T H E   U N I V E R S I T Y   O F   M I C H I G A N

COLLEGE OF LITERATURE, SCIENCE, AND THE ARTS
Department of Mathematics

Technical Report No. 21

SOBOLEV FUNCTIONS AND SOBOLEV SPACES

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ORA Project 024160

submitted for:

UNITED STATES AIR FORCE
AIR FORCE OFFICE OF SCIENTIFIC RESEARCH
GRANT NO. AFOSR-69-1662
ARLINGTON, VIRGINIA

administered through:

OFFICE OF RESEARCH ADMINISTRATION   ANN ARBOR

April 1971

Engn
UHR0954

ADDENDUM VII. SOBOLEV FUNCTIONS AND SOBOLEV SPACES

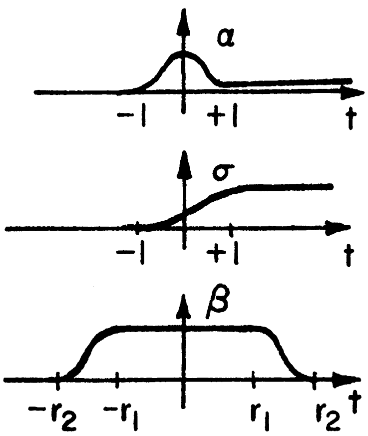
VII 1. THE TEST FUNCTIONS

Given any open subset G of the t-space E_ν, $t = (t^1, \dots, t^\nu)$, $\nu \geq 1$, we shall denote by $C^\infty(G)$ the family of all real valued functions $\varphi(t)$, $t \in G$, which possess (continuous) partial derivatives of all orders in G. We shall denote by $C_0^\infty(G)$ the family of all $\varphi \in C^\infty(G)$ with compact support $K_\varphi \subset G$. The support K_φ of a function $\varphi(t)$, $t \in G$, is the closure of the set of points $t \in G$ where $\varphi(t) \neq 0$.

The following examples are of some interest. For $\nu = 1$ let us define $\alpha(t)$, $-\infty < t < +\infty$, by taking $\alpha(t) = 0$ for $|t| \geq 1$, and $\alpha(t) = \exp(t^2-1)^{-1}$ for $|t| < 1$. Then $\alpha \in C_0^\infty(E_1)$. If we take

$$\sigma(t) = c \int_{-\infty}^{t} \alpha(\tau)\, d\tau, \qquad -\infty < t < +\infty,$$

then $\sigma(t) = 0$ for $t \leq -1$, and by choosing the constant $c > 0$ suitably we can arrange that $\sigma(t) = 1$ for $t \geq 1$. Then, $0 < \sigma(t) < 1$ for all $-1 < t < 1$, and $\sigma \in C^\infty(E_1)$. Finally, given any two numbers r_1, r_2, $0 < r_1 < r_2 < +\infty$, let



let us define $\beta(t)$, $-\infty < t < +\infty$, by taking

$$\beta(t; r_1, r_2) = 1 - \sigma(-1 + 2(|t| - r_1)/(r_2 - r_1)).$$

Then, $\beta(t) = 1$ for $|t| \leq r_1$, $\beta(t) = 0$ for $|t| \geq r_2$, and $0 < \beta(t) < 1$ for $r_1 < |t| < r_2$, and $\beta \in C_0^\infty(E_1)$.

Now, given any open set $G \subset E_\nu$, $\nu \geq 1$, and any point $\bar{t} \in G$, let us take any number ε, $0 < \varepsilon < \text{dist}(\bar{t}, \partial G)$, and let us define $\varphi(t)$, $t \in G$, by taking $\varphi(t) = \alpha(\varepsilon^{-1} |t - \bar{t}|)$. Then $\varphi(t) = 0$ for $|t - \bar{t}| \geq \varepsilon$, $\varphi(t) > 0$ for $|t - \bar{t}| < \varepsilon$, and $\varphi \in C_0^\infty(G)$.

Analogously, let us take any two numbers r_1, r_2, $0 < r_1 < r_2 < \text{dist}(\bar{t}, \partial G)$, and let us define $\psi(t)$, $t \in G$, by taking $\psi(t) = \beta(|t - \bar{t}|; r_1, r_2)$. Then $\psi(t) = 1$ for $t \in G$ with $|t - \bar{t}| \leq r_1$, $\psi(t) = 0$ for $t \in G$ with $|t - \bar{t}| \geq r_2$, $0 < \psi(t) < 1$ for $r_1 < |t - \bar{t}| < r_2$, and $\psi \in C_0^\infty(G)$.

We shall denote by multiindex $\alpha = (\alpha_1,\dots,\alpha_\nu)$ any set of ν nonnegative integers $\alpha_1,\dots,\alpha_\nu$, and by $|\alpha|$ the integer $|\alpha| = \alpha_1+\dots+\alpha_\nu$. If $\varphi \in C^\infty(G)$, we shall denote by $D_j\varphi$ the partial derivative $\partial\varphi/\partial t^\alpha$, $j = 1,\dots,\nu$, and in general by $D^\alpha\varphi$ the partial derivative of order $\alpha = (\alpha_1,\dots,\alpha_\nu)$ of φ in G, $D^\alpha\varphi = D_1^{\alpha_1}\dots D_\nu^{\alpha_\nu}\varphi$.

VII 2. GENERALIZED DERIVATIVES

Let G by any open subset of the t-space E_ν, $t = (t^1,\dots,t^\nu)$, $\nu > 1$. As usual we say that a real-valued function $x(t)$, $t \in G$, is locally integrable in G, and we write $x \in L^{loc}(G)$, or $x \in L_1^{loc}(G)$, if $x \in L(K) = L_1(K)$ for any compact subset $K \subset G$. Analogous definitions hold for L_p-integrability, $1 \le p < +\infty$, and even for $p = +\infty$, where then we understand that x is essentially bounded in every $K \subset G$.

If $x \in L_p(G)$, $1 \le p < +\infty$, then by $\|x\|_{p,G}$ we mean the number

$$\|x\|_{p,G} = [\textstyle\int_G |x(t)|^p dt]^{1/p}.$$

If $x \in L_\infty(G)$, that is, x is essentially bounded in G, then

$$\|x\|_{\infty,G} = \text{Ess Sup } |x(t)| \text{ for } t \in G.$$

Given any function $x(t)$, $t \in G$, $x \in L^{loc}(G)$, we say that another function $y(t)$, $t \in G$, $y \in L^{loc}(G)$, is the first order generalized partial derivative of x with respect to t^i provided

$$\int_G y(t)\,\varphi(t)dt = -\int_G x(t)\,(\partial\varphi/\partial t^i)dt \quad \text{for all } \varphi \in C_0^\infty(G). \quad \text{(VII 2.1)}$$

Analogously, given any multiindex $\alpha = (\alpha_1,\dots,\alpha_\nu)$, we say that a function $y(t)$, $t \in G$, $y \in L^{loc}(G)$, is the generalized derivative of x of order $\alpha = (\alpha_1,\dots,\alpha_\nu)$ provided

$$\int_G y(t)\,\varphi(t)dt = (-1)^{|\alpha|}\int_G x(t)\,D^\alpha\varphi(t)dt \quad \text{for all } \varphi \in C_0^\infty(G). \quad \text{(VII 2.2)}$$

As we shall see these definitions generalize the usual concepts of differentiation, and hence we shall use the same notations $y = D_i x$, or $y = D^\alpha x$. With this notation (VII 2.2) will take the symmetric form

$$\int_G \varphi(t)\, D^\alpha x(t) dt = (-1)^{|\alpha|} \int_G x(t)\, D^\alpha \varphi(t) dt \text{ for all } \varphi \in C_0^\infty(G) \qquad \text{(VII 2.3)}$$

(VII 2.i) If $\nu = 1$, if $x(t)$, $a < t < b$, is AC in every closed interval $[\bar{a}, \bar{b}] \subset (a, b)$, then $x, x' \in L^{loc}(a, b)$, and $x'(t)$ is the generalized derivative of x in (a, b).

Proof: If $\varphi \in C_0^\infty(a, b)$, then the compact support K_φ of φ is contained in some closed interval $[\bar{a}, \bar{b}] \subset (a, b)$, and by integration by parts with Lebesgue integrals we have

$$\int_a^b x'(t)\, \varphi(t) dt = \int_{\bar{a}}^{\bar{b}} x'(t)\, \varphi(t) dt = -\int_{\bar{a}}^{\bar{b}} x(t)\, \varphi'(t) dt$$

$$= -\int_a^b x(t)\, \varphi'(t) dt.$$

For $t = (t^1, \ldots, t^\nu) \in E_\nu$ and $i = 1, \ldots, \nu$, we shall denote by t_i' the $(\nu-1)$-vector $t_i^1 = (t^1, \ldots, t^{i-1}, t^{i+1}, \ldots, t^\nu)$. If $I = [\alpha, \beta]$ is an interval of E_ν, $\alpha = (\alpha^1, \ldots, \alpha^\nu)$, $\beta = (\beta^1, \ldots, \beta^\nu$, $\alpha^i < \beta^i$, $i = 1, \ldots, \nu$, and $i = 1, \ldots, \nu$, we shall denote by I_i' the $(\nu-1)$-dimensional interval $I_i' = [\alpha^s \le t^s \le \beta^s,\ s \ne i,\ s = 1, \ldots, \nu]$. If $dt = dt^1 \ldots dt^\nu$, we shall write $dt_i' = dt^1 \ldots dt^{i-1}\, dt^{i+1} \ldots dt^\nu$.

We shall denote by G_i the projection of G on the t_i'-space $E_{\nu-1}$, that is, the set of all $t_i' \in E_{\nu-1}$ such that $(t_i', t^i) \in G$ for some $t^i \in E_1$. Then G_i is an open subset of $E_{\nu-1}$, $i = 1, \ldots, \nu$. Also, for every $t_i' \in G_i$ we shall denote by $G_i(t_i')$ the open subset of all $t^i \in E_1$ with $(t_i', t^i) \in G$.

(VII 2.ii) For $\nu > 1$, if $x(t), y(t) \in L^{loc}(G)$, and if for a given $i = 1,\dots,\nu$, and almost all $t_i' \in G_i$, the function $x(t_i', t_i)$ of t^i alone is AC in every closed interval $[a, b] \in G_i(t_i')$, with derivative $y = \partial x/\partial t^i$, then y is the generalized partial derivative of x with respect to t^i in G, or $y = \partial x/\partial t^i = D_i x$, $i = 1,\dots,\nu$.

<u>Proof</u>: To simplify notations we shall limit ourselves to the case $\nu = 2$, and we shall write (t, s) instead of $(t^1,\dots,t^\nu)$. Let us prove that

$$\iint_G y(t, s)\, \varphi(t, s) dtds \;=\; -\iint_G x(t, s)\, (\partial\varphi/\varphi t) dtds \text{ for all } \varphi \in C_0^\infty(G). \tag{VII 2.4}$$

Note that, if K_φ is the compact support of φ, then both x and y are in $L(K_\varphi)$ and φ and $\partial\varphi/\partial t$ are continuous on K_φ, hence $y\varphi$ and $x(\partial\varphi/\partial t)$ are integrable on K_φ, identically zero on $G - K_\varphi$, and finally L-integrable in G. Let G_o be the projection of G on the s-axis, that is, let G_o be the set of all $s \in E_1$ such that $(t, s) \in G$ for some t, and thus G_o is open. For every $\bar{s} \in G_o$ let $G(\bar{s})$ denote the set of all t with $(t, s) \in G$ for some t; hence $G(\bar{s})$ is open, and therefore the union of countably many disjoint intervals (α, β). If $K_\varphi(\bar{s})$ has analogous definition, then the intervals (α, β) form an open cover of the compact set $K_\varphi(\bar{s})$; hence, finitely many of such intervals (α, β) cover $K_\varphi(\bar{s})$. By Fubini's theorem we have

$$\iint_G y\, \varphi \, dtds \;=\; \int_{G_o} ds \int_{G(s)} y\, \varphi \, dt \;=\; \int_{G_o} ds \, (\Sigma \int_\alpha^\beta y\, \varphi \, dt),$$

and an analogous relation holds for $x(\partial\varphi/\partial t)$. Here Σ ranges over the finitely many intervals (α, β) above with $(\alpha, \beta) \cap K_\varphi(s) \neq \Phi$. If $[\alpha_o, \beta_o]$ is any closed interval with $(\alpha, \beta) \supset [\alpha_o, \beta_o] \supset (\alpha, \beta) \cap K_\varphi(y)$, then $\varphi(\alpha_o, s) = \varphi(\beta_o, s) = 0$, and by integration by parts

$$\int_\alpha^\beta y\,\varphi\,dt = \int_{\alpha_o}^{\beta_o} y\,\varphi\,dt = -\int_{\alpha_o}^{\beta_o} x(\partial\varphi/\partial t)dt = -\int_\alpha^\beta x(\partial\varphi/\partial t)dt.$$

This holds for any $s \in G_o$ such that $x(t, s)$ is AC with respect to t on each $[\alpha_o, \beta_o]$, that is, for almost all $s \in G_o$. This proves that

$$\iint_G y\,\varphi\,dtds = \int_{G_o} ds(\Sigma \int_\alpha^\beta y\,\varphi\,dt) = -\int_{G_o} ds(\Sigma \int_\alpha^\beta x(\partial\varphi/\partial t)dt)$$

$$= \int_G x(\partial\varphi/\partial t)\,dtds,$$

that is, $y = \partial x/\partial t^i$ is the generalized first order partial derivative of x with respect to t^i according to the definition (VII 2.1).

In particular, if $x,y \in L^{loc}(G)$, and x is continuous in G together with its usual partial derivative $y = \partial x/\partial t^i$ (or $x,y \in L^{loc}(G) \cap C(G)$), then $y = \partial x/\partial t^i = D_i x$ is the generalized first order partial derivative of x with respect to t^i in G according to (VII 2.1).

Analogously, if $x(t)$, $t \in G$, is Lipschitzian in G (or on every compact subset K of G, then the usual first order partial derivatives $D_i x(t)$, $i = 1,\dots,\nu$, of x, which exist a.e. in G and are measurable in G, and bounded in G (or on every compact subset K of G), are also the generalized first order partial derivatives of x in G.

A statement analogous to (VII 2.ii) for partial derivatives of order m is as follows:

(VII 2.iii) For $\nu > 1$, $m \geq 1$, if x is continuous in G with usual continuous partial derivatives of all orders $\leq m - 1$, if for each α with $|\alpha| = m - 1$, and almost all $t_i' \in G_i$, the function $D^\alpha x(t_i', t_i)$ of t^i alone is AC in every closed interval $[a, b] \subset G_i(t_i')$, $i = 1,\dots,\nu$, then the usual partial derivatives of order m certainly exist a.e. in G. If these derivatives are known to be in $L^{loc}(G)$, then all usual partial derivatives $D^\alpha x$ of orders $0 \leq |\alpha| \leq m$ are also generalized partial derivatives (and all are in $L^{loc}(G)$).

A corollary of (VII 1.i) and (VII 2.ii). Formula (VII 2.4) need only be applied $|\alpha|$ times.

A first remark concerning generalized partial derivatives is that if y is the generalized partial derivative of x of some order α, then the same holds for any other two functions $\bar{y}$ and $\bar{x}$, where $\bar{y}$ differs from y and $\bar{x}$ from x at most in sets of measure zero in G. In other words, the relation between x and y defined by (VII 2.2) is a relation between the equivalent classes in Lebesgue integration theory defined by y and x.

(VII 2.iv) If two functions $y_1, y_2 \in L^{loc}(G)$ are such that

$$\int_G y_1 \varphi \, dt = \int_G y_2 \varphi \, dt$$

for all test functions φ, then $y_1 = y_2$ a.e. in G. In particular, generalized derivatives in G, if they exist, are uniquely defined (up to a set of measure zero in G).

Proof. We have here $\int_G (y_1 - y_2)\ \varphi\ dt = 0$ for all $\varphi \in C_o^\infty(G)$. Since $y_1 - y_2 \in L^{loc}(G)$, then for almost all points $\bar{t} \in G$ we have

$$|q|^{-1} \int_G (y_1 - y_2)dt \to y_1(\bar{t}) - y_2(\bar{t})$$

as $\varepsilon \to 0$, where q is the sphere of center $\bar{t}$ and radius $\varepsilon > 0$, and $\bar{t} \in q \subset G$. For every such $\bar{t}$ and $\varepsilon > 0$ such that $\bar{t} \in q \subset G$, we can well determine a number σ, $0 < \sigma \leq 1$, $\sigma = \sigma(\varepsilon)$, so small that

$$\int_{q'-q} |y_1 - y_2|dt \ < \ \varepsilon|q|, \quad \bar{t} \in q \subset q' \subset G,$$

where q' is the sphere of center $\bar{t}$ and radius $\varepsilon(1 + \sigma) = \varepsilon'$. Now let us consider a function $\psi(t;\ \varepsilon,\ \varepsilon')$, $t \in G$, with $\psi \in C_o^\infty$, $\psi = 1$ for $|t - \bar{t}| \leq \varepsilon$, $\psi = 0$ for $|t - \bar{t}| \geq \varepsilon'$, $0 < \psi < 1$ for $\varepsilon < |t - \bar{t}| < \varepsilon'$ (cf. VII 1). Then we have

$$0 \ = \ \int_G (y_1 - y_2)\psi\ dt \ = \ \int_{q'} (y_1 - y_2)\psi\ dt$$

$$= \ \int_q (y_1 - y_2)dt + \theta\ \varepsilon\ |q|$$

for some $-1 < \theta < 1$, and finally

$$0 \ = \ \int_G(y_1 - y_2)(\ |q|^{-1}\ \psi)dt \ = \ |q|^{-1} \int_q (y_1 - y_2)dt + \theta\ \varepsilon.$$

As $\varepsilon \to 0$ we have $0 = y_1(\bar{t}) - y_2(\bar{t})$. This holds for almost all $\bar{t} \in G$. Thus, we have proved that $y_1 = y_2$ a.e. in G.

(VII 2.v) If x, $y \in L^{loc}(G)$, if $y = D^\alpha x$ in G, and $G_o \subset G$, then $y = D^\alpha x$ in G_o.

Proof. Every element $\varphi \in C_o^\infty(G_o)$ can be extended to an element $\varphi \in C_o^\infty(E_\nu)$

(or $\varphi \in C_0^\infty(G)$) by taking $\varphi = 0$ in $E_\nu - G_0$ ($\varphi = 0$ in $G - G_0$). By (   )
we have now

$$\int_{G_0} y\,\varphi\,dt = \int_G y\,\varphi\,dt = (-1)^{|\alpha|}\int_G x(D^\alpha\varphi)dt = (-1)^{|\alpha|}\int_{G_0} x(D^\alpha\varphi)dt, \tag{VII 2.4}$$

and this proves that $y \quad D^\alpha x$ in G_0.

<u>Example 1</u>. We are now in a position to exhibit a function $z(t)$, $0 < t < 1$, ($\nu = 1$), which has no first order generalized derivative $y = z'$ in $(0, 1)$. Consider the usual function $z(t)$, $0 \leq t \leq 1$, defined in association with the ternary Cantor set S in $[0, 1]$. Then, z is continuous and monotone nondecreasing in $[0, 1]$ with $z(0) = 0$, $z(1) = 1$, and $z(t)$ constant on each interval (α, β) of the open set $G = [0, 1] - S$, with $|G| = 1$, $|S| = 0$. Let us prove that z possesses no first order generalized derivative $y = z'$ in $(0, 1)$. Indeed, assume that y is such a function. Then y is a generalized derivative of z also in each interval (α, β) where z is constant, and hence has derivative zero in (α, β). Thus, by force of (VII 2.i) we have $y = 0$ a.e. in each interval (α, β), and hence $y = 0$ a.e. in G, and $y = 0$ a.e. in $(0, 1)$. By (VII 2.1) we have now $\int_0^1 z(t)\,\varphi'(t)dt = 0$ for every $\varphi \in C_0^\infty(0, 1)$. Now take any two intervals of constancy for $z(t)$, say (α, β) and (α', β'), $0 < \alpha < \beta < \alpha' < \beta' < 1$. Then z takes on values c, c', respectively, $0 < c < c' < 1$, in (α, β) and in (α', β'). Take any two intervals of the same length η, say $[a, a+\eta] \subset (\alpha, \beta)$, $[b, b+\eta] \subset (\alpha', \beta')$. Take a function $\sigma(t)$, $-\infty < t < +\infty$, with $\sigma = 0$ for $t \leq 0$, $\sigma = 1$ for $t \geq \eta$, $0 < \sigma < 1$ for $0 < t < \eta$, and $\sigma \in C^\infty(-\infty, -\infty)$ (cfr. VII 1). Finally, let us define $\psi(t)$, $0 < t < 1$, by taking $\psi(t) = 0$ for

$0 < t \leq a$, $\psi(t) = 0$ for $b+\eta \leq t < 1$, $\psi(t) = 1$ for $a+\eta \leq t \leq b$, $\psi(t) = \sigma(t - a)$ for $a \leq t \leq a+\eta$, $\psi(t) = \sigma(b+\eta- t)$ for $b \leq t \leq b+\eta$. Then $\psi \in C_0^\infty(0, 1)$, $\psi'(a + t) = -\psi'(b + \eta - t) = \sigma'(t)$ for $0 \leq t \leq \eta$, $\psi' = 0$ otherwise, and finally

$$0 = \int_0^1 z(t)\, \psi'(t)dt = \left(\int_a^{a+\eta} + \int_b^{b+\eta}\right) z\, \psi'\, dt = (c - c')\int_0^\eta \sigma'(t)dt < 0,$$

a contradiction.

Example 2. We can now exhibit a function $x(t, s)$, $0 < t < 1$, $0 < s < 1$, $(\nu = 2)$, posessing generalized mixed partial derivative $\partial^2 x/\partial t\partial s$ but no first order generalized partial derivatives $\partial x/\partial t$ and $\partial x/\partial s$. Let us take the same function $z(t)$, $0 \leq t \leq 1$, considered in example 1, and define $x(t, s)$ by taking $x(t, s) = z(t) + z(s)$, $(t, s) \in G = [0 < t < 1, 0 < s < 1]$. Let us prove that the generalized mixed partial derivative $y(t,s) = 0$ in G. Indeed, for $y = 0$, we have

$$\iint_G y(t, s)\, \varphi(t, s)dtds = 0 \text{ for all } \varphi \in C_0^\infty(G).$$

On the other hand, for every $\varphi \in C_0^\infty(G)$ we have also

$$\iint_G y(t,s)(\partial^2\varphi/\partial t\partial s) = \int_0^1 z(t)dt \int_0^1 (\partial^2\varphi/\partial t\partial s)ds + \int_0^1 z(s)ds \int_0^1 (\partial^2\varphi/\partial t\partial s)dt = 0$$

because the two interior integrals are both zero for every t and s, respectively. Thus (VII 2.2) holds for $y = 0$ and all $\varphi \in C_0^\infty(G)$; hence $y = 0$ is the generalized mixed derivative $\partial^2 x/\partial t\partial s$. On the other hand, the same argument used in example 1 shows that x has no first order generalized partial derivatives $\partial x/\partial t$, $\partial x/\partial s$.

(VII 2.vi)(Partition of unity).. Given any compact set K and any finite open cover $\bigcup_{i=1}^m G_i$ of K, there are functions $\psi_i(t)$, $t \in E_\nu$, $\psi_i \in C_0^\infty(E_\nu)$, with supp $\psi_i \subset G_i$, $i = 1,\dots,m$, such that $\sum_{i=1}^m \psi_i(t) = 1$ for every $t \in K$.

Proof. First, all we have to do is to construct suitable compact sets C_i, C_i', $i = 1,\dots,m$, such that $K \subset \bigcup_{i=1}^m C_i$, and $C_i \subset \text{int } C_i' \subset C_i' \subset G_i$, $i = 1,\dots,m$. To do this we note that every point $\bar{t} \in K$ belongs to some G_i, and we can take a well determined sphere $S(\bar{t}, 2\bar{\delta})$ of center $\bar{t}$ and some radius $2\bar{\delta}$ such that $S(\bar{t}, 2\bar{\delta}) \subset G_i$. Now the spheres $\{S(\bar{t}, \bar{\delta})\}$ form a cover of K, and hence there is some finite cover $S(t_k, \delta_k)$, $k = 1,\dots,N$, and for every $k = 1,\dots,N$, we can shoose a well determined i, say $i = i(k)$, such that $S(t_k;\delta_k) \subset S(t_k, 2\delta_k) \subset G_{i(k)}$, $k = 1,\dots,N$. Now let C_i be the union of these spheres $S(t_k,\delta_k)$ such that $i(k) = i$, and let C_i' be the union of the corresponding spheres $S(t_k, 2\delta_k)$. Then all sets C_i, C_i', $i = 1,\dots,m$, as finite unions of spheres, are compact, $\bigcup C_i \supset K$, and $C_i \subset \text{int } C_i' \subset C_i' \subset G_i$, $i = 1,\dots,m$.

For every $k = 1,\dots,N$, let $\varphi_k(t) = \beta(t-t_k, \delta_k, \delta_{2k})$ be the function defined in (1.1); hence $\varphi_k \in C_o^\infty(E_\nu)$, $\varphi_k = 1$ on $S(t_k,\delta_k)$, $\varphi_k = 0$ in $E_\nu - S(t_k, 2\delta_k)$, $0 \le \varphi_k \le 1$ otherwise. Now we take $\phi_1 = \varphi_1$, $\phi_2 = (1-\varphi_1)\varphi_2$, $\phi_3 = (1-\varphi_1)(1-\varphi_2)\varphi_3$, $\dots$, $\phi_N = (1-\varphi_1)\dots(1-\varphi_{N-1})\varphi_N$. It is immediately seen that

$$\begin{aligned}\sum_{k=1}^N \phi_k &= \varphi_1 + (1-\varphi_1)\varphi_2 + \dots + (1-\varphi_1)\dots(1-\varphi_{N-1})\varphi_N \\ &= 1 - (1-\varphi_1)(1-\varphi_2)\dots(1-\varphi_N)\end{aligned}$$

for all $t \in E_\nu$. On the other hand, $\phi_k \in C_o^\infty(E_\nu)$, $\phi_k \ge 0$ in E_ν, and $\sum_{k=1}^N \phi_k = 1$ for every $t \in K$, since $t \in K$ implies $t \in S(t_k,\delta_k)$ and $\varphi_k = 1$ for at least one k. Finally, for every $i = 1,\dots,m$, let $\psi_i(t) = \sum \phi_k(t)$, where $\sum$ ranges over all k with $i(l) = l$. Then $\sum_{i=1}^m \psi_i(t) = 1$ for all $t \in K$, $\psi_i \in C_o^\infty(E_\nu)$, $\psi_i \ge 0$ on E_ν.

Also, the support of ψ_i is contained in the union of the spheres $S(t_k, 2\delta_k)$ with $i(k) = i$, all these spheres are in G_i, and supp $\psi_i \subset G_i$, $i = 1,\dots,m$. Statement (VII 2.vi) is thereby proved.

We are now in a position to prove a statement which is essentially the converse of (VII 2.ii).

(VII 2.vii) If $x, y \in L^{loc}(G)$, if there is an open covering $\{\Gamma\}$ of G such that y is the generalized derivative of order α of x in each Γ, then y is the generalized derivative of order α of x in the whole of G.

<u>Proof</u>. Let φ be any element of $C_o^\infty(G)$ with compact support K_φ. Let G_o be an open set with compact closure such that $K_\varphi \subset G_o \subset \mathrm{cl}\, G_o \subset G$. Then $\{\Gamma\}$ is an open cover of cl G_o, and hence there is a finite subcover, say $[\Gamma_s$, $s = 1,\dots,N]$. By force of (VII 2.vi) there are functions $\psi_s(t)$, $t \in G$, with compact support $K_{\psi_s} \subset \Gamma_s$, such that $\psi_s \in C_o^\infty(G)$, $s = 1,\dots,N$, and such that $\sum_{s=1}^N \psi_s(t) = 1$ on cl G_o. Now we have

$$\int_G y\,\varphi\,dt = \int_{\mathrm{cl}\,G_o} y\,\varphi\,dt = \int_{\mathrm{cl}\,G_o} y\Big(\sum_{s=1}^N \psi_s\Big)\varphi$$

$$= \sum_{s=1}^N \int_{\mathrm{cl}\,G_o} y\,\psi_s\,\varphi\,dt = \sum_{s=1}^N \int_{\Gamma_s} y(\varphi\,\psi_s)dt$$

where $\varphi\,\psi_s \in C_o^\infty(\Gamma_s)$. Hence

$$\int_{\Gamma_s} y(\varphi\,\psi_s)dt = (-1)^{|\alpha|} \int_{\Gamma_s} x(D^\alpha(\varphi\,\psi_s))dt$$

because of the property of y to be the weak derivative of order α of x in each Γ_s. Finally,

$$\int_G y\,\varphi\,dt = (-1)^{|\alpha|} \sum_s \int_{\Gamma_s} x\Big(\sum_\beta \binom{\alpha}{\beta} D^\beta\,\psi_s\,D^{\alpha-\beta}\varphi\Big)dt$$

$$= (-1)^{|\alpha|} \int_{G_o} x\,\Big(\sum_\beta \binom{\alpha}{\beta} D^\beta\,\Big(\sum_s\psi_s\Big)\,D^{\alpha-\beta}\varphi\Big)dt$$

Since $\sum_s \psi_s \equiv 1$ in G_o, all derivatives $D^\beta (\sum_s \psi_s)$ with $|\beta| > 0$ are zero in G_o and

$$\int_G y\, \varphi\, dt = (-1)^{|\alpha|} \int_{G_o} x\, (D^\alpha \varphi) dt = (-1)^\alpha \int_G x\, (D^\alpha \varphi)\, dt.$$

Statement (VII 2.vii) is thereby proved.

(VII 2.viii) (Leibnetz rule). If $x \in L_p^{loc}(G)$, $1 \leq p \leq +\infty$, $y \in L_q^{loc}(G)$, $1 \leq q \leq +\infty$, $1/p + 1/q \leq 1$, possess generalized partial derivatives $D^\alpha x$, $D^\alpha y$ of all orders $0 \leq |\alpha| \leq m$, and of classes $L_p^{loc}(G)$, $L_q^{loc}(G)$, respectively, and $1 \leq \lambda \leq +\infty$ is so chosen that $1/p + 1/q \leq 1/\lambda$, then $xy \in L_\lambda^{loc}(G)$ possesses generalized partial derivatives $D^\alpha(xy)$ of all orders $0 \leq |\alpha| \leq m$, all in $L_\lambda^{loc}(G)$, and they are given by the Leibnitz rule

$$D^\alpha(xy) = \sum \binom{\alpha}{\beta} D^\beta x\, D^{\alpha-\beta} y \quad \text{a.e. in } G, \tag{VII 2.5}$$

where $\alpha = (\alpha_1,\ldots,\alpha_\nu)$, $0 \leq |\alpha| \leq m$, $\beta = (\beta_1,\ldots,\beta_\nu)$, where $\sum$ ranges over all $0 \leq \beta \leq \alpha$, that is, $0 \leq \beta_i \leq \alpha_i$, $i = 1,\ldots,\nu$, and

$$\binom{\alpha}{\beta} = \alpha!/\beta!(\alpha - \beta)! = \alpha_1!\ldots\alpha_\nu!/\beta_1!\ldots\beta_\nu!(\alpha_1 - \beta_1)!\ldots(\alpha_\nu - \beta_\nu)!$$

Proof. It is not restrictive to assume $(\lambda/p) + (\lambda/q) = 1$. Then $x \in L_{p/\lambda}(K)$, $y \in L_{q/\lambda}(K)$ for every compact subset K of G. By Hölder inequality with exponents p/λ, q/λ we have

$$(\int_K |xy|^\lambda\, dt \leq (\int_K |x|^p\, dt)^{\lambda/p} (\int_K |y|^q\, dt)^{\lambda/q}.$$

Thus $xy \in L_\lambda(K)$ for every compact $K \subset G$, and $xy \in L_\lambda^{loc}(G)$. The same holds for each of the products $D^{\alpha}x\ D^{\alpha-\beta}y$ which appear in (VII 2.5).

First, let us prove (VII 2.5) for $y \in C_0^{\infty}(G)$. Let φ be any element of $C_0^{\infty}(G)$ and note that

$$\begin{aligned}\int_G \left(\Sigma_\beta \binom{\alpha}{\beta} D^\beta x\ D^{\alpha-\beta}y\right)\varphi\ dt &= \Sigma_\beta \binom{\alpha}{\beta} \int_G (D^\beta x)((D^{\alpha-\beta}y)\varphi)\ dt \\ &= \Sigma_\beta (-1)^{|\beta|}\binom{\alpha}{\beta} \int_G x\ D^\beta((D^{\alpha-\beta}y)\varphi)dt \\ &= \int_G x[\Sigma_\beta \Sigma_\gamma (-1)^{|\beta|}\binom{\alpha}{\beta}\binom{\beta}{\gamma} D^\sigma\varphi D^{\alpha-\sigma}y)dt,\end{aligned} \quad \text{(VII 2.6)}$$

where Σ_β ranges over all $0 \leq \beta \leq \alpha$, and Σ_γ over all $0 \leq \gamma \leq \beta$. In these transformations we have applied the Leibnitz rule on the products $(D^{\alpha-\beta}y)\varphi$ of functions of $C_0^{\infty}(G)$. For $\gamma = \alpha$ we have $\gamma = \beta = \alpha$ and the corresponding term in brackets is $(-1)^{|\alpha|}y\ D^{\alpha}\varphi$. The remaining terms in brackets have sum zero because of the identity $\Sigma_\beta (-1)^{|\beta|}\binom{\alpha}{\beta}\binom{\beta}{\gamma} = 0$. Thus

$$\int_G (\Sigma_\beta \binom{\alpha}{\beta} D^\beta x\ D^{\alpha-\beta}y)\ \varphi\ dt = (-1)^{|\alpha|}\int_G (xy)\ D^{\alpha}\varphi\ dt \qquad \text{(VII 2.7)}$$

for every $\varphi \in C_0^{\infty}(G)$. Thus, by force of (VII 2.iv), the derivative $D^{\alpha}(xy)$ exists and is given by (VII 2.5) a.e. in G. We have proved relation (VII 2.5) for $y \in C_0^{\infty}(G)$, $0 \leq |\alpha| \leq m$.

Now let us assume $y \in L_q^{loc}(G)$. The same argument above applies since now we can use the Leibnitz rule on the products $(D^{\alpha-\beta}y)\varphi$ and partial derivatives of orders $0 \leq \gamma \leq \beta$ which appear in (VII 2.6). We obtain thus relation (VII 2.7), which shows that $D^{\alpha}(xy)$ exists, $0 \leq |\alpha| \leq m$, and is given by formula (VII 2.5) again by force of (VII 2.iv).

(VII 2.ix) If $x \in L_p^{loc}(G)$, if $y = D^\alpha x \in L_p^{loc}(G)$, and $z = D^\beta y \in L_p^{loc}(G)$, then $z = D^{\alpha+\beta}x$.

Proof. For every $\varphi \in C_o^\infty(G)$ we have

$$\begin{aligned}\int_G x(D^{\alpha+\beta}\varphi)dt &= \int_G x\, D^\alpha(D^\beta\varphi)dt = (-1)^{|\alpha|}\int_G (D^\alpha x)(D^\beta\varphi)dt \\ &= (-1)^{|\alpha|}\int_G y(D^\beta\varphi)dt = (-1)^{|\alpha|+|\beta|}\int_G (D^\beta y)\varphi\, dt \\ &= (-1)^{|\alpha+\beta|}\int_G z\,\varphi\, dt.\end{aligned}$$

This proves that $z = D^{\alpha+\beta}x$.

(VII 2.x) Given functions $y_\alpha(t)$, $t \in G$, for every $\alpha = (\alpha_1,\dots,\alpha_\nu)$ with $0 \le |\alpha| \le m$, $y_\alpha \in L_p^{loc}(G)$, and a sequence of functions $x_k(t)$, $t \in G$, $k = 1,2,\dots$, possessing generalized derivatives $D^\alpha x_k(t)$, $t \in G$, $D^\alpha x_k \in L_p^{loc}(G)$, $k = 1,2,\dots$, such that, for every compact subset K of G, $\|D^\alpha x_k - y_\alpha\|_{p,K} \to 0$ as $k \to \infty$, then the function $y = y_o$ has generalized partial derivatives of all orders $\le m$, and $D^\alpha y = y_\alpha$ in G, $0 \le |\alpha| \le m$.

Indeed, if φ is any test function, with compact support $K_\varphi \subset G$, then, for $0 \le |\alpha| \le m$, we have

$$\int_G (D^\alpha x_k)\varphi\, dt = (-1)^{|\alpha|}\int_G x_k(D^\alpha\varphi)dt, \quad k = 1,2,\dots,$$

where the integrations can be made on K_φ. By a passage to the limit on the integrals ranging on K_φ, and then writing G again instead of K_φ, we have

$$\int_G y_\alpha\,\varphi\, dt = (-1)^{|\alpha|}\int_G y(D^\alpha\varphi)dt.$$

Thus, $y_\alpha = D^\alpha y$ for all $0 \le |\alpha| \le m$.

Remark. Statement above holds even if we know only that in every bounded open subset G_o with $G_o \subset \text{cl } G_o \subset G$, we have $D^\alpha x_k \to y_\alpha$ as $k \to \infty$ weakly in $L_p(G_o)$ (or even only weakly in $L_1(G_o)$). The proof is analogous.

VII 3. MOLLIFIERS

We shall consider any function $j(t)$, $t \in E_n$, of class $C_o^\infty(E_\nu)$ such that $j(t) \geq 0$ for all $t \in E_\nu$, $j(t) = 0$ for $|t| \geq 1$, and $\int_{E_\nu} j(t)dt = 1$. An example of such a function is of course $j(t) = c \exp ((|t|^2-1)^{-1}$ for $|t| < 1$, $j(t) = 0$ for $|t| \geq 1$, where c is a suitable constant (cfr. VII 1.1).

For every $\varepsilon > 0$ we shall now define $j_\varepsilon(t)$ by taking $j_\varepsilon(t) = \varepsilon^{-\nu} j(\varepsilon^{-1}t)$. Then $j_\varepsilon(t) \geq 0$ for all $t \in E_\nu$, $j_\varepsilon(t) = 0$ for $|t| \geq \varepsilon$, and $\int_{E_\nu} j_\varepsilon(t)dt = 1$.

We shall now denote by J_ε the operator defined by

$$y(t) = (J_\varepsilon x)(t) = \int_G j_\varepsilon(t-\tau)x(\tau)d\tau,$$

where $x \in L^{loc}(G)$ and where $y(t)$ is defined for all $t \in G$ with dist $(t,\partial G) > \varepsilon$. If $x \in L^{loc}(E_\nu)$, or $x \in L(G)$ for some $G \subset E_\nu$ and we extend x to an element $x \in L^{loc}(E_\nu)$ by taking $x = 0$ in $E_\nu - G$, then y is defined for all $t \in E_\nu$ and $y \in L^{loc}(E_\nu)$.

If $x \in L^{loc}(G)$ and x has compact support $K_x \subset G$, then $x \in L(G)$, and even $x \in L(E_\nu)$ if x is extended to all E_ν by taking $x = 0$ in $E_\nu - G$. Moreover

(VII 3.i) If $x \in L^{loc}(G)$ and has compact support $K_x \subset G$, then for every ε, $0 < \varepsilon <$ dist $(K_x,\partial G)$ we have $y \in C_o^\infty(G)$.

The proof is left as an exercise for the reader.

Below, we shall always assume that x is extended in E_ν by taking $x = 0$ in $E_\nu - G$. With this convention, if $x \in L_1(G)$, then $y = J_\varepsilon x$ is defined for all $t \in G$ (and even for all $t \in E_\nu$), and J_ε: $L_1(G) \to C^\infty(G)$.

(VII 3.ii) If $x \in L_p(G)$, $1 \leq p \leq +\infty$, then $y = J_\varepsilon x \in L_p(G)$ and $\|J_\varepsilon x\|_{p,G} \leq \|x\|_{p,G}$.

Proof. Let us assume $1 < p < +\infty$. By Hölder inequality we have

$$\begin{aligned}
|(J_\varepsilon x)(t)|^p &= |\int_G j_\varepsilon(t-\tau)x(\tau)d\tau|^p \\
&\leq |\int_G (j_\varepsilon(t-\tau))^{1/p}\, x(\tau) \cdot (j_\varepsilon(t-\tau))^{1/q} d\tau|^p \\
&\leq (\int_G j_\varepsilon(t-\tau)|x(\tau)|^p d\tau)(\int_G j_\varepsilon(t-\tau)d\tau)^{p/q} \\
&\leq (\int_G j_\varepsilon(t-\tau)|x(\tau)|^p d\tau)
\end{aligned}$$

where $1/q + 1/p = 1$. Then, by Fubini's theorem we have

$$\begin{aligned}
\|J_\varepsilon x\|_{p,G} &= (\int_G |(J_\varepsilon x)(t)|^p dt)^{1/p} \\
&\leq (\int_G dt \int_G j_\varepsilon(t-\tau)|x(\tau)|^p d\tau)^{1/p} \\
&\leq (\int_G |x(\tau)|^p d\tau \int_G j_\varepsilon(t-\tau)dt)^{1/p} \\
&\leq (\int_G |x(\tau)|^p d\tau)^{1/p} = \|x\|_{p,G}.
\end{aligned}$$

The cases $p = 1$ and $p = \infty$ are left as exercises for the reader.

(VII 3.iii) If $x \in L_p^{loc}(G)$, $1 \leq p \leq +\infty$, then $J_\varepsilon x \to x$ in $L_p(K)$ as $\varepsilon \to 0$ for any compact $K \subset G$; if $x \in L_p(G)$, $1 \leq p \leq +\infty$, then $J_\varepsilon x \to x$ in $L_p(G)$; in any case $J_\varepsilon x(t) \to x(t)$ as $\varepsilon \to 0$ a.e. in G. If a compact set $K \subset G$ is made up of points of continuity for x in G, then $J_\varepsilon x \to x$ uniformly in K. If $x \in L_p(E_\nu)$, then $J_\varepsilon x \to x$ in $L_p(E_\nu)$.

Proof. Again, let us assume first $1 < p < +\infty$. Let K be any compact subset of G and let G_o be any open set with compact closure $\mathrm{cl}G_o$ such that $K \subset G_o \subset \mathrm{cl}G_o \subset G$. Then $x \in L(G_o)$. Let $\delta = \mathrm{dist}\,(K, \mathrm{bd}\, G_o)$, and take $0 < \varepsilon < \delta$. Then $y = J_\varepsilon x$ is certainly defined in all of K. Let $\eta > 0$ by any arbitrary number. By Lebesgue integration theory we know that there exists a continuous function $z(t)$, $t \in \mathrm{cl}G_o$, with $(\int_{G_o} |x-z|^p dt)^{1/p} < \eta/3$. Then $z(t)$ is uniformly continuous in $\mathrm{cl}G_o$ and there is some $\delta' > 0$ such that $t, t' \in \mathrm{cl}G_o$, $|t-t'| \leq \delta'$ implies $|z(t) - z(t')| \leq (3^{-1}\eta|G_o|^{-1})^{1/p}$. Let us assume $0 < \varepsilon < \min[\delta,\delta']$. Now, for $t \in K$, we have

$$(J_\varepsilon z)(t) - z(t) = \int_G j_\varepsilon(t-\tau)z(\tau)d\tau - z(t),$$

and since the sphere q of center t and radius ε is completely contained in G_o and hence in G, we have $\int_G j_\varepsilon(t-\tau)d\tau = 1$ and finally

$$\begin{aligned} |J_\varepsilon z(t) - z(t)| &= \left|\int_G j_\varepsilon(t-\tau)[z(\tau) - z(t)]d\tau\right| \\ &\leq \int_q j_\varepsilon(t-\tau)|z(\tau) - z(t)|d\tau \leq (3^{-1}\eta|G_o|^{-1})^{1/p} \end{aligned}$$

for all $t \in K$. Finally $(\int_{G_o} |J_\varepsilon z-z|^p dt)^{1/p} \leq \eta/3$, and since

$$|J_\varepsilon x-x| \leq |J_\varepsilon(x-z)| + |x-z| + |J_\varepsilon z-z|,$$

by Minkovsky's inequality and (VII 3.ii), we have

$$\begin{aligned} \int_K |J x-x|^p dt &\leq \int_{G_o} |J_\varepsilon x-x|^p dt \leq (\int_{G_o} (J_\varepsilon(x-z)|^p dt)^{1/p} + (\int_{G_o} |x-z|^p dt)^{1/p} \\ &+ (\int_{G_o} |J_\varepsilon z-z|^p dt)^{1/p} \leq \eta/3 + \eta/3 + \eta/3 = \eta \end{aligned}$$

for all $\varepsilon > 0$ sufficiently small. The first part of (VII 3.iii) is proved for $1 < p < +\infty$. We leave the cases $p = 1$ and $p = \infty$ as an exercise for the reader. To prove convergence almost everywhere in G, we note that, given $x \in L^{loc}(G)$, then almost every point $\bar{t} \in G$ has the property that $|q|^{-1}\int_q |x(\tau) - x(t)|d\tau \to 0$ as $\varepsilon \to 0$, where q denotes the sphere of center t and radius ε, and $q \subset G$ for all $\varepsilon > 0$ sufficiently small. Then

$$\begin{aligned}|J_\varepsilon(t) - x(t)| &= |\int_G j_\varepsilon(t-\tau)\, x(\tau)d\tau - x(t)| \\ &\leq |\int_q j_\varepsilon(t-\tau)|x(\tau) - x(t)|d\tau \\ &\leq c\, \varepsilon^{-\nu}\int_q |x(\tau) - x(t)|dt = cc_o|q|^{-1}\int_q|x(\tau) - x(t)|dt,\end{aligned}$$

where c_o is an absolute constant. This proves that $|J_\varepsilon(t) - x(t)| \to 0$ as $\varepsilon \to 0$, and this holds a.e. in G.

The second part of (VII 3.iii) is immediately proved by taking K so large that, if $G' = G-K$, then $\int_{G'}|x|^p dt \leq \eta^p$. Then, by force of (VII 3.ii), we have $\int_{G'}|J_\varepsilon x|^p dt \leq \eta^p$, and finally

$$\begin{aligned}(\int_G |J_\varepsilon x-x|^p dt)^{1/p} &= [(\int_{G'} + \int_K)|J_\varepsilon x-x|^p dt]^{1/p} \\ &\leq [2^p\int_{G'} |J_\varepsilon x|^p dt + 2^p \int_{G'} |x|^p dt + \int_K |J_\varepsilon x-x|^p dt]^{1/p} \\ &\leq (3\eta^p 2^p)^{1/p} = 3^{1/p}(2\eta).\end{aligned}$$

Here we have used the inequality $|\alpha+\beta|^p \leq 2^p(\alpha^p + \beta^p)$ for all α, β real which is immediately proved by taking $\gamma = \max(|\alpha|,|\beta|)$, and noting that $|\alpha+\beta|^p \leq (2\gamma)^p \leq 2^p(|\alpha|^p+|\beta|^p)$.

Now let us prove the third part of (VII 3.iii), which requires a more subtle argument. Let K be a compact subset of G made up of points of continuity for t in G. First x(t) is certainly continuous on K, hence uniformly continuous, and given $\eta > 0$ there is some $\bar{\delta} > 0$ such that $t, t' \in K$, $|t-t'| \leq \bar{\delta}$, implies $|x(t)-x(t')| \leq \eta$. Moreover, for any $\bar{t} \in K$, there is some open sphere $U(\bar{t})$ of center $\bar{t}$ such that $|x(t)-x(\bar{t})| < \eta$ for all $t \in U(\bar{t}) \cap G$. We may well assume that each sphere $U(\bar{t})$ has closure $clU(\bar{t}) \subset G$ and radius $\leq \bar{\delta}/2$. Then finitely many of these spheres, say $U(t_i)$, $i = 1,\dots,N$, cover K, and hence their union is an open set G_o with compact closure and $K \subset G_o \subset clG_o \subset G$. Let $\delta'' = \text{dist}\,(K, \text{bd}\, G_o)$, and let δ_i be the radius of the sphere $U(t_i)$, $i = 1,\dots,N$. Let us assume $0 < \varepsilon < \min[\delta,\bar{\delta}/2,\delta'',\delta_i, i = 1,\dots,N]$. Then, for every $t \in K$ we have as before

$$|(J_\varepsilon x)(t)-x(t)| = |\int_G j_\varepsilon(t-\tau)x(\tau)d\tau - x(t)|$$

$$\leq \int_q j_\varepsilon(t-\tau)|x(\tau)-x(t)|d\tau,$$

where q is the sphere of center t and radius ε. For every $\tau \in q$ we have $|\tau-t| \leq \varepsilon \leq \bar{\delta}/2$. On the other hand $\tau \in G_o$, $\tau \in U(t_i)$ for some $i = 1,\dots,N$, and hence $|\tau-t_i| \leq \delta_i \leq \bar{\delta}/2$. Thus $|t-t_i| \leq |t-\tau| + |\tau-t_i| \leq \bar{\delta}/2 + \bar{\delta}/2 = \bar{\delta}$, with both t and t_i points of K. Then $|x(t)-x(t_i)| \leq \eta$ and

$$|x(\tau)-x(t)| \leq |x(\tau)-x(t_i)| + |x(t_i)-x(t)| \leq \eta + \eta = 2\eta.$$

We have now

$$|(J_\varepsilon x)(t)-x(t)| \leq 2\eta \int_q j_\varepsilon(t-\tau)d\tau = 2\eta,$$

and this relation holds for all $t \in K$ and $\varepsilon > 0$ sufficiently small.

The proof of the fourth part of (VII 3.iii) does not present difficulties. Indeed, given $\eta > 0$, we take $R > 0$ so large that $\int |x(t)|^p dt < \eta/2$ when the integration is performed in $|t| > R/2$. Now for $0 < \varepsilon < R/4$, we have

$$\int_{E_\nu} |J_\varepsilon x - x|^p dt = \left(\int_{|t| \geq 3R/4} + \int_{|t| \leq 3R/4} \right) |J_\varepsilon x - x|^p dt$$

$$\leq 2^p \int_{|t| \geq 3R/4} |x|^p dt + 2^p \int_{|t| \geq R/2} |x|^p dt$$

$$+ \int_{t \leq 3R/4} |J_\varepsilon x - x|^p dt.$$

The first two integrals in the last member are $\leq \eta$, and the last integral approaches zero as $\varepsilon \to 0$ by force of the first part of (VII 3.iii) already proved.

<u>Remark 1</u>. In the third part of (VII 3.iii) the hypothesis that the compact set K is made up of points of continuity for $x(t)$ in G is clearly stronger than the hypothesis that x be continuous on K. The conclusion would not be true under the latter.

<u>Remark 2</u>. We mention here that, if $f \in L_p(E_\nu)$, $1 \leq p \leq +\infty$, and h denotes any vector in E_ν, $h = (h_1, \ldots, h_\nu)$, then $\|f(t+h) - f(t)\|_p \to 0$ as $h \to 0$. In other words,

$$\int_{E_\nu} |f(t+h) - f(t)|^p dt \to 0 \quad \text{as } |h| \to 0.$$

The reader may consult, for instance, E. J. McShane [77q], p. 230, (42.4s). This remark may yield a new proof of the second part of statement (VII 3.iii).

We need the following properties of uniformity.

(VII 3.iv) If $\{f\}$ is a family of functions $f \in L_p(G)$, $1 \leq t \leq +\infty$, all zero in $E_\nu - G$, and such that $\|f(t+h)-f(t)\|_p \to 0$ as $h \to 0$ uniformly with respect to the element $f \in \{f\}$, then $\|J_\varepsilon f-f\|_p \to 0$ as $\varepsilon \to 0$ also uniformly with respect to the element f.

Proof. We assume first $1 < p < +\infty$, and we understand that the smooth functions $J_\varepsilon f(t)$ are defined for all $t \in E_\nu$. Hence, $J_\varepsilon f(t) = 0$ for all $t \in E_\nu - G$ at a distance $\geq \varepsilon$ from $\mathrm{cl}G$. We have now

$$\begin{aligned}\int_{E_\nu} |J_\varepsilon f(t)-f(t)|^p dt &= \int_{E_\nu} [\int_{E_\nu} j_\varepsilon(t-\tau)(f(\tau)-f(t))d\tau]^p dt \\ &\leq \int_{E_\nu} [\int_{E_\nu} j_\varepsilon(u)|f(t-u)-f(t)|du]^p dt \\ &= \int_{E_\nu} [\int_{E_\nu} (j_\varepsilon(u))^{1/p}|f(t-u)-f(t)|(j_\varepsilon(u))^{1/q}du]^p dt \\ &\leq \int_{E_\nu} (\int_{E_\nu} j_\varepsilon(u)|f(t-u)-f(t)|^p du)(\int_{E_\nu} j_\varepsilon(u)du)^{p/q} dt\end{aligned}$$

The last integral is equal to one. Now, given $\eta > 0$, we can determine $\sigma > 0$ such that $\|f(t+h)-f(t)\|_p \leq \eta$ for all $|h| \leq \sigma$. Now we have, for $0 < \varepsilon \leq \sigma$,

$$\|J_\varepsilon f-f\|_p^p \leq \int_{E_\nu} j_\varepsilon(u)du \int_{E_\nu} |f(t-u)-f(t)|^p dt,$$

where $j_\varepsilon(u) = 0$ for all $|u| \geq \varepsilon$. Thus, we may restrict the first integral to the solid ball $|u| \leq \varepsilon \leq \sigma$, and then for any $|u| \leq \varepsilon < \sigma$, we have $\|f(t-u)-f(t)\|_p \leq \eta$. Thus,

$$\|J_\varepsilon f-f\|_p \leq \eta(\int_{E_\nu} j_\varepsilon(u)du)^{1/p} = \eta$$

for all $0 < \varepsilon \leq \sigma$, and any element $f \in \{f\}$. The analogous proofs for $p = 1$ and $p = \infty$ are left to the reader.

(VII 3.v) If $\{f\}$ is a family of functions $f \in L_p(G)$, $1 \leq p \leq +\infty$, all zero in $E_\nu - G$, and such that $\|f(t+h)-f(t)\|_p \to 0$ as $h \to 0$ uniformly with respect to the elements $f \in \{f\}$, then, for every fixed $\varepsilon > 0$, the smooth functions $J_\varepsilon f(t)$, $t \in E_\nu$, are equicontinuous in E_ν.

<u>Proof</u>. Let $\varepsilon > 0$ be a fixed number. We assume first $1 < p < +\infty$. Let K denote the maximum of the function $j(t)$, $t \in E_\nu$, of the first lines of (VII 3). Then $|j_\varepsilon(t)| \leq \varepsilon^{-\nu}$ for all $t \in E_\nu$.

Let $\eta > 0$ be any positive number, and let $\sigma > 0$ be so chosen that $\|f(t+h)-f(t)\|_p \leq \eta(K^{-1}\varepsilon^\nu)^{1/p}$ for all $|h| \leq \sigma$ and $f \in \{f\}$. We have now, for $|h| \leq \sigma$,

$$\begin{aligned}
|J_\varepsilon f(t+h)-J_\varepsilon f(t)| &= |\int_{E_\nu} j_\varepsilon(t+h-\tau)f(\tau)d\tau - \int_{E_\nu} j_\varepsilon(t-\tau)f(\tau)d\tau| \\
&= |\int_{E_\nu} j_\varepsilon(t-\tau)[f(\tau+h)-f(\tau)]d\tau| \\
&= |\int_{E_\nu} j_\varepsilon(u)[f(t+h-u)-f(t-u)]du| \\
&\leq \int_{E_\nu} (j_\varepsilon(u))^{1/p}|f(t+h-u)-f(t-u)|(j_\varepsilon(u))^{1/q}du \\
&\leq (\int_{E_\nu} j_\varepsilon(u)|f(t+h-u)-f(t-u)|^p)^{1/p}(\int_{E_\nu} j_\varepsilon(u)du)^{1/p},
\end{aligned}$$

where $j_\varepsilon(u) = 0$ for $|u| \geq \varepsilon$. Thus, we can restrict the first integral to the solid ball $|u| \leq \varepsilon$, where certainly $|j_\varepsilon(u)| \leq \varepsilon^{-\nu}K$, and then

$$|J_\varepsilon f(t+h) - J_\varepsilon f(t)| \leq (K\varepsilon^{-\nu})^{1/p} \left(\int_{E_\nu} |f(t+h-u)-f(t-u)|^p \, du\right)^{1/p}$$

$$\leq (K\varepsilon^{-\nu})^{1/p} \cdot \eta(K^{-1}\varepsilon^\nu)^{1/p} = \eta$$

for all $|h| \leq \sigma$, and the fixed value of ε. Analogous proof holds for $p = 1$ and $p = \infty$.

(VII 3.vi) If $x \in L_p^{loc}(G)$, $1 \leq p \leq +\infty$, possesses generalized derivative $y = D^\alpha x \in L_p^{loc}(G)$ of some order $\alpha = (\alpha_1, \ldots, \alpha_\nu)$, then for every compact subset K of G and all $0 < \varepsilon < \delta$, where $\delta = \text{dist}\,\{K, \partial G\}$, we have

$$(D^\alpha J_\varepsilon x)(t) = (J_\varepsilon D^\alpha x)(t), \quad t \in K,$$

and $J_\varepsilon x \in C^\infty(K)$.

<u>Proof</u>. For $t \in K$ and $0 < \varepsilon < \delta$ we have

$$(D^\alpha J_\varepsilon x)(t) = D_t^\alpha \int_G j_\varepsilon(t-\tau)x(\tau)d\tau = \int_G (D_t^\alpha j_\varepsilon(t-\tau))x(\tau)d\tau.$$

Here $j_\varepsilon(t-\tau) \in C_o^\infty(G)$ since $j_\varepsilon(t-\tau) = 0$ for $|t-\tau| \geq \varepsilon$, and the solid ball $|t-\tau| \leq \varepsilon$ is completely in the interior of G. By force of (VII 2.2) we have then

$$(D^\alpha J_\varepsilon x)(t) = (-1)^{|\alpha|} \int_G D_\tau^\alpha j_\varepsilon(t-\tau)x(\tau)d\tau$$

$$= (-1)^{2|\alpha|} \int_G j_\varepsilon(t-\tau)(D_\tau^\alpha x(\tau))d\tau$$

$$= (J_\varepsilon D^\alpha x)(t)$$

and this relation holds for $t \in K$ and $o < \varepsilon < \delta = \text{dist}\,(K, \partial G)$.

(VII 3.vii) If $x \in L^{loc}(G)$, $1 \leq p \leq +\infty$, possesses generalized partial derivatives $D^\alpha x \in L_p^{loc}(G)$ of all orders $|\alpha| \leq m$, then for every compact set $K \subset G$, we have $\|D^\alpha x - D^\alpha J_\varepsilon x\|_{p,K} \to 0$ as $\varepsilon \to 0+$ for every $|\alpha| \leq m$ and compact subset $K \subset G$.

If K is made up of points of continuity for x and all $D^\alpha x$ in G, $0 \leq |\alpha| \leq m$, then $J_\varepsilon x \to x$, $D^\alpha J_\varepsilon x \to D^\alpha x$ as $\varepsilon \to 0$ uniformly on K, $0 \leq |\alpha| \leq m$.

A corollary of (VII 3.iii) and (VII 3.vi).

(VII 3.viii) If G is connected, if $x_1(t)$, $x_2(t)$, $t \in G$, are elements of $L^{loc}(G)$ possessing the same generalized first order partial derivatives a.e. in G, that is, $D_i x_1$, $D_i x_2 \in L^{loc}(G)$ and $D_i x_1(t) = D_i x_2(t)$ a.e. in G, $i = 1,\dots,\nu$, then $x_1(t) = x_2(t)+c$ a.e. in G for some constant c.

<u>Proof</u>. By (VII 3.vi) we see that for every closed interval $I = [a,b] \subset G$ and every $\varepsilon > 0$ sufficiently small, $(J_\varepsilon x_1)(t)$, $(J_\varepsilon x_2)(t)$ are of class $C^\infty[a,b]$ and have the same first order partial derivatives on [a,b]. Thus, $(J_\varepsilon x_1)(t) = (J_\varepsilon x_2)(t) + c_\varepsilon$ for all $t \in [a,b]$. Since G is connected, the constant c_ε is independent of [a,b]. By (VII 3.iii) we have $\|J_\varepsilon x_1 - x_1\|_{1,I} \to 0$, $\|J_\varepsilon x_2 - x_2\|_{1,I} \to 0$ as $\varepsilon \to 0$. Hence $c_\varepsilon \to c$ for some constant c as $\varepsilon \to 0$, and $\|x_1 - x_2\|_{1,I} = c \text{ meas}[a,b]$, where c is independent of [a,b]. Thus, $x_1 = x_2 + c$ a.e. in G.

A corollary of (VII 3.viii) is that, if x_1, $x_2 \in L^{loc}(G)$ possess generalized partial derivatives of all orders α, $0 \leq |\alpha| \leq m$ (all in $L^{loc}(G)$), and the derivatives of orders $|\alpha| = m$ coincide a.e. in G, then $x_1 - x_2 = P$ is a polynomial in $t^1,\dots,t^\nu$ of order $\leq m$.

(VII 3.ix) If $x \in L_p^{loc}(G)$, $1 \leq p \leq +\infty$, and y is continuous in G with compact support $K_y \subset G$, with Lipschitzian partial derivatives of all orders $\leq m-1$ (and hence bounded partial derivatives of orders $\leq m$ in G), then for any α with $0 \leq |\alpha| \leq m$ we have

$$\int_G (D^\alpha x) y \, dt = (-1)^{|\alpha|} \int_G x (D^\alpha y) dt \qquad \text{(VII 3.1)}$$

Proof. Let G_o be an open set such that $K_y \subset G_o \subset \mathrm{cl} G_o \subset G$, and let $\varepsilon_o = \mathrm{dist}\,(G_o, \partial G) > 0$. For $0 < \varepsilon < \varepsilon_o$, then we have $D^\alpha(J_\varepsilon x(t)) = J_\varepsilon D^\alpha x(t)$ for all $t \in G_o$. On the other hand relation (VII 3.1) is elementary for $x(t)$ replaced by $(J_\varepsilon x)(t)$. Then

$$\int_{G_o} J_\varepsilon (D^\alpha x) y \, dt = \int_{G_o} (D^\alpha (J_\varepsilon x)) y \, dt = (-1)^{|\alpha|} \int_{G_o} (J_\varepsilon x)(D^\alpha y) dt.$$

Now $\|J_\varepsilon (D^\alpha x) - D^\alpha x\|_p \to 0$, $\|J_\varepsilon x - x\|_p \to 0$ where the L_p norms are taken in G_o. By a passage to the limit we obtain $\int_{G_o} (D^\alpha x) y \, dt = (-1)^{|\alpha|} \int_{G_o} x (D^\alpha y) dt$, and either integral is equal to the corresponding integral in (VII 3.1).

VII 4. SOBOLEV FUNCTIONS AND SOBOLEV SPACES $W_p^m(G)$

For any p and m, $1 \leq p \leq \infty$, $m = 1,2,\ldots$, let us denote by $W_p^m(G)$ the set of all real-valued functions $x(t)$, $t \in G$, possessing generalized partial derivatives $D^\alpha x \in L_p(G)$ of all orders $\alpha = (\alpha_1,\ldots,\alpha_\nu)$, $0 \leq |\alpha| \leq m$. For every $x \in W_p^m(G)$ we denote by $\|x\|_p^m$ the number

$$\|x\|_p^m = \left[\sum_{|\alpha| \leq m} \int_G |D_x^\alpha|^p dt\right]^{1/p} \qquad \text{(VII 4.1)}$$

As we shall see below $W_p^m(G)$ is a Banach space with norm $\|x\|_p^m$. This norm $\|x\|_p^m$ may be indicated also by $\|x\|_{p,G}^m$ or norm $\|x\|_p^m$ in G.

Analogously, for any p and m, we shall denote by W_p^m (loc, G) the set of all $x(t)$, $t \in G$, possessing generalized partial derivatives $D^\alpha x \in L_p^{loc}(G)$ of all orders $0 \leq |\alpha| \leq m$. If G_o is any open subset of G with compact closure cl G_o in G, $G_o \subset \text{cl}\, G_o \subset G$, then every $x \in W_p^m(\text{loc},G)$ has a restriction in G_o which is an element of $W_p^m(G_o)$. Functions $z \in W_p^m(G)$ need not be continuous in G, though we shall see in (VII 8) that this is the case if $mp > \nu$. Also, we shall prove in (VII 9) certain "fine properties" of functions $z \in W_p^m(G)$ which are mild continuity properties with respect to the single coordinates $x^1,\ldots,x^\nu$. Functions $z \in W_1^1(G)$ which are continuous in G are said to be absolutely continuous in the sense of Tonelli, or ACT in G. As examples of functions $z \in W_p^m(G)$ which are not continuous, take $z(x) = |x|^{-h}$, $h > 0$, for $x \in G = [|x| < 1]$. Then $z \in W_p^m(G)$ whenever $(h + m)p < \nu$, $p \geq 1$, $m \geq 1$. For instance, for $\nu = 2$, $z(x) = |x|^{-1/2}$ belongs to $W_1^1(G)$; for $\nu = 4$, $z(x) = |x|^{-1/2}$ belongs to $W_2^1(G)$.

We shall say that a sequence of functions $x_k(t)$, $t \in G$, $k = 1,2,\ldots$, converges in $W_p^m(G)$ toward a function $x(t)$, $t \in G$, if x, $x_k \in W_p^m(G)$, and $x_k \to x$, $D^\alpha x_k \to D^\alpha x$ in $L_p(G)$ for all α, $0 \leq |\alpha| \leq m$, that is, $\|x_k - x\|_p^m \to 0$ as $k \to \infty$.

We shall say that $W_p^m(G)$ is a Sobolev space, and that its elements x are Sobolev functions.

From (VII 2.i) and (VII 2.ii) we deduce immediately the statements: For $\nu = 1$, if $x(t)$, $a < t < b$, is AC in any closed internal $[\bar{a},\bar{b}] \subset (a,b)$, then $x \in W_1^1(\text{loc},(a,b))$. In particular, if $x(t)$, $a \leq t \leq b$, is AC in $[a,b]$, then $x \in W_1^1((a,b))$. Here a, b are finite.

For $\nu \geq 1$, if $x(t)$, $t \in G$, $G \subset E_\nu$, G open, is Lipschitzian in G, then $x \in W_1^1(G)$ if G is bounded, and $x \in W_1^1(\text{loc},G)$ if G is unbounded.

(VII 4.i) $W_p^m(G)$ is a Banach space with norm $\|x\|_p^m$.

<u>Proof.</u> If N is the total number of (distinct) partial derivatives of order α, $0 \leq |\alpha| \leq m$, then obviously

$$W_p^m(G) \subset L_p(G)x\ldots xL_p(G) \quad (N \text{ times}).$$

All we have to prove is that $W_p^m(G)$ is complete. Indeed, if $[x_k]$ is a Cauchy sequence in $W_p^m(G)$, hence $\|x_k - x_h\|_p^m \to 0$ as $h, k \to \infty$, where the norm $\|x\|_p^m$ is defined by (VII 4.1), then $\|x_h - x_k\|_p \to 0$, $\|D^\alpha x_h - Dx_k^\alpha\|_p \to 0$ as $h, k \to \infty$ in $L_p(G)$, for every $0 \leq |\alpha| \leq m$. Since the space $L_p(G)$ is known to be complete, there are elements $x \in L_p(G)$, $y_\alpha \in L_p(G)$, such that $x_k \to x$ and $D^\alpha x_k \to y_\alpha$ in $L_p(G)$, for $0 \leq |\alpha| \leq m$. All we have to prove is that $y_\alpha = D^\alpha x$, $0 \leq |\alpha| \leq m$.

Now $x_k \to x$ in $L_p(G)$ certainly implies $\int_G x_k D^\alpha \varphi \, dt \to \int_G x D^\alpha \varphi \, dt$ for every fixed $\varphi \in C_o^\infty(G)$ and any $|\alpha| \leq m$. Similarly, $D^\alpha x_k \to y_\alpha$ in $L_p(G)$ certainly implies $\int_G (D^\alpha x_k)\varphi \to \int_G y_\alpha \varphi \, dt$. Since $x_k \in W_p^m(G)$ we have

$$\int_G (D^\alpha x_k) \varphi \, dt = (-1)^{|\alpha|} \int_G x_k (D^\alpha \varphi) \, dt, \quad k = 1,2,\dots, \quad |\alpha| \leq m,$$

and hence, as $k \to \infty$, we have also

$$\int_G y_\alpha \varphi \, dt = (-1)^{|\alpha|} \int_G x(D^\alpha \varphi) \, dt, \quad |\alpha| \leq m.$$

That is, $y_\alpha = D^\alpha x$, $0 \leq |\alpha| \leq m$. We have proved that $x \in W_p^m(G)$ and that $W_p^m(G)$ is complete.

(VII 4.ii) (Leibnitz rule in Sobolev spaces). If $x \in W_p^m(G)$ and $y \in W_q^m(G)$, $1/p + 1/q \leq 1$, and $1 \leq \lambda < +\infty$ is so chosen that $1/p + 1/q = 1/h$, then the product $xy \in W_\lambda^m(G)$ and $D^\alpha(xy) = \Sigma_\beta \binom{\alpha}{\beta} D^\beta x \, D^{\alpha-\beta} y$ a.e. in G.

A corollary of (VII 3.viii).

(VII 4.iii) If $x \in W_p^m(G)$, $y = D^\alpha x$, $z = D^\beta y$, $0 \leq |\alpha| \leq m$, $0 \leq |\beta| \leq m$, $0 \leq |\alpha + \beta| \leq m$, then $z = D^{\alpha+\beta} x$ a.e. in G.

A corollary of (VII 2.ix).

The following criterion is often used.

(VII 4.iv) Let $y_\alpha(t)$, $t \in G$, $0 \leq |\alpha| \leq m$, be given functions $y_\alpha \in L_p(G)$, $1 \leq p \leq +\infty$, let R_k, $k = 1,2,\dots$, be a sequence of open subsets of G with $R_k \subset R_{k+1}$, $R_k \uparrow G$ as $k \to \infty$, and let $x_k(t)$, $t \in R_k$, $k = 1,2,\dots$, be a sequence of functions $x_k \in W_p^m(R_x)$ such that $\|y_\alpha - D^\alpha x_k\|_{p,R_x} \to 0$ as $k \to \infty$, $0 \leq |\alpha| \leq m$. Then $y = y_o$ is an element of $W_p^m(G)$, and $y_\alpha = D^\alpha y$ a.e. in G, $0 \leq |\alpha| \leq m$.

A corollary of (VII 2.x).

Let us consider now the class L of all functions $x(t)$, $t \in G$, which are Lipschitzian in G together with all their partial derivatives $D^\alpha x$, $0 \leq |\alpha| \leq m - 1$, and hence possess bounded partial derivatives of order m a.e. in G. Then L is certainly a normed space with the norm $\|x\|_p^m$ defined by (VII 4.1). We shall denote by $H_p^m(G)$ the completion of L with respect to the norm $\|x\|_p^m$. We shall prove in (VII 10), under mild conditions on G, the basic identity

$$W_p^m(G) = H_p^m(G). \tag{VII 4.4}$$

All we can deduce from (VII 4.1) is that $H_p^m(G) \subset W_p^m(G)$ and this is true for every open set $G \subset E_\nu$. Indeed, $L \subset W_p^m(G)$; hence, the completion $H_p^m(G)$ of L with respect to the norm $\|\cdot\|_p^m$, being the smallest complete set containing L, certainly is contained in $W_p^m(G)$, or $H_p^m(G) \subset W_p^m(G)$.

Analogously, we may consider only the class L_o of all those elements x of L having compact support $K_x \subset G$. We shall then denote by H_{op}^m the completion of L_o with respect to the norm $\|x\|_p^m$. We shall prove in (VII 10), under the same mild restrictions on G, that $H_{op}^m(G)$ is the subset of all those functions $x \in W_p^m(G)$ "which are zero on the boundary" ∂G together with all their partial derivatives of orders $0 \leq |\alpha| \leq m - 1$, according to the definition of boundary values we shall discuss in VII 7, 8.

Relation (VII 4.4) is easy to prove for G a half plane, say the half plane E_ν^+ of all $t = (t^1,\dots,t^\nu)$ with $t^1 > 0$. All we have to prove is the following statement:

(VII 4.v) If $x \in W_p^m(E_\nu^+)$, then there is a sequence x_k, $k = 1,2,\dots$, of functions $x_k \in W_p^m(E_\nu^+) \cap C^\infty(E_\nu)$ such that $\|x_k - x\|_p^m \to 0$ as $k \to \infty$.

To prove this statement we first denote by $h = (h^1, 0,\dots,0)$ a vector with $h^1 > 0$, and we note that, if $y(t) = x(t + h)$, then $y \in W_p^m(F)$ where F is the half space $t^1 > -h$; hence, also $y \in W_p^m(E_1^+)$. By Remark 2 in (VII 3) we know that $\|y - x\|_{p,E_\nu^+} \to 0$, $\|D^\alpha y - D^\alpha x\|_{p,E_\nu^+} \to 0$ as $h^1 \to 0+$, $0 \le |\alpha| \le m$. Thus, it is enough to prove our statement for y. Now we take a function $\psi \in C^\infty(E_\nu)$ with the following properties $\psi(t) = \psi(t^1)$ $-\infty < t^1 < t\ \infty$, $\psi(t^1) = 0$ for $t^1 \le -3h^1/4$, $\psi(t^1) = 1$ for $t^1 \ge -h^1/4$, $0 < \psi < 1$ otherwise. We define now z by taking $z(t) = 0$ for $t \le -h^1$, $z(t) = y(t)\,\psi\,(t)$ for $t^1 > -h^1$. Now z is defined in E_ν and we have to prove that z has generalized derivative $D^\alpha z$ in E_ν given by $Y = 0$ for $t^1 \le -h^1$, $Y = \Sigma_\beta \binom{\alpha}{\beta} D^\beta y\, D^{\alpha-\beta}\psi$ for $t^1 > -h^1$, $0 \le |\alpha| \le m$. Indeed, if $\varphi \in C_o^\infty(E_\nu)$, then $\int_{E\nu} Y\,\varphi\,dt = \int_F \Sigma_\beta \binom{\alpha}{\beta}(D^\beta y)(D^{\alpha-\beta}\psi)\varphi\,dt$, where all functions $(D^{\alpha-\beta}\psi)\varphi$ are actually in $C_o(F)$. Then

$$\int_{E_\nu} Y\alpha\ dt = \left[\int_F Y \quad \Sigma_\beta(-1)^{|\beta|}\binom{\alpha}{\beta}\ D^\beta((D^{\alpha-\beta}\psi)\varphi)\right] \quad dt$$

$$= \left[\int_F Y \quad \Sigma_\beta\Sigma_\gamma(-1)^{|\beta|}\binom{\alpha}{\beta}\binom{\beta}{\gamma}\ D^\gamma\varphi\ D^{\alpha-\gamma}\psi\right] \quad dt,$$

where Σ_β ranges over $0 \le \beta \le \alpha$ and Σ_γ over $0 \le \gamma \le \beta$. For $\gamma = \alpha$ we have $\gamma = \beta = \alpha$ and the corresponding term in bracket is $(-1)^{|\alpha|}\psi\,(D^\alpha\varphi)$. All other terms in bracket have sum zero, because of the identity $\Sigma_\beta (-1)^{|\beta|} \binom{\alpha}{\beta}\binom{\beta}{\alpha} = 0$. Thus

$$\int_{E_\nu} Y \varphi \, dt = (-1)^{|\alpha|} \int_{E_\nu} (z \psi)(D^\alpha \varphi) dt,$$

for all $\varphi \in C_0^\infty(E_\nu)$, and we have proved that $Y = D^\alpha z$ in E_ν, $0 \leq |\alpha| \leq m$. By force of (VII 3.iii) (fourth part) we conclude that $\|D^\alpha(J_\varepsilon z) - D^\alpha z\|_{p,E_\nu} \to 0$ as $\varepsilon \to 0+$, $0 \leq |\alpha| \leq m$. In particular, we have $\|D^\alpha(J_\varepsilon z) - D^\alpha z\|_{p,E_\nu^+} \to 0$ as $\varepsilon \to 0$. We can now easily choose h_k^1 and ε_k sufficiently small, and take $x_k = J_{\varepsilon_k} z$.

VII 5. EMBEDDING THEOREMS IN $W_p^m(E_\nu^+)$

The Sobolev embedding theorems are particularly easy to state and prove for Sobolev functions in a half space. We shall introduce below (VII 9) the concept of regions G of class K in E_ν, and then the embedding theorems will be translated immediately in terms of Sobolev functions $x \in W_p^m(G)$ in a region G of class K.

In the τ-space E_ν, $t = (t^1,\dots,t^\nu)$, we shall denote by E_ν^+ the part of E_ν with $t^1 > 0$. Then E_ν^+ is an open subset of E_ν.

We shall often use polar coordinates in E_ν, say (r, ω), $r = |t| \geq 0$, $\omega = (\omega^1,\dots,\omega^\nu) \in S$, where S is the unit sphere $|t| = 1$. We denote by S^+ the half unit sphere $[t|(t) = 1, t^1 \geq 0]$, by dw the usual area measure on S modified by a constant factor so that $\int_{S^+} dw = 1$. We have then, for every $x \in L_1(E_\nu^+)$,

$$\int_{E_\nu^+} x(t)\, dt = \int_0^{+\infty} \int_{S^+} x\, r^{\nu-1}\, dr dw.$$

(VII 5.i) Let $x(t)$, $t \in E_\nu^+$, be an arbitrary function $x \in L_p(E_\nu^+)$, $1 \leq p \leq +\infty$, with compact support, say contained in the half solid ball $[t | t^1 \geq 0, |t| \leq R]$, and possessing generalized first order partial derivatives $D_j x$, $j = 1, \dots, \nu$, in E_ν^+ of class $L_p(E_\nu^+)$. Then, there are functions $h_j(t)$, $t \in E_\nu^+$, $h_j \in C^\infty(E_\nu^+)$, $h_j \in L_s(E_\nu^+)$ for every $1 \leq s < \nu/(\nu-1)$, h_j of compact support $[t | t^1 \geq 0, |t| \leq 4R]$, such that

$$x(t) = -\Sigma_{j=1}^{\nu} \int_{F_\nu^+(t)} (D_j x)(\tau)\, h_j(\tau - t)\, dt \qquad \text{a.e.} \qquad \text{(VII 5.1)}$$

and where $F_\nu^+ = F_\nu^+(t)$ denotes the part of the τ-space where $\tau^1 \geq t^1$. For $\nu = 1$ we have $h = 1$ for $0 < \tau \leq 4R$ and (VII 5.1) reduces to

$$x(t) = -\int_t^{+\infty} Dx(\tau)\, b(\tau)\, d\tau = -\int_\tau^{4R} Dx(\tau)\, d\tau.$$

<u>Proof</u>. Let $b(r)$, $0 \leq r < +\infty$, be a function of class C^∞ on $0 \leq r < +\infty$, equal to one on $[0, 5R]$ and zero on $[6R, +\infty)]$. Let $\varphi(t)$, $t \in E_\nu^+$, be any function $\varphi \in C_0^\infty(E_\nu^+)$ with compact support x contained in $[t | t^1 > 0, |t| < 2R]$. It is convenient to extend φ to the whole t-space by taking $\varphi = 0$ in $E_\nu - E_\nu^+$, so that φ so extended is now in $C_0^\infty(E_\nu)$. For every $t \in E_\nu^t$ with $|t| < 2R$ and $\omega \in S^+$ we certainly have $(\tau - r\omega) = 0$ for $r \geq 4R$; hence

$$\int_{+\infty}^{0} (\partial/\partial r)\varphi(\tau - r\omega))\, b(r) dr = \int_{4R}^{0} (\partial/\partial r)\varphi(t - r\omega) dr = \varphi(t).$$

Then, by integration on S^+ we have also

$$\varphi(\omega) = \int_{S^+} dt \int_{+\infty}^{0} \frac{\partial}{\partial r} \varphi(\tau - r\omega)\, b(r)dr$$

$$= \int_{S^+} d\omega \int_{0}^{+\infty} \Sigma_{j=1}^{\nu} (D_j\varphi)(t - r\omega)\, \omega_j b(r)dr$$

$$= \Sigma_{j=1}^{\nu} \int_{S^+}\int_{0}^{+\infty} (D_j\varphi)(\tau - r\omega)\, (\omega_j\, b(r)r^{-\nu+1})\, r^{\nu-1}dr$$

$$= \Sigma_{j=1}^{\nu} \int_{E_\nu^+} (D_j\varphi)(t-\tau)h_j(\tau)\, d\tau,$$

where $h_j(\tau) = \omega_j\, b(r)r^{-\nu+1}$, $\tau \in E_\nu^+$, $j=1,\dots,\nu$. As we shall see below $h_j \in L_1(E_\nu^+)$. By Fubini's theorem we have now

$$I = \int_{E_\nu^+} x(t)\varphi(t)d\tau = \Sigma_{j=1}^{\nu} \int_{E_\nu^+} x(t)\, d\tau \int_{E_\nu^+} (D_j\varphi)(t-\tau)h_j(\tau)d\tau$$

$$= \Sigma_{j=1}^{\nu} \int_{E_\nu^+} h_j(\tau)d\tau \int_{E_\nu^+} x(t)(D_j\varphi)(t-\tau)d\tau. \qquad \text{(VII 5.2)}$$

Note that φ has compact support certainly contained in a slab $0 < a < t^1 < b < +\infty$ of E_ν^+. hence, for every fixed $\tau = (\tau^1,\dots,\tau^\nu)$ of E_ν^+, $\tau^1 > 0$, and the function $\varphi(t-\tau)$, as a function of t, has compact support contained in the slab defined by $a < t^1 - \tau^1 < b$, or $a + \tau^1 < t^1 < b + \tau^1$, with $a + \tau^1 > 0$. Thus, $\varphi(t-\tau)$ as a function of t, has compact support in E_ν^+, and (VII 5.1) yields

$$I = -\Sigma_{j=1}^{\nu} \int_{E_\nu^+} h_j(\tau)d\tau \int_{E_\nu^+} D_j x(t)\varphi(t-\tau)dt$$

$$= -\Sigma_{j=1}^{\nu} \int_{E_\nu^+} \int_{E_\nu^+} h_j(\tau)\, D_j x(t)\varphi(t-\tau)d\tau dt$$

and the latter is a double integral which can be restricted to the part of $E_\nu^+ \times E_\nu^+$ where $t^1 - \tau^i \geq 0$ (and of course $t^1 \geq 0$, $\tau^1 \geq 0$). By writing

$t - \tau = u$, $t = v$, then the same double integral is transformed into

$$I = -\Sigma_{j=1}^{\nu} \iint h_j (v - u) D_j x(v) y(u) du dv,$$

where now the integration is performed in the part of the uv-space where $u^1 \geq 0$, $v^1 \geq 0$, $v^1 - u^1 \geq 0$. By Fubini's theorem we have now

$$I = -\Sigma_{j=1}^{\nu} \int_{u \in E_\nu^+} \left(\int_{v \in E_\nu^+,\ v^1 \geq u^1} D_j x(v) h_j (v-u) dv \right) \varphi(u) du.$$

By replacing v by τ and u by t, then we have $t \in E^\nu$, $\tau \in E_\nu^+$, $\tau^1 \geq t^1$. If for any $t \in E_\nu^+$ we denote by $F_\nu^+ = F_\nu^+(t)$ the part of the τ-space where $\tau^1 \geq t^1$, then by Fubini's theorem we have

$$\int_{E_\nu^+} x(t) \varphi(t) dt = \int_{E_\nu^+} \left(-\Sigma_{j=1}^{\nu} \int_{F_\nu^+} D_j x(\tau) h_j (\tau - t) d\tau \right) \varphi(t) dt.$$

This relation holds for every $\varphi \in C_o^\infty(E\ )$ with compact support in $[t | t^1 > 0, |t| < 2R]$. By force of (VII 2.iv) we conclude that

$$x(t) = -\Sigma_{j=1}^{\nu} \int_{F_\nu^+} D_j x(\tau) h_j (\tau - t) d\tau$$

at least for almost all t with $t^1 > 0$, $|t| < 2R$.

It remains to prove that $h_j \in L_s(E_\nu^+)$ for every s with $1 \leq s < \nu/(\nu-1)$. Indeed

$$\int_{E_\nu^+} |h_j(t)|^s dt = \int_{S^+} \int_o^{+\infty} | \omega_j\ b(r) r^{-\nu+1} |^s r^{\nu-1} dr\, d\omega$$

$$K \int_o^{4R} r^{(\nu-1)(1-s)} dr,$$

and $(\nu-1)(1-s) > -1$ reduces to the assumed inequality $1 \leq s < \nu/(\nu-1)$.

Note that in the integral (VII 5.1) we may assume $\tau^1 \geq 0$, $|\tau| \leq 2R$, $\tau^1 - t^1 > 0$, $|\tau - t| \leq 2R$. In any case, $0 \leq r = |\tau - t| \leq 4R$.

(VII 5.ii) If $1 \leq p \leq +\infty$, if $x \in W_p^1(E_\nu^+)$ has compact support contained in the half solid ball $[t|t^1 \geq 0, |t| \leq R]$, then $x \in L_q(E_\nu^+)$ for every q, $1 \leq q \leq +\infty$, with $1/q > 1/p - 1/\nu$, and there is a constant K depending only on R, p, q such that

$$\|x\|_q \leq K \Sigma_{j=1}^{\nu} \|D_j x\|_p. \qquad \text{(VII 5.3)}$$

In particular for $p > \nu$, $q = \infty$, $x \in L_\infty(E_\nu^+)$ and

$$\text{ess sup } |x| \leq K \Sigma_{j=1}^{\nu} \|D_j x\|_p \qquad \text{(VII 5.4)}$$

In other words, if $W_{p,R}(E_\nu^+)$ denotes the set of all elements $x \in W_p^m(E_\nu^+)$ with compact support in the solid half ball $[t|t^1 \geq 0, |t| \leq R]$, then the identity transformation carrying an element $x \in W_{p,R}(E_\nu^+)$ into the same function x as an element of $L_q(E_\nu^+)$ is a bounded map $W_{p,R}(E_\nu^+) \to L_q(E_\nu^+)$. We shall see in (VII 5.v) that the same map is also compact (for $p > 1$, and even for $p = 1$ under restrictions).

Proof. First, let us assume $1 < p < +\infty$, take $q < \infty$, and define p' by means of $1/p + 1/p' = 1$. Note that, for $\lambda = \nu - 1$, we have

$$2\varepsilon = (\nu/q) - (\lambda - \nu/p') = 1 + \nu/q - \nu/p > 0.$$

Also, note that p can replace q in the relation $1/q > 1/p - 1/\nu$; hence, we can assume $p < q$. Then, for $r = |t-\tau|$, we have $|h_j| \leq Kr^{-\nu+1} = Kr^{-\lambda}$ for some constant K, and then, by (VII 5.1), we also have

$$|x(t)| \leq K \Sigma_{j=1}^{\nu} \int \left(|D_j x|^{p/q} r^{\varepsilon-\nu/q}\right)\left(|D_j x|^{p(1/p-1/q)}\right)\left(r^{\varepsilon-\nu/p'}\right) d\tau, \tag{VII 5.5}$$

where $D_j x = (D_j x)(\tau)$, $r = |t-\tau|$, and the integration is performed in the solid ball $|r| \leq 2R$. If we take $\lambda_1 = 1/q$, $\lambda_2 = 1/p - 1/q$, $\lambda_3 = 1/p'$, we have $\lambda_1, \lambda_2, \lambda_3 > 0$, $\lambda_1 + \lambda_2 + \lambda_3 = 1$. By Hölder inequality for three factors we have then

$$|x(t)| \leq K \Sigma_{j=1}^{\nu} \left[\int |D_j x|^p \, r^{-\nu+\varepsilon q} d\tau\right]^{1/q} \left[\int |D_j x|^p d\tau\right]^{1/p-1/q} \left[\int r^{-\nu+\varepsilon p'} \, d\tau\right]^{1/p'}, \tag{VII 5.6}$$

where again the integrals are taken in $|r| \leq 2R$. Since $-\nu + \varepsilon p' > -\nu$ the third factors are below a fixed constant. The second factors are also finite and equal to $\|D_j x\|_p^{1-p/q'}$. By taking powers q in (VII 5.6), by integration in G, and interchanging the order of integration, we have

$$\int_{r\leq R} |x(t)|^q dt \leq K^1 \Sigma_{j=1}^{\nu} \|D_j x\|_p^{q'-q} \int_{r\leq R} |D_j x|^p d\tau \int_{r\leq R} r^{-\nu+\varepsilon q} d\tau$$

Since $-\nu+\varepsilon q > -\nu$ the last integral is below a fixed number. By Torelli's theorem then $|x(t)|^q$ is L-integrable in G, and by Fubrini's theorem the transformation above are valid. We obtain now

$$\int_{r\leq R} |x(t)^q \, dt \leq K'' \; \Sigma_{j=1}^{\nu} \|D_j x\|_p^q$$

for some constant K", and (D5.4) follows.

If $\nu < p$, $q = \infty$, then we take $2\varepsilon = 1 - \nu/p > 0$, and instead of (VII 5.5) we write

$$|x(t)| \leq K \ \Sigma_{j=1}^{\nu} \ \int |D_j x| r^{2\varepsilon - \nu/p'} d\tau$$

and by Hölder inequality

$$|x(t)| \leq K \ \Sigma_{j=1}^{\nu} \left(\int |D_j x|^p d\tau \right)^{1/p} \left(\int r^{-\nu + 2\varepsilon p'} d\tau \right)^{1/p'} .$$

The argument is now similar to the one above.

Let us consider the case $p = 1$. Again we take $\lambda = \nu - 1$. Finally, the relation $2\varepsilon = 1 + \nu/q - \nu/p > 0$ reduces to $2\varepsilon = 1 + \nu/q - \nu$, and

$$|x(t)| \leq K \ \Sigma_{j=1}^{\nu} \ \int \left(|D_j x|^{1/q} \ r^{2\varepsilon - \nu/q} \right) \left(|D_j x|^{1-1/q} \right) d\tau .$$

If we take $\lambda_1 = 1/q$, $\lambda_2 = 1-1/q$, we have $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_1 + \lambda_2 = 1$, and, by Hölder inequality,

$$|x(t)| \leq K \ \Sigma_{j=1}^{\nu} \left[\int |D_j x| r^{-\nu + 2\varepsilon q} \right]^{1/q} \left[\int |D_j x| d\tau \right]^{1-1/q} .$$

As before, by taking powers q, by integration in $r \leq R$, and interchanging the order of integrations, we have

$$\int_{r \leq R} |x(t)|^q dt \leq K^1 \ \Sigma_{j=1}^{\nu} \ \|D_j x\|_1^{q-1} \ \int_{r \leq R} |D_j f| d\tau \ \int_{r \leq R} r^{-\nu + 2\varepsilon q} d\tau ,$$

where $-\nu + 2\varepsilon q > -\nu$. The last integral therefore, is below a fixed number, and the estimate above shows, by force of Tonelli's and Fubini's theorems, that $|x(t)|^q$ is L-integrable in E_ν^+. Finally,

$$\int_{E_\nu^+} |x(t)|^q dt \leq K'' \ \Sigma_{j=1}^{\nu} \ \|D_j x\|_1^q$$

and (VII 5.2) follows for $p = 1$.

(VII 5.iii) If $\nu \geq 1, 1 \leq p \leq +\infty$, $m \geq 1$, if $x \in W_p^m(E_\nu^+)$ with compact support contained in the solid half ball $[t|t^1 \geq 0,\ |t| \leq R]$, then $x \in L_q(E_\nu^+)$ for every $1 \leq q \leq +\infty$ with $1/q > 1/p - m/\nu$, and there is a constant K depending only on R, p, ν, m, q such that $\|x\|_q \leq K\ \Sigma_{|\beta| = m} \|D^\beta x\|_p$. Also, every generalized partial derivative $D^\alpha x$ of order $0 \leq |\alpha| \leq m - 1$ is of class $L_q(E_\nu^+)$ for every $1 \leq q < +\infty$ with $1/q > 1/p - (m-|\alpha|)/\nu$, and there is a constant K depending only on R, p, ν, m, q, α such that

$$\|D^\alpha x\|_q \leq K\ \Sigma_{|\beta_\cdot| = m} \|D^\beta x\|_p, \quad 0 \leq |\alpha| \leq m - 1. \qquad \text{(VII 5.7)}$$

In other words, we have here bounded maps $W^m_{p,R}(E_\nu^+) \to L_q(E_\nu^+)$ as mentioned after statement (VII 5.ii), and, as we shall see further, these maps are also compact (for $p > 1$ and even for $p = 1$ under restrictions).

Proof. If $\varepsilon = 1/q - 1/p - m/\nu$, let $p_o = p, p_1, \dots, p_m = q$ be the numbers defined by

$$1/p_{s+1} = 1/p_s - 1/\nu + \varepsilon/m, \qquad s = o,1,\dots, m - 1.$$

Then, by (VII 5.ii) we conclude that all derivatives $D^\alpha x$ with $|\alpha| = m - 1$ are in L_{p_1}, all derivatives $D^\alpha x$ with $|\alpha| = m - 2$ are in L_{p_2}, and so on. Thus x is in L_{p_m}, or $x \in L_q$, as stated. The corresponding statement holds for the derivatives since $\varepsilon > 0$ above can be any positive number. The remaining part is a corollary of (VII 5.ii).

(VII 5.iv) If $1 \leq p \leq +\infty$, $\nu \geq 1$, $m \geq 1$, $mp > \nu$, if $x \in W_p^m(E_\nu^+)$ has compact support contained in half solid ball $[t|t^1 \geq 0,\ |t| \leq R]$, then x and all partial derivatives $D^\alpha x$ of orders $0 \leq |\alpha| < m - \nu/p$ are continuous in the closure of E_ν^+. Also, there are constants K, depending only on R, p, q, m, ν, α, such that

$$|D^\alpha x(t)| \leq K \, \Sigma_{|\beta| = m} \|D^\beta x\|_p, \quad 0 \leq |\alpha| < m - \nu/p, \quad t \in \text{cl}\, E_\nu^+. \qquad \text{(VII 5.8)}$$

Also, there is a function $\chi(h) \geq 0$, $h = (h_1, \ldots, h\nu) \in E_\nu$, depending only on R, p, q, m, ν, α, such that, for all $t \in E_\nu^+$, $t + h \in E_\nu^+$, we have

$$|D^\alpha x(t + h) - D^\alpha x(t)| \leq K \;\; \chi(h) \, \Sigma_{|\beta| = m} \|D^\beta x\|_p. \qquad \text{(VII 5.9)}$$

This is Sobolev's imbedding theorem for $W_p^m(E_\nu^+)$. It will be translated in terms of arbitrary regions $G \subset E_\nu$ (of class K) in (VII 11). This statement shows that, not only x and all partial derivatives $D^\alpha x$ with $0 \leq |\alpha| < m-\nu/p$ are continuous on the closure of E_ν^+, but also that the identity transformation carrying an element $x \in W_{pR}^m(E_\nu^+)$ into the same function x, or into its derivatives $D^\alpha x$, $0 \leq |\alpha| < m-\nu/p$, as an element of C, is a bounded map $W_{pR}^m(E_\nu^+) \to C(\text{cl}E_\nu^+)$. Moreover, the same map is compact. Namely, the functions $x \in W_{pR}^m(E_\nu^+)$ with $\|x\|_p^m \leq M$ for some constant M, certainly are continuous and equicontinuous functions on cl E_ν^+ (and so are their derivatives $D^\alpha x$, $0 \leq |\alpha| < m-\nu/p$.

<u>Proof</u> of (VII 5.iv) For any multiindex $\alpha = (\alpha_1^1, \ldots, \alpha^\nu)$ with $0 \leq |\alpha| < m-\nu/p$. $0 \leq |\alpha| \leq m - 1$, let $g = D^\alpha x$. Then the ν first order partial derivatives $D_j g$, $j = 1, \ldots, \nu$, are derivatives of order $|\alpha| + 1 \leq m$ of x. If $|\alpha| \leq m - 2$, then $D_j g \in L_q(E_\nu^+)$ $j = 1, \ldots, \nu$, for every $1 \leq q \leq +\infty$ with $1/q > 1/p - (m - |\alpha| - 1)/\nu$. If $1/q' + 1/q = 1$, and we take $q > p$ with $1/q > 1/p -$

$(m - |\alpha| - 1)/\nu$, then

$$1/q' = 1 - 1/q < 1 - 1/p + (m - |\alpha| - 1)/\nu$$
$$= (1/\nu)(m - |\alpha| - \nu/p) + (\nu - 1)/\nu,$$

where $m - |\alpha| - \nu/p > 0$. Thus, if we take $q > p$ with $1/q$ larger than and sufficiently close to $1/p - (m - |\alpha| - 1)/\nu$, then $1/q' > (\nu - 1)/\nu$, and $1 \leq q' < (\nu - 1)/\nu$.

If $|\alpha| = m - 1$, then $|\alpha| + 1 = m$, and $D_j g \in L_q(E_\nu)$ for $q = p$, $j = 1,\dots,\nu$. Again, for $1/q' + 1/q = 1$, then

$$1/q' = 1 - 1/q = 1 - 1/p = 1 - 1/p + (m - |\alpha| + 1)/\nu$$
$$= (1/\nu)(m - |\alpha| - \nu/p) + (\nu - 1)/\nu,$$

and again $1/q' > (\nu - 1)/\nu$, and $1 \leq q' < \nu/(\nu - 1)$. In any case we have determined q, $q' \geq 1$, with $1/q + 1/q'$ with $1 \leq q' < \nu/(\nu - 1)$, and this shows that $q > 1$.

The functions h_j are known to be in $L_s(E_\nu^+)$ for every $1 \leq s < \nu/(\nu - 1)$, hence $h_j \in L_{q'}(E^+)$, and on the other hand $D_j g \in L_q(E_\nu^+)$, for the chosen $q \geq p$ and $1/q' + 1/q = 1$. By (VII 5.1) we have now, a.e. in E_ν^+

$$g(t) = \Sigma - {}_{j=1}^{\nu} \int_{F_\nu^+(t)} D_j g(\tau)\, h_j(\tau - t) d\tau,$$

and by Hölder inequality also

$$|g(t)| \leq \Sigma_{j=1}^{\nu} \left(\int_{F_\nu^+} |D_j g(\tau)|^q \, d\tau\right)^{1/q} \left(\int_{F_\nu^+} |h_j(\tau - t)|^{q'} \, d\tau\right)^{1/q'}$$
$$K \, \Sigma_{j=1}^{\nu} \|D_j g\|_q$$

By force of (VII 5.iii) we have then

$$|D^\alpha x(t)| = |g(t)| \leq K^1 \, \Sigma_{|\beta| = m} \|D^\beta x\|_p .$$

This proves relations (VII 5.7).

For $t \in E_\nu^+$ and every $h = (h^1,\dots,h^\nu) \in E_\nu$ with $t + h \in E_\nu^+$, we have now

$$g(t+h) - g(t) = -\Sigma_{j=1}^{\nu} \left[\int_{F_\nu^+(t+h)} D_j g(\tau) h_j(\tau-t-h)d\tau - \int_{F_\nu^+(t)} D_j g(\tau) h_j(\tau-t)d\tau \right],$$

$$F_\nu^+(t) = \left[\tau | \tau \in E_\nu,\ \tau^1 \geq t^1\right], \quad F_\nu^+(t+h) = \left[\tau | \tau \in E_\nu,\ \tau^1 \geq t^1 + h^1\right].$$

Let $\bar{t}^1 = \min\,[t^1,\ t^1 + h^1,\ \bar{\bar{t}}^1] = \max\,[t^1\ \ t^1 + h^1]$, and let H_1, H_2 be the sets $H_1 = [t | t \in E_\nu^+,\ \bar{t}^1 \leq t^1 \leq \bar{\bar{t}}^1]$ $H_2 = [t | t \in E_\nu^+\ \bar{\bar{t}}^1 < t^1]$. Then

$$|g(t+h) - g(t)| \leq \Sigma_{j=1}^{\nu} \left| \int_{H_1} D_j\, g(\tau) h_j(\tau-h)d\tau \right|$$

$$+ \Sigma_{j=1}^{\nu} \left| \int_{H_2} D_j g(\tau) \left[h_j(\tau-t-h) - h_j(\tau-t)\right] d\tau \right| = \Sigma_{j=1}^{\nu} (J_{j1} + J_{j2}).$$

We have now

$$|J_{j2}| \leq \left(\int_{E_\nu^+} |D_j g(\tau)|^q d\tau \right)^{1/q} \left(\int_{E_\nu^+} |h_j(\tau-t-h) - h_j(\tau-t)|^{q'} d\tau \right)^{1/q'}$$

$$\leq \|D_j g\|_q \, \|h_j(u-h) - h_j(u)\|_{q'} = \|D_j g\|_q \, \chi_j(h).$$

By Remark 2 in (VII 4) we know that $\chi_i(h) \to 0$ as $|h| \to 0$, $j = 1 \ldots \nu$. To estimate J_{j1} we may well observe that we can replace the domain of integration H_1 by the subset H_o of all points $t \in H$ with $|t| \leq 2R$. Then $|H_o| \leq 2^{\nu-1} R^{\nu-1} (\bar{\bar{t}}^1 - \bar{t}^1) < 2^{\nu-1} R^{\nu-1} |h|$. Also, we shall take numbers s' and λ so that $q' < s' < \nu/(\nu-1)$, $\lambda \geq 1$, $1/\lambda = 1/q' - 1/s'$. Then $s' \geq 1$, $q' \geq 1$, $\lambda' \geq 1$, $1/\lambda + 1/q + 1/s' = 1$, and by Hölder inequality we have

$$|J_{j1}| \leq \left(\int_{H_o} 1^\lambda d\tau \right)^{1/\lambda} \left(\int_{H_o} |D_j g(\tau)|^q d\tau \right)^{1/q} \left(\int_{H_o} |h_j(\tau-h)|^{s'} d\tau \right)^{1/s'}$$

$$\leq K \|D_j g\|_q \, |H_o|^{1/\lambda} \leq \|D_j o_j\|_q \, K(2^{\nu-1} R^{\nu-1} |h|)^{1/\lambda}.$$

These estimates for J_{j1} and J_{j2}, together with (VII 5.iii) yield (VII 5.8). Statement (VII 5.iv) is thereby proved.

Remark 1. Statement (VII 5.iv) is certainly not valid without the assumption $mp > \nu$. For instance, for $\nu = 1$, $m = 1$, $p = 1$ the AC functions $x_k(t)$, $t \in I = [0,1]$, defined by $x_k(t) = 1 - kt$ for $0 \leq t \leq k^{-1}$, $x_k(t) = 0$ for $k^{-1} \leq t \leq 1$, are all in $W_1^1(I)$ with $\|x\|_1 = (2k)^{-1}$, $\|x'\|_1 = 1$, $k = 1,2,\ldots$. Clearly, they are not equicontinuous on $[0,1]$.

(VII 5.v) If $\nu \geq 1$, $m \geq 1$, $p \geq 1$, if $x \in W_p^m(E_\nu^+)$ is any function with compact support contained in the half solid ball $[t|t^1 \geq 0, |t| \leq R]$, then there are functions $\chi(h) \geq 0$ $h = (h^1 \ldots h^\nu) \in E_\nu^+$ with $X(h) \to 0$ as $|h| \to 0$ depending only on R ν, m, p, q, such that

$$\|D^\alpha x(t \epsilon h) - D^\alpha x(t)\|_q \leq \chi(h)\, \Sigma_{|\beta|=m} \|D^\beta x\|_p, \quad 0 \leq |\alpha| \leq m - 1 \quad \text{(VII 5.10)}$$

provided, $1 \leq q \leq +\infty$, $1/q > 1/p - (m-|\alpha|)/\nu$ and either $p > 1$ or $p \geq 1$ and $|\alpha| \leq m - 2$.

For $p = 1$, $|\alpha| = m - 1$, let $\omega_\beta(\sigma)$ $\sigma \geq 0$, $|\beta| = m$ denote the supremum of $\int_X |D^\beta x(\tau)| d\tau$ for all measurable subsets X of E_ν^+ with $|X| \leq \sigma$. Then there are functions $\chi(h) \geq 0$ as above and constants K_1, K_2 depending only on R, q ν, m, such that

$$\|D^\alpha x(t+h) - D^\alpha x(t)\|_q \leq \chi(h) \quad {}_{|\beta|=m}\|D^\beta x\|_1 + K_1 \quad \Sigma_{|\beta|=m}\omega_\beta(K_2|h|), \quad \text{(VII 5.11)}$$

provided $1 \leq q \leq +\infty$, $1/q > 1/p - (m-|\alpha|)/\nu = 1 - 1/\nu$.

Proof. We assume first $p > 1$, $m = 1$, $\alpha = 0$, $1/q > 1/p - 1/\nu$ and also $h = (h^1, \dots h^\nu)$ with $h^1 \geq 0$. Take $1/p' + 1/p = 1$, $\eta = \nu/(\nu-1)$ and note that $1/q + 1/p' - 1/\eta = 1/q - 1/p + 1/\nu > 0$. If we define ε by taking $\varepsilon(1/q + 1/p') = \eta(1/q + 1/p' - 1/\eta)$, then we see that $0 < \varepsilon < \eta$, and that $(\eta - \varepsilon)/q + (\eta - \varepsilon)/p' = 1$. From (VII 5.1) we have

$$x(t+h) - x(t) = - \Sigma_{j=1}^{\nu} \left[\int_{F_\nu^+(t+h)} D_j x(\tau)\, h_j(\tau-t-h)d\tau - \int_{F_\nu^+(t)} D_j x(\tau) h_j(\tau-t)d\tau\right],$$

$$F_\nu^+(t) = [\tau | \tau \in E_\nu,\ \tau^1 \geq t^1],\quad F_\nu^+(t+h) = [\tau | \tau \in E_\nu,\ \tau^1 \geq t^1 + h^1].$$

Let $\bar{t}^1 = \min[t^1, t^1 + h^1]$, $\bar{\bar{t}}^1 = \max[t^1, t^1 + h^1]$, and let H_1, H_2 be the sets $H_1 = [t | t \in E_\nu^+,\ \bar{t}^1 \leq t^1 \leq \bar{\bar{t}}^1]$, $H_2 = [t | t \in E_\nu^+,\ \bar{\bar{t}}^1 < t^1]$. Then

$$|x(t+h| - x(t)| \leq \Sigma_{j=1}^{\nu} \left| \int_{H_1} D_j x(\tau)\, h_j(\tau-h)d\tau \right|$$

$$+ \Sigma_{j=1}^{\nu} \left| \int_{H_2} D_j x(\tau)\, [h_j(\tau-t-h) - h_j(\tau-t)]d\tau \right| = \Sigma_{j=1}^{\nu} (J_{j1}(t) + J_{j2}(t)).$$

In J_{j1} the range of integration H_1 can be restricted to the set H_o of all points $t \in H_1$ with $|t| \leq R]$: hence, $|H_o| \leq K|h^1| \leq K|h|$ for some constant K.

Note that $1/q > 1/p - 1/\nu$, that we can well assume $1 < p < q$, and that a relation $1/q > 1/\hat{p} - 1/\nu$ must hold for some number $\hat{p}$, $1 < \hat{p} < p < q$ sufficiently close to p.

As in the first part of the proof of (VII 5.ii) we take $1/p'' = 1 - 1/\hat{p}$, $\lambda = \nu - 1$, $2\varepsilon = (\nu/q) - (\lambda - \nu/p'') = 1 + \nu/q - \nu/\hat{p} > 0$, and

$$|J_{j1}(t)| \leq \int_{H_o} \left(|D_j x|^{\hat{p}/q} r^{\varepsilon - \nu/q}\right)\left(|D_j x^{\hat{p}(1/\hat{p} - 1/q)}\right)\left(r^{\varepsilon - \nu/p''}\right) d\tau.$$

If we take $\lambda_1 = 1/q$, $\lambda_2 = 1/\hat{p} - 1/q$, $\lambda_3 = 1/p''$, we have $\lambda_1, \lambda_2, \lambda_3 > 0$, $\lambda_1 + \lambda_2 + \lambda_3 = 1$, and by Hölder inequality also

$$|J_{j1}(t)| \leq \left[\int_{H_o} |D_j x|^{\hat{p}} r^{-\nu+\varepsilon q} d\tau\right]^{1/q} \left[\int_{H_o} |D_j x|^{\hat{p}} d\tau\right]^{1/\hat{p} - 1/q} \left[\int_{H_o} r^{-\nu+\varepsilon p''} d\tau\right]^{1/p''},$$

where $|t| \leq R$. Since $-\nu + \varepsilon p'' > -\nu$, the third factor is below a fixed constant. By taking powers q, integrating with respect to t in $|t| \leq R$, and noting that then $r = |t-\tau|$ varies in $[o,2R]$, we have

$$\int_{t \leq R} |J_{j1}(t)|^q dt \leq K_1' \left[\int_{H_o} |D_j x|^{\hat{p}} d\tau\right]^{q(1/\hat{p} - 1/q)} \left[\int_{H_o} |D_j x|^{\hat{p}} d\tau \int_{r \leq 2R} r^{-\nu+\varepsilon q} d\tau\right].$$

Since $-\nu + \varepsilon q > -\nu$, the last integral is below a fixed number, and thus

$$\int_{t \leq R} |J_{j1}(t)|^q dt \leq K_1'' \left[\int_{H_o} |D_j x|^{\hat{p}} d\tau\right]^{q/\hat{p}}.$$

Again, by Hölder inequality with exponents $\lambda_1 = \hat{p}/p$, $\lambda_2 = 1 - \hat{p}/p$, $\lambda_1, \lambda_2 > 0$, we have

$$\int_{t \leq R} |J_{j1}(t)|^q dt \leq K_1'' |H_o|^{q(1/\hat{p} - 1/p)} \left[\int_{H_o} |D_j x|^p d\tau\right]^{q/p},$$

and finally, since $|H_o| \leq K|h|$, also

$$\left(\int_{|t| \leq R} |J_{j1}(t)|^q dt\right)^{1/q} \leq K_1''' \|D_j x\|_p |h|^{1/\hat{p} - 1/p}.$$

We shall now consider J_{j2}. We have

$$|J_{j2}(t)| \leq \int_{H_2} |D_j x(\tau)|\ |h_j(\tau-t-h) - h_j(\tau-t)|\,d\tau$$

$$\leq \int_{H_2} (|D_j x(\tau)|^{p/q}\ |h_j(\tau-t-h) - h_j(\tau-t)|^{(\eta-\varepsilon)/q}$$

$$(|D_j x(\tau)|^{p(1/p - 1/q)}\ (|h_j(\tau-t-h) - h_j(\tau-t)|^{(\eta-\varepsilon)/p'})\,d\tau.$$

If we take $\lambda_1 = 1/q$, $\lambda_2 = 1/p - 1/q$, $\lambda_3 = 1/p'$, we have $\lambda_1, \lambda_2, \lambda_3 > 0$, $\lambda_1 + \lambda_2 + \lambda_3 = 1$, and by Hölder inequality with three factors, also

$$|J_{j2}(t)| \leq \left(\int_{H_2} |D_j x(\tau)|^p\ |h_j(\tau-t-h) - h_j(\tau-t)|^{\eta-\varepsilon} d\tau\right)^{1/q}$$

$$\left(\int_{H_2} |D_j x(\tau)|^p d\tau\right)^{1/p - 1/q} \left(\int_{H_2} |h_j(\tau-t-h) - h_j(\tau-t)|^{\eta-\varepsilon} d\tau\right)^{1/p'}.$$

The second factor is equal to $\|D_j x\|_p^{(q-p)/q}$. Since $\eta = \nu/(\nu-1)$, and $h_j \in L_s$ for every $s < \eta$, we see that the last factor is finite. If we take $\tau - t = u$, the last factor is $\leq \|h_j(u-h) - h_j(u)\ \|_{\eta-\varepsilon}^{(\eta-\varepsilon)/p'}$. Thus, by taking powers q and integration, we have, for some constant K,

$$\int_{|t| \leq R} |J_{j2}(t)|^q dt \leq K_2\ \|D_j x\|_p^{q-p}\ \|h_j(u-h) - h_j(u)\|_{\eta-\varepsilon}^{(\eta-\varepsilon)q/p'}$$

$$\int_{E_\nu^+} |D_j x(\tau)|^p d\tau \int_{E_\nu} |h_j(\tau-t-h) - h_j(\tau-t)|^{\eta-\varepsilon} dt.$$

If we take $u = \tau - t$ in the last factor we have

$$\int_{|t| \leq R} |J_{j2}(t)|^q dt \leq K_2'\ \|D_j x\|_p\ \|h_j(u-h) - h_j(u)\|_{\eta-\varepsilon}^{(\eta-\varepsilon)(q/p'+1)}.$$

Since $\eta = \nu/(\nu-1)$, and $h_j \in L_s$ for all $s < \eta$, we see that the last factor is a function $\chi_j(h)$ with $\chi_j(h) \to 0$ as $|h| \to 0$, and χ_j depends only on R, p, ν, q.

By taking $\chi(h) = \max[K_1''' \, |h|^{1/\hat{p} - 1/p} + K_2' \, \chi_j(h)]$ we have relation (VII 5.9), and statement (VII 5.vi) has been proved for $m = 1$, $p > 1$.

Let us assume now $m = 1$, $\alpha = 0$, $p = 1$. Hence, q is any number such that $1/q > 1-1/\nu$, or $1 = p < q < \nu/(\nu-1)$. Concerning J_{j1} we have the estimate

$$\begin{aligned}
|J_{j1}(t)| &\leq \int_{H_o} |D_j x(\tau)| \; r^{-\nu+1} d\tau \\
&= \int_{H_o} (|D_j x(\tau)|^{1/q} \; r^{-\nu+1})(|D_j x(\tau)|^{1-1/q}) d\tau \\
&\leq \left[\int_{H_o} |D_j x(\tau)| \; r^{(-\nu+1)q} d\tau\right]^{1/q} \left[\int_{H_o} |D_j x(\tau)| d\tau\right]^{1-1/q}.
\end{aligned}$$

By taking powers q and integrating on $[|t| \leq R, \, t' > 0]$, we have

$$\int_{|t| \leq R} |J_{j1}(t)|^q dt \leq \left[\int_{H_o} |D_j x(\tau)| d\tau\right]^{q-1} \left[\int_{H_o} |D_j x(\tau)| d\tau \int_{r \leq 2R} r^{(-\nu+1)q} dr\right].$$

Since $q < \nu/(\nu-1)$, hence $(-\nu+1)q > -\nu$, the last integral is below a fixed constant, and

$$\begin{aligned}
\int_{|t| \leq R} |J_{j1}(t)|^q dt &\leq K_1 \left[\int_{H_o} |D_j x(\tau)| d\tau\right]^q \\
&\leq K_1 \; [\omega_j(|H_o|)]^q,
\end{aligned}$$

$$\left(\int_{|t| \leq R} |J_{j1}(t)|^q dt\right)^{1/q} \leq K_1' \; \omega_j(K|h|).$$

Concerning J_{j2} we have the estimate

$$|J_{j2}(t)| \le \int_{H_2} |D_j x(\tau)|\ |h_j(\tau-t-h) - h_j(\tau-t)|\,d\tau$$

$$\le \int_{H_2} (|D_j x(\tau)|^{1/q}\ |h_j(\tau-t-h) - h_j(\tau-t)|)(|D_j x(\tau)|^{1-1/q})\,d\tau$$

$$\le \left[\int_{H_2} |D_j x(\tau)|\ |h_j(\tau-t-h) - h_j(\tau-t)|^q d\tau\right]^{1/q}$$

$$\left[\int_{H_2} |D_j x(\tau)|\,d\tau\right]^{1-1/q}$$

and, by taking powers q and integration, also

$$\int_{|t|\le R} |J_{j2}(t)|^q dt \le \left[\int_{H_2} |D_j x(\tau)|\,d\tau\right]^{q-1}$$

$$\left[\int_{H_2} |D_j x(\tau)|\,d\tau \int_{r\le 2R} |h_j(\tau-t-h) - h_j(\tau-t)|^q dt\right].$$

Since $h_j \in L_\eta(E_\nu^+)$ for $\eta = \nu/(\nu-1)$, and $1 < q < \nu/(\nu-1)$, we see that $h_j \in L_q(E_\nu^+)$ and hence $\chi_j(h) = \|h_j(u-h) - h_j(u)\|_q \to 0$ as $|h| \to 0$. Thus,

$$\left(\int_{|t|\le R} |J_{j2}(t)|^q dt\right)^{1/q} \le \chi_j^{1/q}(h)\ \|D_j x\|_1.$$

From these estimates for J_{j1} and J_{j2} we immediately obtain (VII 5.10) for $m = 1$, $\alpha = 0$, $p = 1$. So far we have proved (VII 5.v) for $m = 1$ and $p \ge 1$.

Let us assume $m > 1$ and $0 \le |\alpha| \le m-1$. Let $g = D^\alpha x$, and note that the first order partial derivatives $D_j g$, $j = 1,\dots,\nu$, are derivatives of order $|\alpha|+1$ of x. If $|\alpha| = m-1$, then $|\alpha|+1 = m$, and we can apply (VII 5.v) to $g \in W_p^1(E_\nu^+)$. Then either (VII 5.9) or (VII 5.10) hold according as $p > 1$ or $p = 1$. If $|\alpha| \le m-2$, then $D_j g \in L_{\bar p}(E_\nu^+)$ for every $\bar p$ with $1 \le \bar p \le +\infty$ and $1/\bar p > 1/p - (m-|\alpha|-1)/\nu$, and we can take $\bar p$ to be > 1 and as close to $1/p - (m-|\alpha|-1)/\nu$ as we want. We now apply (VII 5.v) to $g \in W_{\bar p}^1(E_\nu^+)$ with $\bar p > 1$.

$$\|g(t+h) - g(t)\|_q \leq \chi(h) \Sigma_{j=1}^{\nu} \|D_j g\|_{\overline{p}}$$

for every q with $1/q > 1/\overline{p} - 1/\nu$. Thus, for every q with $1/q > 1/p - (m-|\alpha|)/\nu$ and by the use of (VII 5.9) and (VII 5.iii) we have

$$\|D^{\alpha}x(t+h) - D^{\alpha}x(t)\| \leq \chi(h) \Sigma_{|\gamma| = |\alpha|+1} \|D^{\gamma}x\|_{\overline{p}}$$

$$\leq \chi(h) K \Sigma_{|\beta| = m} \|D^{\beta}x\|_p.$$

Statement (VII 5.v) is thereby proved.

<u>Remark 2</u>. Relation (VII 5.10) of (VII 5.v) is not valid for $|\alpha| = m-1$, $p = 1$, as the following example shows. In other words, the exceptional case $p = 1$, $|\alpha| = m-1$, for which we have proved (VII 5.10) instead of (VII 5.9) cannot be disregarded. In the example we consider below we have $m = 1$, $p = 1$, $\alpha = 0$, $\nu = 1$, and hence we can take for q any number $q > 1$. We shall denote by $x_k(t)$, $0 \leq t \leq 1$, $k = 1,2,\ldots$, the usual piece-wise linear functions which converge uniformly to the ternary Cantor function $x(t)$, $0 \leq t \leq 1$, on the interval [0,1]. Namely, if $I_{11}, I_{21}, I_{22}, I_{31}, I_{32}, I_{33}, I_{34},\ldots, I_{k1}, I_{k2},\ldots,I_{k2^{k-1}}$ are the intervals of constancy of x of lengths $1/3$, $1/3^2,\ldots,1/3^k$, respectively, let $x_k(t) = x(t)$ for t in these intervals, and let $x_k(t)$ vary linearly in the 2^k complementary intervals $J_{k1}, J_{k2},\ldots,J_{k2^k}$. Then x_k has a variation $j_k = 1/2^k$ on each interval J_{ks}, $s = 1,\ldots,2^k$, all of length $1/3^k$. Then each $x_k(t)$, $0 \leq t \leq 1$, $k = 1,2,\ldots$, is absolutely continuous and nondecreasing in [0,1] with $x_k(0) = 0$, $x_k(1) = 1$, hence $\|x_k\|_1 \leq 1$, $\|x_k'\|_1 = |x(1) - x(0)| = 1$, for all k. If we define each $x_k(t)$ for $t < 0$ by

taking $x_k(t) = 0$, then each x_k is of class $W_1^1(E)$ where E is the half interval $(-\infty, 1)$.

For any given h, $0 \leq |h| < 3^{-2}$, let k denote any integer with $1/3^k \leq |h|$. Then the displacement operation $t \to t+h$ takes each point $t \in J_{ks} = [\alpha, \beta]$ into a point t+h either $t+h \geq \beta$ or $t+h \leq \alpha$. Thus, each t of the interval J'_{ks} concentric to J_{ks} and length $(1/3)|J_{ks}| = 1/3^{k+1}$ is mapped into an interval either at the right of β or at the left of α. In any case we have $|x_k(t+h) - x_k(t)| \geq (1/3)j_k = (1/3)(1/2^k)$ for all $t \in J'_{ks}$ with $|J'_{ks}| = 1/3^{k+1}$. Thus, for any k with $1/3^k < |h|$ we certainly have

$$\|x_k(t+h) - x_k(t)\|_q \geq (2^k \cdot 3^{-k-1}(3^{-1}2^{-k})^q)^{1/q}$$

$$= 3^{-1}\, 3^{-(k+1)/q}\, 2^{(1-1/q)k}.$$

Let us prove that it is not possible that

$$\|x_k(t+h) - x_k(t)\|_q \leq \chi(h),$$

for all k with $3^{-k} < |h|$, for some function $\chi(h)$, $0 < |h| \leq 3^{-2}$, with $\chi(h) > 0$, $\chi(h) \to 0$ as $|h| \to 0$. It is enough to prove that it is not possible that

$$3^{-1}\, 3^{-(k+1)/q}\, 2^{(1-1/q)k} \leq \chi(h) \qquad \text{(VII 5.12)}$$

for all k with $3^{-k} \leq |h|$. Indeed for the minimum k for which this relation holds, we have $3^{-k} \leq |h| < 3^{-k+1}$, hence

$$k \log 3 \geq -\log |h| > (k-1) \log 3, \qquad \text{(VII 5.13)}$$

while (VII 5.12) can be written in the form

$$A = -\log 3 - ((k+1)/q)\log 3 + ((q-1)/q)k \log 2 \leq \log \chi(h) = B.$$

In view of (VII 5.13) we have

$$A = -\log 3 - (1/q)\log 3 - (1/q)(\log 3)((\log 3)^{-1} \log |h|)$$

$$+ ((q-1)/q) \log 2 (\log 3 - \log |h|)$$

$$= (-\log 3 - (1/q) \log 3 + ((q-1)/q \log 6) - \log |h|,$$

and the last expression is certainly positive for all $|h| > 0$ sufficiently small. On the other hand $\chi(h) > 0$, $\chi(h) \to 0$ as $|h| \to 0$, hence $B = \log \chi(h)$ must be negative for all $|h| > 0$ sufficiently small. Thus, $A > 0$, $B < 0$, $A < B$, a contradiction. We have proved that relation (VII 5.10) does not hold for the elements $x_k \in W_1^1(E)$.

The following variant of (VII 5.i) is relevant.

(VII 5.vi) Let $x(t)$, $t \in E_\nu^+$, be an arbitrary function $x \in L_p(E_\nu)$, $1 \leq p \leq +\infty$, with compact support contained in the half solid ball $[t | t' \geq 0, |t| \leq R]$, and possessing generalized partial derivatives $D^\alpha x$ in E_ν^+ of all orders $|\alpha| \leq m$ all of class $L_p(E_\nu^+)$. Then, there are functions $h_\alpha(t)$, $t \in E_\nu^+$, for every $|\alpha| = m$, $h_\alpha \in C^\infty(E_\nu^+)$, h_α of compact support contained in $[t | t' \geq 0, |t| \leq 4R]$ such that

$$x(t) = \Sigma_{|\alpha|=m} \int_{F_\nu^+(t)} (D^\alpha x)(\tau) h_\alpha(\tau - t) d\tau \quad \text{a.e.} \qquad \text{(VII 5.14)}$$

where $F_\nu^+ = F_\nu^+(t)$ denotes the part of the τ-space with $\tau^1 > t^1$. In addition $h_\alpha \in L_s(E_\nu^+)$ for every $1 \leq s < \nu/(\nu-m)$ if $\nu > m$, and $h_\alpha \in C^\infty(\text{cl } E_\nu^+)$ if $\nu \leq m$.

Proof. We use the same notations as in the proof of (VII 5.i). Then for φ as assigned in that proof, $t \in E^+_\nu$, $|t| < 2R$, and $\omega \in S^+$, we have

$$((m-1!) \int_{+\infty}^{o} r^{m-1}((\partial^m/\partial r^m) \varphi(t-r\omega)) b(r)dr$$

$$= ((m-1)!)^{-1} \int_{4R}^{o} r^{m-1}(\partial^m/\partial r^m) \varphi(t-r\omega)dr$$

$$= ((m-2)!)^{-1} \int_{4R}^{o} r^{m-2} (\partial^{m-1}/\partial r^{m-1}) \varphi(t-r\omega)dr = \dots$$

$$= \int_{4R}^{o}(\partial/\partial r) \varphi(t-r\omega)dr = \varphi(t).$$

As in the proof of (VII 5.i) we have now

$$\varphi(t) = ((m-1)!)^{-1} \Sigma_{|\alpha|=m} \int_{E^+_\nu}^{\varphi} \int_{S^+} (D^\alpha\varphi)(t-r\omega)(\omega^\alpha b(r) r^{-\nu+m}) r^{\nu-1} dr\, d\omega$$

$$= \Sigma_{|\alpha|=m} \int_{E^+_\nu} (D^\alpha\varphi)(t-\tau)\, h_\alpha(\tau) d\tau,$$

where $h_\alpha(\tau) = \omega^\alpha b(r)\, r^{-\nu+m}$, $\tau \in E^+_\nu$, $\alpha = (\alpha_1,\dots,\alpha_\nu)$, $|\alpha| = m$, $\omega^\alpha = (\omega_1)^{\alpha_1}\dots(\omega_\nu)^{\alpha_\nu}$. The proof proceeds now exactly as for (VII 5.i). It remains to prove that $h_\alpha \in L_s(E^+_\nu)$ for the stated s. We assume $\nu > m$. We have

$$\int_{E^+_\nu} |h_\alpha(t)|^s\, ds = \int_{S^+} \int_{o}^{+\infty} |\omega^\alpha\, b(r)\, r^{-\nu+m}|^s\, r^{\nu-1}\, dr\, d\omega$$

$$\leq K \int_{o}^{6R} r^{-\nu s+ms+\nu-1}\, dr,$$

and $-\nu s + ms + \nu - 1 > -1$ reduces to the assigned inequality $1 \leq s < \nu/(\nu-m)$.

We are now in a position to prove the following useful variant of statement (VII 5.ii):

(VII 5.vii) If $x \in W^m_p(E^+_\nu)$ has compact support contained in the half solid ball $[t|t^1 \geq 0,\ |t| \leq R]$, if Λ_σ is any hyperplane of dimension σ in E_ν, $1 \leq \sigma \leq \nu$, and $G_\sigma \subset \Lambda_\sigma \cap \mathrm{cl}\, E^+_\nu$ is an open set in Λ_σ contained in $[t|t' \geq 0,\ |t| \leq R]$, then the restriction x^* of x on G_σ belongs to $L_{q^*}(G_\sigma)$ and

$$\|x^*\|_{L_{q^*(G_\sigma)}} \leq K\, \Sigma_{|\alpha|=m} \|D^\alpha x\|_p, \qquad \text{(VII 5.15)}$$

where K is constant depending only on R, p, σ, Λ_σ, m, q^*, provided $\nu \geq mp$, $\sigma > \nu - mp$, and $q^* < \sigma p/(\nu - mp)$. If $\nu < mp$, then x is continuous on cl E^+_ν, and so is x^* on cl G_σ.

In other words, the identity transformation carrying an element $x \in W^m_{p,R}(E^+_\nu)$ into its restriction x^* in $G_\sigma \subset \Lambda_\sigma$ is a bounded map $W^m_{p,R}(E^+_\nu) \to L_{q^*}(G_\sigma)$. The same map is also compact with a few restrictions as we shall mention below.

Proof. We may replace G_σ be a region, say still G_σ, well contained in the solid ball $|t| \leq 2R$. Let us assume first $1 < p < +\infty$, $1/p + 1/p' = 1$, $q^* < +\infty$, and note that, for $\lambda = \nu - m$, we have

$$2\varepsilon = (\sigma/q^*) - (\lambda - \nu/p') = \sigma(1/q^* - (\nu - mp)/\sigma p) > 0.$$

Also note that $p < \sigma p/(\nu - mp)$, and therefore we can certainly take $p < q^* < \sigma p/(\nu - mp)$. Then, for $r = |t - \tau|$ we have $|h_\alpha| \leq K\, r^{-\nu+m} = K\, r^{-\lambda}$ for some constant K and then, by (VII 5.14) we also have

$$|x(t)| \leq K\, \Sigma_{|\alpha|=m} \int (|D^\alpha x|^{p/q^*}\, r^{\varepsilon - \sigma/q^*})(|D_\alpha x|^{p(1/p - 1/q^*)})(r^{\varepsilon - \nu/p'})\, d\tau, \qquad \text{(VII 5.16)}$$

where $D^\alpha x = (D^\alpha x)(\tau)$, $r = |t-\tau|$, and the integration is performed in the solid ball $|r| \leq 4R$. If we take $\lambda_1 = 1/q^*$, $\lambda_2 = 1/p - 1/q^*$, $\lambda_3 = 1/p'$, we have $\lambda_1 + \lambda_2 + \lambda_3 = 1$, $\lambda_1, \lambda_2, \lambda_3 > 0$. By Hölder inequality for three factors we have then

$$|x(t)| \leq K \sum_{|\alpha|=m} [\int |D^\alpha x|^p \, r^{-\sigma+\varepsilon q^*} \, d\tau]^{1/q^*} \, [\int |D^\alpha x|^p \, d\tau]^{1/p - 1/q^*}$$

$$[\int r^{-\nu+\varepsilon p'} \, d\tau]^{1/p'},$$

where again the integrals are taken in $|r| \leq 4R$. Since $-\nu+\varepsilon p' > -\nu$, the third factors are below a fixed constant. The second factors are also finite and equal to $\|D^\alpha x\|_p^{1-p/q^*}$. By taking powers q^*, integrating on G_σ, and interchanging the order of integration, we have

$$\int_{G_\sigma} |x(t)|^{q^*} dt \leq K' \sum_{|\alpha|=m} \|D^\alpha x\|_p^{q^*-p} \int_{r\leq 4R} |D^\alpha x|^p \, d\tau \int_{G_\sigma} r^{-\sigma+\varepsilon q^*} d\tau.$$

Since $-\sigma+\varepsilon q^* > -\sigma$, the last integral is below a fixed constant. By Tonelli's theorem the multiple integral in $dt d\tau$ above exists, and by Fubini's theorem the change of order of integration performed above is valid. We obtain now

$$\int_{G_\sigma} |x(t)|^{q^*} dt \leq K'' \sum_{|\alpha|=m} \|D^\alpha x\|_p^{q^*}$$

for some constant K", and (VII 5.15) follows.

The remaining cases, in particular the case $p = 1$, can now be treated analogously.

A theorem analogous to (VII 5.v) holds here, too, and guarantees that the map $W^m_{pR}(E^+_\nu) \to L_{q^*}(G_\sigma)$ is compact. A relation analogous to (VII 5.10) must be proved, and we leave the proof to the reader. Again the case $m = 1$, $p = 1$ is exceptional, and for this case the reader will be able to prove a relation analogous to (VII 5.11).

The case $m = 1$, $p = 1$ is actually exceptional as we can see from the following example with $\nu = 2$, $m = 1$, $p = 1$. Let $\varphi(u)$, $-\infty < u < +\infty$, $\varphi(0) = 1$, be a function of class $C_0^\infty(-\infty, \infty)$, and let $z_k(t, u)$, $(t, u) \in E = [-\infty < t < 1, -\infty < u < +\infty]$ be the functions defined by taking $z_k(t, u) = x_k(t)\, \varphi(u)$ for $0 \leq t \leq 1$, $-\infty < u < +\infty$, and $z_k(t, u) = 0$ for $-\infty < t < 0$, where $x_k(t)$, $0 < t < 1$, are the functions defined in Remark 2. Then $z_k \in W_1^1(E) \cap C(\mathrm{cl}\, E)$, $\|z_k\|_1 \leq M$, $\|\partial z_k/\partial t\|_1 \leq M$, $\|\partial z_k/\partial u\|_1 \leq M$ for some constant M, and actually $z_k \in W_{1R}^1(E)$ for a suitable R. On the other hand, the restriction z_k^* of z_k on the hyperplane $u = 0$, is the function of $x_k(t)$ for which no relation analogous to (VII 5.10) holds.

VII 6. SOBOLEV FUNCTIONS AS THE INTEGRALS OF THEIR DERIVATIVES

We begin with a statement for $\nu = 1$ to the effect that any function x of one real variable possessing first order generalized derivative y coincides almost everywhere with a function which is locally AC, and $x' = y$ almost everywhere. This statement will be the converse of what we proved in (VII 2).

(VII 6.i) If $\nu = 1$, if $x(t)$, $a < t < b$, is an element of $W_1^1(\text{loc}, (a, b))$ with generalized derivative $y(t)$, $a < t < b$ (thus, $x, y \in L^{\text{loc}}(a, b)$), then there is a function $f(t)$, $a < t < b$, which is continuous in (a, b) and AC in every closed interval $[\bar{a}, \bar{b}] \subset (a, b)$, such that $x(t) = f(t)$, $y(t) = f(t)$ a.e. in (a, b). In particular, if $x \in W_1^1(a, b)$ (hence, $x, y \in L(a, b)$), then $f(t)$, $a \leq t \leq b$, is continuous and AC in $[a, b]$.

Proof. Let α, β be any two points $a < \alpha < \beta < b$, and denote by n any integer sufficiently large so that $n^{-1} < \alpha - a$, $n^{-1} < b - \beta$. Let $\phi(t)$, $t \in E_1$, be defined by taking $\phi = 1$ in $[\alpha, \beta]$, $\phi = 0$ otherwise. Let $\phi_n(t)$, $t \in E_1$, be defined by taking $\phi_n = 1$ in $[\alpha, \beta]$, $\phi_n = 0$ outside $[\alpha - 1/n, \beta + 1/n]$, $\phi_n = 1 + n(t - \alpha)$ in $[\alpha - 1/n, \alpha]$, and $\phi_n = 1 + n(\beta - t)$ in $[\beta, \beta + 1/n]$. Then, ϕ, ϕ_n are in $L_1(E_1)$, ϕ_n is continuous, and both have compact support. Also, ϕ_n is AC with bounded derivative $\phi_n' = n$ in $(\alpha - 1/n, \alpha)$, $\phi_n' = -n$ in $(\beta, \beta + 1/n)$, and $\phi_n' = 0$ outside $[\alpha - 1/n, \alpha]$ and $[\beta, \beta + 1/n]$. Also, $\phi_n \to \phi$ in $L_1(E_1)$ as $n \to \infty$, since

$$\int_{E_1} |\phi_n - \phi| dt = 2(1/2n) = 1/n.$$

Note that, for any $x \in L_1(E_1)$ we have

$$\int_{E_1} x \, \phi_n \, dt \to \int_{E_1} x \, \phi \, dt \text{ as } n \to \infty. \tag{VII 6.1}$$

Indeed,

$$\left| \int_{E_1} x(\phi_n - \phi \, dt \right| \leq \left(\int_{\alpha-1/n}^{\alpha} + \int_{\beta}^{\beta+1/n} \right) |x| dt \tag{VII 6.2}$$

and the last expression certainly approaches zero as $n \to \infty$.

Let us prove that

$$\int_a^b y \, \phi_n \, dt = -\int_a^b x \, \phi_n' \, dt \tag{VII 6.3}$$

First, we know that $\int_a^b y \, \psi \, dt = -\int_a^b x \, \psi' \, dt$ for every $\psi \in C_0^\infty(a, b)$. Thus, for any fixed n and ε, $0 < \varepsilon \leq 1$, sufficiently small we have $\int_a^b y(J_\varepsilon \phi_n) dt = -\int_a^b x(J_\varepsilon \phi_n)' \, dt$. As $\varepsilon \to 0$ we know that $J_\varepsilon \phi_n \to \phi_n$, $(J_n \phi_n)' \to \phi_n'$ a.e. in (a,b) as $n \to \infty$, with $0 \leq (J_\varepsilon \phi_n)(t) \leq 1$, $0 \leq (J_\varepsilon \phi_n)'(t) \leq n$ for all $t \in (a, b)$. Thus, by dominated convergence theorem, we have

$$\int_a^b y(J_\varepsilon \phi_n) dt \to \int_a^b y \, \phi_n \, dt, \quad \int_a^b x(J_\varepsilon \phi_n)' dt \to \int_a^b x \, \phi_n' \, dt \text{ as } \varepsilon \to 0,$$

and (VII 6.3) is proved.

As stated by (VII 6.1) the first member of (VII 6.3) approaches $\int_a^b y \, \phi \, dt$ as $n \to \infty$, or $\int_\alpha^\beta y \, dt$. The second member of (VII 6.3) equals

$$n^{-1} \int_\beta^{\beta+1/n} x \, dt - n^{-1} \int_{\alpha-1/n}^{\alpha} x \, dt;$$

hence, the second member of (VII 6.3) approaches $x(\beta) - x(\alpha)$ for all α, β

outside a possible set E of measure zero, or $\alpha, \beta \in (a, b) - E$. Thus, as $n \to \infty$, and $\alpha, \beta \in (a, b) - E$, we obtain from (VII 6.3)

$$\int_{\alpha}^{\beta} y \, dt = x(\beta) - x(\alpha).$$

In other words, if $\bar{t} \in (a, b) - E$, we have

$$x(t) = x(\bar{t}) + \int_{\bar{t}}^{t} y(t) \, dt$$

for all $t \in (a, b) - E$. Thus, x coincides a.e. in (a, b) with the AC function

$$f(t) = x(\bar{t}) + \int_{\bar{t}}^{t} y(t) dt,$$

or $x(t) = f(t)$, $y(t) = f'(t)$ a.e. in (a, b). If $y \in L(a, b)$, then the last expression defines f as an AC function in [a, b].

A property (P) is said to hold for almost all intervals $I = [\alpha, \beta] \subset G$, $\alpha = (\alpha^1,\dots,\alpha^\nu)$, $\beta = (\beta^1,\dots,\beta^\nu)$, $\alpha^i < \beta^i$, $i = 1,\dots,\nu$, if P holds for all intervals $[\alpha, \beta]$ as above with $(\alpha, \beta) \in E_\nu \times E_\nu - E$ where E is a subset of $E_\nu \times E_\nu$ of measure zero.

To make this definition more precise, one may observe that the set of points $(\alpha, \beta) \in E_\nu \times E_\nu$ such that $I = [\alpha, \beta] \subset G$, is an open set, namely, $G^* \subset G \times G \subset E_\nu \times E_\nu$, and thus E is a subset of measure zero of G^*.

(VII 6.ii) If $\nu \geq 2$, if $x(t)$, $t \in G$, $t = (t^1,\dots,t^\nu)$, $G \subset E_\nu$, is any function in $L^{loc}(G)$ with generalized first order partial derivatives $y_i = D_i x$, $i = 1,\dots,\nu$, (also in $L^{loc}(G)$), then for almost all closed intervals $[\alpha, \beta] \subset G$ the following relation holds:

$$\int_{\alpha_i'}^{\beta_i'} [x(\beta_i, t_i') - x(\alpha_i, t_i')] = \int_{\alpha}^{\beta} y_i(t)dt, \quad i = 1,\dots,\nu. \qquad \text{(VII 6.4)}$$

Proof. If is enough to prove this statement for $\nu = 2$, and then we can write (t, s) for (t^1, t^2), $[a, b; c, d]$ for (α, β), x_t, x_s for y_1, y_2. Then the proof is analogous to the one for (VII 6.i) where intervals $I = [a, b, c, d] \subset G$, $I_n = [a - 1/n, b + 1/n, c - 1/n, d + 1/n] \subset G$ are used, and functions Φ, Φ_n defined by $\Phi = 1$ for $t \in I$, $\Phi = 0$ for $t \in E_\nu - I$; $\Phi_n = 1$ for $t \in I$, $\Phi_n = 0$ for $t \in E_\nu - I_n$, $\Phi_n = 1 - n(t - b)$ for $(t, s) \in T_{1n} = [(t, s) | b \leq t \leq b + 1/n, c - (t - b) \leq s \leq d + (t - b)]$, etc.

(VII 6.iii) Lemma. Let $x(t)$, $y_1(t),\dots,y_\nu(t)$, $t \in G$, be functions in $L^{loc}(G)$ such that relations (VII 6.4) hold for almost all closed intervals $[\alpha,\beta] \subset G$. Let $[A, B]$ be any closed interval $[A, B] \subset G$. Then for each $i = 1,\dots,\nu$, there is a function $f_i(t)$, $t \in [A, B]$ such that (a) $f_i(t) = x(t)$ a.e. in $[A, B]$; (b) for almost all $t_i' \in [A_i', B_i']$, $f_i(t_i', t^i)$ is AC in t^i on the linear interval $[A_i, B_i]$; (c) for almost all $t_i' \in [A_i', B_i']$ we have $\partial f_i(t_i', t^i)/\partial t^i = y_i(t_i', t^i)$ where $\partial f_i/\partial t^i$ is the usual partial derivative of $f_i(t_i', t^i)$ with respect to t^i.

Proof. Let $[A, B]$ and i be fixed. Then the relations

$$\int_{a_i'}^{b_i'} [x(t_i', b_i) - x(t_i', a_i)]\, dt_i' = \int_a^b y_i(t)\, dt, \quad i = 1,\dots,\nu, \qquad \text{(VII 6.5)}$$

for all intervals $[a, b] \subset [A, B]$ such that the point (a, b) does not belong to a certain set E or 2ν-measure zero in $[A, B] \times [A, B]$. By Fubini's theorem, if the point (a_i, b_i), $A_i \leq a_i < b_i \leq B_i$, is not in a certain set F of 2-measure zero, then the set $E(a_i,b_i)$ of points (a_i',b_i') such that $(a,b) \in E$ is of

$(2\nu - 2)$-measure zero. Let H be the set of values t^i, $A^i < t^i < B^i$, for which $x(t_i', t^i)$ is not integrable in t_i' on $[A_i', B_i']$. By Fubini's theorem H is of linear measure zero. Let I be the union of F and of the set of points (a_i, b_i), $A_i < a_i < b_i < B_i$, such that either a_i or b_i is in H. Then, I is still of 2-measure zero. Moreover, if $(a_i, b_i) \in [A,B] - I$, then the integrals on both sides of relation (VII 6.5) are continuous in (a_i', b_i') on $[A_i', B_i']$, and hence relation (VII 6.3) holds for every interval $[a, b] \subset [A, B]$ such that (a_i, b_i) is not in I.

Now let $\bar{a}_i$ be any number, $A_i \leq \bar{a}_i \leq B_i$, such that the set of values b_i for which $(\bar{a}_i, b_i) \in I$ is of linear measure zero, (such an $\bar{a}_i$ exists by Fubini's theorem). Then $\bar{a}_i$ is not in H, and we define f_i by taking

$$f_i(t_i', t^i) = x(t_i', \bar{a}_i) + \int_{\bar{a}_i}^{t^i} y_i(t_i', t^i)dt^i \qquad \text{(VII 6.6)}$$

for each $t_i' \in [A_i', B_i']$ for which $y_i(t_i', t_i)$ is integrable in t^i on $[A_i, B_i]$, (this being the case, by Fubini's theorem, for almost all t_i' in $[A_i', B_i']$. We define $f_i = 0$ otherwise. Clearly, f_i is measurable on $[A, B]$ and AC in t^i on $[A_i, B_i]$ for almost all $t_i' \in [A_i', B_i']$. Furthermore, for almost all $t_i' \in [A_i', B_i']$, the first order derivative $\partial f_i(t_i', t^i)/\partial t^i$ exists and is equal to $y_i(t_i',t^i)$ for almost all $t_i \in [A_i,B_i]$. Hence, we need only to show that $f_i(t) = x(t)$ a.e. in $[A,B]$.

By integrating the right-hand side of relation (VII 6.6) with respect to t_i (this is possible by the choice of $\bar{a}_i$), we see that $f_i(t_i', t^i)$ is integrable in t_i' for each t^1, and that the relation

$$\int_{A_i'}^{B_i'} |f_i(t_i', t^i)|dt_i' \leq \int_{A_i'}^{B_i'} |x(t_i', a_i|dt_i' + \int_A^B y_i(t)|dt$$

holds independently of t^i. Thus, by Fubini's theorem, f_i is integrable on $[A, B]$. Moreover, relation (VII 6.5), with x replaced by f_i, holds for every interval $[a, b] \subset [A, B]$. Hence, if b_i is not in a certain set of linear measure zero, then $f_i(t_i', t^i)$ and $x(t_i', t^i)$ are (by Fubini's theorem) integrable in t_i'. Thus, by taking $a_i = \bar{a}_i$ and b_i as above in relation (VII 6.5), we have, by (VII 6.5), that

$$\int_{a_i'}^{b_i'} x(t_i', b_i)dt_i' = \int_{a_i'}^{b_i'} f_i(t_i', b_i)\, dt_i'$$

for every $[a_i', b_i'] \subset [A_i', B_i']$. Hence, for such a b_i, we have $x(t_i', b_i) = f_i(t_i', b_i)$ for almost all $t_i' \in [A_i', B_i']$. Since x and f_i are both integrable on $[A, B]$, it follows that $f_i(t) = x(t)$ a.e. on $[A, B]$, and lemma (VII 6.iii) is thereby proved.

(VII 6.iv) Lemma. Let $x(t), y_1(t),\dots,y_\nu(t)$, $t \in G$, be functions in $L^{loc}(G)$ such that relations (VII 6.4) hold for almost all closed intervals $[\alpha, \beta] \subset G$. Let $[a, b]$, $[A, B]$ be closed intervals with $[a, b] \subset \text{int}\, [A, B] \subset [A, B] \subset G$. For each $i = 1,\dots,\nu$, let $f_i(t)$, $t \in [A, B]$, be the function defined in (VII 6.iii). Then (a) $(J_\varepsilon x)(t) \to x(t) = f_i(t)$, $(\partial(J_\varepsilon x)/\partial t^i)(t) \to y_i(t)$ as $\varepsilon \to 0$ a.e. in $[a, b]$, $i = 1,\dots,\nu$; (b) for almost all $t_i' \in [a_i', b_i']$ we have $J_\varepsilon x \to f_i(t)$ as $\varepsilon \to 0$ uniformly for $t^i \in [a_i, b_i]$.

<u>Proof</u>. From (VII 3.iii) we know that $J_\varepsilon x \to x$ as $\varepsilon \to 0$ a.e. in $[A, B]$. Since $x = f_i$ a.e. in $[A, B]$ we have also $J_\varepsilon x \to f_i$ as $\varepsilon \to 0$ a.e. in $[A, B]$.

Now $(J_\varepsilon x)(t) = \int_A^B j_\varepsilon(t - \tau)x(\tau)dt$, for $t \in [a, b]$ and $\varepsilon > 0$ sufficiently small. Hence

$$\begin{aligned} \partial J_\varepsilon x(t)/\partial t^i &= \int_A^B(\partial j_\varepsilon(t-\tau)/\partial t^i)x(\tau)dt = \int_A^B(\partial j_\varepsilon(t-\tau)/\partial t^i)f_i(\tau)dt \\ &= -\int_A^B(\partial j_\varepsilon(t-\tau)/\partial \tau^i)f_i(\tau)dt \\ &= -\int_{A_i'}^{B_i'} d\tau_i' \int_{A_i}^{B_i}(\partial j_\varepsilon(t_i' - \tau_i', t^i - \tau^i)/\partial \tau^i)f_i(\tau_i', \tau^i)d\tau^i \\ &= \int_A^B j_\varepsilon(t-\tau)\, y_i(\tau)dt = (J_\varepsilon y_i)(t), \end{aligned}$$

where we have integrated by parts in the interior integral by using (VII 5.iii). Now, by (VII 3.iii), we conclude that $\partial J_\varepsilon x(t)/\partial t^i \to y_i(t)$ as $\varepsilon \to 0$ a.e. in $[a, b]$, $i = 1,\dots,\nu$. Part (a) of (VII 6.iv) is thereby proved.

To prove part (b) let us assume first that $y_i(t)$ is of constant sign on $[A, B]$, say $y_i(t) \geq 0$. Then $f_i(t_i', t^i)$ is continuous and monotone nondecreasing with respect to t^i for each $t_i' \in [A_i', B_i']$ not in a certain set Z_1 of $(\nu - 1)$-dimensional zero. By (VII 3.iii) and (VII 6.iii, part (a)), $J_\varepsilon x(t) \to f_i(t_i', c_i)$ for all t_i' not in a certain set $Z(c_i)$ of $(\nu - 1)$-measure zero. Let S be a countable set of such values c_i such that S is dense in some interval $[\bar{a}_i, \bar{b}_i]$, $A_i < \bar{a}_i < \bar{b}_i < B_i$, with $\bar{a}_i, \bar{b}_i \in S$. Let Z_2 be the union of Z_1 and of the sets $Z(c_i)$ with $c_i \in S$. Then Z_2 is still of $(\nu - 1)$-measure zero. By (VII 6.6) we see that for each $t_i' \in [A_i', B_i']$ not in Z_2, the functions $J_\varepsilon(t_i', t^i)$ as well as $f_i(t_i', t^i)$ are continuous and monotone nondecreasing in t^i. Moreover, for $t_i' \notin Z_2$, we have $J_\varepsilon(t_i', t^i) \to f_i(t_i', t^i)$ for all $t_i \in S$, and hence, by the monotonicity and continuity, the convergence is uniform for $t \in [a_i, b_i]$.

If $D_i x(t)$ changes sign on $[A, B]$, then choose $\bar{c}_i$, $A_i \le \bar{c}_i \le B_i$, so that $f_i(t_i', t^i)$ is integrable with respect to t_i' on $[A_i', B_i']$, and define

$$f_i^+(t_i', t^i) = f_i(t_i', c_i) + \int_{c_i}^{t^i} g_i^+(t_i', \tau^i)d\tau^i,$$

$$f_i^-(t_i', t^i) = \int_{c_i}^{t^i} g_i^-(t_i', \tau^i)d\tau^i,$$

for each t_i' for which $f_i(t_i', t^i)$ is AC with respect to t^i, and where

$$g_i^+ = 2^{-1}(|g_i| + g_i), \quad g_i^- = 2^{-1}(|g_i| - g_i), \quad g_i = \partial f_i/\partial t^i.$$

Let $f_i^+ = f_i^- = 0$ otherwise. Then $f_i = f_i^+ - f_i^-$ on $[A, B]$, and as in the first part of the proof, both f_i^+ and f_i^- are AC and monotome nondecreasing with respect to t^i for almost all t_i'. Since $J_\varepsilon x = J_\varepsilon f_i = J_\varepsilon f_i^+ - J_\varepsilon f_i^-$, the proof reduces to the case of the preceding paragraph.

(VII 6.v) <u>Theorem</u>. If $x(t)$, $t \in G$, is an element of $W_1^1(\text{loc},G)$, then there is a function $x_o(t)$, $t \in G$, such that (a) $x_o \in W_1^1(\text{loc},G)$; (b) $x_o(t) = x(t)$ a.e. in G; (c) for every $i = 1,\dots,\nu$ and for almost all $\bar{t}_i' \in G_i$ the function $x_o(\bar{t}_i',t^i)$ is AC with respect to t^i on the linear open set $G_i(\bar{t}_i')$; and (d) $\partial x_o(\bar{t}_i',t^i)/\partial t^i = D_i x(\bar{t}_i',t^i)$ a.e. in G. In particular, if $w \in W_1^1(G)$, then $x_o \in W_1^1(G)$.

<u>Proof</u>. Let $x_o(t) = \lim (J_\varepsilon x)(t)$ as $\varepsilon \to 0$ whenever this limit exists and finite, and set $x_o = 0$ otherwise. By (VII 3.iii) we have $x_o(t) = x(t)$ a.e. in G, and parts (a) and (b) of (VII 6.v) are thereby proved. Let R_m, R_m', $m = 1,2,\dots$, be two sequences of closed intervals $R_m = [a_m, b_m]$, $R_m' = [A_m, B_m]$, $R_m \subset \text{int } R_m' \subset R_m' \subset G$, with $\bigcup_m R_m = G$, and for each $i = 1,\dots,\nu$ let $f_{mi}(t)$,

$t \in R'_m$, be the functions defined in (VII 6.iv) in correspondence of the interval R'_m and the given function x. Then, for each m and i, by (VII 6.iii) and (VII 6.iv), for almost all $t'_i \in [a'_{mi}, b'_{mi}]$, $x_o(t'_i, t^i) = f_{mi}(t_i, t^i)$ is AC in t^i on $[a_{mi}, b_{mi}]$, and $\partial x_o(t'_i, t^i)/\partial t^i = \partial f_{mi}(t'_i, t^i) = D_i(t'_i, t^i)$ for almost all $t^i \in [a_{mi}, b_{mi}]$. Also, $f_{mi} = f_{m_1 i}$ a.e. on $R_m \cap R_{m_1}$ for any two m, m_1 with $R_m \cap R_{m_1} \neq \Phi$, and hence by continuity we have $f_{mi}(t'_i, t^i) = f_{m_1 i}(t'_i, t^i)$ for <u>all</u> $t^i \in [a_{mi}, b_{mi}] \cap [a_{m_1 i}, b_{m_1 i}]$ for almost all $t'_i \in [a'_{mi}, b'_{mi}] \cap [a'_{m_1 i}, b_{m_1 i}]$. Since G is covered by the countably many intervals R_m, we conclude that, for almost all t'_i, $x_o(t'_i, t^i)$ is AC on $G_i(t'_i)$, and $\partial x_i(t'_i, t^i)/\partial t^i = D_i x(t'_i, t^i)$ for almost all $t^i \in G_i(t'_i)$.

<u>Remark</u>. We are now in a position to state and prove the following statement which is the converse of (VII 6.ii): If $x(t), y_1(t), \dots, y_\nu(t)$, $t \in G$, are functions in $L^{loc}(G)$ such that relations (VII 6.4) hold for almost all closed intervals $[\alpha, \beta] \subset G$, then $x \in W_1^1(loc, G)$ with generalized first order derivatives $D_i x = y_i$, $i = 1, \dots, \nu$, a.e. in G.

Indeed, by repeating the argument in (VII 2.ii), we see that relation (VII 2.1) can now be proved with x replaced by f_i and y replaced by $\partial f_i/\partial t^i = y_i$ a.e. in G; hence relation (VII 2.1) holds for x and y_i, $i = 1, \dots, \nu$, that is, y_i is the generalized first order partial derivative of x with respect to t^i in G, according to the definitions of (VII 2), $i = 1, \dots, \nu$.

VII 7. BOUNDARY VALUES OF SOBOLEV FUNCTIONS ON THE BOUNDARY OF INTERVALS

We initiate here the study of boundary values of Sobolev functions. For the sake of simplicity it is convenient to begin with boundary values of such functions on the boundary of intervals. This will apply immediately to functions defined in E_ν^+ and their boundary values on the hyperplane $t^1 = 0$, by considering arbitrary intervals $[0 < t^1 < b, a^i < t^i < b^i, i = 2,\dots,\nu] \subset E_\nu$. In the next section we shall introduce the concept of regions of class K, and then we will be able to define boundary values of Sobolev functions in such regions.

Let $R = [a,b]$ be a closed interval in E_ν, and let R^o denote the interior of R. Let $x(t)$, $t \in R^o$, be a function of class $W_p^1(R^o)$, $p \geq 1$, and let $x_o(t)$, $t \in R^o$, be the corresponding function defined in (VII 5.v). Then, for each $i = 1,\dots,\nu$, and almost all $t_i \in [a_i',b_i']$, the function $x_o(t_i',t_i)$ is AC in t^i on the linear interval (a_i,b_i) and the limits exist

$$\phi_{i1}(t_i') = x_o(t_i',a_i+o), \qquad \phi_{i2}(t_i') = x_o(t_i',b_i-o). \qquad \text{(VII 7.1)}$$

Any change of values of x in a set of measure zero in R^o may imply a change of values of x_o also in a set of measure zero, but—as one could retrace from the proof of (VII 6.v) and previous lemmas, and as we shall prove independently below—the limits (VII 7.1) may be altered at most in a set of $(\nu-1)$-measure zero in $[a_i',b_i']$.

The 2ν functions ϕ_{i1}, ϕ_{i2}, $i = 1,\dots,\nu$, define—up to a set of measure zero on ∂R—a function $\phi(t)$, $t \in \partial R$, on the boundary of R. We think of ϕ as defining an equivalent class on bd R. We say that ϕ is the set of boundary values of x on ∂R. If Φ coincides a.e. on ∂R with a function which is continuous on ∂R, we say that x has continuous boundary values.

(VII 7.i) If $x(t)$, $t \in R^o$, is an element of $W_p^1(R^o)$, $p \geq 1$, then the boundary values $\Phi(t)$, $t \in \partial R$, of x are in $L_p(\partial R^o)$. The function Φ will be often denoted by γx.

Proof. It is enough to prove that $\Phi(t_i', a_i)$ is L_p-integrable on the face $F_{i1} = \{t_i = a_i,\ t_i' \in [a_i', b_i']\}$ of R. To prove this, let us consider two numbers t_{i1}, t_{i2}, $a_i < t_{i1} < t_{i2} < b_i$. Then for almost all $t_i' \in [a_i', b_i']$ we have

$$x_o(t_i', t_{i2}) - x_o(t_i', t_{i1}) = \int_{t_{i1}}^{t_{i2}} D_i x(t_i', \tau^i) d\tau^i, \qquad \text{(VII 7.2)}$$

where x_o is the usual function defined in (VII 6.v). For p = 1 we have

$$\int_{a_i'}^{b_i'} |x_o(t_i', t_{i2}) - x_o(t_i', t_{i1})| dt_i'$$

$$= \int_{a_i'}^{b_i'} dt_i' \left| \int_{t_{i1}}^{t_{i2}} D_i x(t_i', \tau^i) d\tau^i \right| \leq \int_{a_i'}^{b_i'} \int_{t_{i1}}^{t_{i2}} |D_i x(t)| dt. \qquad \text{(VII 7.3)}$$

Since $D_i x$ is L-integrable in R, the last member certainly approaches zero as t_{i1}, $t_{i2} \to a_i$. This proves that the limit $x_o(t_i', t^i) \to \Phi_{i1}(t_i')$ as $t^i \to a_i$ occurs, not only pointwise almost everywhere on F_{i1}, but also strongly in $L_1(F_{i1})$. Thus, as $t_{i1} \to a_i$, we deduce from (VII 7.3) that

$$\int_{a_i'}^{b_i'} |x_o(t_i', t_{i2}) - \Phi_{i1}(t_i')| dt_i' \leq \int_{a_i'}^{b_i'} \int_{a_i}^{t_{i2}} |D_i x(t)| dt. \qquad \text{(VII 7.4)}$$

For $p > 1$, we deduce from (VII 7.2) and Hölder inequality that

$$\int_{a_i'}^{b_i'} |x_o(t_i',t_{i2}) - x_o(t_i',t_{i1})|^p dt$$

$$= \int_{a_i'}^{b_i'} dt_i' \left|\int_{t_{i1}}^{t_{i2}} D_i x(t_i',\tau^i)d\tau^i\right|^p \leq |t_{i2}-t_{i1}|^{p-1} \int_{a_i'}^{b_i'} \int_{t_{i1}}^{t_{i2}} |D_i x(t)|^p dt.$$

Thus, $x_o(t_i',t^i) \to \phi_{i1}(t_i')$ as $t^i \to a_i$ strongly in $L_p(F_{i1})$. As $t_{i1} \to a_i$, we deduce as before

$$\int_{a_i}^{b_i'} |x_o(t_i',t_{i2}) - \phi_{i1}(t_i')|^p dt_i' \leq |t_{i2}-a_i|^{p-1} \int_{a_i'}^{b_i'} \int_{a_i}^{t_{i2}} |D_i x(t)|^p dt. \qquad \text{(VII 7.5)}$$

(VII 7.ii) If $x(t)$, $x_k(t)$, $t \in R^o$, $k = 1,2,\ldots$, are functions in $W_p^1(R^o)$, $p > 1$, if $\int_R |D_i x_k(t)|^p dt \leq M$ for some constant M and all $m = 1,2,\ldots$, if $x_k \to x$ strongly in $L_p(R)$, then $\phi_k \to \phi$ strongly in $L_p(\partial R)$, where $\phi(t)$, $\phi_k(t)$, $t \in \partial R$, denote the boundary values of x, x_k. The same result is true if $p = 1$ provided the generalized derivatives $D_i x_k(t)$, $t \in R^o$, $k = 1,2,\ldots$, $i = 1,\ldots,\nu$, are known to be equiabsolutely integrable on R^o.

<u>Proof</u>. Assume $p > 1$ and note that relation (VII 7.5) holds for x_m as well as for x. Then from (VII 7.5) and the uniform boundedness of the numbers

$$\int_a^b |D_i x(t)|^p dt, \int_a^b |D_i x_k(t)|^p dt, \quad i = 1,\ldots,\nu,\ k = 1,2,\ldots,$$

we see that, given $\varepsilon > 0$, there is some $\delta > 0$ such that for each t_{i2} with $a_i < t_{i2} \leq a_i+\delta$, we have

$$\left[\int_{a_i'}^{b_i'} |x(t_i',t_{i2}) - \phi_{i1}(t_i')|^p dt_i'\right]^{1/p} < \varepsilon, \quad i = 1,\ldots,\nu, \text{ and}$$

$$\left[\int_{a_i'}^{b_i'} |x_k(t_i',t_{i2}) - \phi_{ki1}(t_i')|^p dt_i'\right]^{1/p} < \varepsilon, \quad i = 1,\ldots,\nu,\ k = 1,2,\ldots.$$

Since $x_k \to x$ strongly in R^o, there is a k_o such that

$$\left[\int_a^b |x_k(t)-x(t)|^p dt\right]^{1/p} < \delta^{1/p}\varepsilon$$

for all $k \geq k_o$. By Minkovski's inequality we have now

$$\left[\int_{a_i'}^{b_i'} |\phi_{il}(t_i') - \phi_{kil}(t_i')|^p dt_i'\right]^{1/p}$$

$$= \left[\delta^{-1} \int_{a_i}^{a_i+\delta} \int_{a_i'}^{b_i'} |\phi_{il}(t_i') - \phi_{kil}(t_l')|^p dt_i' dt^i\right]^{1/p}$$

$$\leq \delta^{-1/p} \left\{\int_{a_i}^{a_i+\delta} \int_{a_i'}^{b_i'} \left[|\phi_{il}(t')-x(t_i',t^i)| + |x(t_i',t^i)-x_k(t_i',t^i)| \right.\right.$$

$$\left. + |x_k(t_i',t^i) - \phi_{kil}(t_i')|\right] dt_i' dt^i$$

$$\leq \delta^{-1/p} \left\{\int_{a_i}^{a_i+\delta} \int_{a_i'}^{b_i'} |\phi_{il}(t_i')-x(t_i',t^i)|^p dt_i' dt^i\right\}^{1/p}$$

$$+ \delta^{-1/p} \left\{\int_{a_i}^{a_i+\delta} \int_{a_i'}^{b_i'} |x(t_i',t^i)-x_k(t_i',t^i)|^p dt_i' dt^i\right\}^{1/p}$$

$$+ \delta^{-1/p} \left\{\int_{a_i}^{a_i+\delta} \int_{a_i'}^{b_i'} |x_k(t_i',t^i) - \phi_{kil}(t_i')|^p dt_i' dt^i\right\}^{1/p}$$

$$\leq \delta^{-1/p} \{\delta\varepsilon^p\}^{1/p} + \delta^{-1/p} \delta^{1/p}\varepsilon + \delta^{-1/p} \{\delta\varepsilon^p\}^{1/p} = 3\varepsilon,$$

and this relation holds for all $k \geq k_o$. We have proved that, for $p > 1$, $\phi_{kil} \to \phi_{il}$ as $k \to \infty$ strongly in $L_p(F_{il})$.

If $p = 1$ and the functions $D_i x_k$, $i = 1,\dots,\nu$, $k = 1,2,\dots$, are equi-absolutely integrable in R^o, then from relation (VII 7.4) which holds for x as well as for x_k, we see that given $\varepsilon > 0$ there is a $\delta > 0$ such that for each t_{i2}, $a_i < t_{i2} \le a_i + \delta$, we have

$$\int_{a_i'}^{b_i'} |x(t_i', t_{i2}) - \phi_{i1}(t_i')| dt_i' < \varepsilon,$$

$$\int_{a_i'}^{b_i'} |x_k(t_i', t_{i2}) - \phi_{ki1}(t_i')| dt_i' < \varepsilon.$$

Also, for some k_o and all $k \ge k_o$ we have

$$\int_a^b |x_k(t) - x(t)| dt < \delta\varepsilon.$$

The details are now analogous to those of the previous case, and the statement (VII 7.ii) is thereby proved.

The case p = 1 in statement (VII 7.ii) is actually exceptional, as it can be seen by the following example. Let $\nu = 1$, $p = 1$, $x_k(t)$, $0 \le t \le 1$, be defined by taking $x_k(t) = 1 - kt$ for $0 \le t \le k^{-1}$, $x_k(t) = 0$ for $k^{-1} \le t \le 1$. Then $\|x_k\|_1 \le 1$, $\|x_k'\|_1 = 1$, $k = 1,2,\ldots$, and $x_k \in W_1^1(I)$ where $I = (0,1)$. If $x(t) = 0$, $0 \le t \le 1$, then $x_k \to x$ strongly in $L_1(I)$. On the other hand $\phi_k(o) = 1$, $\phi(o) = 0$, and ϕ_k does not converge to $\phi(o)$.

Statements (VII 7.i) and (VII 7.ii) extend immediately to functions $z \in W_p^m(R^o)$, $p \ge 1$. Indeed we have

(VII 7.iii) If $z(t)$, $t \in R^o$, is an element of $W_p^m(R^o)$, $p \ge 1$, $m \ge 1$, then z and each of the generalized partial derivatives $D^\alpha z$ with $0 \le |\alpha| \le m-1$, possesses boundary values $\phi^\alpha(t)$, $t \in R$, and $\phi^\alpha \in L_p(R)$.

(VII 7.iv) If $z(t)$, $z_k(t)$, $t \in R^o$, $k = 1,2,\ldots$, are functions in $W_p^m(R^o)$, $p > 1$, $m \ge 1$, if $\int_{R^o} |D^\alpha z_k(t)|^p dt \le M$ for some constant M and all $0 \le |\alpha| \le m$, and $k = 1,2,\ldots$, if $D^\alpha z_k \to D^\alpha z$ as $k \to \infty$ strongly in $L_p(R^o)$ for all $0 \leq |\alpha| \leq m-1$, then $\phi^\alpha_k \to \phi^\alpha$ strongly in $L_p(\partial R)$ where $\phi^\alpha(t)$, $\phi^\alpha_k(t)$, $t \in \partial R$, denote the boundary values of z, z_k. The same result is true if $p = 1$, provided the generalized partial derivatives $D^\alpha z_k(t)$, $t \in R^o$, with $|\alpha| = m$ and all $k = 1,2,\ldots$, are known to be equiabsolutely integrable in R^o.

We are now in a position to prove a statement similar to (VII 4.iv) which we shall use in (VII 10):

(VII 7.v) If $x \in W^m_p(E^+_\nu)$ and all partial derivatives $D^\alpha x$, $0 \leq |\alpha| \leq m$, have boundary values $\gamma D^\alpha x = 0$ on $t^1 = 0$, then there is a sequence of functions $x_k \in W^m_p(E^+_\nu) \cap C^\infty_o(E^+_\nu)$ with $\|x_k - x\|^m_p \to 0$ as $k \to \infty$.

Proof. As in the proof of (VII 4.iv) we take $h = (h^1, 0, \ldots, 0)$ with $h^1 > 0$, and we denote by y the function defined by $y(t) = 0$ for $t' \leq h^1$, $y(t) = x(t-h)$ for $t' > h^1$. Let us prove that $y \in W^m_p(E^+_\nu)$ with $D^\alpha y(t) = 0$, for $t < t^1$, $D^\alpha y(t) = D^\alpha x(t-h)$ for $t^1 > h^1$. It is enough to prove that corresponding relations (VII 6.5) hold for all $D^\alpha y$ $0 \leq |\alpha| \leq m-1$, and their first order partial derivatives. This is trivial, of course, up to a displacement, and the use of the boundary values $\gamma D^\alpha y$ which are all zero on $t^1 = h^1$. Now we know that $\|D^\alpha y - D^\alpha x\|_{p,E_\nu} \to 0$ as $h^1 \to 0+$, $0 \leq |\alpha| \leq m$, and, on the other hand, for any $h^1 > 0$ fixed, we also have $\|D^\alpha(J_\varepsilon y) - D^\alpha y\|_p \to 0$ as $\varepsilon \to 0+$, $0 \leq |\alpha| \leq m$. For suitable values of $h^1 = h^1_k$, $\varepsilon = \varepsilon_k > 0$, we can now take $x_k = J_\varepsilon y$.

VII 8. INVARIANCE OF SOBOLEV FUNCTIONS WITH RESPECT TO TRANSFORMATIONS OF CLASS K

(VII 8.i) <u>Theorem.</u> If $z(x)$, $x = (x^1,\dots,x^\nu) \in G$, is a given function $z \in W^1_p(G)$, $p \geq 1$, and $T : U \to G$, or $T : x = x(u)$, $u = (u^1,\dots,u^\nu) \in U$, is a transformation of class K (see (VII 2)), that is, T is one to one and Lipschiztian together with its inverse $T^{-1} : u = u(x)$, $x \in G$, then $Z(u) = z(x(u))$, $u \in U$, is a function $Z \in W^1_p(U)$, and

$$D_{u^i} Z(u) = \sum_{j=1}^{\nu} D_{x^j} x(x(u))\, D_{u^i} x^j(u), \quad i = 1,\dots,\nu, \text{ a.e. in } U.$$

<u>Proof.</u> First assume $p = 1$. We have $|Tu - Tv| \leq K|u - v|$ for all $u, v \in U$ and some constant M; hence the functions $x^j(u)$, $j = 1,\dots,\nu$, are Lipschitzian in u with the same constant K, and finally $|D_{u^i} x(u)| \leq K$ a.e. in U, $i,j = 1,\dots,\nu$ (see (VII 1.i)). Then we have also $|dx/du| \leq M$ a.e. in G, where dx/du is the Jacobian, and the constant M is certainly $\leq \nu! K^\nu$. Let us consider a sequence R_k, $k = 1,2,\dots$, of open sets invading G, that is, $R_k \subset R_{k+1}$, $R_k \uparrow G$, the closure of each R_k being the finite union of closed intervals in G. For each k we shall consider also an analogous open set R'_k such that $R_k \subset \text{cl } R_k \subset R'_k \subset \text{cl } R'_k \subset G$, the closure of each R'_k being again the finite union of closed intervals in G. By (VII 4.iv) we know that there is also a sequence $z_k(x)$, $x \in R_k$, $k = 1,2,\dots$, of functions, each z_k continuous in R_k with its first order partial derivatives $D_{x^i} z_k(x)$, $x \in R_k$, $i = 1,\dots,k$, such that

$$\int_{R_k} [|z - z_k| + \sum_{i=1}^{\nu} |D_{x^i} z - D_{x^i} z_k|]dx \leq 1/k, \quad k = 1,2,\dots,$$

and actually each z_k is continuous with its partial derivatives in the open neighborhood R'_k of R_k (indeed, we can take $z_k = J_\varepsilon z$ for $\varepsilon < \mathrm{dist}(R'_k, \partial G)$) and ε sufficiently small.

If $Z_k(u) = z_k(x(u))$, $u \in S'_k = T^{-1}(R'_k)$, then $Z_k(u)$ is a continuous function of u, and for every i and every u'_i the function $Z_k(u'_i, u^i)$ of u^i only is AC with respect to u^i in the one-dimensional open set $S'_k(u'_i)$ of all u^i such that $(u'_i, u^i) \in S'_k$. Indeed, the same functions are AC as superpositions of a function which is continuous with all its first order partial derivatives, and a Lipschitzian function, and $\partial Z_k(u'_i, u^i)/\partial u^i$ exists for almost all $u^i \in S'_k(u'_i)$ and admits of a bound which is independent of u_i and u^i (but may depend on k). Thus, $Z_k \in W^1_1(S'_k)$, and if $S_k = T^{-1}(R_k)$, we have $S_k \subset \mathrm{cl}\, S_k \subset S'_k \subset \mathrm{cl}\, S'_k \subset U$. Since $\mathrm{cl}\, S_k$ is a compact subset of S'_k, by (VII 4.iv) we know that $Z_k(u)$ and its first order partial derivatives are the uniform limit on S_k of the mollified functions $J_\varepsilon Z_k(u)$ and their first order partial derivatives. For every k we shall take, therefore, an $\varepsilon = \varepsilon(k) > 0$ sufficiently small so that, if $V_k = J_\varepsilon Z_k(u)$, $\varepsilon = \varepsilon(k)$, $u \in S_k$, we have

$$\int_{S_k} [|V_k - Z_k| + \Sigma_i |D_{u^i} V_k - D_{u^i} Z_k|] du < 1/k, \quad k = 1,2,\ldots, \tag{VII 8.1}$$

where, by force of (VII 4.i), we have

$$D_{u^i} Z_k = D_{u^i} z_k(x(u)) = \Sigma_j D_{x^j} z_k(x(u)) D_{u^i} x^j(u)$$

$$= \Sigma_j D_{x^j} z(x(u)) D_{u^i} x^j(u) + \Sigma_j [D_{x^j} z_k(x(u)) - D_{x^j} z(x(u))] D_{u^i} x^j(u).$$

Hence, for $k \leq p \leq q$,

$$D_{u^i} Z_p(u) - D_{u^i} Z_q(u) = \Sigma_j \, [D_{x^j} z_p(x(u)) - D_{x^j} z_q(x(u))] D_{u^i} x^j(u),$$

$$\int_{S_k} |D_{u^i} Z_p(u) - D_{u^i} Z_q(u)| du \leq K \Sigma_j \int_{S_k} |D_{x^j} z_p(x(u)) - D_{x^j} z_q(x(u))| du$$

$$= K \Sigma_j \int_{R_k} |D_{x^j} z_p(x) - D_{x^j} z_q(x)| \, |du/dx| dx$$

$$\leq K M \Sigma_j \int_{R_k} [|D_{x^j} z_p - D_{x^j} z| + |D_{x^j} z_q - D_{x^j} z|] dx$$

$$\leq K M(1/p + 1/q) \leq 2K M/p. \qquad \text{(VII 8.2)}$$

Analogously we have, for $k \leq p \leq q$,

$$\int_{S_k} |V_p(u) - V_q(u)| \leq \int_{S_k} [|V_p(u) - Z_p(u)| + |V_q(u) - Z_q(u)|$$

$$+ |Z_p(u) - Z_q(u)|] du$$

$$\leq 1/p + 1/q + \int_{R_k} |z_p(x) - z_q(x)| \, |du/dx| dx$$

$$\leq 1/p + 1/q + M \int_{R_k} |z_p(x) - z_q(x)| dx$$

$$\leq 1/p + 1/q + M \int_{R_k} [|z_p - z| + |z_q - z|] dx$$

$$\leq 1/p + 1/q + M(1/p + 1/q). \qquad \text{(VII 8.3)}$$

By combining (VII 8.1), (VII 8.2), (VII 8.3) we see that for every $k \leq p \leq q$ we have

$$\int_{S_k} [|V_p - V_q| + \Sigma_i |D_{u^i} V_p - D_{u^i} V_q|] du$$

$$\leq \int_{S_k} [|V_p - V_q| + \Sigma_i |D_{u^i} V_p - D_{u^i} Z_p| + \Sigma_i |D_{u^i} V_q - D_{u^i} Z_q|$$

$$+ \Sigma_i |D_{u^i} Z_p - D_{u^i} Z_q|] du$$

$$\leq (M + 1)(1/p + 1/q) + 1/p + 1/q + K\, M(1/p + 1/q)$$

$$\leq 2(M + 2 + KM)/p = M_o/p.$$

Hence, there are functions $V(u)$, $Q_i(u)$, $u \in U$, $i = 1,\dots,\nu$, such that

$$\int_{S_k} [\,|V_p - V| + \Sigma_i |D_{u^i} V_p - Q_i|\,]du \leq M_o/p \quad \text{for all } p \geq k. \qquad \text{(VII 8.4)}$$

By combining (VII 8.4) with $p = k$ and (VII 8.1) we have

$$\int_{S_k} [\,|Z_k - V| + \Sigma_i |D_{u^i} Z_k - Q_i|\,]du \leq (M_o + 1)/k,$$

for all $k = 1,2,\dots$, where

$$D_{u^i} Z_k(u) = D_{u^i} z_k(x(u)) = \Sigma_j D_{x^j} z_k(x(u)) D_{u^i} x^j(u) \quad \text{a.e. in } S_k,$$

and $i = 1,\dots,\nu$. Relation (VII 8.4) for $p = k$ and $k = 1,2,\dots$, implies by force of (VII 4.iv) that V is an element of $W_1^1(U)$ and that Q_i, $i = 1,\dots,\nu$, are the generalized first order partial derivatives of V, or $Q_i = D_{u^i}V$, $i = 1,\dots,\nu$. On the other hand, for every interval $S \subset U$ and m sufficiently large so that $S \subset S_k$ we have now

$$\int_S |Q_i(u) - \Sigma_j D_{x^j} z(x(u))\, D_{u^i} x^j(u)|du$$

$$\leq \int_S |Q_i - D_{u^i} Z_k|du + \int_S |D_{u^i} Z_k - \Sigma_j D_{x^j} z_k(x(u))\, D_{u^i} x^j(u)|du$$

$$+ \int_S \Sigma_j |D_{x^j} z_k(x(u)) - D_{x^i} z(x(u))|\, D_{u^j} x^j(u) du$$

$$\leq (M_o + 1)/k + 0 + K\, \Sigma_j \int_S |D_{x^j} z_k(x(u)) + D_{x^j} z(x(u))|du$$

$$\leq (M_o + 1)/k + KM/k = (KM + M_o + 1)/k,$$

where m can be as large as we want. Thus, the integral in the first member is zero and

$$D_{u^i} z(x(u)) = Q_i(u) = \sum_j D_{x^k} z(x(u)) \, D_{u^i} x^j(u)$$

a.e. in S and hence a.e. in U. Statement (VII 8.i) for $p = 1$ is thereby proved. The proof for $p > 1$ is analogous, and is left as an exercise for the reader.

Theorem (VII 8.i) can be expressed by saying that functions of classes $W_p^1(G)$ are invariant with respect to transformations of class K. In particular, they are invariant with respect to transformation of class C_1^1, or of change of orthogonal coordinates, and finally, by force of (VII 4.iv), with respect to passage from Cartesian to polar coordinates.

We shall now state an extension of (VII 8.i) to functions $z \in W_p^m(G$. We shall consider one to one transformations $T : U \to G$, or $T : x = x(u)$, $u = (u^1,\dots,u^\nu) \in U$, with inverse $T^{-1} : u = u(x)$, $x = (x^1,\dots,x^\nu) \in G$, such that all functions $D^\alpha x^j(u)$, $u \in U$, and $D^\alpha u^j(x)$, $x \in G$, $j = 1,\dots,\nu$, $0 \le |\alpha| \le m-1$, exist and are continuous in U and G respectively, and the same functions with $|\alpha| = m-1$ are uniformly Lipschitzian in U and G, respectively. We shall say that T is a transformation of class K_m. Thus, the transformation of class K_1 are the usual transformations of class K considered above. The following theorem holds:

(VII 8.ii) If $z(x)$, $x = (x^1,\dots,x^\nu) \in G$, is a given function $z \in W_p^m(G)$, $m \ge 1$, $p \ge 1$, and $T : U \to G$, or $T : x = x(u)$, $u = (u^1,\dots,u^\nu) \in U$, is a transformation of class K_m, then $Z(u) = z(x(u))$, $u \in U$, is a function $Z \in W_p^m(U)$ and usual formulas for the partial derivatives $D^{\alpha}Z$, $0 \leq |\alpha| \leq m$, hold almost everywhere in U.

VII 9. OPEN SETS OF CLASS K

We shall now introduce the concept of open set of class K, or K_1, and of class K_m, $m \geq 1$, in the t-space E_ν, $t = (t^1,\dots,t^\nu)$. We shall often denote these sets as regions of class K or K_m.

A bounded open set $G \subset E_\nu$ is said to be an open set, or region, of class K or K_1, if (a) G can be covered by finitely many open sets γ_s, $s = 1,\dots,N$; (b) If I denotes the interval $I = [0 < u^1 < 1, -1 < u^i < 1, i = 2,\dots,\nu]$, there is an N', $0 \leq N' \leq N$, and for each $s = 1,\dots,N'$ a positive transformation T_s of class K (see (VII 2)) defined on I such that $\gamma_s = T_s(I)$, $s = 1,\dots,N'$; (c) If λ denotes the segment $\lambda = [u^1 = 0, -1 < u^i < 1, i = 2,\dots,\nu]$ and I' denotes the interval $I' = [0 \leq u^1 < 1, -1 < u^i < 1, i = 2,\dots,\nu] = I \cup \lambda$, then for each $s = N'+1,\dots,N$, there is a positive transformation T_s of class K defined on I' such that $\gamma_s = T_s(I)$, $\Gamma_s = T_s(\lambda)$, and $\Gamma_s \subset \partial G$; (d) the sets Γ_s, $s = N'+1,\dots,N$, form a finite cover of ∂G.

Note that the sets $\{\gamma_s = T_s(I), s = 1,\dots,N', \gamma'_s = T_s(I') = \gamma_s \cup \Gamma_s, s = N'+1,\dots,N\}$ form a finite cover of cl G. Note that each part Γ_s of the boundary of G is in one-to-one correspondence with the(ν-1)-dimensional cell λ and this correspondence is certainly Lipschitzian with its inverse.

If G is of class K, then we can say that cl G is a ν-dimensional manifold with boundary of class K, and that ∂G is a (ν-1)-dimensional compact manifold of class K. In both cases "of class K" expresses the fact that only transformations of class K are used (one-to-one and uniformly Lipschitzian with their inverse. Any system $[\gamma_s, T_s, s = 1,\dots,N]$ as above will be said to be a typical representation of G as a "region of class K."

If in all definitions above we use only transformations of class K_m, $m \geq 1$, (see (VII 2)), then we say that G is a region of class K_m in E_ν.

Let G be a region $G \subset E_\nu$ of class K_m, $m \geq 1$, and let $[\gamma_s, T_s, s = 1,\dots,N]$ be a typical representation of G. Thus, $[\gamma_1,\dots,\gamma_N]$ is a covering of G, and $[\gamma_i,\ i = 1,\dots,N',\ \gamma_i \cup \Gamma_i,\ i = N'+1,\dots,N]$ a covering of clG. We can always think of $\gamma_i \cup \Gamma_i$, $i = N'+1,\dots,N$, as a part of an open set γ_i'' with $\gamma_i'' - \Gamma_i - \gamma_i \subset E_\nu - G$, $i = N'+1,\dots,N$, so that $[\gamma_1,\dots,\gamma_{N'}, \gamma_{N'+1}'',\dots,\gamma_N'']$ is a covering of clG. We shall now consider a partition of unity $[\psi_1,\dots,\psi_N]$, where $\gamma_s \in C_0^\infty(E_\nu)$, γ_s has compact support $K_s \subset \gamma_s$ for $s = 1,\dots,N'$, γ_s has compact support $K_s \subset \gamma_s' = \gamma_s + \Gamma_s$ for $s = N'+1,\dots,N$, and $\sum_{s=1}^N \psi_s(t) = 1$ for every $t \in$ clG.

If $x(t)$, $t \in G$, is of class $W_p^m(G)$, then we have

$$x(t) = \sum_{s=1}^N x(t)\, \psi_s(t), \quad t \in G, \qquad \text{(VII 9.1)}$$

and the same relation holds even on ∂G whenever $x(t)$ has boundary values on ∂G. Note that $x(t)\, \psi_s(t)$, $t \in G$, has compact support $\bar{K}_s \subset K_s \subset \gamma_s$ if $s = 1,\dots,N'$, and compact support $\bar{K}_s \subset \mathrm{cl}\, G \cap \gamma_s' = \gamma_s \cup \Gamma_s$ if $s = N'+1,\dots,N$. By force of (VII 4.ii) the functions $x(t)\, \psi_s(t)$ are of class $W_p^m(G)$ and their derivatives are given by the usual rule

$$D^\alpha(x\, \psi_s) = \sum_\beta \binom{\alpha}{\beta} D^\beta x\, D^{\alpha-\beta} \psi_s,$$

where $\alpha = (\alpha_1,\dots,\alpha_\nu)$, $0 \leq |\alpha| \leq m$, $\beta = (\beta_1,\dots,\beta_\nu)$ and $\sum$ ranges over all β with $0 \leq \beta_i \leq \alpha_i$, $i = 1,\dots,\nu$.

Thus, if we choose a given representation of G and a corresponding partition of unity, then we can find constants K, K' such that

$$|D^{\alpha}(x\,\psi_s)(t)| \le K\,\Sigma_{|\beta| \le |\alpha|}|D^{\beta}x(t)| \qquad \text{a.e. in } G,$$

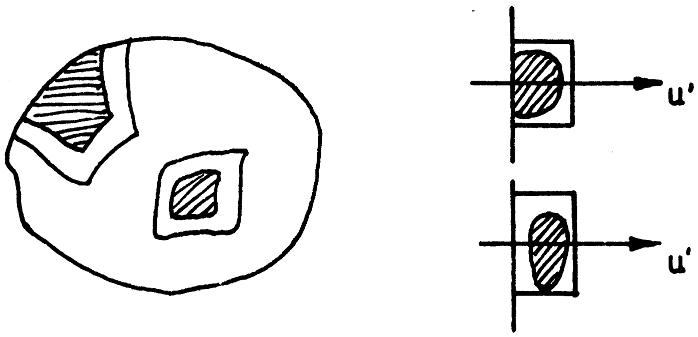
$$\|D^{\alpha}(x\,\psi_s)\|_p \le K'\,\Sigma_{|\beta| \le |\alpha|}\|D^{\alpha}x\|_p$$

for all $s = 1,\dots,N$.

Finally, $T_s : I \to \gamma_s$, $s = 1,\dots,N'$, $T_s : I' \to \gamma_s \cup \Gamma_s$, $s = N'+1,\dots,N$, and we consider the functions

$$Z_s(u) = T_s^{-1}(x(t)\,\psi_s(t)), \tag{VII 9.2}$$

$s = 1,\dots,N$. For $1 \le s \le N'$ these functions Z_s are of class $W_p^m(I)$, and have compact support $\tilde{K}_s \subset I$. For $N'+1 \le s \le N$ these functions Z_s are of class $W_p^m(I)$ but have compact support $\tilde{K}_s \subset I' = I \cup \lambda$. In either case we can extend these functions in all of E_ν^+ by taking them equal to zero in $E_\nu^+ - I$, or $E_\nu^+ - I'$, and $Z_s \in W_p^m(E_\nu^+)$.



If the functions $Z_s(u) \in W_p^m(E_\nu^+)$ have continuous boundary values γZ_s on the cell λ, then the corresponding functions $x(t)\,\gamma_s(t)$ have continuous boundary values $\gamma(x(t)\,\psi_s(t))$ on Γ, and so $x(t)$ has continuous boundary values γx on ∂G. The same holds for the derivatives $D^{\alpha}x$, $0 \le |\alpha| \le m-1$.

If these boundary values are not continuous functions, we must show that they are measurable with respect to the natural hyperarea measure σ on the boundary ∂G of G. This hyperarea σ can be introduced rather straightforwardly by means of the following statement.

(VII 9.i) If G is a class of K_m in E_ν, $m \geq 1$, then there are completely additive set functions $S_i(E)$, $V_i(E)$, $i = 1,\dots,\nu$, and $\sigma(E)$, defined on a suitable class $\mathcal{E}$ of sets $E \subset \partial G$, with the following properties:

(a) $V_i(E) \geq 0$, $\sigma(E) \geq 0$, $|S_i(E)| \leq V_i(E) \leq \sigma(E)$, $i = 1,\dots,\nu$, for all $E \in \mathcal{E}$.

(b) If $\sigma(E) = 0$ and E_i denotes the projection of E on the hyperplane $X_i = [t^i = 0]$ of E_ν, then $|E_i| = 0$, where $|E_i|$ denotes the $(\nu-1)$-dimensional Lebesgue measure on X_i, $i = 1,\dots,\nu$.

(c) If $[\gamma_s, T_s, s = 1,\dots,N]$ is any representation of G, and $T_s : I \cup \lambda \to \gamma_s \cup \Gamma_s$, $s = N'+1,\dots,N$, hence T_s maps λ onto $\Gamma_s \subset \partial G$, and F is any measurable subset of λ, then $E = T_s(F) \in \mathcal{E}$, and

$$S_i(E) = \int_F (dt_i'/du_1')du_1', \quad V_i(E) = \int_F |dt_i'/du_1'|du_1', \quad i = 1,\dots,\nu,$$

$$\sigma(E) = \int_F [\textstyle\sum_{i=1}^{\nu}(dt_i'/du_1')^2]^{1/2}du_1', \qquad \text{(VII 9.3)}$$

where $t_i' = (t',\dots,t^{i-1}, t^{i+1},\dots,t^\nu)$, $u_1' = (u^2,\dots,u^\nu)$ and dt_i'/du_1', $i = 1,\dots,\nu$, are the usual Jacobians of the transformation $T_s : \lambda \to \Gamma_s$.

(d) If T: cl G $\to$ cl H is a transformation of class K_m, then sets measurable with respect to σ on ∂G and to σ' on ∂H correspond, and there exists a constant $K \geq 1$, depending only on the transformation, such that

$$K^{-1}\,\sigma(E) \le \sigma'(E') \le K\,\sigma(E)$$

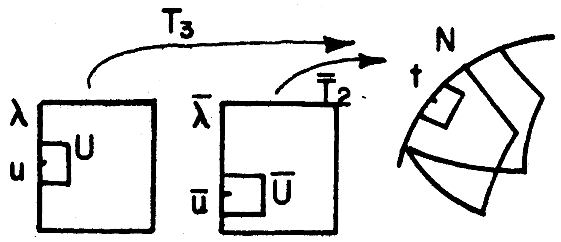
whenever E on ∂G and E' on ∂H are corresponding measurable sets.

Proof. If E, F, T_s are as in (c) above, let τ_i denote the natural projection operation of E_ν onto the hyperplane $X_i = [t \mid t^i = 0]$ of E_ν. Then $\tau_i T_s : \lambda \to X_i$, or $t_i' = t_i'(u)$, $u \in \lambda$, certainly is a continuous mapping non-necessity one-one, from the $(\nu-1)$-dimensional cell $\lambda = [u^1 = 0, -1 < u^j < 1, j = 2,\dots,\nu]$ into the $(\nu-1)$-dimensional hyperplane $X_i = [-\infty < t^h < \infty, h \ne i, h = 1,\dots,\nu, t^i = 0]$, and $\tau_i T_s$ is certainly Lipschitzian on λ of the same constant M as T_s. By force of (VII 2.iii) with $\nu-1$ replacing ν, the set $E_i' = \tau_i(E) = \tau_i T_s(F)$ is measurable. We take $\sigma_i(E)$ as given by first relation (VII 9.3), as defining a signed measure function. We must show, however, that such a definition does not depend upon the representation. Indeed, if $[\bar\gamma_h, \bar T_h, h = 1,\dots,\bar N]$ is any other representation of G, and we assume that for a given subset E of ∂G, we have

$$E \subset \Gamma_s \subset \partial G, \quad E \subset \bar\Gamma_h \subset \partial G, \quad E_i = \tau_i E = \tau_i T_s(F), \quad E_i = \tau_i E = \tau_i \bar T_h(\bar F),$$

$$\tau_i T_s : t_i' = t_i'(u),\ u \in \lambda, \qquad \tau_i \bar T_h : t_i' = \xi_i(u),\ u \in \bar\lambda.$$

If t is any point of E, then t is an interior point both of Γ_s and $\bar\Gamma_h$, and we denote by $u = T_s^{-1} t$, $\bar u = \bar T_h^{-1} t$ the counterimages on λ and $\bar\lambda$, respectively. We



can even take a neighborhood N of t in cl G, such that $N \subset \gamma_s \cup \Gamma_s$, $N \subset \gamma_h \cup \bar\Gamma_h$, and we denote by U and $\bar U$ the corresponding counterimages $U = T_s^{-1}N$, $\bar{U} = \bar{T}_h^{-1}N$. Now $\bar{T}_h^{-1}T_s$ is a one-one transformation of U onto $\bar{U}$, certainly of class K (and $T_s^{-1}\bar{T}_h$ is a one-one tranformation of class K of $\bar{U}$ onto U). Then $\bar{T}_h^{-1}T_s$ and $T_s^{-1}\bar{T}_h$ are positive transformations (as products of positive transformations), and they certainly induce positive transformations between λ and $\bar{\lambda}$, or $du_1'/d\bar{u}_1' > 0$ a.e. in $\bar{\lambda}$, and $d\bar{u}_1'/du_1' > 0$ a.e. on λ, and the same must hold, in view of (VII 4.iii), a.e. in F and $\bar{F}$ respectively. By (VII 5.i) we have then

$$S_i(E) = \int_{\bar{F}} \frac{dt_i'}{du_i'} du_1' = \int_{\bar{F}} \frac{dt_i'}{du_1'} \left|\frac{du_1'}{d\bar{u}_1'}\right| d\bar{u}_1' = \int_{\bar{F}} \frac{dt_i'}{du_1'} \frac{du_1'}{d\bar{u}_1'} d\bar{u}_1'$$

$$= \int_F \frac{dt_i'}{d\bar{u}_1'} d\bar{u}_1' = S_i(\bar{E}).$$

This shows that $S_i(E)$ does not depend upon the representation (and even for the same representation we may choose any of the overlapping neighborhood elements Γ_s). It is clear now that all these set functions join up to form a completely additive set function $S_i(E)$ over the whole of ∂G. The same argument, with obvious simplifications, holds for $V_i(E)$, $i = 1,\dots,\nu$, and for $\sigma(E)$. The formulas in (a) for any representation follow by addition. Part (a) of (VII 9.i) is thereby proved.

If E is as above and $\sigma(E) = 0$, then the ν Jacobians dt_i'/du_1', $i = 1,\dots,\nu$, must be zero a.e. on F, and this in view of (VII 4.iii) again, implies $|F| = 0$, and finally $|E_i| = 0$, in view of (VII 2.iii). This proves part (b) of (VII 9.i). Part (c) was proved above. Part (d) is a consequence of (VII 4.ix) on the multiplication rule for Jacobians. Statement (VII 9.i) is thereby proved.

We are now in a position to prove the identity $H_p^m(G) = W_p^m(G)$ for any region G of class K_m, and to prove for such regions the interpretation of $H_p^m(G)$ we have mentioned after relation (VII 4.4). All this is a consequence of the remarks already made at the end of (VII 4), and of the following statement.

(VII 9.ii) If $x \in W_p^m(G)$ with G of class K_m, $1 \leq p \leq +\infty$, then there is a sequence of functions $x_k \in W_p^m(G)$, $k = 1,2,\ldots$, and each x_k is Lipschitzian in G together with all generalized partial derivatives $D^\alpha x_k$, $0 \leq |\alpha| \leq m-1$, hence the derivatives $D^\alpha x_k$ with $|\alpha| = m$ are bounded, and $\|x_k - x\|_p^m \to 0$ as $k \to \infty$.

If in addition, the boundary values $\gamma D^\alpha x$ are known to be zero a.e. on the boundary ∂G of G for $0 \leq |\alpha| \leq m-1$, then we can choose the sequence x_k so as each x_k is identically zero on and near ∂G.

Proof. If we consider as at the beginning of this section any given representation $[\gamma_s, T_s, s = 1,\ldots,N]$ of G and a corresponding partition of unity ψ_s, $s = 1,\ldots,N$, then we know that the corresponding functions $Z_s(u)$ belong to $W_p^m(E_\nu^+)$. We can now apply (VII 4.v) to each function $Z_s(u)$, $u \in E_\nu^+$, to obtain a sequence $Z_{sk}(u)$, $u \in E_\nu^+$, $k = 1,2,\ldots$, of functions of class $C^\infty(E_\nu)$ such that $\|Z_{ks} - Z_s\|_p^m \to 0$ as $k \to \infty$. Finally, the functions $x_{sk} = T_s(Z_{ks})$ have the required properties on each γ_s and Γ_s and we define them to be zero everywhere else on cl G. Finally, the functions $x_k = \sum_s x_{ks} \psi_s$ have the required properties on $G \cup \partial G$.

If the boundary values $\gamma D^\alpha x$, $0 \leq |\alpha| \leq m-1$, are all zero, then the functions $Z_s(u)$, $u \in E_\nu^+$, have boundary values zero on the straight line $u^1 = 0$ and so have the derivatives $D^\alpha Z_s(u)$, $0 \leq |\alpha| \leq m-1$. We can apply (VII 7.vi), and thus we can choose each sequence Z_k, $k = 1,2,\ldots$, so that all $D^\alpha Z_{sk}$, $0 \leq |\alpha| \leq m-1$, are identically zero on $u^1 = 0$. Then the functions z_{sk} and their derivatives $D^\alpha z_{sk}$, $0 \leq |\alpha| \leq m-1$, are zero on and near ∂G.

VII 10. WEAK COMPACTNESS IN L_p FOR $p \geq 1$

As seen in Chapter 4, we need compactness theorems in $L_p(G)$ and $W_p^m(G)$ for $p > 1$ as well as for $p = 1$. For the convenience of the reader we state and prove below some of these theorems for $L_p(G)$ with $p > 1$ and with $p = 1$, and in (VII 11) we shall state and prove corresponding compactness theorems for $W_p^m(G)$ with $p > 1$ and with $p = 1$.

We shall denote by t the real vector variable $t = (t^1,\dots,t^\nu) \in E_\nu$, and by $z(t)$, $t \in G$, a real-valued function defined on a subset G of E_ν. By the notation $z \in L_p(G)$, $p \geq 1$, we shall mean, as usual, that z is measurable and that $|z|^p$ is L_1-integrable in G.

(VII 10.i) Let G be a measurable bounded subset of E_ν, and $z_k(t)$, $t \in G$, $k = 1,2,\dots$, a sequence of real-valued measurable functions such that (α) $z_k \in L_p(G)$, $k = 1,2,\dots$, (β) $\int_G |z_k|^p dt \leq M$, $k = 1,2,\dots$, for some constants $p > 1$, $M \geq 0$. Then there is a measurable function $z(t)$, $t \in G$, and a subsequence $[z_{k_s}]$ such that

$$z \in L_p(G); \tag{VII 10.1}$$

$$\int_G |z|^p dt \leq \varliminf_{s\to\infty} \int_G |z_{k_s}|^p dt;$$

$$\int_G |z|\, dt \leq \varliminf_{s\to\infty} \int_G |z_{k_s}|\, dt; \tag{VII 10.2}$$

$$\int_G z\varphi\, dt = \lim_{s\to\infty} \int_G z_{k_s} \varphi\, dt; \tag{VII 10.3}$$

for every real-valued measurable function $\varphi(t)$, $t \in G$, with $\varphi \in L_q(G)$, $1/p + 1/q = 1$. All integrals above are finite.

Remark 1. This theorem is usually proved as a consequence of general statements of functional analysis. Indeed the space $L_p(G)$ with the usual L_p-norm is a uniformly convex normed space and hence symmetric by remarks of J. A. Clarkson, and consequently any strongly bounded sequence is weakly compact by a theorem of L. Alaog concerning weak topologies in normed linear spaces (see E. Rothe, Pacific Math. Journ. 3, 1953, 493-499). Nevertheless, there are direct proofs of statement (VII 10.i) and of the corresponding statement for p = 1 (see (VII 10.ii) below and its proof), and these direct proofs are based on the remark that hypothesis (β) implies that the function x_k are equiabsolutely integrable in G, that is, given $\varepsilon > 0$, there is some $\delta = \delta(\varepsilon) > 0$ such that $H \subset G$, H measurable, $|H| < \delta$, implies $\int_H |z_k| dt < \varepsilon$, $k = 1,2,\ldots$. Indeed, by Hölder's inequality

$$\int_H |z_k| dt \leq (\int_H 1^q dt)^{1/q} (\int_H |z_k|^p dt)^{1/p} \leq M^{1/p} |H|^{1/q},$$

and it is enough to assume $\delta = \varepsilon^q M^{-q/p}$.

Remark 2. There is a statement underlying (VII 10.i), namely that, for G bounded, (or at least G with finite measure $|G| < +\infty$), the integrability of $|z|^p$, $p > 1$, implies the integrability of every power $|z|^r$, $1 \leq r \leq p$. This statement is not valid for G unbounded, as simple examples show (see for instance [24r]). All statements of this Section VII 10 have a slightly modified counterpart for the case in which G is unbounded. For the sake of simplicity we limit ourselves to the case G bounded, and we refer the interested reader to [24r] for extensions.

Remark 3. For $p = 1$ statement (VII 10.i) is not true, as the following well known example shows. Take $\nu = 1$, $G = [o,1]$, $x_k(t) = k$ for $0 \leq t \leq k^{-1}$, $x_k(t) = 0$ for $k^{-1} < t \leq 1$, $k = 1,2,\ldots$. Then we can take $x(t) = 0$ for all $0 \leq t \leq 1$, and now for $\varphi = 1$, $0 \leq t \leq 1$, we have $\int_o^1 z_k dt = 1$, $\int_o^t z\,dt = 0$, and (VII 10.3) is not valid. For $p = 1$ and G bounded, statement (VII 10.i) can be replaced by the following statement (VII 10.ii).

(VII 10.ii) Let G be any measurable bounded subset of E_ν, and $x_k(t)$, $t \in G$, $k = 1,2,\ldots$, a sequence of real-valued measurable functions such that (γ) the functions x_k are equiabsolutely integrable in G. Then there is a measurable function $x(t)$, $t \in G$, and a subsequence $[x_{k_s}]$ such that

$$x \in L_1(G), \tag{VII 10.4}$$

$$\int_G |x|\,dt \leq \underline{\lim}_{s\to\infty} \int_G |x_{k_s}|\,dt, \tag{VII 10.5}$$

$$\int_G x\varphi\,dt = \lim_{s\to\infty} \int_G x_{k_s}\varphi\,dt, \tag{VII 10.6}$$

for every measurable bounded functions $\varphi(t)$, $t \in G$. All integrals above are finite.

Remark 4. Since G is bounded and has, therefore, finite measure, condition (γ) certainly implies $\int_G |x_k|\,dt \leq M'$, $k = 1,2,\ldots$, for some constant M', and thus a condition analogous to (β) of (VII 10.i) is superfluous here.

Proofs of (VII 10.ii) and of (VII 10.i). It is note restrictive to assume that G is contained in the hypercube $0 \leq t^i \leq N$, $i = 1,\ldots,\nu$, for some integer

N. Let us define each function x_k in E_ν by taking $x_k(t) = 0$ for $t \in E_\nu - G$. Then the function x_k are L_1-integrable in every interval $R_o \subset E_\nu$, and $\int_{R_o} |x_k(t)|dt \leq M$, $k = 1,2,\ldots$. Given any interval $R = [a,b]$, $a = (a^1,\ldots,a^\nu)$, $b = (b^1,\ldots,b^\nu)$, by $\int_a^b x_k(t)dt$ we shall denote the integral of x_k in R with the usual conventions concerning signs. Let R_o be the interval $[o,N]$, $0 = (o,\ldots,o)$, $N = (N,\ldots,N)$, $N > 0$. For every $k = 1,2,\ldots$, let us consider the function $X_k(t) = \int_o^t x_k(\tau)d\tau$, defined for every $t = (t^1,\ldots,t^\nu) \in R_o$, and where the integral ranges over the interval $[o,t]$. Then, for every interval $R \subset R_o$ the interval functions

$$\Psi_k(R) = \int_R x_k(\tau)d\tau, \quad k = 1,2,\ldots, \quad R = [a,b] \subset R_o,$$

can be expressed in terms of the usual differences of order ν of the function X_k with respect to the 2^ν vertices of R, say

$$\Psi_k(R) = \Delta_R Z_k = Z_k(b) - Z_k(a) \quad \text{for } \nu = 1,$$

$$\Psi_k(R) = \Delta_R Z_k = Z_k(b^1,b^2) - Z_k(a^1,b^2) - Z_k(b^1,a^2) + Z_k(a^1,a^2)$$

for $\nu = 2$, etc.

As a consequence of (γ) the interval functions $\Psi_k(R) = \Delta_R Z_k$, $k = 1,2,\ldots$, are equiabsolutely continuous in the usual sense, that is, given $\varepsilon > 0$, there is some $\delta = \delta(\varepsilon) > 0$ such that, for every finite system $R_1,\ldots,R_J$ of nonoverlapping intervals $R_j \subset R_o$, $j = 1,\ldots,J$, with $\Sigma_j |R_j| \leq \delta$, we have $\Sigma_j |\Psi_k(R_j)| \leq \varepsilon$.

If $t, t' \in R_o$, $t = (t^1,\ldots,t^\nu)$, $t' = (t'^1,\ldots,t'^\nu)$, and $|t-t'| = d$, let

p_i, $i = 0,1,\dots,\nu$, be the $\nu+1$ points $p_o = t$, $p_\nu = t'$, $p_i = (t^1,\dots,t^{\nu-i}, t'^{\nu-i+1},\dots,t'^{\nu})$, $i = 1,\dots,\nu-1$. Note that

$$X_k(t') - X_k(t) = \sum_{i=1}^{\nu}[X_k(p_i) - X_k(p_{i-1})]$$
$$= \sum_{i=1}^{\nu}\left[\int_o^{p_i} - \int_o^{p_{i-1}}\right] x_k(\tau)d\tau,$$

and that the two intervals $[o,p_i]$, $[o,p_{i-1}]$ differ by the single interval $r_i = (q_i,p_i)$, where

$$q_i = (o,\dots,o,t^{\nu-i+1},o,\dots,o),$$
$$p_i = (t^1,\dots,t^{\nu-1},t'^{\nu-i+1},\dots,t'^{\nu}),$$

and hence

$$X_k(t') - X_k(t) = \sum_{i=1}^{\nu} \int_{q_i}^{p_i} x_k(\tau)d\tau,$$

where $|r_i| \leq N^{\nu-1}|t^{\nu-i+1} - t'^{\nu-i+1}| \leq N^{\nu-1}|t-t'| = N^{\nu-1}d$. Thus, given $\varepsilon > 0$, whenever $|t-t'| = d < \delta/N^{\nu-1}$, we have $|X_k(t) - X_k(t')| \leq N^{\nu-1}(\varepsilon/N^{\nu-1}) = \varepsilon$ for every $k = 1,2,\dots$. This shows that the functions $X_k(t)$, $t \in R_o$, $k = 1,2,\dots$, are equicontinuous in R_o. Since $X_k(o) = 0$, the same functions are equibounded in R_o. By Ascoli's theorem there is, therefore, a subsequence X_{k_s}, $s = 1,2,\dots$, with $k_s \to \infty$, which is uniformly convergent in R_o toward a continuous function $X(t)$, $t \in R_o$. Since $X_k(t) = 0$ for every $t = (t^1,\dots,t^{\nu})$ with $t \in R_o$ and $t^1t^2\dots t^{\nu} = 0$, we deduce that $X(t) = 0$ for the same t.

For any interval $R = [a,b] \subset R_o$, we have

$$\int_R z_{k_s}(\tau)d\tau = \Delta_R X_{k_s} = \Sigma_j \pm X_{k_s}(\alpha_j),$$

where Σ_j ranges over the 2^ν vertices α_j of R with the usual sign conventions as mentioned above. As $s \to \infty$ we deduce

$$\Psi(R) = \lim_{s\to\infty} \int_R z_{k_s}(\tau)d\tau = \Delta_R X = \Sigma_j \pm X(\alpha_j),$$

and the convergence is uniform with respect to $R \subset R_o$. Since the interval functions $\Psi_k(R) = \Delta_R Z_k$ are equiabsolutely continuous in R_o, then the interval function $\Psi(R) = \Delta_R X$ has the same property.

By Banach's theorem (V 6.ii) there is a measurable and L_1-integrable function $x(t)$, $t \in R_o$ with

$$\Psi(R) = \Delta_R X = \int_R x(\tau)d\tau$$

for every $R \subset R_o$, and

$$\lim_{s\to\infty} \int_R x_{k_s}(\tau)d\tau = \int_R x(\tau)d\tau.$$

This relation proves (VII 10.6) for every function φ which is the characteristic function of an interval.

Thus (VII 10.6) is proved also for functions φ which are characteristic functions of a finite union of nonoverlapping intervals. If E is any measurable set, we can approach E in measure by means of a sequence of finite unions of nonoverlapping intervals, and then (VII 10.6) can be proved for functions φ which are the characteristic functions of measurable sets. Then (VII 10.6) is proved also for measurable step functions φ. Finally, any measurable bounded function φ can be approached by means of a sequence of measurable step functions with the same bound and thus (VII 10.6) can be proved in general. Relation (VII 11.5) is now a consequence of (VII 10.6). Indeed, if $\varphi(t)$, $t \in G$, is defined by taking $\varphi = 1$ if $x \geq 0$, $\varphi = -1$ if $x < 0$, then φ is bounded and measurable, hence by (VII 10.6)

$$\int_G |x|\,dt = \int_G x\varphi\,dt = \lim_{s\to\infty} \int_G x_{k_s} \varphi\,dt \leq \underline{\lim}_{s\to\infty} \int_G |x_{k_s}|\,dt.$$

Thus, statement (VII 10.ii) is completely proved.

Let us now prove (VII 10.i). First, let us prove that for every $\varphi \in L_q(G)$ the product $x\varphi$ is L_1-integrable. Let $\varphi_N(t)$, $t \in G$, be defined by taking $\varphi_N = \varphi$ if $-N \leq \varphi \leq N$, $\varphi_N = N$ if $\varphi \geq N$, $\varphi_N = -N$ if $\varphi \leq -N$, where N is any integer. Then φ_N is measurable and bounded in G. Also, let $\psi(t)$, $t \in G$, be defined by taking $\psi = +1$ if $x\varphi \geq 0$, $\psi = -1$ if $x\varphi < 0$. Thus, ψ also is measurable and bounded in G, and so is $\varphi_N\psi$. Note that φ_N and φ have the same sign, that $|x|\,|\varphi_N| = |x\varphi_N| = x\varphi_N\psi$, and hence, by force of (VII 10.6)

$$\int_G |x|\,|\varphi_N|\,dt = \int_G x\varphi_N\psi\,dt = \lim_{s\to\infty} \int_G x_{k_s}\,\varphi_N\psi\,dt,$$

where

$$\left|\int_G x_{k_s}\,\varphi_N\psi\,dt\right| \leq \int_G |x_{k_s}|\,|\varphi|\,dt \leq \left(\int_G |x_{k_s}|^p dt\right)^{1/p} \left(\int_G |\varphi|^q dt\right)^{1/q}$$
$$\leq M^{1/p} \left(\int_G |\varphi|^q dt\right)^{1/p}.$$

Thus $\int_G |z|\,|\varphi_N|\,dt$ is below a fixed number independent of N. By Lebesgue monotone convergence theorem, we conclude that $z\varphi$ is L-integrable and that

$$\int_G |x\varphi|\, dt \leq M^{1/p} \left(\int_G |\varphi|^q dt\right)^{1/q}.$$

If E^+, E^- denote the subsets of G where $z\varphi \geq 0$, or $z\varphi < 0$, respectively, then E^+ and E^- are measurable. By Lebesgue monotone convergence theorem we deduce

$$\int_{E^\pm} \varphi\, dt = \lim_{N\to\infty} \int_{E^\pm} x\varphi_N dt,$$

and by addition also

$$\int_G x\varphi\, dt = \lim_{N\to\infty} \int_G x\varphi_N dt.$$

Thus, given $\varepsilon > 0$ there is some N_o such that

$$\left|\int_G x\varphi\, dt - \int_G x\varphi_N dt\right| < \varepsilon \quad \text{for } N \geq N_o. \tag{VII 10.7}$$

Also

$$\left|\int_G x_{k_s} \varphi\, dt - \int_G x_{k_s} \varphi_N\, dt\right| = \left|\int_G x_{k_s}(\varphi-\varphi_N) dt\right|$$

$$\leq \left(\int_G |x_{k_s}|^p dt\right)^{1/p} \left(\int_G |\varphi-\varphi_N|^q dt\right)^{1/q} \leq M\left(\int_G |\varphi-\varphi_N|^q dt\right)^{1/q},$$

where the last expression certainly approaches zero as $N \to \infty$. Thus, we can take N_o such that

$$\left|\int_G x_{k_s} \varphi\, dt - \int_G x_{k_s} \varphi_N\, dt\right| < \varepsilon \quad \text{for all } N \geq N_o \text{ and all } s. \tag{VII 10.8}$$

Finally, from (VII 10.6) we have

$$\int_G x\varphi_N\, dt = \lim_{s\to\infty} \int_G x_{k_s} \varphi_N\, dt$$

for every N, in particular for $N = N_o$, and hence there is some s_o such that

$$\left|\int_G x\varphi_{N_o}\,dt - \int_G x_{k_s}\varphi_{N_o}\,dt\right| < \varepsilon, \tag{VII 10.9}$$

We have now, from (VII 10.7), (VII 10.8), (VII 10.9),

$$\left|\int_G x\varphi\,dt - \int_G x_{k_s}\varphi\,dt\right| < 3\varepsilon \quad \text{for all } s \geq s_o.$$

We have proved that relation (VII 10.3) holds for every $\varphi \in L_q(G)$ with $1/p + 1/q = 1$. Let us prove that

$$\left|\int_G x\varphi\,dt\right| \leq M^{1/p}\left(\int|\varphi|^q dt\right)^{1/q}$$

for every $\varphi \in L_q(G)$. Indeed, for any such φ,

$$\left|\int_G x\varphi\,dt\right| = \left|\lim_{s\to\infty}\int_G x_{k_s}\varphi\,dt\right| \leq \varliminf_{s\to\infty}\left(\int_G |x_{k_s}|^p dt\right)^{1/p}\left(\int_G|\varphi|^q dt\right)^{1/q}$$

$$\leq M^{1/p}\left(\int_G|\varphi|^q dt\right)^{1/q}.$$

This shows that $\int_G x\varphi\,dt$ is a continuous linear operator in $L_q(G)$ (and thus $x \in L_p(G)$). This can be seen as follows. For every $N > 0$ let $\varphi_N(t)$, $t \in G$, be defined by taking $\varphi_N = |x(t)|^{p-1}\operatorname{sgm} x(t)$ if $|x(t)| \leq N$; $\varphi_N(t) = N^{p-1}\operatorname{sgm} x(t)$ if $|x(t)| \geq N$. These functions φ_N are bounded and measurable, hence

$$\left|\int_G x\varphi_N\,dt\right| \leq M^{1/p}\left(\int_G|\varphi_N|^q dt\right)^{1/q}.$$

On the other hand

$$\left|\int_G x\varphi_N dt\right| = \int_G x\varphi_N\, dt = \int_G |x|\ |\varphi_N| dt \geq \int_G |\varphi_N|\ |\varphi_N|^{\frac{1}{p-1}} dt$$

$$= \int_G |\varphi_N|^{\frac{p}{p-1}} dt = \int_G |\varphi_N|^q dt.$$

Therefore

$$\int_G |\varphi_N|^q dt \leq M^{1/p} \left(\int_G |\varphi_N|^q dt\right)^{1/q}, \left(\int_G |\varphi_N|^q dt\right)^{1/p} dt \leq M^{1/p},$$

for every N. Since $|\varphi_N| \to |x|^{p-1}$, by Lebesgue monotone convergence theorem we conclude that $|x|^{(p-1)q}$ is L-integrable in G, that is, $|x|^p$ is L-integrable, and

$$\int_G |x|^p dt \leq M.$$

Now we can prove (VII 10.2). Indeed, if we take $\varphi = |x|^{p-1}$ sgm x, then φ is measurable, and $|\varphi|^q = |x|^{q(p-1)} = |x|^p$ is L-integrable, that is, $\varphi \in L_q(G)$, and by (VII 10.6)

$$\int_G |z|^p dt = \int_G z\varphi\, dt = \lim_{s\to\infty} \int_G z_{k_s} \varphi\, dt$$

$$\leq \underline{\lim}_{s\to\infty} \left(\int_G |z_{k_s}|^p dt\right)^{1/p} \left(\int_G |x|^p dt\right)^{1/q},$$

and finally, by algebraic manipulation,

$$\int_G |z|^p dt \leq \underline{\lim}_{s\to\infty} \int_G |z_{k_s}|^p dt.$$

VII 11. WEAK COMPACTNESS IN $W_p^m(G)$ FOR $p \geq 1$ AND G OF CLASS K_m, AND SUMMARY OF THE EMBEDDING THEOREMS FOR $W_p^m(G)$

We have seen in (VII 9) that every function $x \in W_p^m(G)$, $1 \leq p \leq +\infty$, where G is a region of class K_m (see VII 9), can be decomposed into the finite sum $x = \Sigma_s z_s = \Sigma_s x\psi_s$ of functions z_s whose transformations $Z_s = T_s^{-1}(x\psi_s)$ can be interpreted as elements Z_s of $W_p^m(E_\nu^+)$. It is now clear that all properties of the spaces $W_p^m(E_\nu^+)$, $1 \leq p \leq +\infty$, of (VII 5), (VII 6), (VII 7), can be transferred to the spaces $W_p^m(G)$. We simply summarize below the main results.

(VII 11.i) If $x(t)$, $t \in G$, is an element of $W_p^m(G)$ and G is a region of class K_m in E_ν, $\nu \geq 1$, $p \geq 1$, $m \geq 1$, then the generalized partial derivatives $D^\alpha x$, $0 \leq |\alpha| \leq m$, are certainly of class $L_p(G)$, and the boundary values $\gamma D^\alpha x$ for $0 \leq |\alpha| \leq m-1$, are defined σ-a.e. on ∂G and are certainly of class $L_p(\partial G)$. However:

(a) Each derivative $D^\alpha x$, $0 \leq |\alpha| \leq m-1$, is actually at least of class $L_q(G)$ for every q, $1 \leq q \leq +\infty$, with $1/q > 1/p - (m-|\alpha|)/\nu$, and

$$\|D^\alpha x\|_q \leq K \Sigma_{|\alpha| \leq |\beta| \leq m} \|D^\beta x\|_p,$$

where the constant K depends only on G, m, p, q, α. In particular, for $\alpha = 0$, we have $x \in L_q(G)$ for every q, $1 \leq q \leq +\infty$, with $1/q > 1/p - m/\nu$, and

$$\|x\|_q \leq K \|x\|_p^m.$$

Thus, the identity transformation carrying an element x of $W_p^m(G)$ into the same function x as an element of $L_q(G)$ is a bounded transformation $W_p^m(G) \to L_q(G)$.

(b) Each derivative $D^\alpha x$, $0 < |\alpha| \leq m-1$, is actually continuous on clG, provided $1/p < (m-|\alpha|)/\nu$, and then

$$|D^\alpha x(t)| \leq K \sum_{|\alpha| \leq |\beta| \leq m} \|D^\beta x\|_p, \quad t \in \text{clG},$$

where the constant K depends only on G, m, p, α.

In particular, for $\alpha = 0$, x is continuous in clG provided $1/p < m/\nu$, and

$$|x(t)| \leq K \|x\|_p^m.$$

Thus, the identity transformation carrying an element x of $W_p^m(G)$ into the same function x as an element of C is a bounded transformation $W_p^m(G) \to C(\text{clG})$.

If $p > 1$, the transformations defined in (a) and (b) are not only bounded, but compact. In other words, under the condition of (a), $\alpha = 0$, $W_p^m(G) \to L_q(G)$, and if $[x_k]$ is a sequence of functions $x_k \in W_p^m(G)$, $\|x_k\|_p^m \leq M$, then a suitable subsequence converges (strongly) in $L_q(G)$. Under the conditions of (b), $\alpha = 0$, $W_p^m(G) \to C(G)$ and $[x_k]$ as above, then the functions x_k are equicontinuous in clG. These statements hold even for $p = 1$, provided the derivatives of maximal order $D^\alpha x_k$, $|\alpha| = m$, $k = 1,2,\ldots$, are known to be equiabsolutely integrable in G. A more precise form of these results will be given below in (VII 11.ii).

(VII 11.ii) If $p > 1$, G is a region of class K_m in E_ν, $\nu \geq 1$, $m \geq 1$, if $[x_k]$ is a sequence of elements $x_k \in W_p^m(G)$ with $\|x_k\|_p^m \leq M$, $k = 1,2,\dots,$ for some constant M, then there is a subsequence $[k_s]$ and an element $x \in W_p^m(G)$ such that:

(a) If $\nu \geq mp$, then $x_{k_s} \to x$ strongly in $L_q(G)$ for every q with $1/q > 1/p - m/\nu$.

(b) If $\nu < mp$, then all x_k and x are continuous on clG and $x_{k_s} \to x$ uniformly on clG.

(c) For every $0 \leq |\alpha| \leq m-1$, if $\nu \geq (m-|\alpha|)p$, then $D^\alpha x_{k_s} \to D^\alpha x$ strongly in $L_q(G)$ for every $1 \leq q \leq +\infty$ with $1/q > 1/p - (m-|\alpha|)/\nu$.

(d) For every $0 \leq |\alpha| \leq m-1$, if $\nu < (m-|\alpha|)p$, then all $D^\alpha x_k$ and $D^\alpha x$ are continuous on clG, and $D^\alpha x_{k_s} \to D^\alpha x$ uniformly on clG.

(e) For every $|\alpha| = m$, $D^\alpha x_{k_s} \to D^\alpha x$ weakly in $L_p(G)$ as $s \to \infty$. Concerning the boundary values γx_k, γx, $\gamma D^\alpha x_k$, $\gamma D^\alpha x$, $0 \leq |\alpha| \leq m-1$, we have also

(f) If $\nu \geq mp > 1$, then $\gamma x_{k_s} \to \gamma x$ strongly in $L_{q^*}(\partial G)$ for every $1 \leq q^* \leq +\infty$ with $1 \leq q^* < (\nu-1)p/(\nu-mp)$. (The case $\nu < mp$ is included in (b).)

(g) For every $0 \leq |\alpha| \leq m-1$, if $\nu \geq (m-|\alpha|)p > 1$, then $\gamma D^{\alpha} x_{k_s} \to \gamma D^{\alpha} x$ strongly in $L_{q^*}(\partial G)$ for every $1 \leq q^* < (\nu-1)p/(\nu-(m-|\alpha|)p)$.

Statements (abcdefg) above hold even for $p = 1$ provided we know the partial derivatives $D^{\alpha} x_k$ of maximal order $|\alpha| = m$ are equiabsolutely integrable in G.

(h) For $p > 1$, in all cases it is true that the sequence $[k_s]$ can be so chosen that $x_{k_s} \to x$, $D^{\alpha} x_{k_s} \to D^{\alpha} x$ strongly in $L_p(G)$ for all $0 \leq |\alpha| \leq m-1$, and $D^{\alpha} x_{k_s} \to D^{\alpha} x$ weakly in $L_p(G)$ for $|\alpha| = m$.

(k) For $p > 1$, in all cases it is true that the sequence $[k_s]$ can be so chosen that $\gamma x_{k_s} \to \gamma x$, $\gamma D^{\alpha} x_{k_s} \to \gamma D^{\alpha} x$ strongly in $L_p(G)$ for all $0 \leq |\alpha| \leq m-1$.

Statements (h), (k) hold even for $p = 1$ provided we know that the partial derivatives $D^{\alpha} x_k$ of maximal order $|\alpha| = m$ are equiabsolutely integrable in G.

VII 12. BIBLIOGRAPHICAL NOTES

The initial idea of what today are called Sobolev functions can be traced in B. Levi [123] (1906), who used them in problems of the calculus of variations and elliptic differential equations. A more systematic theory of functions $x \in W_1^1(G)$ can be seen in G. C. Evans [119] (1920), who used them in potential theory. The absolute continuous functions of L. Tonelli [ ] (1926), or functions A C T, are the functions $x \in W_1^1(G) \cap C(G)$, that is, those Sobolev functions $x \in W_1^1(G)$ which are continuous in G, or in clG. Tonelli used these functions systematically in two dimensional problems of the calculus of variations (free problems) and in surface area [ ] theory. The full use of Sobolev functions in the calculus of variations and partial differential equations started with C. B. Morrey [73bc] (1934-42), and continued with G. Stampacchia [95abcd], and many others. Meanwhile S. L. Sobolev [94ab] discovered the embedding theorems, and his work was soon continued by V. I. Kondrosov [122].

In the presentation above section VII 6 contains essentially the ideas of G. C. Evans and C. B. Morrey. Concerning the embedding theorems we have essentially used the pattern of N. Dunford and J. T. Schwartz [121], proving these theorems first for a half space (section VII 5 above), and then extending to arbitrary open sets (section VII 11) (of Morrey class K here, of class C^∞ in N. Dunford and J. T. Schwartz. Nevertheless we have improved the presentation, eliminating a few oversights, and including the weak compactness theorems for $p = 1$, which have been left out both in S. L. Sobolev as well as in N. Dunford and J. T. Schwartz. The case $p = 1$ is particularly relevant in the calculus of variations(Tonelli's existence theorems, and some of the present extensions to Lagrange problems.)