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SEMINORMALITY AND UPPER SEMICONTINUITY IN OPTIMAL CONTROL

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In the present paper we discuss properties of upper semicontinuity of variable convex closed sets in Euclidean spaces, taking into consideration the modification of Kuratowski's concept of upper semicontinuity [4] which we denoted in [1a,b] as property (Q). We have used this property in the proof of lower closure theorems in Lagrange and Mayer problems of optimal control. These theorems reduce to well known lower semicontinuity statements for usual free problems of the calculus of variations. Lower closure theorems are used to prove existence theorems for Lagrange and Mayer problems of optimal control [1a,b]. The same property (Q) mentioned above was used again in recent studies by J. R. LaPalm [5], A. Lasota and C. Olech [6], C. Olech [8], L. Cesari, T. Nishiura, and J. R. LaPalm [2], and in recent papers by Cesari [1c,d,e,f] concerning existence theorems for Lagrange problems with multiple integrals and partial differential equations. In these papers we requested property (Q) for variable sets $\tilde{\tilde{Q}}(x)$ of the form $\tilde{\tilde{Q}}(x) = [(z^o,z)|z^o \geq f_o(x,u), z = f(x,u), u \in U(x)] \subset E_{n+1}$, where f_o and $f = (f_1,\dots,f_n)$ are given continuous functions.

In the present paper we give criteria (Sections 2, 3, 6, and 8) for property (Q) of the sets $\tilde{\tilde{Q}}(x)$ in addition to those already stated in [1a,b]. In particular we show (Sections 3 and 8) that a slight particularization of

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property (Q) for these sets $\tilde{\tilde{Q}}(x)$ can be expressed in a form which is similar to Tonelli's seminormality condition [9] for free problems of the calculus of variations. Thus, property (Q) of the sets $\tilde{\tilde{Q}}(x)$ is shown here to represent a generalization—for Lagrange problems—of the well known seminormality condition for free problems. In Sections 4, 5, and 7 we state a number of properties of convex real-valued functions on a convex subset of E_n related to the concept of seminormality, and we use these results in Section 8. In Section 6 we prove another criterion for property (Q) of the sets $\tilde{\tilde{Q}}(x)$ when f is linear in u and f_o is convex and seminormal in u.

1. PROPERTIES (U) AND (Q) OF VARIABLE SETS

Let A be a given subset of the x-space E_r, for every $x \in A$ let U(x) be a given subset of the u-space E_m, and let M be the set of all (x,u) with $x \in A$, $u \in U(x)$. Thus, M is the graph of U(x) in the space $E_r \times E_m$. For every $\bar{x} \in A$ and $\delta > 0$ let $N_\delta(\bar{x})$ denote the set of all $x \in A$ at a distance $\leq \delta$ from $\bar{x}$. For every $\bar{x} \in A$ and $\delta > 0$ let $U(\bar{x};\delta)$ denote the union of all U(x) with $x \in N_\delta(\bar{x})$, or $U(\bar{x};\delta) = [u \in E_m | u \in U(x),\ x \in N_\delta(\bar{x})]$. We say that the sets U(x) have property (U) at a point $\bar{x} \in A$ if

$$U(\bar{x}) = \bigcap_\delta \text{cl } U(\bar{x};\delta) \quad . \tag{1.1}$$

We say that the sets U(x) have property (Q) at $\bar{x} \in A$ if

$$U(\bar{x}) = \bigcap_\delta \text{cl co } U(\bar{x};\delta) \quad . \tag{1.2}$$

Here cl and co denote the closure and the convex hull respectively of the sets under consideration. We say that the sets U(x) have property (U) [(Q)] in A if this property holds at every point $\bar{x} \in A$. Property (U) is Kuratovski's concept of uppersemicontinuity of sets [4], and was used, for instance, by G. Choquet [3] and E. Michael [7].

Note that in (1.1) and (1.2) the sign $\subset$ holds trivially, and thus the actual requirements can be written in the form

$$U(\bar{x}) \supset \cap_\delta \text{ cl } U(\bar{x};\delta) \quad , \text{ or}$$

$$U(\bar{x}) \supset \cap_\delta \text{ cl co } U(\bar{x};\delta) \ ,$$

respectively. The following statements are easily proved:

(1.i) If $U(x)$ has property (U) at $\bar{x}$, then $U(\bar{x})$ is closed.

(1.ii) If $U(x)$ has property (Q) at $\bar{x}$, then $U(\bar{x})$ is closed and convex.

(1.iii) If A is closed, then $U(x)$ has property (U) in A if and only if M is closed.

A number of other statements concerning properties (U) and (Q) have been stated in [1a,b], and will not be repeated here.

If $f(x,u) = (f_1,\dots,f_n)$, $(x,u) \in M$, is a given vector function, $f: M \to E_n$, we shall denote by $Q(x) \subset E_n$ the set $Q(x) = f(x,U(x))$, or $Q(x) = [z = (z^1,\dots, z^n) | z = f(x,u),\ u \in U(x)]$.

2. THE SETS $\tilde{\tilde{Q}}(x)$ AND A FIRST CRITERION FOR PROPERTY (Q)

In Lagrange problems of optimal control and the calculus of variations, besides the vector function $f(x,u) = (f_1,\dots,f_n)$, also a scalar function $f_o(x,u)$ is given, $f_o = M \to E_1$.

If $\tilde{f}(x,u) = (f_o,f) = (f_o,f_1,\dots,f_n)$, then we may denote by $\tilde{Q}(x) \subset E_{n+1}$ the set $f(x,U(x))$, or $\tilde{Q}(x) = [(z^o,z)\,|\,z^o = f_o(x,u),\ z = f(x,u),\ u \in U(x)]$. Also, we shall denote by $\tilde{\tilde{Q}}(x)$ the set

$$\tilde{\tilde{Q}}(x) \;=\; [(z^o,z)\,|\,z^o \geq f_o(x,u),\ z = f(x,u),\ u \in U(x)] \quad .$$

We may say that $\tilde{Q}(x)$ is the "figurative," and that $\tilde{\tilde{Q}}(x)$ is the set of points "above the figurative." Note that for every $x \in A$ the set $Q(x)$ is the projection on the z-space E_n of the set $\tilde{\tilde{Q}}(x) \subset E_{n+1}$. Thus, if $\tilde{\tilde{Q}}(x)$ is convex, then certainly $Q(x)$ is also convex.

We shall say that a function $g(x,u)$ is "of slower growth than $f_o(x,u)$ as $|u| \to \infty$ uniformly in some subset A_o of A, provided given $\varepsilon > 0$ there is some $\bar{u} = \bar{u}(\varepsilon,A_o) \geq 0$ such that $x \in A_o$, $u \in U(x)$, $|u| \geq \bar{u}$ implies $|g(x,u)| \leq \varepsilon\, f_o(x,u)$.

(2.i) (a criterion for property (Q) under a growth condition) Given A closed, M closed, $f_o(x,u)$ and $f(x,u) = (f_1,\dots,f_n)$ continuous on M, assume that 1 and f are of slower growth than f_o as $|u| \to +\infty$ uniformly on some neighborhood A_o of a point $\bar{x} \in A$. If the set $\tilde{\tilde{Q}}(\bar{x})$ is convex, then the sets $\tilde{\tilde{Q}}(x)$ satisfy property (Q) at $\bar{x}$.

A proof of (2.i) has been given in [1b].

3. A SECOND CRITERION FOR PROPERTY (Q)

Note that if the sets $\tilde{\tilde{Q}}(x)$ satisfy property (Q) at a point $\bar{x} \in A$ then

$$\tilde{\tilde{Q}}(\bar{x}) \supset \cap_\delta \text{ cl co } \tilde{\tilde{Q}}(\bar{x};\delta)$$

This means that, if a point (z^o,z) belongs to the set $\bigcap_\delta$ cl co $\tilde{\tilde{Q}}(\bar{x};\delta)$, then $(z^o,z) \in \tilde{\tilde{Q}}(\bar{x}) = [(z^o,z) | z^o \geq f_o(\bar{x},u),\ z = f(\bar{x},u),\ u \in U(\bar{x})]$; hence

$$z \in Q(\bar{x}) = [z = f(\bar{x},u),\ u \in U(\bar{x})] \quad .$$

In other words, the following property (α) is a necessary condition for the sets $\tilde{\tilde{Q}}(x)$ to have property (Q) at $\bar{x}$:

(α) If $(z^o,z) \in \bigcap_\delta$ cl co $\tilde{\tilde{Q}}(\bar{x};\delta)$, then $z \in Q(\bar{x})$.

For free problems of the calculus of variations $n = m$, $f = u$, or $f_i = u^i$, $i = 1,\ldots,n$, and $U(x) = E_n$. For these problems then the sets under consideration reduce to $Q(x) = E_n$ and

$$\tilde{Q}(x) = [(z^o,u) | z^o = f_o(x,u),\ u \in E_n] \subset E_{n+1} \quad ,$$

$$\tilde{\tilde{Q}}(x) = [(z^o,u) | z^o \geq f_o(x,u),\ u \in E_n] \subset E_{n+1} \quad .$$

Thus, property (α) is trivially satisfied for free problems.

We shall now introduce the following "condition" (X), at a point $\bar{x} \in A$:

(X) for every $\bar{z} \in Q(\bar{x})$, there is at least one point $\bar{u} \in (U(\bar{x})$ with $\bar{z} = f(\bar{x},\bar{u})$ and the following property: given $\varepsilon > 0$ there are numbers $\delta > 0$, and r, $b = (b_1,\ldots,b_n)$ real, such that

$$(X')\ f_o(x,u) \geq r + \textstyle\sum_j b_j f_j(x,u) \text{ for all } x \in N_\delta(\bar{x}) \text{ and } u \in U(x);$$

$$(X'')\ f_o(\bar{x},\bar{u}) \leq r + \textstyle\sum_j b_j f_j(\bar{x},\bar{u}) + \varepsilon \quad .$$

For free problems (that is, $m = n$, $f = u$, $U = E_n$) the present property (X) reduces to the following one concerning the function f_o only:

(X_f) For every $\bar{u} = (\bar{u}^1,\dots,\bar{u}^m) \in E_m$ and $\varepsilon > 0$ there are numbers $\delta > 0$ and r, $b = (b_1,\dots,b_m)$ real such that

$$(X_f') \; f_o(x,u) \geq r + \textstyle\sum_j b_j u^j \text{ for all } x \in N_\delta(\bar{x}) \text{ and all}$$

$$u = (u^1,\dots,u^m) \in E_m \quad ;$$

$$(X_f'') \; f_o(\bar{x},\bar{u}) \leq r + \textstyle\sum_j b_j \bar{u}^j + \varepsilon \quad .$$

As we shall see in no. 5 below, this condition (X_f) is the well known weak seminormality condition of the function f_o at $(\bar{x},\bar{u})$ for all $\bar{u} \in E_m$.

(3.i) (a criterion for property (Q) under conditions (α) and (X)) If conditions (α) and (X) hold at the point $\bar{x} \in A$, then $\tilde{\tilde{Q}}(\bar{x})$ is closed and convex, and the sets $\tilde{\tilde{Q}}(x)$ satisfy property (Q) at the point $\bar{x}$.

<u>Proof</u>. To prove that the sets $\tilde{\tilde{Q}}(x)$ satisfy property (Q) at $\bar{x}$ (and hence $\tilde{\tilde{Q}}(\bar{x})$ is closed and convex), we have only to prove that, if $\tilde{\bar{z}} = (\bar{z}^o,\bar{z}) \in \bigcap_\delta$ cl co $\tilde{\tilde{Q}}(\bar{x};\delta)$, then $\tilde{\bar{z}} = (\bar{z}^o,\bar{z}) \in \tilde{\tilde{Q}}(\bar{x})$. From condition (α) all we know is that $\bar{z} \in Q(\bar{x})$. Hence, there is some $\bar{u} \in U(\bar{x})$ such that $\bar{z} = f(\bar{x},\bar{u})$, (hence $(\bar{x},\bar{u}) \in M$), and statements (X'), (X") hold.

For every $\delta > 0$ we have $\tilde{\bar{z}} = (\bar{z}^o,\bar{z}) \in$ cl co $\tilde{\tilde{Q}}(x;\delta)$, and thus for every $\delta > 0$ there are points $\tilde{z} = (z^o,z) \in$ co $\tilde{\tilde{Q}}(\bar{x};\delta)$ at a distance as small as we want from $\tilde{\bar{z}} = (\bar{z}^o,\bar{z})$. Thus, there is a sequence of numbers $\delta_k > 0$ and of points $\tilde{z}_k = (z_k^o,z_k) \in$ co $\tilde{\tilde{Q}}(\bar{x};\delta_k)$ such that $\delta_k \to 0$, $\tilde{z}_k \to \tilde{\bar{z}}$ as $k \to \infty$. In other words, for every integer k there is a system of points $x_k^\gamma \in N_{\delta_k}(\bar{x})$, $\gamma = 1,\dots,\nu$, say $\nu = n + 2$, corresponding points $\tilde{z}_k^\gamma = (z_k^{o\gamma},z_k^\gamma) \in \tilde{\tilde{Q}}(x_k^\gamma)$, points $u_k^\gamma \in U(x_k^\gamma)$, and numbers $\lambda_k^\gamma \geq 0$, $\gamma = 1,\dots,\nu$, such that

$$1 = \sum \lambda_k^\gamma,\ \tilde{z}_k = \sum \lambda_k^\gamma \tilde{z}_k^\gamma,\ z_k^o = \sum \lambda_k^\gamma z_k^{o\gamma},\ z_k = \lambda_k^\gamma z_k^\gamma,$$

$$z_k^{o\gamma} \geq f_o(x_k^\gamma, u_k^\gamma),\ z_k^\gamma = f(x_k^\gamma, u_k^\gamma) \quad , \tag{3.1}$$

where $\gamma = 1,\ldots,\nu$, where $\sum$ ranges over all $\gamma = 1,\ldots,\nu$, and $z_k^\gamma \to \bar{x}$, $\tilde{z}_k \to \tilde{\bar{z}}$, $z_k^o \to \bar{z}^o$, $z_k \to \bar{z}$ as $k \to \infty$, $\gamma = 1,\ldots,\nu$.

By condition (X') there is a neighborhood $N_\delta(\bar{x})$ of $\bar{x}$ in A, and numbers r, $b = (b_1,\ldots,b_\nu)$ real, such that

$$\bar{f}_o(x,u) = f_o(x,u) - r - b\cdot f(x,u) \geq 0 \text{ for all } x \in N_\delta(\bar{x}) \text{ and } u \in U(x); \tag{3.2}$$

$$\bar{f}_o(\bar{x},\bar{u}) = f_o(\bar{x},\bar{u}) - r - b\cdot f(\bar{x},\bar{u}) \leq \varepsilon \quad . \tag{3.3}$$

For k sufficiently large, so that $|x_k^\gamma - \bar{x}| < \delta$, $\gamma = 1,\ldots,\nu$, we have now, from (3.1) and (3.2),

$$z_k^o = \sum \lambda_k^\gamma z_k^{o\gamma} \geq \sum \lambda_k^\gamma f_o(x_k^\gamma, u_k^\gamma) \geq \sum \lambda_k^\gamma [r + b\cdot f(x_k^\gamma, u_k^\gamma)]$$

$$= \sum \lambda_k^\gamma [r + b\cdot z_k^\gamma] = r + b\cdot \sum \lambda_k^\gamma z_k^\gamma = r + b\cdot z_k \quad .$$

As $k \to \infty$ we obtain $\bar{z}^o \geq r + b\cdot\bar{z}$; hence, from (3.3),

$$\bar{z}^o \geq r + b\cdot\bar{z} = r + b\cdot f(\bar{x},\bar{u})$$

$$= f_o(\bar{x},\bar{u}) - \bar{f}_o(\bar{x},\bar{u}) \geq f_o(\bar{x},\bar{u}) - \varepsilon \quad .$$

Here $\varepsilon > 0$ is arbitrary; hence $\bar{z}^o \geq f_o(\bar{x},\bar{u})$, while $\bar{z} = f(\bar{x},\bar{u})$. This shows that $\tilde{\bar{z}} = (\bar{z}^o,\bar{z}) \in \tilde{\tilde{Q}}(\bar{x})$. We have proved that the sets $\tilde{\tilde{Q}}(x)$ satisfy property (Q) at the point $\bar{x} \in A$. Statement (3.i) is thereby proved.

4. SOME PROPERTIES OF CONVEX FUNCTIONS

If U is a given subset of E_n and F(u), u ε U, a real-valued function, then F(u) is said to be convex in u provided U is convex, and $u_1, u_2 \in U$, $0 \leq \alpha \leq 1$, implies $F(\alpha u_1 + (1 - \alpha)u_2) \leq \alpha F(u_1) + (1 - \alpha) F(u_2)$. The following statements are well known:

(4.i) If U is a convex subset of E_n and F(u), u ε U, a given real-valued function, then F(u) is convex if and only if $u_j \in U$, $\lambda_j \geq 0$, $j = 1,\ldots,\nu$, ν finite, $\lambda_1 + \ldots + \lambda_\nu = 1$, $u_o = \sum_{j=1}^{\nu} \lambda_j u_j$, implies $F(u_o) \leq \sum_{j=1}^{\nu} \lambda_j F(u_j)$.

(4.ii) If U is a convex subset of E_n, and F(u), u ε U, a given real-valued function, then F(u) is convex if and only if the set $\tilde{\tilde{Q}} = [(z,u) | z \geq F(u), u \in U] \subset E_{n+1}$ is convex.

A linear scalar function

$$z(u) = r + b_1 u^1 + \ldots + b_n u^n = r + b \cdot u , \quad u \in E_n,$$

$r, b_1,\ldots,b_n$ real, is said to be a <u>supporting plane</u> of F(u), u ε U, at a point $\bar{u} \in U$, provided $F(\bar{u}) = z(\bar{u})$ and $F(u) \geq z(u)$ for all u ε U.

(4.iii) If U is a convex subset of E_n, and F(u), u ε U, a given real-valued convex function, then F(u) has a supporting plane at every interior point $\bar{u}$ of U.

<u>Proof</u>. We know already that the set $\tilde{\tilde{Q}} = [(z,u) | z \geq F(u), u \in U] \subset E_{n+1}$ is convex, and hence there exists some supporting hyperplane V at $(\bar{z},\bar{u})$, $\bar{z} = F(\bar{u})$. If $V = [(z,u) | p^o z + p \cdot u - c \geq 0]$, p^o, $p = (p^1,\ldots,p^n)$, c real, then

$p^o\bar{z} + p\cdot\bar{u} - c = 0$ and $p^o z + p\cdot u - c \geq 0$ for all $u \in U$ and $z \geq F(u)$. Let us prove that $p^o \neq 0$. Indeed, if $p^o = 0$, then we have $p\cdot\bar{u} - c = 0$, $p\cdot u - c \geq 0$ for all $u \in U$. If $u_1 \in E_n$ is a point where $p\cdot u_1 - c > 0$, and ε real, then for $u = u(\varepsilon) = \varepsilon u_1 + (1 - \varepsilon)\bar{u}$ we have $p\cdot u(\varepsilon) - c > 0$ for all $\varepsilon > 0$, and $pu(\varepsilon) - c < 0$ for all $\varepsilon < 0$, with $u(\varepsilon) \to \bar{u}$ as $\varepsilon \to 0$. Since $\bar{u} \in \text{int } U$, then both $u(\varepsilon)$, $u(-\varepsilon)$ both belong to U for $\varepsilon > 0$ sufficiently small, and $pu(-\varepsilon) - c < 0$, a contradiction. We have proved that $p_o \neq 0$. Actually, we must have $p_o > 0$ since $p^o z + p\cdot\bar{u} - c \geq 0$ for all $z \geq F(\bar{u})$. Finally, if we take $z(u) = (-p\cdot u + c)/p^o$, then $z(\bar{u}) = F(\bar{u})$ and $F(u) \geq z(u)$ for all $u \in U$.

Given a set U we denote as usual by int U the subset of its interior points.

If U has no interior points, that is, int $U = \phi$, statement (4.iii) has the following inplication. First, let us denote by R the hyperspace of E_n of minimum dimension r containing U. Then $U \subset R \subset E_n$, $0 \leq r \leq n$. If U is reduced to a single point then $R = U$ and $r = 0$. Otherwise, $1 \leq r \leq n$, and we denote by Rint U the certainly nonempty set of points of U which are interior to U with respect to R. Thus, int $U \subset$ Rint $U \subset U \subset R \subset E_n$. Statement (4.iii) has now the following corollary:

(4.iv) Under the same hypotheses as in (4.iii), F(u) has a supporting plane at every point $\bar{u} \in$ Rint U.

The following statement also is relevant:

(4.v) Under the same hypotheses as in (4.iii), F(u) is continuous at every point $\bar{u} \in$ Rint U.

Proof. We may well assume that U is not a single point, that $1 \leq r \leq n$, and Rint $U \neq \Phi$. Let $\bar{u}$ be any point $\bar{u} \in$ R int U, and let $z = c + p\cdot u$ the supporting plane at $\bar{u}$, so that $F(\bar{u}) = c + p\cdot\bar{u}$. Assume, if possible that for some $\sigma > 0$ and sequence of points $u_k \in$ Rint U, $u_k \to \bar{u}$ as $k \to \infty$, we have $F(u_k) - F(\bar{u}) \leq -\sigma$ for all k. Then, $F(u_k) \geq c + p\cdot u_k$, and hence $-\sigma \geq F(u_k) - F(\bar{u}) \geq p(u_k - \bar{u})$. As $k \to \infty$, we have $-\sigma \geq 0$, a contradiction. Assume now, if possible, that for some $\sigma > 0$ and sequence of points $u_k \in$ Rint U, $u_k \to \bar{u}$ as $k \to \infty$, we have $F(u_k) - F(\bar{u}) \geq \sigma$ for all k. Then the points $u'_k = 2\bar{u} - u_k = \bar{u} - (u_k - \bar{u})$ are also points of Rint U for k sufficiently large, and the relations hold

$$\bar{u} = 2^{-1}u_k + 2^{-1}u'_k \;,\; F(\bar{u}) \geq 2^{-1}F(u_k) + 2^{-1}F(u'_k) \;.$$

Then

$$F(u'_k) \leq 2F(\bar{u}) - F(u_k) \leq F(\bar{u}) - \sigma$$

with $u'_k \to \bar{u}$ as $k \to \infty$, and, as we have seen, we are led to a contradiction. This proves that F is continuous at every point $\bar{u} \in$ Rint U.

(4.vi) Under the same hypotheses as in (4.iii), F(u) is bounded below on every bounded part K of U.

Proof. Indeed, if K contains more than one point, then K contains some point $\bar{z} \in$ Rint U, and if $z(u) = c + p\cdot u$ is a supporting plane at $\bar{u}$, then $F(u) \geq c + p\cdot u$ for all $u \in K \subset U$, and $c + p\cdot u$ has a finite lower bound on K.

(4.vii) Under the same hypotheses as in (4.iii), $F(u)$ is upper semicontinuous at every point $\bar{u} \in U -$ Rint U along any segment s issued from $\bar{u}$ and contained in U.

Proof. Let s be the segment $s = \bar{u}u_o$, $s \subset U$. Assume if possible, that there is a sequence of points $u_k \in s \subset U$, $u_k \to \bar{u}$ as $k \to \infty$, with $F(u_k) - F(\bar{u}) \geq \sigma$ for all k. Then, all points interior to the segment s are certainly points of Rint U, say $u = (1 - \alpha)\bar{u} + \alpha u_o$, $0 < \alpha < 1$, and since $F(u) \leq (1 - \alpha) F(\bar{u}) + \alpha F(u_o)$, we see that F is bounded above on s. Since $h_k = u_k - \bar{u} \to 0$ as $k \to \infty$, there is a sequence of numbers $\beta_k > 1$ with $\beta_k \to \infty$, $\beta_k h_k \to 0$ as $k \to \infty$. Hence, the points $u_k' = \bar{u} + \beta_k(u_k - \bar{u})$, $k = 1,2,\dots$, are on the straight line from $\bar{u}$ containing s, and $u_k' \to \bar{u}$ as $k \to \infty$. Thus $u_k' \in s$, $u_k' \in$ Rint U for all k sufficiently large, and the relations hold

$$u_k = \beta_k^{-1}u_k' + \beta_k^{-1}(\beta_k - 1)\bar{u} \quad ,$$

$$F(u_k) \leq \beta_k^{-1}F(u_k') + \beta_k^{-1}(\beta_k - 1) F(\bar{u}) \quad ,$$

$$F(u_k') \geq \beta_k F(u_k) - (\beta_k - 1) F(\bar{u}) \geq F(\bar{u}) + \beta_k\sigma \quad .$$

Hence, $F(u_k') \to +\infty$ as $k \to +\infty$, a contradiction since F is bounded above on s. We have proved that F is upper semicontinuous at $\bar{u}$ along s.

A function $F(u)$, $u \in U$, convex on a convex set U may not be continuous at the points of $U -$ Rint U, as the following example shows. Take $U = [u|0 \leq u \leq 1]$, and $F(u) = 0$ for $0 < u < 1$, $F(u) = 1$ for $u = 0$ and $u = 1$.

(4.viii) If U is a convex subset of E_n, if $F(u)$, $u \in U$, is a given real-valued convex function on U, and the set $\approx{Q} = [(z^o,u) | z^o \geq F(u),\ u \in U]$ is (convex and) closed, then the function F(u) is lower semicontinuous at every point $\bar{u} \in U -$ Rint U, and even continuous on each segment s issued from $\bar{u}$ and contained in U.

<u>Proof</u>. Assume, if possible, that there is a number $\sigma > 0$ and points $\bar{u}$, u_k, $k = 1,2,\ldots$, with $\bar{u} \in U -$ Rint U, $u_k \in s \subset U$, $F(u_k) < F(\bar{u}) - \sigma$ for all k. Take $\bar{z}^o = F(\bar{u})$, and note that all points $(\bar{z}^o - \sigma,\ u_k)$ are in $\approx{Q}$. Then, as $k \to \infty$, we see that $(z^o - \sigma, \bar{u})$ is in the closed set $\approx{Q}$, a contradiction, since $(z,\bar{u}) \in \approx{Q}$ if and only if $z \geq \bar{z}^o = F(\bar{u})$. The last part of the statement is a consequence of (4.vii).

A function F(u), $u \in U$, convex on a convex set U may not be continuous at the points of U - Rint U, even if the set $\approx{Q}$ is closed, as the following example shows. Take $U = [(u,v) | 0 \leq u \leq 1,\ 0 \leq v \leq 1 - (1 - (1 - u)^2)^{1/2}$, $F(u,v) = v$ if $0 \leq u \leq 1$, $0 \leq v \leq u$, $F(u,v) = (2u)^{-1}(u^2 + v^2)$ if $0 < u < 1$, $u \leq v \leq 1 - (1 - (1 - u)^2)^{1/2}$. Obviously, U is convex, F is convex in (u,v), but F is not continuous at (0,0) since $F(0,0) = 0$, $F(u,(1 - (1 - u)^2)^{1/2}) = 1$ for all $0 < u < 1$.

Given a convex set $U \subset E_n$ and a scalar function F(u), $u \in U$, we say that F(u) is convex at the point $\bar{u} \in U$ provided $F(\bar{u}) \leq \sum_{j=1}^{\nu} \lambda_j F(u_j)$ for any convex combination $\bar{u} = \sum_{j=1}^{\nu} \lambda_j u_j$ of points $u_j \in U$.

(4.ix) If U is a convex subset of E_n, and F(u), $u \in U$, a given real-valued function, then F(u) is convex at an interior point $\bar{u}$ of U if and only if F(u) has a supporting plane at $\bar{u}$.

A proof of this statement can be found in L. Turner [10]. We repeat here the proof for the convenience of the reader.

Proof. Suppose F convex at the point $u \in \text{int}\, U$. Then the smallest convex set co $\tilde{\tilde{Q}}$ containing $\tilde{\tilde{Q}} = [(z,u)|z \geq F(u),\ u \in U] \subset E_{n+1}$ is the set of all points $(z,u) = \sum_{j=1}^{\nu} \lambda_j(z_j,u_j)$ with $(z_j,u_j) \in \tilde{\tilde{Q}}$, $\lambda_j \geq 0$, $\lambda_1 + \dots + \lambda_\nu = 1$, ν finite. Now $(z,\bar{u}) \notin \text{co}\, \tilde{\tilde{Q}}$ if $z < F(\bar{u})$ since, for every convex combination $(z,u) = \sum_{j=1}^{\nu} \lambda_j(z_j,d_j)$ with $u = \bar{u}$, $\bar{u} = \sum_j^{\nu} \lambda_j u_j$, we have $z = \sum_j \lambda_j z_j \geq \sum_j \lambda_j F(u_j) \geq F(\bar{u})$. Hence $z > z_o$, and therefore $(F(\bar{u}),\bar{u})$ is a boundary point of co $\tilde{\tilde{Q}}$. Then, there is a hyperplane $V = [(z,u)|p_o z + p\cdot u - c = 0] \subset E_{n+1}$ such that $p_o F(\bar{u}) + p\cdot\bar{u} - c = 0$ and $p_o z + p\cdot u - c \geq 0$ for all $(z,u) \in \text{co}\, \tilde{\tilde{Q}}$.

For every convex combination $\bar{z} = \sum \lambda_j u_j$ and numbers $z_j \geq F(u_j)$, we have $(z_j,u_j) \in \text{co}\, \tilde{\tilde{Q}}$, and $p_o z_j + p\cdot u_j - c \geq 0$. Therefore, $p_o[\sum_j \lambda_j z_j] + p\cdot\bar{u} - c \geq 0$, $p_o F(\bar{u}) + p\cdot\bar{u} - c = 0$, and $p_o[\sum \lambda_j z_j - F(\bar{u})] \geq 0$. Since this is true for arbitrary large z_j and $\lambda_j \geq 0$, we conclude that $p_o \geq 0$. But $p_o = 0$ implies $p\cdot u - c \geq 0$ for all $u \in U$, which is impossible, as in the previous proof. Thus $p_o > 0$ and this hyperplane V can be written now in the form $z = b\circ u + r$, with $b = -p/p_o$, $r = -c/p_o$, and $z \geq b\cdot u + r$ for all $(z,u) \in \text{co}\, \tilde{\tilde{Q}}$, $F(\bar{u}) = b\cdot\bar{u} + r$. Thus $z(u) = b\cdot u + r$ is a supporting plane for $F(u)$ at $u = \bar{u}$.

Conversely, if $F(u)$ has a supporting plane $z(u) = b\cdot u + r$ at $\bar{u} \in U$, then for every convex combination $\bar{u} = \sum_j \lambda_j u_j$ of points $u_j \in U$ we have $\sum_j \lambda_j F(u_j) \geq \sum_j \lambda_j z(u_j) = \sum_j \lambda_j [b\cdot u_j + r] = b\cdot\bar{u} + r = F(\bar{u})$, and $F(u)$ is convex at $\bar{u}$.

The following statement (4.v) concerns the case where $U = E_n$, $F(u)$ is convex in u in E_n, and (4.v) gives a characterization of those F which are linear on no straight line of E_n.

(4.x) If $F(u)$, $u \in E_n$, is convex in u, then there are no points u_o, $u_1 \in E_n$ with $u_1 \neq 0$ such that

$$F(u_o) = 2^{-1}[F(u_o) + \lambda u_1) + F(u_o - \lambda u_1)] \text{ for all real } \lambda, \qquad (4.1)$$

if and only if there is a linear function $w(u) = \bar{r} + \bar{b}\cdot u$, $u \in E_n$, $\bar{r}$, $\bar{b} = (\bar{b}_1,\dots,\bar{b}_n)$ real, such that $F(u) \geq w(u)$ for all $u \in E_n$, and $f(u) - w(u) \to +\infty$.

This statement was essentially proved by L. Tonelli [9] under smoothness conditions on F. The proof below, based only on continuity and convexity properties, can be found in L. Turner [10], and is repeated here for the convenience of the reader.

Proof. (a) Let us prove the sufficiency. Assume, if possible, that such a linear function $w(u)$ as above exists and that also (4.1) holds for some u_o, $u_1 \in E_n$, $u_1 \neq 0$. Let $z(u) = r + b\cdot u$, $u \in E_n$, r, $b = (b_1,\dots,b_n)$ real, be a supporting plane of $F(u)$ at u_o. Then $F(u) \geq z(u)$ for all $u \in E_n$, and

$$F(u_o + \lambda u_1) \geq r + b\cdot(u_o + \lambda u_1) \quad ,$$

$$F(u_o - \lambda u_1) \geq r + b\cdot(u_o + \lambda u_1) \quad ,$$

$$F(u_o) = r + b\cdot u_o \quad .$$

By difference then we have

$$F(u_o + \lambda u_1) - F(u_o) \geq b\cdot(\lambda u_1) \quad ,$$

$$F(u_o - \lambda u_1) - F(u_o) \geq b\cdot(-\lambda u_1) \quad ,$$

and by using (4.1) also

$$2^{-1}[F(u_o + \lambda u_1) - F(u_o - \lambda u_1)] \geq b\cdot(\lambda u_1) \quad ,$$

$$2^{-1}[F(u_o - \lambda u_1) - F(u_o + \lambda u_1)] \geq b\cdot(-\lambda u_1) \quad .$$

Since the sum of these relations is 0 = 0, we conclude that = sign holds in both; hence

$$F(u_o + \lambda u_1) - F(u_o) = 2^{-1}[F(u_o + \lambda u_1) - F(u_o - \lambda u_1)] = b\cdot(\lambda u_1) \quad ,$$

$$F(u_o - \lambda u_1) - F(u_o) = 2^{-1}[F(u_o - \lambda u_1) - F(u_o + \lambda u_1)] = b\cdot(-\lambda u_1) \quad ,$$

and finally

$$F(u_o + \lambda u_1) = F(u_o) + b\cdot(\lambda u_1) = r + b\cdot(u_o + \lambda u_1) \quad .$$

From $F(u) \geq w(u)$ we deduce now

$$F(u_o + \lambda u_1) = r + b\cdot(u_o + \lambda u_1) \geq \bar{r} + \bar{b}\cdot(u_o + \lambda u_1) \quad ,$$

and hence

$$r - \bar{r} + (b - \bar{b})u_o \geq \lambda(\bar{b} - b)\cdot u_1$$

for all λ real. Since the first member is a constant, we must have $(b - b)\cdot u_1 = 0$, and then

$$F(u_o + \lambda u_1) - w(u_o + \lambda u_1) = r + b\cdot(u_o + \lambda u_1) - \bar{r} - \bar{b}(u_o + \lambda u_1)$$

$$= r - \bar{r} + (b - \bar{b})\cdot u_o \ ,$$

where the last member is a constant. This contradicts that $F(u) - w(u) \to \infty$ as $|u| \to +\infty$. We have proved the sufficiency of the condition.

(b) Let us prove the necessity. First assume that $F(u) \geq 0$ for all $u \in E_n$, with $F(0) = 0$. Let T be the set of all real vectors $b = (b_1,\ldots,b_n)$ for which there is some real number r such that $F(u) \geq r + b\cdot u$ for all $u \in E_n$. If b_1, $b_2 \in T$ and r_1, r_2 are the corresponding numbers, then for $0 \leq \alpha \leq 1$

$$F(u) - [\alpha r_1 + (1 - \alpha)r_2 - (\alpha b_1 + (1 - \alpha)b_2)\cdot u] =$$

$$= \alpha[F(u) - (r_1 + b_1\cdot u)] + (1 - \alpha)[F(u) - (r_2 + b_2\cdot u)] \geq 0$$

for all $u \in E_n$. Hence, $\alpha b_1 + (1 - \alpha)b_2 \in T$, and T is convex. Moreover, T contains the origin since $F(u) \geq 0$ for all $u \in E_n$.

Let us prove that T is not contained in any (n - 1) dimensional subspace of E_n. If it were, there would be a unit vector e such that $e\cdot b = 0$ for all $b \in T$. Since $F(\lambda e) + F(-\lambda e) > 0$ for some $\lambda \neq 0$, then either $F(\lambda e) > 0$ or $F(-\lambda e) > 0$. Suppose $F(\lambda e) > 0$ to be concrete. Let $z(u) = F(\lambda e) + b\cdot(u - \lambda e)$ be a supporting plane for $F(u)$ at the point λe. This supporting plane exists by force of (4.iii). Then $F(u) \geq z(u)$ for all u, so $b \in T$, $e\cdot b = 0$ and $z(\gamma e) = F(\lambda e) + b\cdot(\gamma e - \lambda e) = F(\lambda e) > 0$ for all γ real. Thus, in the directions $\pm$ e the function $z(u)$ is constant and positive. But $z(0) \leq F(0) = 0$, a contradiction. Thus, T is n-dimensional.

We know that a convex set in E_n contained in no (n - 1)-dimensional manifold has an interior point. Therefore, let $\bar{b}$, $\varepsilon > 0$, be such that $\bar{b} \in T$ and $|b - \bar{b}| \leq \varepsilon$ implies $b \in T$. Let $\bar{r}$ be a constant such that $F(u) \geq w(u) = \bar{r} + \bar{b} \cdot u$ for all $u \in E_n$. Suppose that lim inf $[F(u) - w(u)] \nrightarrow +\infty$, where lim inf is taken as $|u| \to +\infty$. Then, there is a constant $a > 0$ and a sequence $[u_k]$ such $|u_k| \to +\infty$, $F(u_k) - w(u_k) < a$ for all k. Without loss of generality we can assume that $u_k/|u_k|$ converges to a unit vector u as $k \to \infty$. Then $\bar{b} + \varepsilon u \in T$, and there is a constant r_1 such that $z(u) = r_1 + (b + \varepsilon u)\cdot u \leq F(u)$ for all u. Thus

$$F(u_k) - w(u_k) \geq r_1 + (\bar{b} + \varepsilon u)\cdot u_k - \bar{r} - \bar{b}\cdot u_k$$

$$= r_1 - \bar{r} + \varepsilon u \cdot u_k$$

$$= r_1 - \bar{r} + \varepsilon |u_k| u\cdot(u_k/|u_k|) \to +\infty$$

as $k \to \infty$, a contradiction. Thus, $F(u) - w(u) \to +\infty$ as $|u| \to +\infty$.

We have proved the statement for functions F with $F(u) \geq 0$ and $F(0) = 0$. For an arbitrary $F(u)$ let $z(u) = F(0) + b_1 \cdot u$ be a supporting plane for $F(u)$ at the origin. Let $G(u) = F(u) - z(u)$. Then $G(u) \geq 0$ for all $u \in E_n$ and $G(0) = 0$. Thus, G satisfies the hypotheses assumed at the beginning, and there exists $w_2(u) = r_2 + b_2 \cdot u$ such that $G(u) \geq w_2(u)$ for all u and $G(u) - w_2(u) \to +\infty$. Let $w(u) = z(u) + w_2(u)$. Then $F(u) - w(u) = G(u) - w_2(u) > 0$ for all $u \in E_n$, and lim $[F(u) - w(u)] =$ lim $[G(u) - w_2(u)] = +\infty$, where both limits are taken as $|u| \to +\infty$. Statement (4.v) is thereby proved.

5. SEMINORMALITY OF CONVEX FUNCTIONS

As usual let A be a closed subset of the x-space, and $f_o(x,u)$ a given scalar function continuous on $A \times E_n$.

The function $f_o(x,u)$ is said to be weakly seminormal in u at the point $(\bar{x},\bar{u}) \in A \times E_m$ provided, given $\varepsilon > 0$, there are numbers $\delta > 0$, and r, $b = (b_1,\ldots,b_n)$ real, such that

$$(X_f') \quad f_o(x,u) \geq r + b\cdot u \quad \text{for all } x \in N_\delta(\bar{x}),\ u \in E_m;$$

$$(X_f'') \quad f_o(\bar{x},\bar{u}) \leq r + b\cdot u + \varepsilon \quad .$$

The function $f_o(x,u)$ is said to be weakly seminormal in u at the point $\bar{x} \in A$ if it has the just mentioned property at $(\bar{x},\bar{u}) \in A \times E_m$ for every $\bar{u} \in E_m$.

The function $f_o(x,u)$ is said to be seminormal in u at the point $(\bar{x},\bar{u}) \in A \times E_m$ provided, given $\varepsilon > 0$, there are numbers $\delta > 0$, $\nu > 0$, and r, $b = (b_1,\ldots,b_n)$ real, such that

$$(SN') \quad f_o(x,u) \geq r + b\cdot u + \nu|u - \bar{u}| \quad \text{for all } x \in N_\delta(\bar{x}),\ u \in E_m;$$

$$(SN'') \quad f_o(\bar{x},\bar{u}) \leq r + b\cdot\bar{u} + \varepsilon \quad .$$

The function $f_o(x,u)$ is said to be seminormal in u at the point $\bar{x} \in A$ if it has the just mentioned property at $(\bar{x},\bar{u}) \in A \times E_m$ for every $\bar{u} \in E_m$. These concepts of seminormality are essentially due to L. Tonelli [9].

Requirement (SN") is often stated in the stronger form (SN"*) $f_o(x,u) \leq r + b\cdot u + \varepsilon$ for all $x \in N_\delta(\bar{x})$, $u \in E_m$, $|u - \bar{u}| \leq \delta$. As we shall see, statement (5.i) below holds for both form (SN") and (SN"*).

(5.i) If $f_o(x,u)$ is continuous in $A \times E_m$, then f_o is seminormal in u at $\bar{x}$ if and only if $f_o(\bar{x},u)$ is convex in u, and for no $\bar{u}$, $u_1 \in E_m$, $u_1 = 0$, it occurs that $f_o(\bar{x},\bar{u}) = 2^{-1}[f_o(\bar{x},\bar{u} + \lambda u_1) + f_o(\bar{x},\bar{u} - \lambda u_1)]$ for all $\lambda \geq 0$.

This statement was proved by L. Tonelli [9] under smoothness conditions on F. The proof below, based only on continuity and convexity properties, can be found in L. Turner [10], and is repeated here for the convenience of the reader.

Proof. (a) Suppose $f_o(x,u)$ seminormal in u at the point $\bar{x} \in A$. Then for every $\bar{u} \in E_m$ there are constants r, $b = (b_1,\dots,b_m)$ real and $\nu > 0$ such that (SN') and (SN") hold. Let $\zeta(u)$ denote $\zeta(u) = r + b\cdot u$. Then, if $\sum_j \lambda_j u_j$ is any convex combination of points $u_j \in E_m$, with $\bar{u} = \sum_j \lambda_j u_j$, then

$$f_o(\bar{x},\bar{u}) \leq \zeta(\bar{u}) + \varepsilon = \sum_j \lambda_j\, z(u_j) + \varepsilon < \sum \lambda_j\, f_o(\bar{x},u_j) + \varepsilon ,$$

where ε is arbitrary. Thus $f_o(\bar{x},u)$ is convex in u at the point $u = \bar{u}$. If there were points $\bar{u}$, $u_1 \in E_m$ with $u_1 \neq 0$ such that

$$2^{-1}[f_o(\bar{x},\bar{u} + \lambda u_1) + f_o(\bar{x},\bar{u} - \lambda u_1)] = f_o(\bar{x},\bar{u}) \tag{5.1}$$

for all real λ, then by force of (SN')

$$f_o(\bar{x},\bar{u}) = 2^{-1}[f_o(\bar{x},\bar{u} + \lambda u_1) + f_o(\bar{x},\bar{u} - \lambda u_1)] \geq 2^{-1}[\zeta(\bar{u} + \lambda u_1) + \zeta(\bar{u} - \lambda u_1)]$$

$$+ 2\nu|\lambda|\ |u_1| = \zeta(\bar{u}) + 2\nu|\lambda|\ |u_1| .$$

This is impossible since λ can be arbitrarily large. We have proved the necessity of the condition.

(b) Let us assume that $f_o(\bar{x},u)$ is convex in u and that for no points $\bar{u}$, $u_1 \in E_m$, $u_1 \neq 0$, the relation $f_o(\bar{x},\bar{u}) = 2^{-1}[f_o(\bar{x},\bar{u} + \lambda u_1) + f_o(\bar{x},\bar{u} - \lambda u_1)]$

holds for all $\lambda \geq 0$, and let us prove that f_o is seminormal in u at $\bar{x} \in A$. Let $\bar{u}$ be any point of E_m, and let $v(u) = r_1 + b_1 \cdot u$ be the supporting plane of $f_o(\bar{x},u)$ at $u = \bar{u}$. Let $w(u) = \bar{r} + \bar{b} \cdot u$ be the function satisfying the requirements of (4.v) for $f_o(\bar{x},u)$ thought of as a function of u alone. Then, for $0 \leq \alpha \leq 1$, and all u we have

$$f_o(\bar{x},u) - [\alpha w(u) + (1 - \alpha) v(u)] = \alpha[f_o(\bar{x},u) - w(u)] + (1 - \alpha)[f_o(\bar{x},u) - v(u)] \geq 0.$$

Let α_o be so small that $\alpha_o |w(\bar{u}) - v(\bar{u})| < \varepsilon/4$, and let

$$z(u) = \alpha_o w(u) + (1 - \alpha_o) v(u) - \varepsilon/4 .$$

Then

$$f_o(\bar{x},u) - z(u) = \alpha_o[f_o(\bar{x},u) - w(u)] + (1 - \alpha_o)[f_o(\bar{x},u) - v(u)] - \varepsilon/4$$

$$\geq \varepsilon/4 \text{ for all } u \in E_n; \tag{5.2}$$

$$\lim[f_o(\bar{x},u) - z(u)] = +\infty \quad \text{as } |u| \to +\infty; \tag{5.3}$$

$$f_o(\bar{x},\bar{u}) - z(\bar{u}) = v(\bar{u}) - z(\bar{u}) = \alpha_o[v(\bar{u}) - w(\bar{u})] + \varepsilon/4 \leq \varepsilon/2 . \tag{5.4}$$

From (5.3) we conclude that, for some $m > 0$ we have

$$\operatorname*{Inf}_{|u-u_o|=m} [f_o(\bar{x},u) - z(u)] > 2\varepsilon .$$

Now define $\eta(x) = \operatorname{Inf}[f_o(x,u) - z(u)]$, where Inf is taken for $|u - u_o| = m$. Then $\eta(x)$ is a continuous function of x for $x \in A$, and $\eta(\bar{x}) > 2\varepsilon$.

Then (5.2) and (5.3) above, and the continuity of $\eta(x)$, imply that there is $\delta > 0$ such that

$$f(x,u) - z(u) > \varepsilon/8 \quad \text{for } |x - \bar{x}| \leq \delta, \ |u - \bar{u}| = m; \tag{5.5}$$

$$\eta(x) > 9\varepsilon/8 \quad \text{for } |x - \bar{x}| < \delta \ ; \tag{5.6}$$

$$f_o(x,u) < z(u) + \varepsilon \quad \text{for } |x - \bar{x}| < \delta, \ |u - \bar{u}| < \delta. \tag{5.7}$$

Relation (5.7) is requirement (SN") (actually, the stronger statement (SN"*)).

If $\nu = \varepsilon/8m$, then (5.5) implies $f_o(x,u) - z(u) - \nu|u - \bar{u}| > (\varepsilon/8) - \nu|u - \bar{u}| > \varepsilon/8 - \varepsilon/8 = 0$ for $|x - \bar{x}| < \delta$, $|u - \bar{u}| \leq m$. For $|u - \bar{u}| > m$, let $\alpha = m/|u - u_o|$, so $0 < \alpha < 1$, and let us define $u_1 = \alpha(u - \bar{u}) + \bar{u}_o$. Then $|u_1 - \bar{u}| = (m/|u - \bar{u}|)(u - \bar{u}) = m$, $u_1 = \alpha u + (1 - \alpha)\bar{u}$, and thus, for $|x - \bar{x}| < \delta$,

$$f_o(x,u_1) \leq \alpha f_o(x,u) + (1 - \alpha) f_o(x,\bar{u}) \quad ,$$

$$f_o(x,u_1) - z(u_1) \leq \alpha[f_o(x,u) - z(u)] + (1 - \alpha)[f_o(x,\bar{u}) - z(\bar{u})] \quad ,$$

$$f_o(x,u) - z(u) \geq f_o(x,\bar{u}) - z(\bar{u}) + (1/\alpha)[f_o(x,u_1) - z(u_1)] - [f_o(x,\bar{u}) - z(\bar{u})]$$

$$> 0 + (1/\alpha)[\eta(x) - \varepsilon] > (1/\alpha)(9\varepsilon/8 - \varepsilon) = \varepsilon/8a \quad .$$

Since $\alpha\nu|u - \bar{u}| = \varepsilon/8$, we have

$$f_o(x,u) - z(u) - \nu|u - u_o| > (\varepsilon/8a) - (\varepsilon/8a) = 0 \ ,$$

or $f_o(x,u) > z(u) + \nu|u - u_o|$ for all u and $|x - \bar{x}| < \delta$. This is requirement (SN'). Statement (5.i) is thereby proved.

6. A THIRD CRITERION FOR PROPERTY (Q)

We give here a simple criterion for property (Q) of the sets $\tilde{\tilde{Q}}(x)$ of no. 2 for the case in which f is linear in u.

(6.i) (a criterion for property (Q) for f linear in u). If A is closed, $U = E_m$, $M = A \times E_m$, if $f_o(x,u)$ is continuous on M, convex in u, and seminormal in u at a point $\bar{x} \in A$, if $f(x,u) = B(x)u + C(x)$ where the matrices B,C have entries continuous on A, then the sets $\tilde{\tilde{Q}}(x)$ staisfy property (Q) at $\bar{x}$.

Proof. By seminormality we know that there is a neighborhood $N_{\delta_o}(\bar{x})$ of $\bar{x}$ in A and real numbers r, $b = (b_1,\dots,b_n)$ such that $f_o(x,u) - r - b\cdot u \geq 0$ for all $x \in N_{\delta_o}(\bar{x})$, $u \in E_m$. By replacing f_o by $f_o - r - b\cdot u$ if necessary, we see that it is not restrictive to assume $f_o \geq 0$ for all $x \in N_{\delta_o}(\bar{x})$ and $u \in E_m$.

Thus, $f_o(x,u) \geq 0$ for all $x \in N_{\delta_o}(\bar{x})$, $u \in E_m$, and the sets $\tilde{\tilde{Q}}(x)$ are defined by $[(z^o,z) | z^o \geq f_o(x,u),\ z = f(x,u),\ u \in E_m]$. We have to prove that $\tilde{z} = (z^o,z) \in \bigcap_\delta \text{cl co } \tilde{\tilde{Q}}(\bar{x};\delta)$ implies $\tilde{z} \in \tilde{\tilde{Q}}(\bar{x})$. Let $\tilde{\bar{z}}$ be a given point $\tilde{\bar{z}} = (z^o,\bar{z}) \in \bigcap_\delta \text{cl co } \tilde{\tilde{Q}}(\bar{x};\delta)$ and let us prove that $\tilde{\bar{z}} \in \tilde{\tilde{Q}}(\bar{x})$. For every $\delta > 0$ we have $\tilde{\bar{z}} = (\bar{z}^o,\bar{z}) \in \text{cl co } \tilde{\tilde{Q}}(\bar{x};\delta)$ and thus, for every $\delta > 0$ there are points $\tilde{z} = (z^o,z) \in \text{co } \tilde{\tilde{Q}}(\bar{x};\delta)$ at a distance as small as we want from $\tilde{z} = (\bar{z}^o,\bar{z})$. Thus, there is a sequence of numbers $\delta_k > 0$ and of points $\tilde{z}_k = (z^o_k,z_k) \in \text{co } \tilde{\tilde{Q}}(\bar{x};\delta_k)$ such that $\delta_k \to 0$, $\tilde{z}_k \to \tilde{\bar{z}}$ as $k \to \infty$. In other words, for every integer k there is a system of points $x^\gamma_k \in N_{\delta_k}(\bar{x})$, $\gamma = 1,\dots,\nu$, say $\nu = n + 2$, corresponding points $\bar{z}^\gamma_k = (z^{o\gamma}_k, z^\gamma_k) \in \tilde{\tilde{Q}}(x^\gamma_k)$, points $u^\gamma_k \in E_m$ and numbers λ^γ_k, $0 \leq \lambda^\gamma_k \leq 1$, $\gamma = 1,\dots,\nu$, such that

$$1 = \sum \lambda^\gamma_k,\quad \tilde{z}_k = \sum \lambda^\gamma_k \tilde{z}^\gamma_k,\quad z^o_k = \sum \lambda^\gamma_k z^{o\gamma}_k,\quad z_k = \sum \lambda^\gamma_k z^\gamma_k ,$$

$$z^{o\gamma}_i \geq f_o(x^\gamma_k,u^\gamma_k),\quad z^\gamma_k = f(x^\gamma_k,u^\gamma_k) = B(x^\gamma_k)u^\gamma_k + C(x^\gamma_k) , \qquad (6.4)$$

where $\gamma = 1,\ldots,\nu$, $k = 1,2,\ldots$, where $\sum$ ranges over all $\gamma = 1,\ldots,\nu$, $x_k^\gamma \in N_{\delta_k}(\bar{x})$, and $x_k^\gamma \to \bar{x}$, $\bar{z}_k \to \tilde{\bar{z}}$, $z_k^o \to \bar{z}^o$, $z_k \to \bar{z}$ as $k \to \infty$, $\gamma = 1,\ldots,\nu$.

By seminormality of f_o at the point $\bar{x}$ there are numbers δ', $0 < \delta' \le \delta_o$, $\nu > 0$, r real, so that $f_o(x,u) \ge r + \nu|u|$ for all $x \in N_\delta(\bar{x})$. If k is sufficiently large so that $\delta_k < \delta' \le \delta_o$, and hence $|x_k^\gamma - \bar{x}| \le \delta_k < \delta'$, and because $\zeta = r + b|u|$ is a convex function of u we have

$$z_k^o = \sum \lambda_k^\gamma z_k^{o\gamma} \ge \sum \lambda_k^\gamma f_o(x_k^\gamma, u_k^\gamma)$$

$$\ge \sum \lambda_k^\gamma [r + \nu|u_k^\gamma|] \ge r + \nu|\sum \lambda_k^\gamma u_k^\gamma| \quad .$$

Thus $|\sum \lambda_k^\gamma u^\gamma{}_k| \le \nu^{-1}[z_k^o - r]$ where $z_k^o \to \bar{z}^o$ as $k \to \infty$. This proves that $\sum \lambda_k^\gamma u_k^\gamma$, $k = 1,2,\ldots$, is a bounded sequence of points of E_m. By a suitable extraction, there is a subsequence, say still $[k]$, such that $u_k = \sum \lambda_k^\gamma u_k^\gamma \to \bar{u} \in E_m$ as $k \to \infty$.

From the third relation of (6.4), where $z_k^o \to \bar{z}^o$, $z_k^{o\gamma} \ge 0$, $0 \le \lambda_k^\gamma \le 1$, we deduce that each of the ν sequences $[\lambda_k^\gamma z_k^{o\gamma}$, $k = 1,2,\ldots]$, $\gamma = 1,\ldots,\nu$, is bounded. From the fifth relation (6.4) we then deduce

$$\lambda_k^\gamma z_k^{o\gamma} \ge f_o(x_k^\gamma, u_k^\gamma) \ge \lambda_k^\gamma (r + \nu|u_k^\gamma|) \quad ,$$

and hence $\lambda_k^\gamma |u_k^\gamma| \le \nu^{-1}[\lambda_k^\gamma z_k^{o\gamma} + |r|]$. Thus each of the ν sequences $[\lambda_k^\gamma u_k^\gamma$, $k = 1,2,\ldots,]$, $\gamma = 1,2,\ldots,\nu$, is bounded.

If we denote by Δ_k^γ the expression

$$\Delta_k^\gamma = \lambda_k^\gamma [(B(x_k^\gamma)u_k^\gamma + C(x_k^\gamma)) - (B(\bar{x})u_k^\gamma + C(\bar{x}))] \quad ,$$

or

$$\Delta_k^\gamma = (B(x_k^\gamma) - B(\bar{x}))(\lambda_k^\gamma u_k^\gamma) + \lambda_k^\gamma(C(x_k^\gamma) - C(\bar{x})) \quad ,$$

and because of the continuity of B and C, and of $x_k^\gamma \to \bar{x}$, $0 \leq \lambda_k^\gamma \leq 1$, we conclude that $\Delta_k^\gamma \to 0$ as $k \to \infty$, $\gamma = 1,\dots,\nu$.

Given $\varepsilon > 0$, by the seminormality of $f_o(x,u)$ at $\bar{x}$ we can determine new numbers $\delta'' > 0$, and r, $b = (b_1,\dots,b_n)$ real so that

$$f_o(x,u) \geq r + b\cdot u \text{ for all } x \in N_{\delta''}(\bar{x}),\ u \in E_m, \text{ and } f_o(\bar{x},\bar{u}) \leq r + b\cdot\bar{u} + \varepsilon \quad .$$

Now we have

$$z_k^o = \sum \lambda_k^\gamma z_k^o \geq \sum \lambda_k^\gamma f_o(x_k^\gamma, u_k^\gamma) \geq \sum \lambda_k^\gamma [r + b\cdot u_k^\gamma] = r + b\cdot u_k$$

$$= r + b\cdot\bar{u} + b\cdot(u_k - \bar{u}) \leq f_o(\bar{x},\bar{u}) + b(u_k - \bar{u}) - \varepsilon \quad ;$$

$$z_k = \sum \lambda_k^\gamma z_k = \sum \lambda_k [B(x_k)u_k^\gamma + C(x_k)] = \sum \lambda_k [B(\bar{x})u_k^\gamma + C(\bar{x})]$$

$$+ \Delta_k = B(\bar{x})u_k + C(\bar{x}) + \Delta_k \quad .$$

At the limit as $k \to \infty$ we obtain

$$\bar{z}^o \geq f_o(\bar{x},\bar{u}) - \varepsilon, \ \bar{z} = B(\bar{x})\bar{u} + C(\bar{x}) \quad ,$$

and because $\varepsilon > 0$ is arbitrary, also $\bar{z}^o \geq f_o(\bar{x},\bar{u})$, $\bar{z} = f(\bar{x},\bar{u})$; hence $\tilde{\bar{z}} = (\bar{z}^o,\bar{z}) \in \tilde{\tilde{Q}}(\bar{x})$. Statement (6.i) is thereby proved.

7. THE FUNCTION T

Given A, $U(x)$, M, f_o, $f = (f_1,\dots,f_n)$, $Q(x)$, $\tilde{\tilde{Q}}(x)$ as usual (A,M closed,

f_o, f continuous on M), let us remind here that the sets $Q(x)$ are the projections of the sets $\tilde{\tilde{Q}}(x) \subset E_{n+1}$ on the z-space E_n. Hence, for every $(z^o, z) \in \tilde{\tilde{Q}}(x)$ we have $z \in Q(x)$, and $z = f(x,u)$, $z^o \geq f_o(x,u)$ for some $u \in U(x)$. Conversely, if $z \in Q(x)$, hence $z = f(x,u)$ for some $u \in U(x)$, all points (z^o, z) with $z^o \geq f_o(x,u)$ certainly are in $\tilde{\tilde{Q}}(x)$.

For any fixed $\bar{x} \in A$ let us consider the scalar function defined on $\tilde{\tilde{Q}}(\bar{x})$:

$$T(z;\bar{x}) = \operatorname{Inf}[f_o(\bar{x},u) \mid z = f(\bar{x},u),\ u \in U(\bar{x})]$$

$$= \operatorname{Inf}[z^o \mid (z^o, z) \in \tilde{\tilde{Q}}(\bar{x})],\ z \in Q(\bar{x}) \quad .$$

Then, for $\bar{x} \in A$, we have $-\infty \leq T(z;\bar{x}) < +\infty$ for all $z \in Q(\bar{x})$. We shall consider $T(z;\bar{x})$ as a function of z in $Q(\bar{x})$.

Note that the convexity of $\tilde{\tilde{Q}}(\bar{x}) \subset E_{n+1}$ implies the convexity of $Q(\bar{x}) \subset E_n$, but $Q(\bar{x})$ may be not closed even if $\tilde{\tilde{Q}}(\bar{x})$ is closed. Also, we shall denote by R the linear manifold in E_n containing $Q(\bar{x})$ of minimum dimension r, thus, $Q(\bar{x}) \subset R \subset E_n$, $0 \leq r \leq n$. As usual, we shall denote by int $Q(\bar{x})$ the set of all $z \in E_n$ which are interior to $Q(\bar{x})$ (with respect to E_n), and by Rint $Q(\bar{x})$ the set of all points z which are interior to $Q(\bar{x})$ with respect to R; thus int $Q(\bar{x}) \subset$ Rint $Q(\bar{x}) \subset Q(\bar{x}) \subset R \subset E_n$.

(7.i) If $\tilde{\tilde{Q}}(\bar{x})$ is convex, then either $T(z;\bar{x}) = -\infty$ for all $z \in$ Rint $Q(\bar{x})$, or $T(z;\bar{x}) > -\infty$ for all $z \in Q(\bar{x})$. In the latter case, $T(z,\bar{x})$ is finite everywhere and a convex function of z in $Q(\bar{x})$, $T(z;\bar{x})$ is bounded below on every bounded subset of $Q(\bar{x})$, and $T(z;\bar{x})$ is continuous in the convex set Rint $Q(\bar{x})$ (open with respect to R). Finally, if $\tilde{\tilde{Q}}(\bar{x})$ is convex and closed, and $T(z,\bar{x}) >$

$-\infty$ for all $z \in Q(\bar{x})$, then $T(z;\bar{x})$ is lower semicontinuous at every point $z \in Q(\bar{x})$ - Rint $Q(\bar{x})$.

Proof. If $Q(\bar{x})$ is a single point, then $r = 0$, Rint $Q(\bar{x}) = \phi$, and nothing has to be proved. Assume that $Q(\bar{x})$ is not a single point. Then $1 \leq r \leq n$, and Rint $Q(\bar{x}) \neq \phi$. Let $\bar{z}$ be any point $\bar{z} \in$ Rint $Q(\bar{x})$. Assume that at some point $z_1 \in Q(\bar{x})$ we have $T(z_1,\bar{x}) = -\infty$, and let us prove that $T(\bar{z};\bar{x}) = -\infty$. For any integer k there are points $(z_k^o, z_1) \in \tilde{\tilde{Q}}(\bar{x})$ with $z_k^o < -k$, $k = 1,2,\ldots$ Take $\lambda = z_1 - \bar{z}$, and choose $\delta > 0$ so small that $z_2 = \bar{z} - \lambda\delta \in$ Rint $Q(\bar{x})$. Take any point $(z_2^o, z_2) \in \tilde{\tilde{Q}}(\bar{x})$, and note that all points

$$(\alpha z_2^o + (1 - \alpha)z_k^o,\ \alpha z_2 + (1 - \alpha)z_1),\ 0 \leq \alpha \leq 1,$$

belong to $\tilde{\tilde{Q}}(\bar{x})$. In particular, for $\alpha = (1 + \delta)^{-1}$, we have

$$\begin{aligned} \alpha z_2 + (1 - \alpha)z_1 &= \alpha(\bar{z} - \lambda\delta) + (1 - \alpha)z_1 \\ &= \bar{z} - (1 - \alpha)(\bar{z} - z_1) - \alpha\lambda\delta \\ &= \bar{z} + \lambda(1 - \alpha - \alpha\delta) = \bar{z} \ , \end{aligned}$$

$$T(\bar{z};\bar{x}) \leq \alpha z_2^o + (1 - \alpha)z_k^o \leq (1 + \delta)^{-1}z_2^o - (1 - (1 + \delta)^{-1})k \ ,$$

where the last term approaches $-\infty$ as $k \to \infty$, hence $T(\bar{z};\bar{x}) = -\infty$. Since $\bar{x}$ is any point of Rint $Q(\bar{x})$, we have proved the first part of (7.i).

The remaining parts of (7.i) are now a consequence of the definitions and of statements (4.v), (4.vi), and (4.viii).

The first of the two cases mentioned in (7.i) may actually occur, even in situations where the sets $\tilde{\tilde{Q}}$ have property (Q) at $\bar{x}$. Indeed, take $m = n = 1$, $f_o = u$, $f = 0$, $U = E_1$. Then $Q = [z|z = 0]$, $\tilde{\tilde{Q}} = [(z^o, z)|z^o \in E_1, z = 0]$, and $T = -\infty$. As another case, take $n = 1$, $m = 2$, u, v control variables, $f_o = u$, $f = \sin v$, $U = [(u,v) \in E_2]$. Then $Q = [z| -1 \leq z \leq 1]$, $\tilde{\tilde{Q}} = [(z^o, z)|z^o \in E_1, -1 \leq z \leq 1]$, and $T(z) = -\infty$ for all $-1 \leq z \leq 1$. In both cases, Q and $\tilde{\tilde{Q}}$ are fixed, closed, convex sets, and certainly have property (Q). As a third example take $n = 1$, $m = 2$, u, v control variables, $f_o = (1 - \sin^2 v)u$, $f = \sin v$, $U = [(u,v) \in E_2]$. Then $Q = [z|-1 \leq z \leq 1]$, and $\tilde{\tilde{Q}} = [(z^o, z)|z^o \in E_1$ if $-1 < z < 1$, and $z^o = 0$ if $z = \pm 1]$. Finally, $T(z) = -\infty$ for $-1 < z < 1$, $T(z) = 0$ for $z = \pm 1$.

The following example proves that $T(z;\bar{x})$ may not be lower semicontinuous on $Q(\bar{x})$ if the set $\tilde{\tilde{Q}}(\bar{x})$ is not closed. As usual, we shall denote by $[g(P)]_h$ the function of P which has the value $g(P)$ if $g(P) \geq h$, and the value h if $g(P) < h$. Now take $n = 1$, $m = 2$, (u,v) control variables, $f_o = [(1 - \sin^2 v)u]_{-1}$, $f = \sin v$, $U = [(u,v) \in E_2]$. Then $Q = [z| -1 \leq z \leq 1]$, and $\tilde{\tilde{Q}} = [(z^o, z)| z^o \geq -1$ if $-1 < z < 1$, $z^o \geq 0$ if $z = \pm 1]$. Finally, $T(z) = -1$ for $-1 < z < 1$, $T(z) = 0$ for $z = \pm 1$.

The following example shows that, even if the set $\tilde{\tilde{Q}}(\bar{x})$ is closed and convex, the function $T(z;\bar{x})$ may not be continuous at the points $z \in Q(\bar{x})$ - Rint $Q(\bar{x})$. Let Q be the convex set $[(\xi,\eta)|0 \leq \xi \leq 1, 0 \leq \eta \leq (1 - (1 - \xi)^2)^{1/2}]$, and let $T(\xi,\eta)$ be defined by taking $T = \eta$ for $0 \leq \xi \leq 1$, $0 \leq \eta \leq \xi$, $T = (2\xi)^{-1}(\xi^2 + \eta^2)$ for $0 < \xi < 1$, $\xi \leq \eta \leq (1 - (1 - \xi)^2)^{1/2}$. As we have seen in Section 4, $T(\xi,\eta)$ is convex and bounded in Q, and continuous in Q except at the point $(\xi = 0, \eta = 0)$. Now let us define the set $\tilde{\tilde{Q}}$. To this purpose, let U be the union of the two closed disjoint sets $U_1 = [(u,v,w)|0 \leq u \leq 1, -1 \leq v \leq u-1, w \geq 0]$ and $U_2 = [(u,v,w)|0 \leq u \leq 1, 0 \leq v \leq (1-(1-u)^2)^{1/2}, w \geq 0]$. Let $\eta(w,u)$ be the function defined for all (u,w), $0 \leq u \leq 1$, $w \geq 0$, by taking $\eta(w,u) = (w+1)^{-1} - u$ for $0 \leq u \leq (w+1)^{-1}$, $\eta(w,u) = 0$ for $(w+1)^{-1} \leq u \leq 1$. Finally, let us define the functions $f_o(u,v,w)$, $f_1(u,v,w)$, $f_2(u,v,w)$ continuous on $U = U_1 \cup U_2$, by taking $f_1 = u$, $f_2 = v+1$, $f_o = v$ on U_1, and

$$\xi = f_1 = v + (u-1)(1+v)^{-1/2}[(1+v)^{1/2} - (1-v)^{1/2}]$$

$$\eta = f_2 = v$$

$$f_o = 2(\xi + \eta(w,u))^{-1}[2\eta(w,u) + \xi^2 + \eta^2]$$

on U_2. Then, if $\tilde{\tilde{Q}}$ denotes the set $\tilde{\tilde{Q}} = [(z^o,\xi,\eta)|z^o \geq f_o, \xi = f_1, \eta = f_2, (u,v,w) \in U = U_1 \cup U_2]$, and $T(\xi,\eta) = \text{Inf}[z^o|(z^o,\xi,\eta) \in \tilde{\tilde{Q}}]$, then T is exactly the convex function defined above on Q, and $\tilde{\tilde{Q}}$ is convex and closed.

The following example shows that at points $\bar{z} \in Q(\bar{x})$ - Rint $Q(\bar{x})$ the supporting plane of $\tilde{\tilde{Q}}(\bar{x})$ may be vertical, even if $\tilde{\tilde{Q}}(\bar{x})$ is convex and closed, $Q(\bar{x})$ is convex and compact, and $T(z;\bar{x})$ continuous on $Q(\bar{x})$. Indeed, take $Q = [(u,v)|u^2 + v^2 \leq 1]$, $T = -(1-u^2-v^2)^{1/2}$, $U = Q$, $f_1 = u$, $f_2 = v$, $f_o = T$, $\tilde{\tilde{Q}} = [(z^o,u,v)|z^o \geq T, (u,v) \in U]$.

8. A CHARACTERIZATION OF PROPERTY (Q) FOR THE SETS $\tilde{\tilde{Q}}(t,x)$

For fixed $\bar{x} \in A$ and $\delta > 0$, let us consider the set

$$\tilde{\tilde{Q}}^*(\bar{x};\delta) = \text{co}\, \tilde{\tilde{Q}}(\bar{x};\delta) = \text{co}\{\bigcup_{x \in N_\delta(\bar{x})} \tilde{\tilde{Q}}(x)\} \subset E_{n+1} , \tag{8.1}$$

and its projection on the z-space E_n

$$Q^*(\bar{x};\delta) = \text{co}\, Q(\bar{x};\delta) = \text{co}\{\bigcup_{x\in N_\delta(\bar{x})} Q(x)\} \subset E_n \quad .$$

Both sets $\tilde{\tilde{Q}}^*(\bar{x};\delta)$ and $Q^*(\bar{x},\delta)$ are convex, and

$$E_{n+1} \supset \tilde{\tilde{Q}}(\bar{x};\delta) \supset \tilde{\tilde{Q}}(\bar{x}),\ E_n \supset Q^*(\bar{x};\delta) \supset Q(\bar{x}) \quad . \tag{8.2}$$

As before, we shall consider the scalar function defined on $Q^*(\bar{x};\delta)$:

$$T^*(z;\bar{x},\delta) = \text{Inf}\{z^o | (z^o,z) \in \tilde{\tilde{Q}}^*(x,\delta)\},\ z \in Q^*(\bar{x};\delta) \quad .$$

Thus, we have again $-\infty \leq T^*(z;\bar{x},\delta) < \infty$ for all $z \in Q^*(\bar{x};\delta)$. Also, for every $z \in Q(\bar{x})$, we have $T^*(z;\bar{x},\delta) \leq T(z;\bar{x})$. We have now the following characterization of property (Q):

(8.i) If $T(z;\bar{x}) > -\infty$ in $Q(\bar{x})$, then the sets $\tilde{\tilde{Q}}(x)$ have property (Q) at $\bar{x}$ if and only if properties (α) and (X) hold at the point $\bar{x}$.

Proof. We have already proved in (3.i) that the union of (α) and (X) implies (Q). We need only prove that, if $T(z;\bar{x}) > -\infty$ in $Q(\bar{x})$, and $\tilde{\tilde{Q}}(x)$ has property (Q) at $\bar{x}$, then both (α) and (X) hold at $\bar{x}$. We know already that (α) is a necessary condition for property (Q) and thus (α) holds. Also, $\tilde{\tilde{Q}}(\bar{x})$ is closed and convex. Since $T(z;\bar{x}) > -\infty$ by hypothesis, we know from (7.i) that $T(z;\bar{x})$ is a lower semicontinuous convex function of z in the convex set $Q(\bar{x})$. We have already noticed that $-\infty \leq T^*(z;\bar{x},\delta) \leq T(z,\bar{x}) \subset +\infty$ for all $z \in Q(\bar{x})$ and $\delta > 0$.

Now take any point $\bar{z} \in Q(\bar{x})$, and let $\bar{z}^o = T(\bar{z},\bar{x})$. Then by (7.i) the point $(\bar{z}^o,\bar{z})$ belongs to $\tilde{\tilde{Q}}(\bar{x})$, and hence there is some $\bar{u} \in U(\bar{x})$ with $\bar{z}^o = T(\bar{z},\bar{x}) = f_o(\bar{x},\bar{u})$, $\bar{z} = f(\bar{x},\bar{u})$. Given $\varepsilon > 0$, the point $\bar{P} = (\bar{z}^o - \varepsilon,\bar{z})$ is not on the closed set $\tilde{\tilde{Q}}(\bar{x})$, and hence has a minimum distance η from this set, with $0 < \eta \leq \varepsilon$. Since $T(z;\bar{x})$ is lower semicontinuous at $\bar{z}$, there is some η', $0 < \eta' \leq \eta$, such that $T(z;\bar{x}) > T(z;\bar{x}) - \eta/3$ for all $z \in Q(\bar{x})$ with $|z - \bar{z}| \leq \eta'$.

Let σ be the closed ball in E_{n+1} of center $\bar{P} = (\bar{z}^o - \varepsilon,\bar{z})$ and radius $\eta'/3$. Let σ_o denote the projection of σ on the z-space; thus σ_o is the closed ball in E_n of center $\bar{z}$ and radius $\eta'/3$. We shall denote also by σ_1 the closed ball in E_n of center $\bar{z}$ and radius $2\eta'/3$.

Now let us consider the convex sets $\tilde{\tilde{Q}}^*(x;\delta) = \text{co}\, \tilde{\tilde{Q}}(\bar{x},\delta)$ defined in (8.1) and their relative functions $T^*(z;\bar{x},\delta)$ defined in (8.2). Let us prove that there is some $\delta_o > 0$ such that

$$0 \leq T(\bar{z};\bar{x}) - T^*(z;\bar{x},\delta) < \eta/3 \tag{8.3}$$

for all $0<\delta\leq\delta_o$ and $z \in \sigma_1 \cap Q^*(\bar{z},\delta)$. Indeed in the contrary case, there would be numbers $\delta_k > 0$ and points $z_k \in \sigma_1 \subset E_n$, $k = 1,2,\dots$, with $\delta_k \to 0$ as $k \to \infty$ and $T^*(z_k;\bar{x},\delta_k) \leq T(\bar{z},\bar{x}) - \eta/3$, and hence points $(z_k^o,z_k) \in \text{co}\, \tilde{\tilde{Q}}(\bar{x},\delta_k)$ with $z_k^o \leq T(\bar{z};\bar{x}) - \eta/3 = \bar{z}^o - \eta/3$. Hence, for every $\delta > 0$ we have $(z_k^o,z_k) \in \text{co}\, \tilde{\tilde{Q}}(x,\delta)$ for all k sufficiently large, and then also $(\bar{z}^o - \eta/3,z_k) \in \text{co}\, \tilde{\tilde{Q}}(\bar{x};\delta)$. If $\bar{z}'$ is any point of accumulation of $[z_k]$ we have then $\bar{z}' \in \sigma_1$, $(\bar{z}^o - \eta/3,\bar{z}') \in \text{cl co}\, \tilde{\tilde{Q}}(\bar{x},\delta)$, and by property (Q) also $(\bar{z}^o - \eta/3,\bar{z}') \in \tilde{\tilde{Q}}(\bar{x}) = \bigcap_\varepsilon \text{cl co}\, \tilde{\tilde{Q}}(x,\delta)$. This implies $T(\bar{z}';\bar{x}) \leq \bar{z}^o - \eta/3$ with $\bar{z}' \in \sigma_1$, $|\bar{z}' - \bar{z}| \leq 2\eta'/3 < \eta'$, a contradiction. We have proved that for some $\delta_o > 0$ relation (8.3) holds for all $0 < \delta \leq \delta_o$ and $z \in \sigma_1 \cap Q^*(\bar{x},\delta)$.

Let us prove that any two points $P = (z^o, z) \in \sigma$ and $P' = (z'^o, z') \in Q^*(\bar{x};\delta_o)$ have a distance $\{P,P'\} \geq \eta'/3$. Indeed, either P' is outside the cylinder ($z^o \in E_1$, $z \in \sigma_1$) and then

$$\{P',P\} \geq |z' - z| \geq |z' - \bar{z}| - |z - \bar{z}| \geq 2\eta'/3 - \eta'/3 = \eta'/3 ,$$

or P' is inside the cylinder above, and then by (8.3) for $0 < \delta \leq \delta_o$,

$$z'^o \geq T^*(z';\bar{x},\delta_o) > T(\bar{z},\bar{x}) - \eta/3 = \bar{z}^o - \eta/3 ,$$

$$\{P',P\} \geq z'^o - z^o = [\bar{z}^o - (\bar{z}^o - \varepsilon)] + [z'^o - \bar{z}^o] + [\bar{z}^o - \varepsilon, z^o]$$

$$\geq \varepsilon - \eta/3 - \{\bar{P},P\} \geq \eta - \eta/3 - \eta/3 = \eta/3 .$$

Thus, the convex sets σ and $\tilde{\tilde{Q}}^*(\bar{x};\delta)$ have a distance $\geq \eta/3$, and the same occurs for the convex closed sets σ and cl $\tilde{\tilde{Q}}^*(\bar{x},\delta)$, ($\sigma$ compact). We conclude that there is some hyperplane Π in E_{n+1} separating the two convex sets σ and cl $\tilde{\tilde{Q}}^*(\bar{x};\delta)$.

This hyperplane Π must intersect the vertical segment $[\bar{z}^o - \varepsilon + \eta/3 \leq z^o \leq \bar{z}^o, z = \bar{z}]$ at some point ($z^o = r$, $z = \bar{z}$), and Π cannot be parallel to the z^o-axis, otherwise all points of the straight line $z = \bar{z}$ would be on Π, in particular the center $\bar{P}$ of the ball σ, and not all points of σ could be separated from $\tilde{\tilde{Q}}^*(\bar{x};\delta)$. Thus, Π is of the form

$$\Pi: \quad z = r + b\cdot(z - \bar{z}) = (r - b\cdot\bar{z}) + b\cdot z ,$$

$\tilde{\tilde{Q}}(\bar{x})$ as well as cl $\tilde{\tilde{Q}}^*(x;\delta)$ are above Π, and thus, $(z^o,z) \in$ cl $\tilde{\tilde{Q}}^*(x;\delta)$ implies $z^o \geq (r - b\cdot\bar{z}) + b\cdot z$. In other words, for $0 < \delta \leq \delta_o$, $x \in N_\delta(\bar{x})$, $x \in A$,

$u \in U(x)$, we have $f_0(x,u) \geq (r - b \cdot \bar{z}) + b \cdot f(x,u)$. On the other hand, $f_0(\bar{x},\bar{u}) = \bar{z}^0 = (\bar{z}^0 - \varepsilon) + \varepsilon < r + \varepsilon = (r - b \cdot \bar{z}) + b \cdot \bar{z} + \varepsilon$. We have proved that property (X) holds at the point $\bar{x} \in A$. Statement (8.i) is thereby proved.

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