}(z) \, Y_{20}^{2}(z) - (-2i\omega)^{-1} e^{-2i\omega t} \, Y_{20}^{3}(z)]$$

$$+ p\omega\,[(4i\omega)^{-1}\,(e^{2i\omega t + i\alpha} - e^{i\alpha}) + 2^{-1}\,t\,e^{-i\alpha}]$$

$$+ (2i\omega)^{-1}\beta\,[(2i\omega)^{-1}\,e^{2i\omega t}\,Y_{1oz} + t\,Y_{2oz}(z)\,] + \dots \}\,,$$

where we have written only the terms in ε^o and ε. Only these same terms in ε^o and ε^1 are in all successive approximations $Y_1^{(k)}(t,z)$, $Y_2^{(k)}(t,z)$, and thus in the limits $Y_1(t,z)$, $Y_2(t,z)$ as $k \to \infty$.

The equations $\varepsilon^{-1}[Y_1(T,z) - Y_{1o}(z)\,] = o$, $\varepsilon^{-1}[Y_2(T,z) = Y_{2o}(z)] = o$, yield now

$$Y_{1o}(z) + (2\omega)^{-2}Y_{1o}^{2}(z)Y_{2o}(z) + p\omega e^{i\alpha} + (i\omega)^{-1}\beta\,Y_{1oz} + O(\varepsilon) = o,$$

$$Y_{2o}(z) + (2\omega)^{-2}Y_{1o}(z)Y_{2o}^{2}(z) - p\omega e^{-i\alpha} - (i\omega)^{-1}\beta\,Y_{2oz} + O(\varepsilon) = o\,.$$

If we take $Y_{1o}(z) = \lambda(z)e^{i\theta(z)}$, $Y_{2o}(z) = -\lambda(z)e^{-i\theta(z)}$, λ, θ real, then $Y_{1oz} = (\lambda_z + i\lambda\theta_z)e^{-i\theta}$, $Y_{2oz} = (-\lambda_z + i\lambda\theta_z)e^{-i\theta}$, and we obtain the real equations

$$\begin{aligned} &\lambda^3 - 4\omega^2\lambda - 4p\omega^3\cos(\alpha-\theta) - 4\omega\beta\lambda\theta_z + O(\varepsilon) = o\,, \\ &p\omega\sin(\alpha-\theta) - \omega^{-1}\beta\,\lambda_z + O(\varepsilon) = o\,. \end{aligned} \tag{23}$$

For $\varepsilon = o$, $\beta = o$ the equations reduce to the usual ones for periodic solutions of the van der Pol equation with forcing term ([1a]), (8.5.27), p. 133).

Note that the equation $\lambda^3 - 4\omega^2\lambda - 4p\omega^3 = o$ has certainly a simple positive root λ_o. If β is small, we shall think θ close to α, $\sin(\alpha-\theta)$ close to zero, $\cos(\alpha-\theta)$ close to 1, and λ close to λ_o.

For every vector $\Lambda = (\Lambda_1,\Lambda_2)$ in a suitable neighborhood V of (o,o) let us denote by $\lambda(\Lambda)$ and $\theta(\Lambda)$ the roots close to λ_o and α respectively of the equations

$$\lambda^3 - 4\omega^2\lambda - 4p\omega^3\cos(\alpha-\theta) = 4\omega\beta\lambda\Lambda_1 ,$$

$$p\omega \sin(\alpha-\theta) = \omega^{-1}\beta\Lambda_2 .$$

We shall now denote by $W = (W_1, W_2)$ in vectorial form the functional defined, for $|z| \leq s\sigma$, $|\varepsilon| \leq \varepsilon_o$, by the first members of the equations

$$W_1(z,\varepsilon;\ \Lambda(\xi), |\varepsilon| \leq \sigma) = \lambda^3 - 4\omega^2\lambda - 4p\omega^3\cos(\alpha-\theta) - 4\omega\beta\lambda\Lambda_1 + O(\varepsilon) ,$$

$$W_2(z,\varepsilon;\ \Lambda(\xi), |\xi| \leq \sigma) = p\omega\sin(\alpha-\theta) - \omega^{-1}\beta\Lambda_2 + O(\varepsilon) ,$$

where $\lambda = \lambda(\Lambda)$, $\theta = \theta(\Lambda)$. For $z = o$, $\varepsilon = o$ these equations reduce to the same equations without the terms $O(\varepsilon)$, they can be satisfied by taking $\lambda(o,o) = \lambda(\Lambda)$, $\theta(o,o) = \theta(\Lambda)$, and the functional determinant of $W(o,o,\cdot)$ with respect to Λ is then $(-4\omega\beta\lambda(o,o))(-\omega^{-1}\beta) = 4\beta^2\lambda(\Lambda)$, a number close to $4\beta^2\lambda_o$, and hence $\neq o$. By the use of the same implicit function theorems mentioned in example 2 we conclude that there is a solution $\Lambda_1(z,\varepsilon)$, $\Lambda_2(z,\varepsilon)$ or $\lambda(z,\varepsilon)$, $\theta(z,\varepsilon)$, to equations (23), for all $|z| \leq s'\sigma$, $|\varepsilon| \leq \varepsilon_1$, and some s', ε_1, $o < s_1 \leq s \leq 1$, $o < \varepsilon_1 \leq \varepsilon_o$. The solutions Y_1, λ_2 then have the form

$$Y_1(t,z) = \lambda(z,\varepsilon)e^{i\theta(z,\varepsilon)} + O(\varepsilon), \quad Y_2(t,z) = -\lambda(z,\varepsilon)e^{-i\theta(z,\varepsilon)} + O(\varepsilon) ,$$

and hence

$$x(t,\varepsilon) = \omega^{-1}\lambda(z,\varepsilon)\sin[\theta(z,\varepsilon) + \omega t] + O(\varepsilon) ,$$

is a periodic solution of period $T = 2\pi/\omega$ of equation (22), with $|z| \leq s'\sigma$, $|\varepsilon| \leq \varepsilon_1$ and where $\Lambda_1(o,o) = \lambda_z(o,o)$, $\Lambda_2(o,o) = \theta_z(o,o)$, are arbitrary numbers, $(\Lambda_1, \Lambda_2) \in V$.

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